4 vs 7 sparse undirected unweighted Diameter is SETH-hard at time $n^{4/3}$

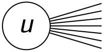
Édouard Bonnet

ENS Lyon, LIP

July 2021, ICALP

Diameter

diam(G) = largest distance between a pair of vertices of G

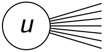


$\text{longest}_{u,v}$ shortestPath(u, v)?

- In weighted graphs, no better known than APSP
- In unweighted graphs, solvable in $\tilde{O}(n^{\omega})$

Diameter

diam(G) = largest distance between a pair of vertices of G



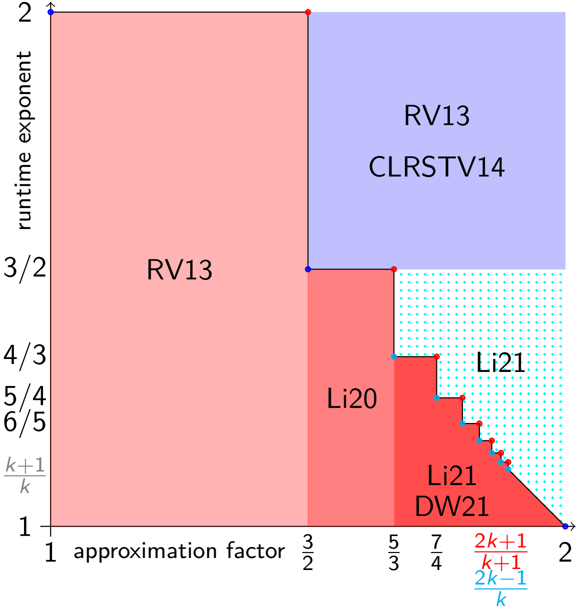
$\text{longest}_{u,v}\ \text{shortestPath}(u, v)$?

- In weighted graphs, no better known than APSP
- In unweighted graphs, solvable in $\tilde{O}(n^\omega)$

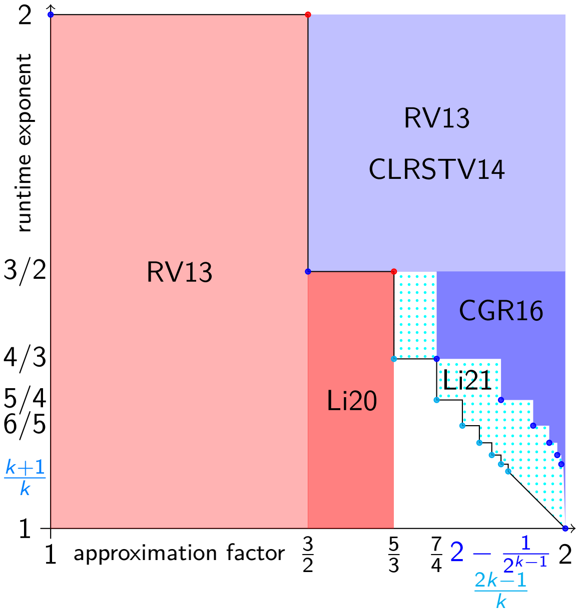
Our scope:

sparse graphs, with $\tilde{O}(n)$ edges
Time vs Approximation trade-offs

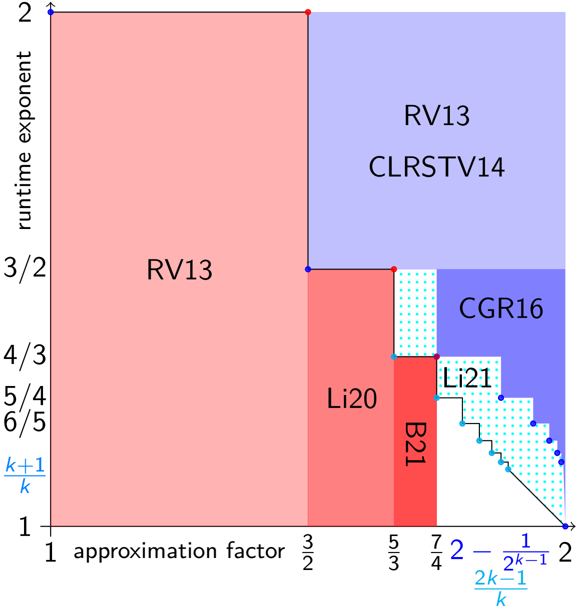
Current situation for *directed* unweighted DIAMETER



Undirected unweighted DIAMETER



Undirected unweighted DIAMETER



SETH

$\forall k, \exists \varepsilon > 0$, no classical algorithm solves n-var k-SAT in $(2 - \varepsilon)^n$

In 1999, Impagliazzo and Paturi introduce ETH and mention a stronger version of it in their conclusion

$$\text{SETH} \Rightarrow \text{ETH} \Rightarrow \text{P} \neq \text{NP}$$

- ETH and SETH are then mainly used for NP-hard problems

SETH

$\forall k, \exists \varepsilon > 0$, no classical algorithm solves n-var k-SAT in $(2-\varepsilon)^n$

In 1999, Impagliazzo and Paturi introduce ETH and mention a stronger version of it in their conclusion

$$\text{SETH} \Rightarrow \text{ETH} \Rightarrow \text{P} \neq \text{NP}$$

- ETH and SETH are then mainly used for NP-hard problems
- In 2005, SETH is used for the first time for a problem in P

ORTHOGONAL VECTORS,

SETH

$\forall k, \exists \varepsilon > 0$, no classical algorithm solves n-var k-SAT in $(2-\varepsilon)^n$

In 1999, Impagliazzo and Paturi introduce ETH and mention a stronger version of it in their conclusion

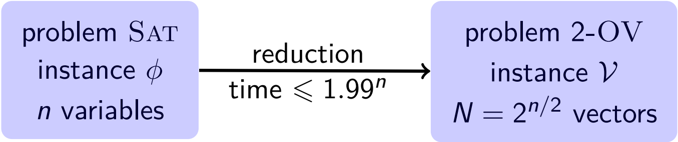
$$\text{SETH} \Rightarrow \text{ETH} \Rightarrow \text{P} \neq \text{NP}$$

- ETH and SETH are then mainly used for NP-hard problems
- In 2005, SETH is used for the first time for a problem in P
- 2014-, dozens of papers show SETH-hardness of problems in P

Orthogonal Vectors, Diameter, Fréchet Distance, Edit Distance, Longest Common Subsequence, Furthest Pair, dynamic problems, problems from Machine Learning, Model Checking, Language Theory etc.

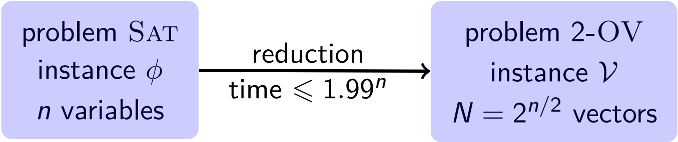
Reduction from SAT to a problem in P

2-ORTHOGONAL VECTORS (2-OV):
Are there two orthogonal vectors in a given set of N $0, 1$-vectors?



Reduction from Sat to a problem in P

2-Orthogonal Vectors (2-OV):
Are there two orthogonal vectors in a given set of N $0, 1$-vectors?



$\rightarrow$ Solving 2-OV in $N^{1.99}$ solves Sat $1.99^n + 2^{\frac{1.99n}{2}}$, refuting SETH

Sat → 2-Orthogonal Vectors [W05]

arbitrary equipartition of X: $x_1, x_2, \ldots, x_{\frac{n}{2}}, x_{\frac{n}{2}+1}, x_{\frac{n}{2}+2}, \ldots, x_n$

Find an assignment

- A of the red variables and
- B of the blue variables

such that all the clauses are satisfied by A or by B

SAT → 2-ORTHOGONAL VECTORS [W05]

arbitrary equipartition of X: $x_1, x_2, \ldots, x_{\frac{n}{2}}$, $x_{\frac{n}{2}+1}, x_{\frac{n}{2}+2}, \ldots, x_n$

Find an assignment

- A of the red variables and
- B of the blue variables

such that all the clauses are satisfied by A or by B

	R	B	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1										
A_2										
A_3										
A_4										
B_1										
B_2										
B_3										
B_4										

SAT → 2-ORTHOGONAL VECTORS [W05]

arbitrary equipartition of X: $x_1, x_2, \ldots, x_{\frac{n}{2}}$, $x_{\frac{n}{2}+1}, x_{\frac{n}{2}+2}, \ldots, x_n$

Find an assignment

- A of the red variables and
- B of the blue variables

such that all the clauses are satisfied by A or by B

	R	B	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	1	0								
A_2										
A_3										
A_4										
B_1										
B_2										
B_3										
B_4										

A_1 assigns red variables

Sat → 2-Orthogonal Vectors [W05]

arbitrary equipartition of X: $x_1, x_2, \ldots, x_{\frac{n}{2}}, x_{\frac{n}{2}+1}, x_{\frac{n}{2}+2}, \ldots, x_n$

Find an assignment

- A of the red variables and
- B of the blue variables

such that all the clauses are satisfied by A or by B

	R	B	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	1	0	1							
A_2										
A_3										
A_4										
B_1										
B_2										
B_3										
B_4										

A_1 does *not* satisfy C_1

Sat → 2-Orthogonal Vectors [W05]

arbitrary equipartition of X: $x_1, x_2, \ldots, x_{\frac{n}{2}}$, $x_{\frac{n}{2}+1}, x_{\frac{n}{2}+2}, \ldots, x_n$

Find an assignment

- A of the red variables and
- B of the blue variables

such that all the clauses are satisfied by A or by B

	R	B	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	1	0	1	0						
A_2										
A_3										
A_4										
B_1										
B_2										
B_3										
B_4										

A_1 satisfies C_2

SAT → 2-ORTHOGONAL VECTORS [W05]

arbitrary equipartition of X: $x_1, x_2, \ldots, x_{\frac{n}{2}}$, $x_{\frac{n}{2}+1}, x_{\frac{n}{2}+2}, \ldots, x_n$

Find an assignment

- A of the red variables and
- B of the blue variables

such that all the clauses are satisfied by A or by B

	R	B	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	1	0	1	0	0	1	0	0	1	0
A_2										
A_3										
A_4										
B_1										
B_2										
B_3										
B_4										

first vector $(1, 0, 1, 0, 0, 1, 1, 0, 1, 0)$

SAT → 2-ORTHOGONAL VECTORS [W05]

arbitrary equipartition of X: $x_1, x_2, \ldots, x_{\frac{n}{2}}$, $x_{\frac{n}{2}+1}, x_{\frac{n}{2}+2}, \ldots, x_n$

Find an assignment

- A of the red variables and
- B of the blue variables

such that all the clauses are satisfied by A or by B

	R	B	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	1	0	1	0	0	1	0	0	1	0
A_2	1	0	0	0	0	1	1	1	0	1
A_3	1	0	0	1	0	1	0	0	1	1
A_4	1	0	0	0	1	1	0	1	1	1
B_1	0	1	1	1	0	0	1	1	1	0
B_2	0	1	0	1	0	1	0	1	0	0
B_3	0	1	1	1	1	1	0	0	0	1
B_4	0	1	0	1	0	0	1	0	0	1

SAT → 2-ORTHOGONAL VECTORS [W05]

arbitrary equipartition of X: $x_1, x_2, \ldots, x_{\frac{n}{2}}$, $x_{\frac{n}{2}+1}, x_{\frac{n}{2}+2}, \ldots, x_n$

Find an assignment

- A of the red variables and
- B of the blue variables

such that all the clauses are satisfied by A or by B

	R	B	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	1	**0**	1	**0**	**0**	1	**0**	**0**	1	**0**
A_2	1	0	0	0	0	1	1	1	0	1
A_3	1	0	0	1	0	1	0	0	1	1
A_4	1	0	0	0	1	1	0	1	1	1
B_1	0	1	1	1	0	0	1	1	1	0
B_2	0	1	0	1	0	1	0	1	0	0
B_3	0	1	1	1	1	1	0	0	0	1
B_4	**0**	1	**0**	1	**0**	**0**	1	**0**	**0**	1

Consequence for 2-OV

From a SAT-instance on n variables and m clauses, we created $N := 2^{\frac{n}{2}+1}$ vectors in dimension $d := m + 2$

Consequence for 2-OV

From a SAT-instance on n variables and m clauses, we created $N := 2^{\frac{n}{2}+1}$ vectors in dimension $d := m + 2$

An algorithm solving 2-OV in time $2^{o(d)} N^{2-\varepsilon}$ would solve SAT in $2^{o(m)} 2^{n(1-\varepsilon/2)} \rightarrow$ breaking SETH

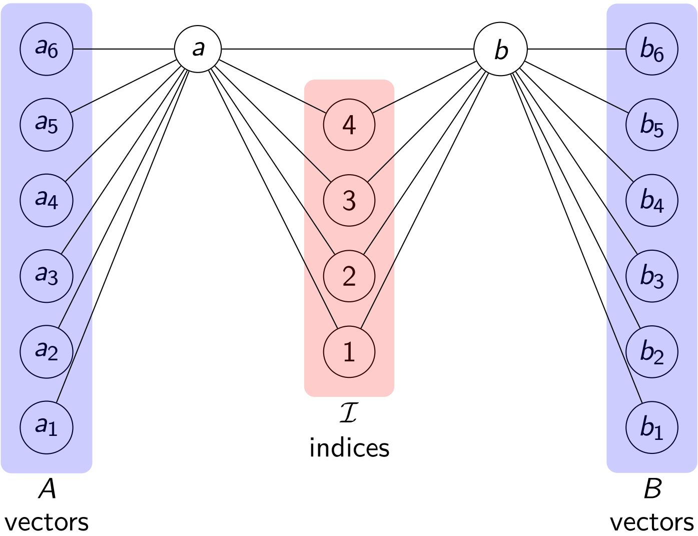
Consequence for 2-OV

From a $\textsc{Sat}$-instance on n variables and m clauses, we created $N := 2^{\frac{n}{2}+1}$ vectors in dimension $d := m + 2$

An algorithm solving 2-OV in time $2^{o(d)} N^{2-\varepsilon}$
would solve $\textsc{Sat}$ in $2^{o(m)} 2^{n(1-\varepsilon/2)}$ $\rightarrow$ breaking SETH

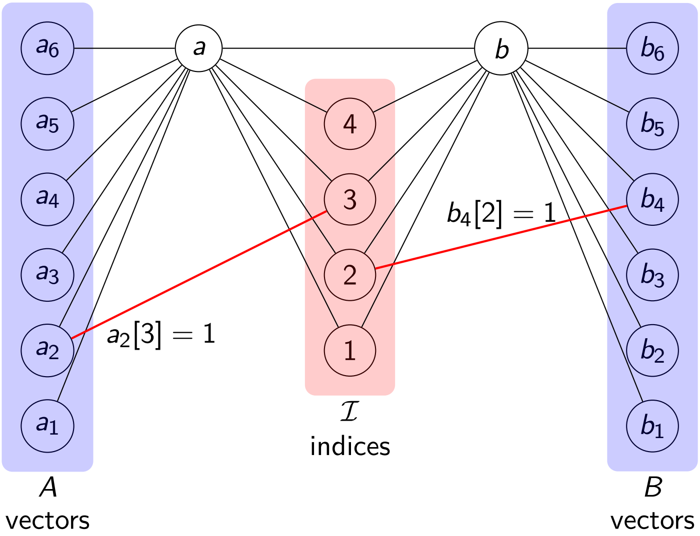
Most useful consequence here:
$N^{2-o(1)}$-time is required even if $d = \log^{O(1)} N$
Same for k-OV and $N^{k-o(1)}$-time

2-Orthogonal Vectors → Diameter [RV13]



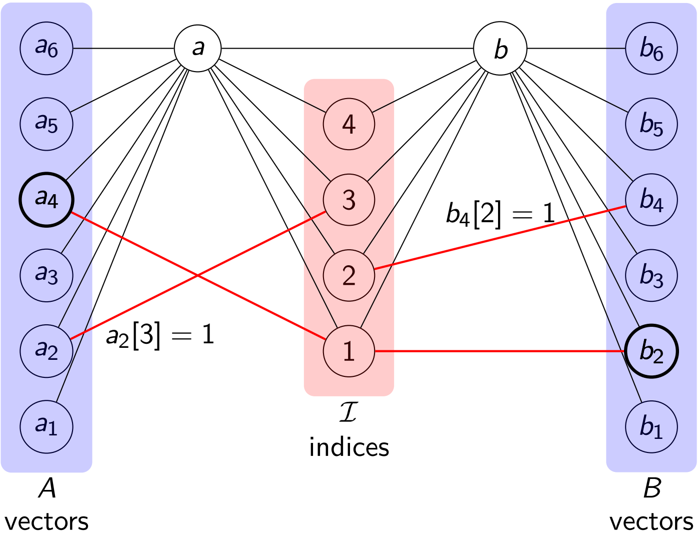
So far, all the pairs but of $A \times B$ are at distance $\leqslant 2$

2-Orthogonal Vectors → Diameter [RV13]



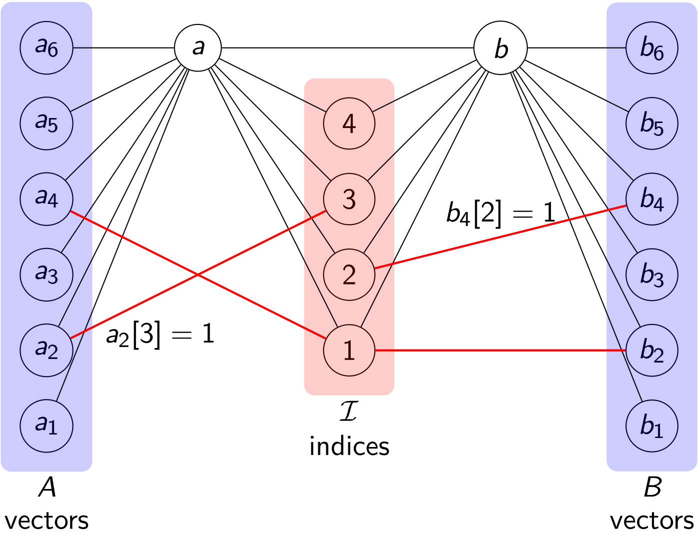
we put an edge between vector v and index i iff $v[i] = 1$

2-Orthogonal Vectors → Diameter [RV13]



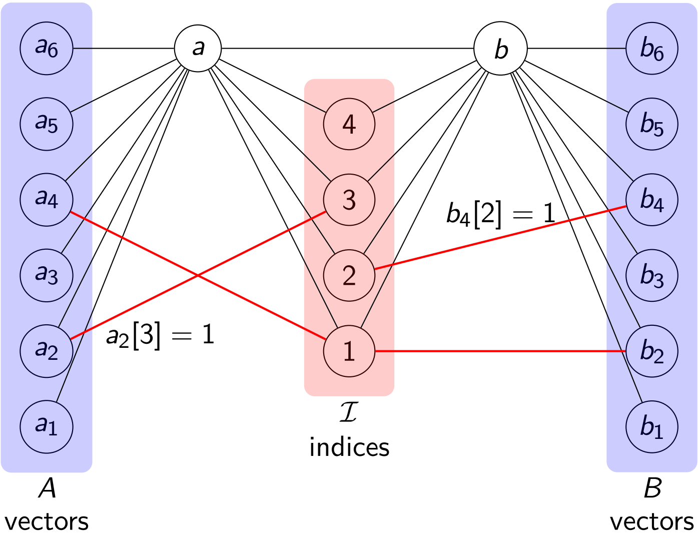
A pair (a4,b2) is at distance 2 $\Leftrightarrow \langle a_4, b_2 \rangle \neq 0$

2-Orthogonal Vectors → Diameter [RV13]



$\mathrm{diam}(G) = 3 \Leftrightarrow \exists(a_i, b_j)$ at distance 3 $\Leftrightarrow$ orthogonal pair

2-Orthogonal Vectors → Diameter [RV13]



If no orthogonal pair, $\text{diam}(G) = 2$

Our result

Theorem

Approximating sparse undirected unweighted DIAMETER *within factor better than* $\frac{7}{4}$ *requires time* $n^{\frac{4}{3}-o(1)}$*, unless SETH fails.*

Our result

Theorem

Approximating sparse undirected unweighted DIAMETER *within factor better than* $\frac{7}{4}$ *requires time* $n^{\frac{4}{3}-o(1)}$*, unless SETH fails.*

Plan: hardness of 4 vs 7 DIAMETER from N-vector 4-OV to $O(N^3)$-vertex $\tilde{O}(N^3)$-edge DIAMETER-instances.

Our result

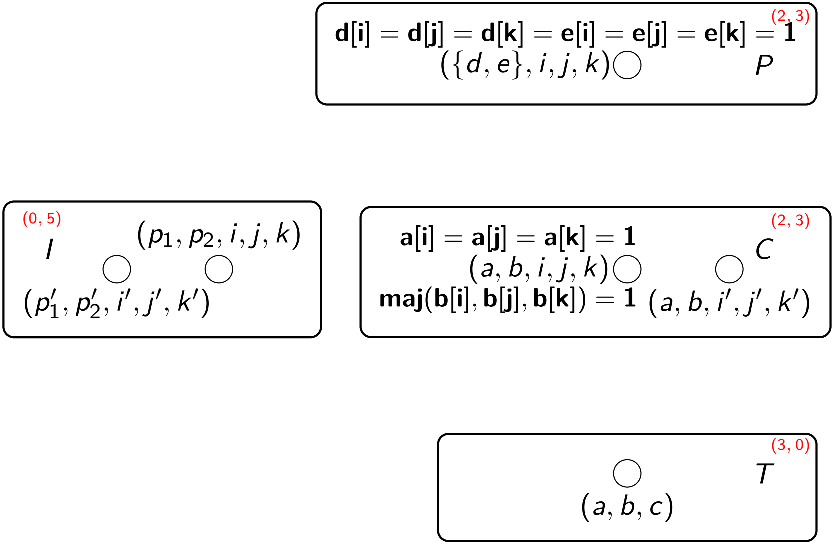
Theorem

Approximating sparse undirected unweighted DIAMETER *within factor better than* $\frac{7}{4}$ *requires time* $n^{\frac{4}{3}-o(1)}$*, unless SETH fails.*

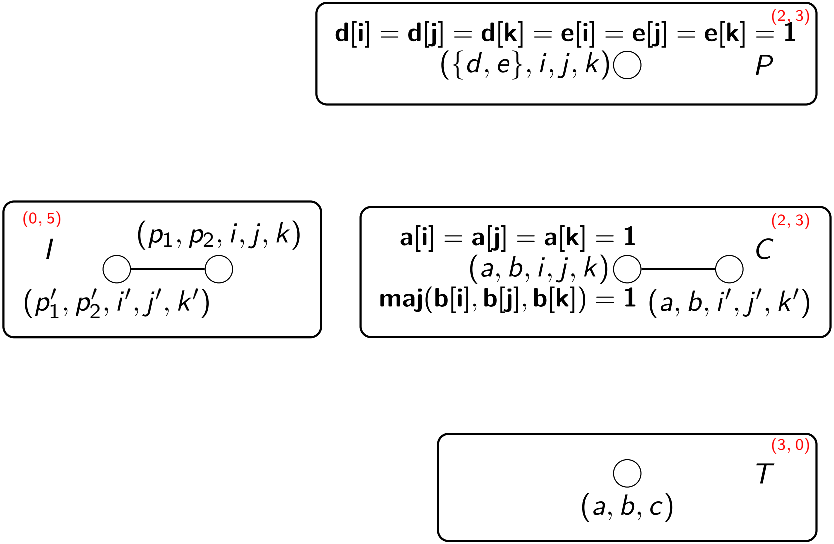
Plan: hardness of 4 vs 7 DIAMETER from N-vector 4-OV to $O(N^3)$-vertex $\tilde{O}(N^3)$-edge DIAMETER-instances.

2 vs 3 hardness from 2-OV [RV13]
3 vs 5 hardness from 3-OV [Li20]

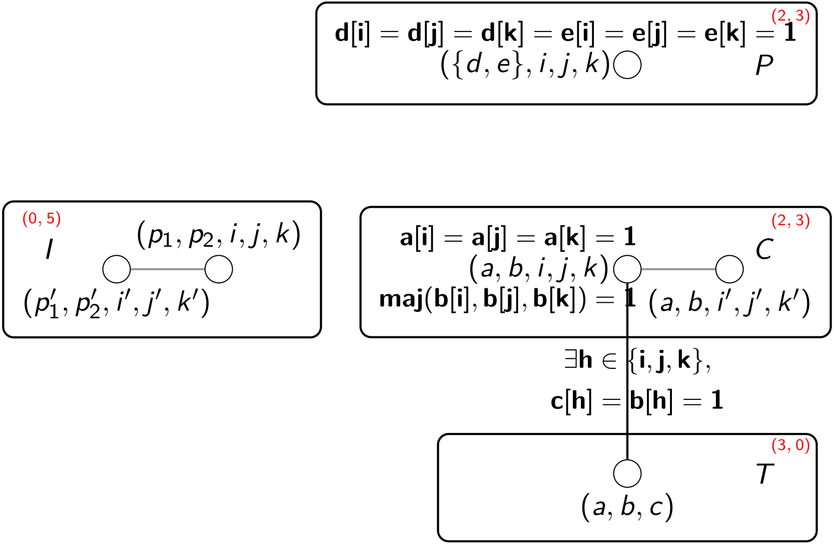
Construction *with weights*



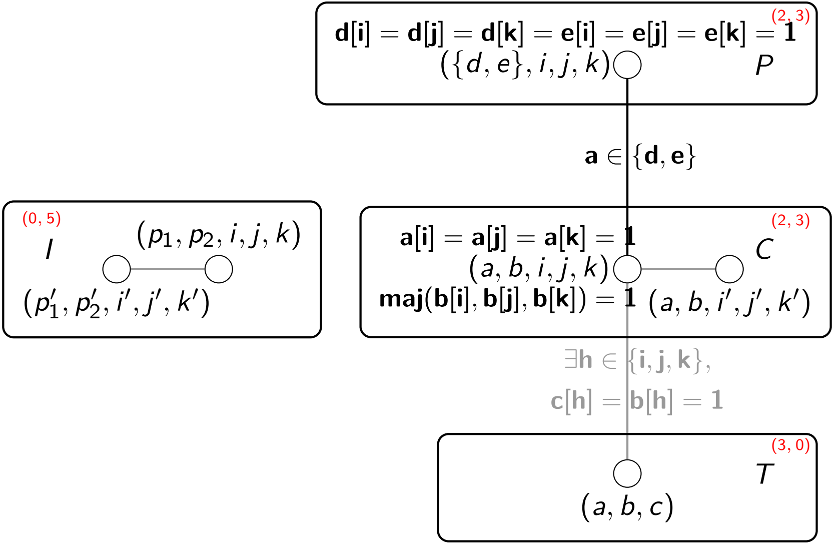
Construction *with weights*



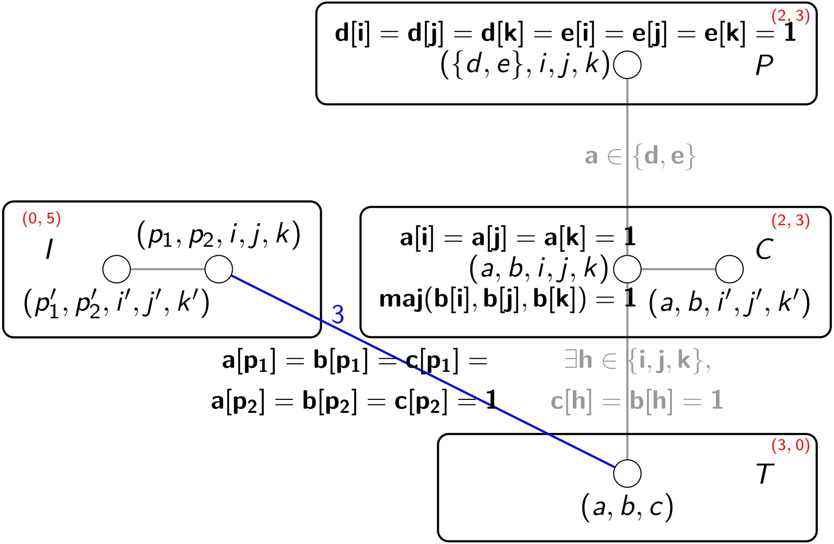
Construction *with weights*



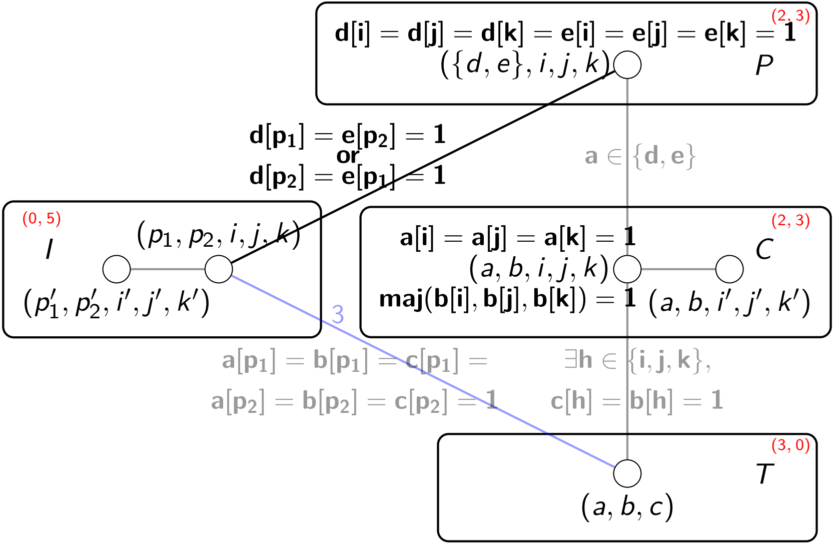
Construction *with weights*



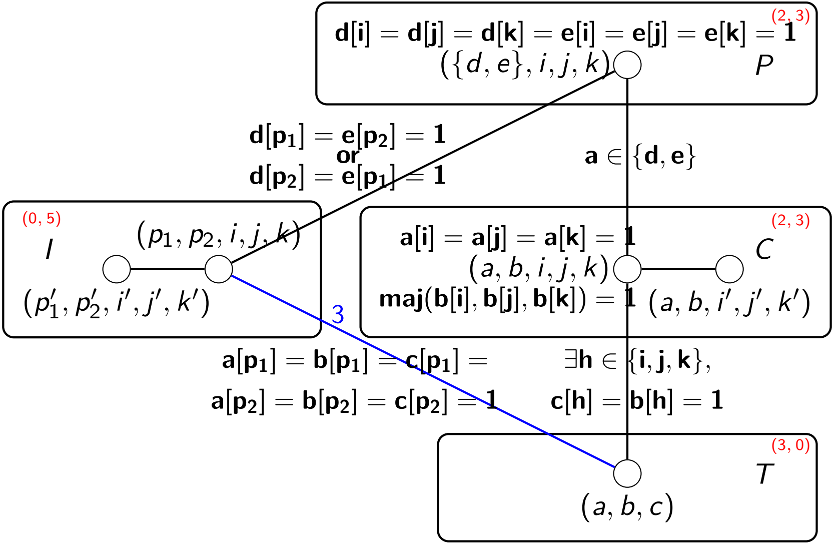
Construction *with weights*



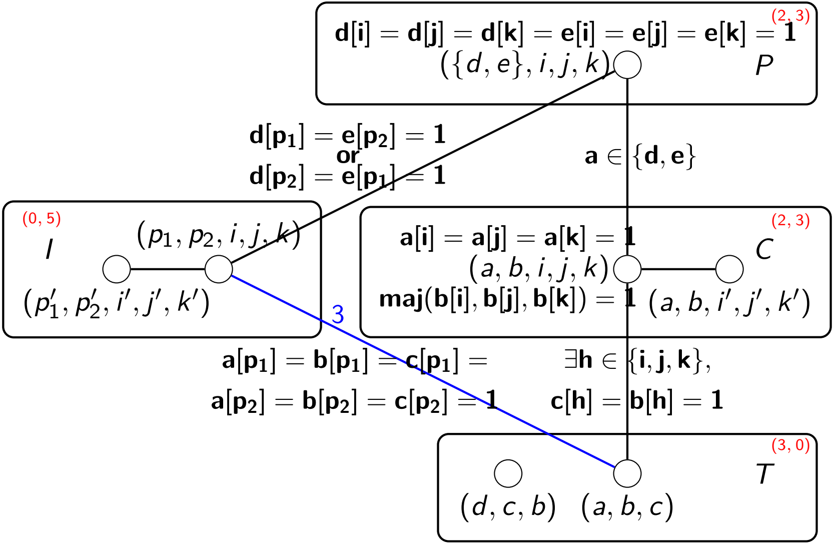
Construction *with weights*



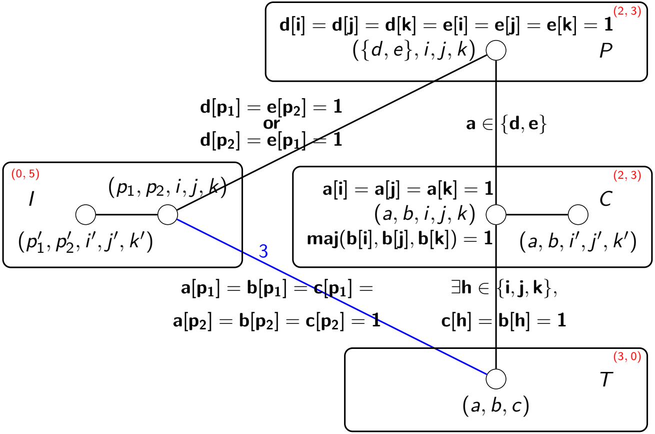
Construction *with weights*



Construction *with weights*

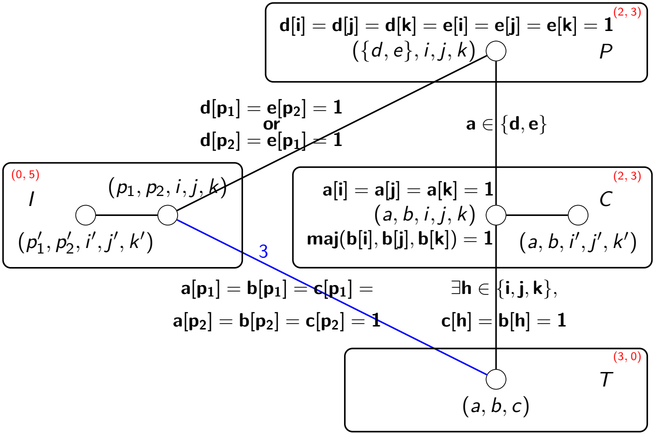


No orthogonal quadruple ⇒ diameter at most 4



Automatic paths of length at most 4, except for T-T, T-C, T-P, and C-C

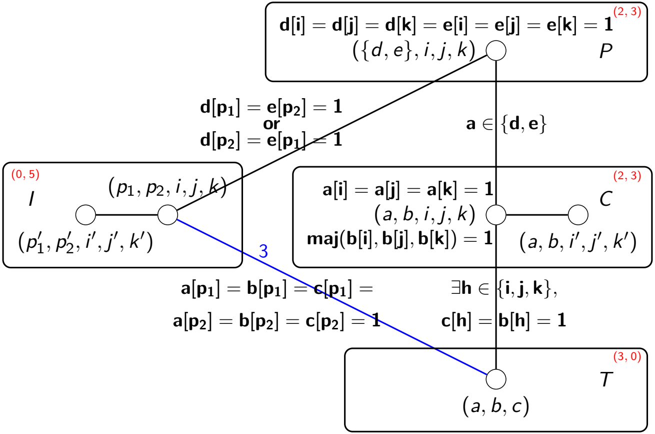
No orthogonal quadruple ⇒ diameter at most 4



T-T, T-C, C-C: (a, b, c) or $(a, b, i', j', k') - (a, b, i, j, k) - (\{a, d\}, i, j, k) - (d, e, i, j, k) - (d, e, f)$ or (d, e, i'', j'', k'')

with $i = \text{ind}(a, b, c, d)$, $j = \text{ind}(a, b, d, e)$, $k = \text{ind}(a, d, e, f)$

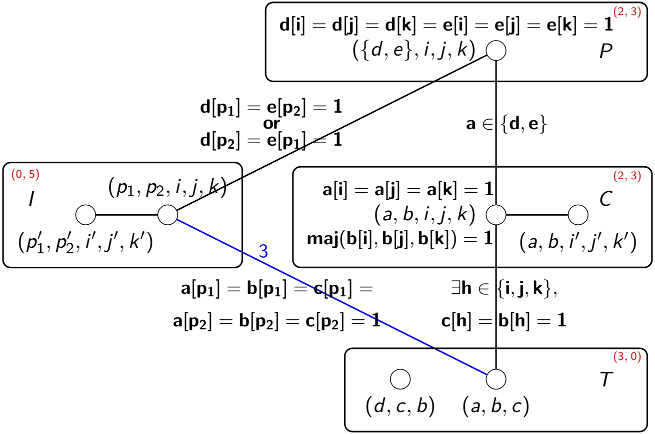
No orthogonal quadruple ⇒ diameter at most 4



T-P: $(a, b, c) - (p_1, p_2, i, j, k) - (\{d, e\}, i, j, k)$

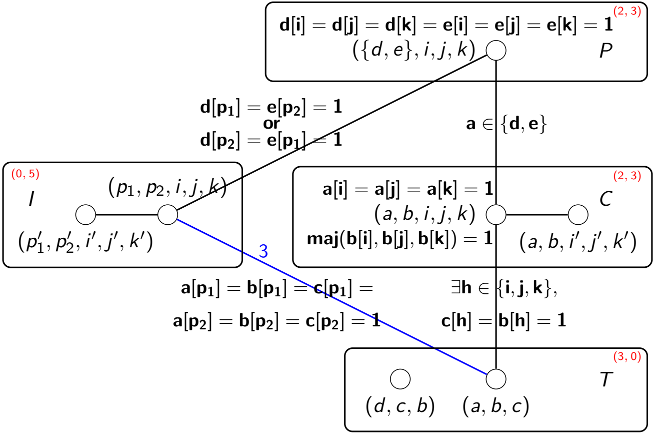
with $p_1 = \text{ind}(a, b, c, d)$, $p_2 = \text{ind}(a, b, c, e)$

a, b, c, d orthogonal $\Rightarrow d((a, b, c), (d, c, b)) \geqslant 7$



Set I cannot help for a path of length 6

a, b, c, d orthogonal $\Rightarrow d((a,b,c),(d,c,b)) \geqslant 7$



(a, b, i, j, k) and (d, c, i, j, k) have to be part of the path

Removing the weights