Inapproximability of Diameter in super-linear time: Beyond the 5/3 ratio

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SETH

$\forall k, \exists \varepsilon > 0$, no classical algorithm solves n-var k-SAT in $(2-\varepsilon)^n$

In 1999, Impagliazzo and Paturi introduce ETH and mention a stronger version of it in their conclusion

$$\text{SETH} \Rightarrow \text{ETH} \Rightarrow \text{P} \neq \text{NP}$$

- ETH and SETH are then mainly used for NP-hard problems

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ORTHOGONAL VECTORS,

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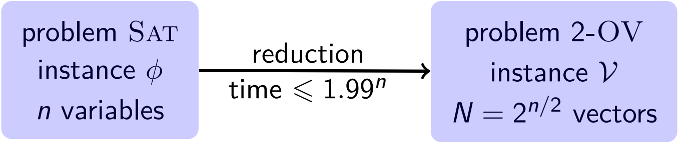
- In 2005, SETH is used for the first time for a problem in P
- 2014-, dozens of papers show SETH-hardness of problems in P

Orthogonal Vectors, Diameter, Fréchet Distance, Edit Distance, Longest Common Subsequence, Furthest Pair, dynamic problems, problems from Machine Learning, Model Checking, Language Theory etc.

Reduction from SAT to a problem in P

2-ORTHOGONAL VECTORS (2-OV):
Are there two orthogonal vectors in a given set of N $0,1$-vectors?



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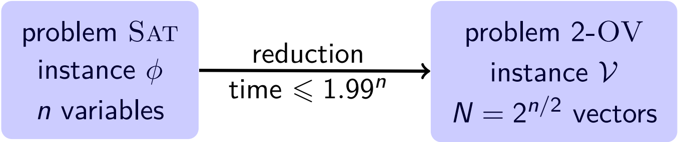

$\rightarrow$ Solving 2-OV in $N^{1.99}$ solves SAT $1.99^n + 2^{\frac{1.99n}{2}}$, refuting SETH

SAT → 2-ORTHOGONAL VECTORS [W05]

arbitrary equipartition of X: $x_1, x_2, \ldots, x_{\frac{n}{2}}$, $x_{\frac{n}{2}+1}, x_{\frac{n}{2}+2}, \ldots, x_n$

Find an assignment

- A of the red variables and
- B of the blue variables

such that all the clauses are satisfied by A or by B

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	R	B	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1										
A_2										
A_3										
A_4										
B_1										
B_2										
B_3										
B_4										

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	R	B	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	1	0								
A_2										
A_3										
A_4										
B_1										
B_2										
B_3										
B_4										

A_1 assigns red variables

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A_1	1	0	1							
A_2										
A_3										
A_4										
B_1										
B_2										
B_3										
B_4										

A_1 does *not* satisfy C_1

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A_1	1	0	1	0						
A_2										
A_3										
A_4										
B_1										
B_2										
B_3										
B_4										

A_1 satisfies C_2

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	R	B	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	1	0	1	0	0	1	0	0	1	0
A_2										
A_3										
A_4										
B_1										
B_2										
B_3										
B_4										

first vector $(1, 0, 1, 0, 0, 1, 1, 0, 1, 0)$

Sat → 2-Orthogonal Vectors [W05]

arbitrary equipartition of X: $x_1, x_2, \ldots, x_{\frac{n}{2}}, x_{\frac{n}{2}+1}, x_{\frac{n}{2}+2}, \ldots, x_n$

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	R	B	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	1	0	1	0	0	1	0	0	1	0
A_2	1	0	0	0	0	1	1	1	0	1
A_3	1	0	0	1	0	1	0	0	1	1
A_4	1	0	0	0	1	1	0	1	1	1
B_1	0	1	1	1	0	0	1	1	1	0
B_2	0	1	0	1	0	1	0	1	0	0
B_3	0	1	1	1	1	1	0	0	0	1
B_4	0	1	0	1	0	0	1	0	0	1

SAT → 2-ORTHOGONAL VECTORS [W05]

arbitrary equipartition of X: $x_1, x_2, \ldots, x_{\frac{n}{2}}$, $x_{\frac{n}{2}+1}, x_{\frac{n}{2}+2}, \ldots, x_n$

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	R	B	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	1	**0**	1	**0**	**0**	1	**0**	**0**	1	**0**
A_2	1	0	0	0	0	1	1	1	0	1
A_3	1	0	0	1	0	1	0	0	1	1
A_4	1	0	0	0	1	1	0	1	1	1
B_1	0	1	1	1	0	0	1	1	1	0
B_2	0	1	0	1	0	1	0	1	0	0
B_3	0	1	1	1	1	1	0	0	0	1
B_4	**0**	1	**0**	1	**0**	**0**	1	**0**	**0**	1

Consequence for 2-OV

From a Sat-instance on n variables and m clauses, we created $N := 2^{\frac{n}{2}+1}$ vectors in dimension $d := m + 2$

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An algorithm solving 2-OV in time $2^{o(d)} N^{2-\varepsilon}$
would solve SAT in $2^{o(m)} 2^{n(1-\varepsilon/2)} \rightarrow$ breaking SETH

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would solve Sat in $2^{o(m)} 2^{n(1-\varepsilon/2)} \rightarrow$ breaking SETH

Most useful consequence here:
$N^{2-o(1)}$-time is required **even if** $\mathbf{d} = \log^{\mathbf{O(1)}} \mathbf{N}$
Same for k**-OV and** $\mathbf{N^{k-o(1)}}$**-time**

Diameter

diam(G) = largest distance between a pair of vertices of G

- In weighted graphs, no better known than APSP in $O(n^3)$
- In unweighted graphs, solvable in $\tilde{O}(n^\omega)$
- In unweighted sparse ($m = \Theta(n)$) graphs, solvable in $O(n^2)$
- $\frac{3}{2}$-approximable in $\tilde{O}(m^{1.5})$
- In sparse graphs, $\frac{3}{2}$-approximable in $\tilde{O}(n^{1.5})$

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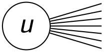
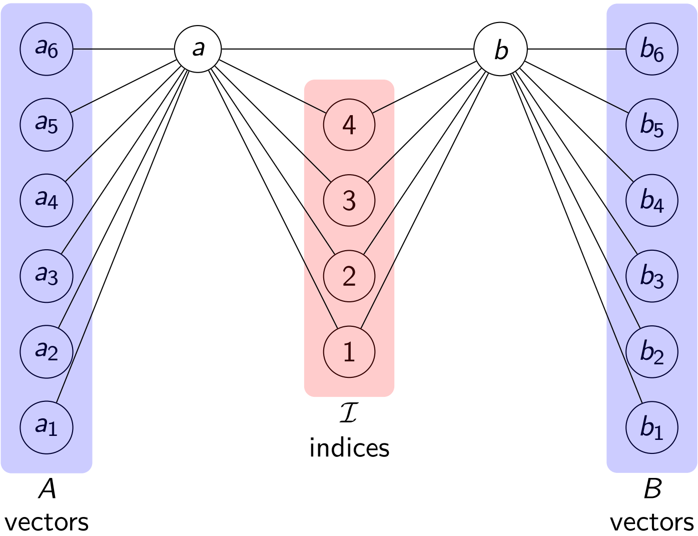


longest$_{u,v}$ shortestPath(u, v)?

- In weighted graphs, no better known than APSP in $O(n^3)$
- In unweighted graphs, solvable in $\tilde{O}(n^\omega)$
- **In unweighted sparse graphs, solvable in $\mathbf{O(n^2)}$**
- $\frac{3}{2}$-approximable in $\tilde{O}(m^{1.5})$
- **In sparse graphs, $\frac{3}{2}$-approximable in $\tilde{\mathbf{O}}(\mathbf{n}^{1.5})$**

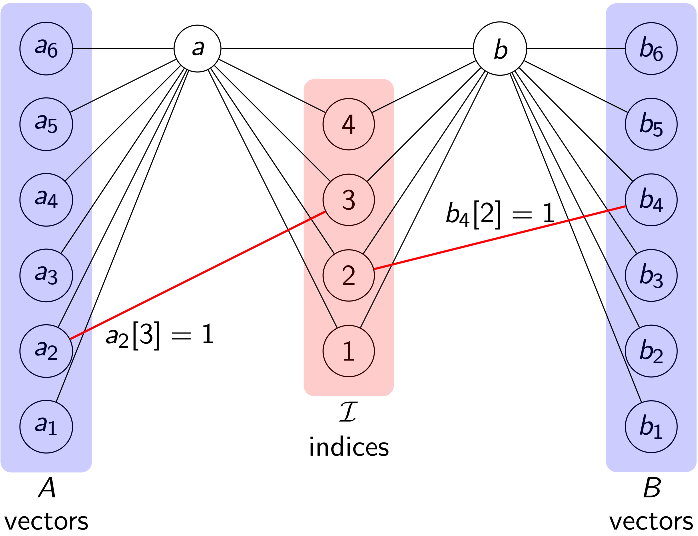
Linear reduction from Orthogonal Vectors:
no $\mathbf{n}^{1.99}$ algorithm even to $(\frac{3}{2} - \varepsilon)$-approximate Diameter
on unweighted sparse instances, assuming the SETH.

2-Orthogonal Vectors → Diameter [RV13]



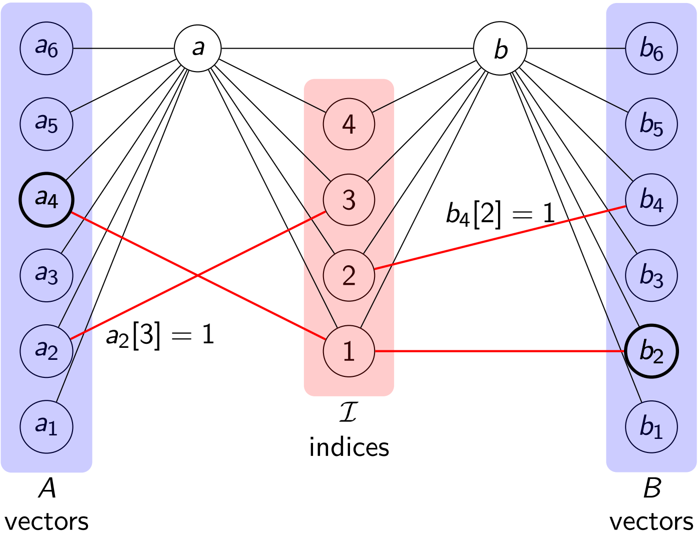
So far, all the pairs but of $A \times B$ are at distance $\leqslant 2$

2-Orthogonal Vectors → Diameter [RV13]



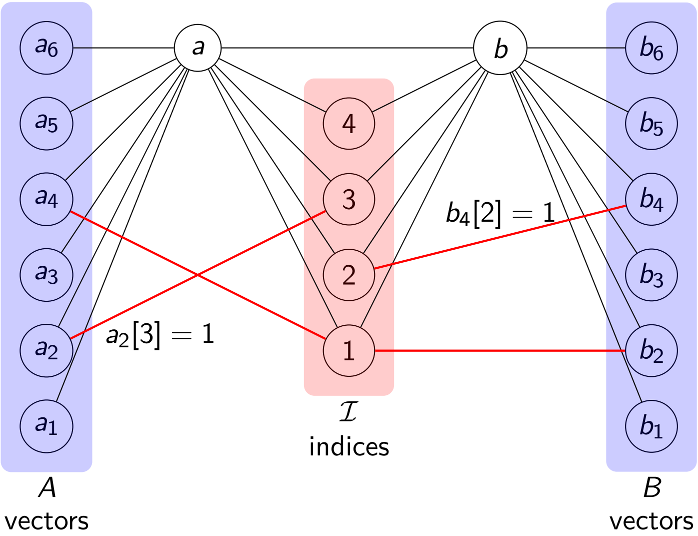
we put an edge between vector v and index i iff $v[i] = 1$

2-Orthogonal Vectors → Diameter [RV13]



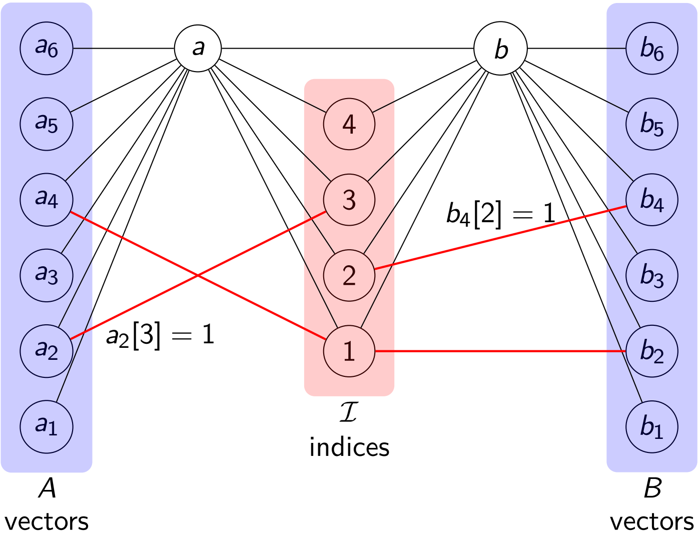
A pair (a4,b2) is at distance 2 $\Leftrightarrow \langle a_4, b_2 \rangle \neq 0$

2-Orthogonal Vectors → Diameter [RV13]



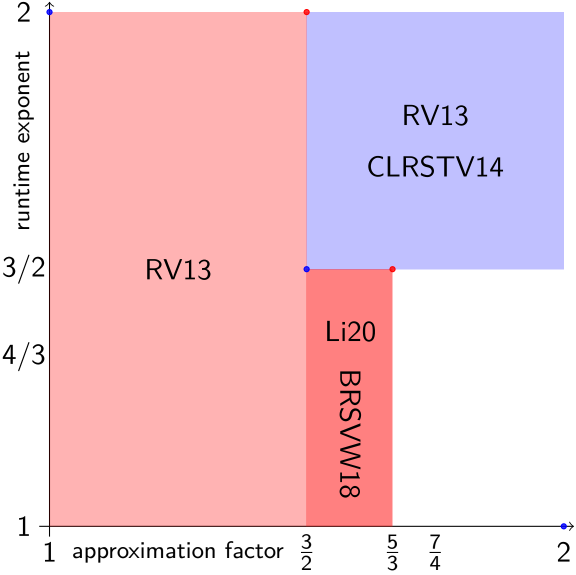
$\text{diam}(G) = 3 \Leftrightarrow \exists(a_i, b_j)$ at distance 3 $\Leftrightarrow$ orthogonal pair

2-Orthogonal Vectors → Diameter [RV13]

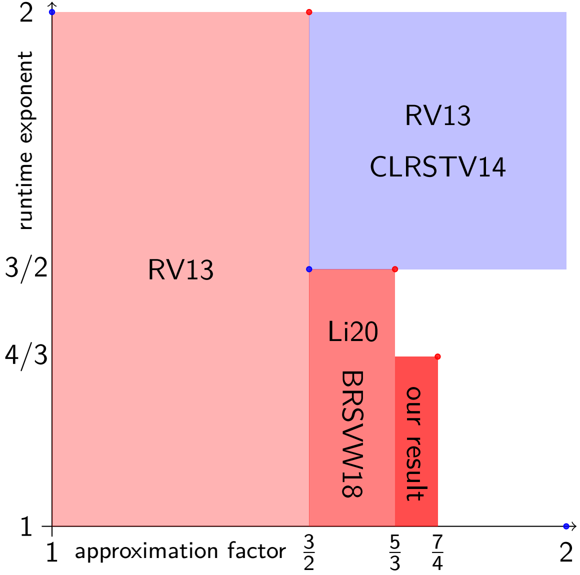


If no orthogonal pair, $\mathrm{diam}(G) = 2$

Time-approximation trade-offs of directed weighted DIAMETER



Time-approximation trade-offs of directed weighted Diameter



Our result

Theorem

Approximating weighted directed DIAMETER *within factor better than* $\frac{7}{4}$ *requires time* $n^{\frac{4}{3}-o(1)}$*, unless SETH fails.*

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Plan: hardness of 4 vs 7 DIAMETER from N-vector 4-OV to $O(N^3)$-vertex $\tilde{O}(N^3)$-arc DIAMETER-instances.

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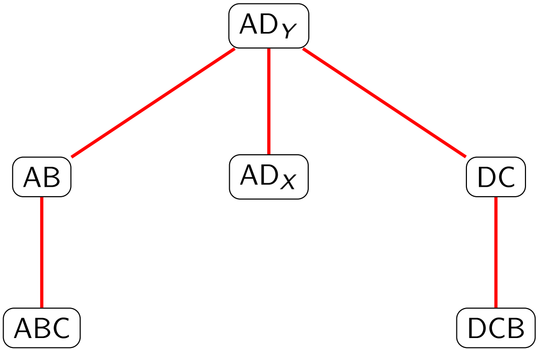
Plan: hardness of 4 vs 7 DIAMETER from N-vector 4-OV to $O(N^3)$-vertex $\tilde{O}(N^3)$-arc DIAMETER-instances.

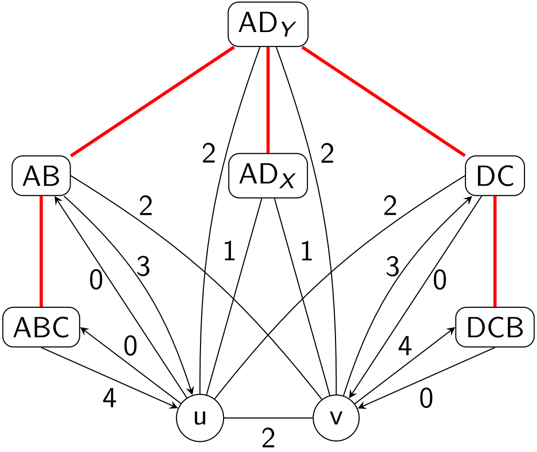
2 vs 3 hardness from 2-OV [RV13]
3 vs 5 hardness from 3-OV [Li20]

Global structure



Will define the six sets $ABC, AB, AD_Y, AD_X, DC, DCB$ later
and the variable edge sets

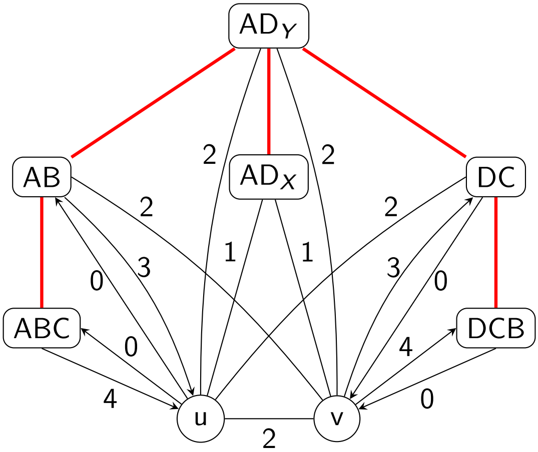
Global structure



(1) Yes(4-OV) $\Rightarrow \exists (u, v) \in$ ABC $\times$ DCB with $d(u, v) = 7$

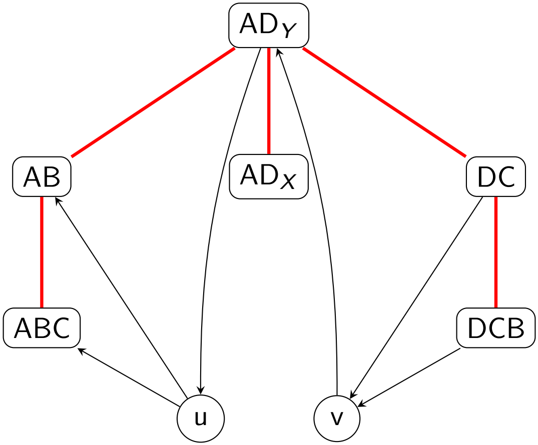
(2) No(4-OV) $\Rightarrow \forall u, v,\ \ d(u, v) \leqslant 4$

Global structure



(2) is satisfied except for $ABC \times AD_Y$, $ABC \times DC$, $ABC \times DCB$, $AB \times DC$, $AB \times DCB$, $AD_Y \times DCB$

Global structure



Dalirrooyfard and Wein observed that these unweighted arcs suffice

Vertex sets

$S = A = B = C = D$ is the input set of $0, 1$-vectors in $\{0, 1\}^\ell$

Vertices:

- $(a, b, c) \in \mathrm{ABC}$ for every $(a, b, c) \in A \times B \times C$
- $(d, c, b) \in \mathrm{DCB}$ for every $(d, c, b) \in D \times C \times B$

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- $(a, b, i, j, k) \in \text{AB}$ for every $(a, b) \in A \times B$, $i, j, k \in [\ell]$ and $a[i] = a[j] = a[k] = 1$ and $\text{maj}(b[i], b[j], b[k]) = 1$
- same for DC

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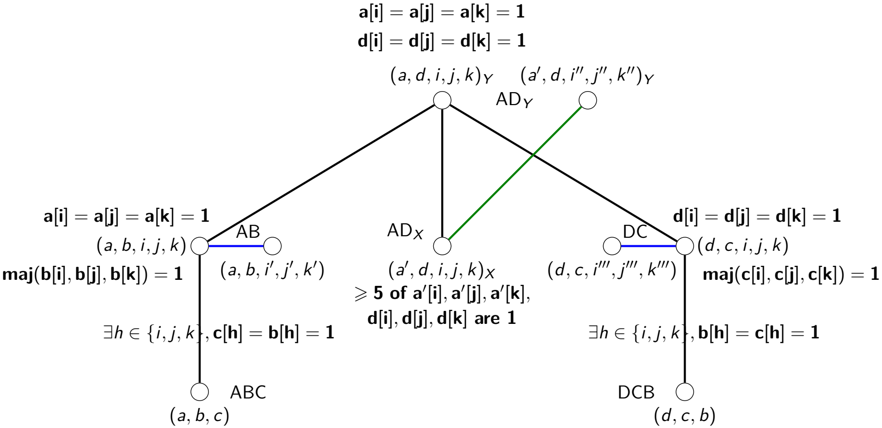
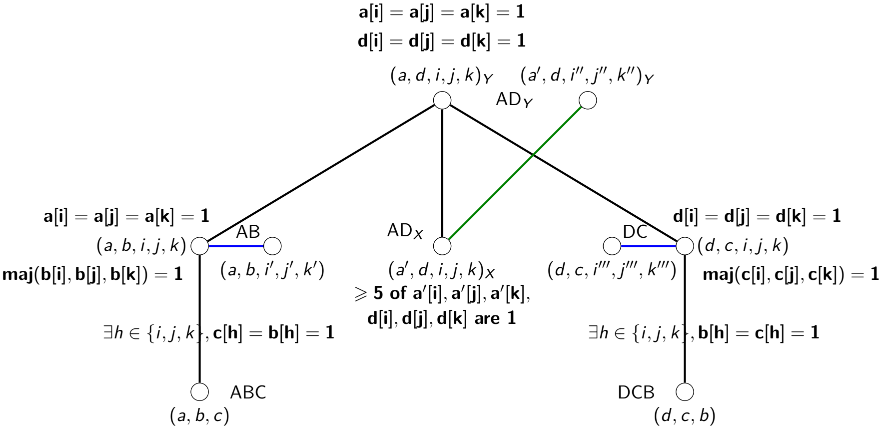
- same for DC
- $(a, d, i, j, k)_Y \in \text{AD}_Y$ for every $(a, b) \in A \times D$, $i, j, k \in [\ell]$ and $a[i] = a[j] = a[k] = d[i] = d[j] = d[k] = 1$
- $(a, d, i, j, k)_X \in \text{AD}_X$ for every $(a, b) \in A \times D$, $i, j, k \in [\ell]$ and $a[i], a[j], a[k], d[i], d[j], d[k]$ equal 1 but at most 1.

Edge sets



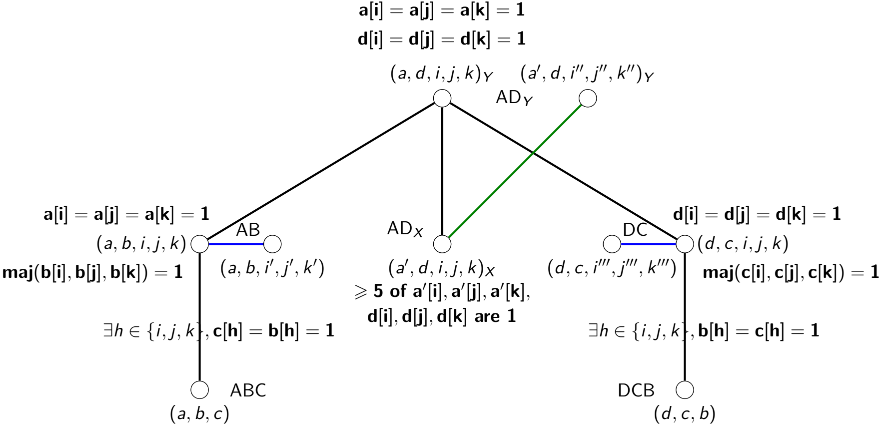
Regular edges,

Edge sets



Regular edges, index-switching edges,

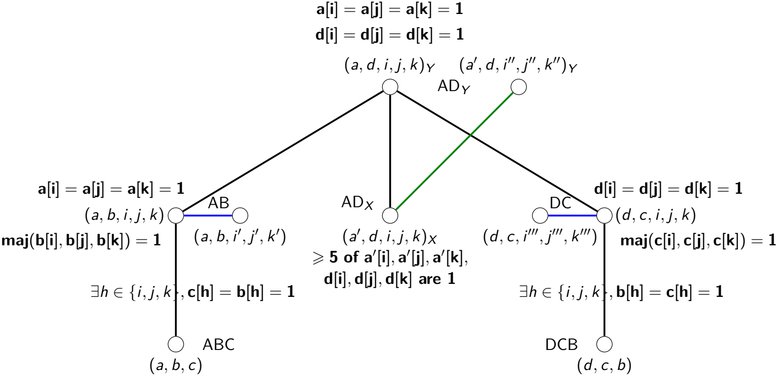
Edge sets



Regular edges, index-switching edges, skew edges

No orthogonal quadruple in $S \Rightarrow \forall u, v,\ d(u,v) \leqslant 4$

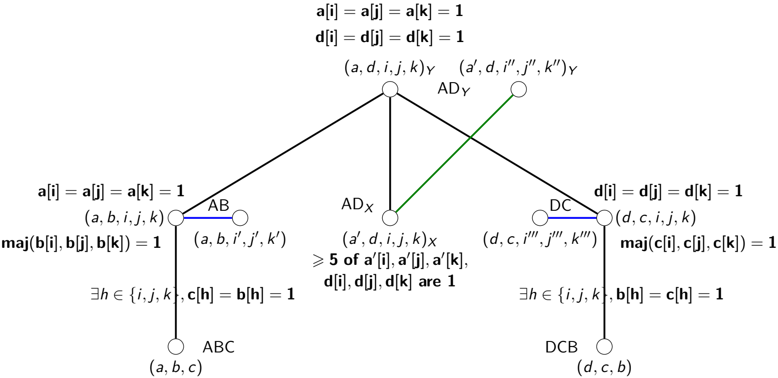
Important fact: For every $a, b, c, d \in S$, $\exists i = \mathrm{ind}(a,b,c,d) \in [\ell]$, $a[i] = b[i] = c[i] = d[i] = 1$



$$\mathrm{ABC} \times \mathrm{DCB}:\ i = \mathrm{ind}(a,b,c,d), j = \mathrm{ind}(a,b,c',d), k = \mathrm{ind}(a,b',c',d)$$

$$(a,b,c) \to (a,b,i,j,k) \to (a,d,i,j,k)_Y \to (d,c',i,j,k) \to (d,c',b')$$

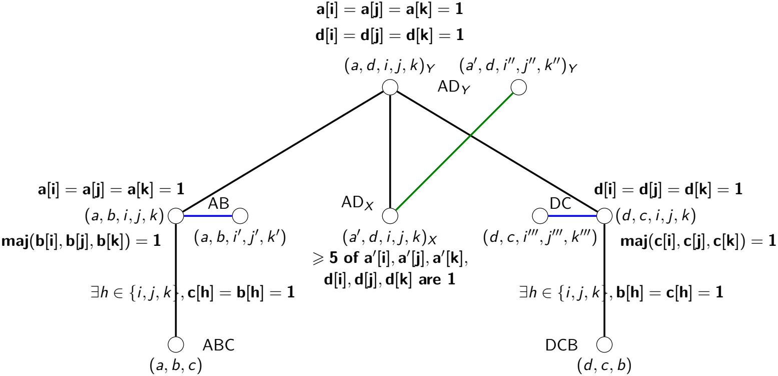
No orthogonal quadruple in $S \Rightarrow \forall u, v,\ d(u,v) \leqslant 4$



ABC $\times$ DC: $i = \mathrm{ind}(a, b, c, d), j = \mathrm{ind}(a, b, c', d), k = \mathrm{ind}(a, c', d)$

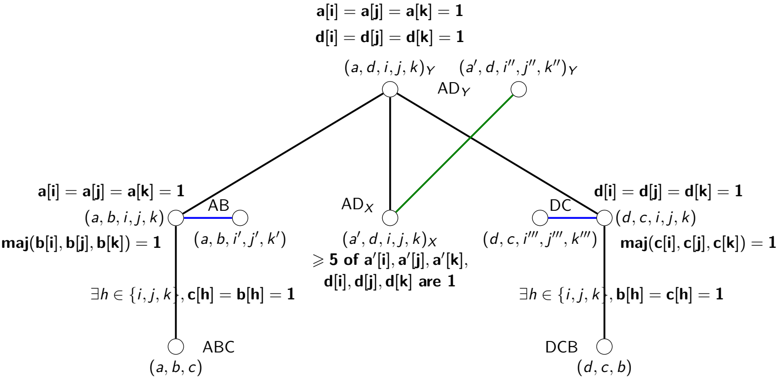
$(a, b, c) \to (a, b, i, j, k) \to (a, d, i, j, k)_Y \to (d, c', i, j, k) \to (d, c', i', j', k')$

No orthogonal quadruple in $S \Rightarrow \forall u, v,\ d(u,v) \leqslant 4$



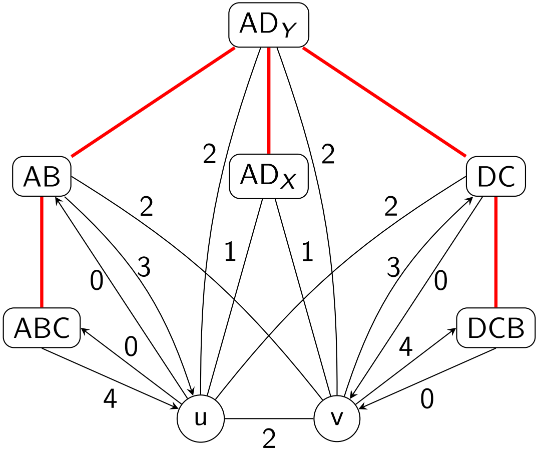
$$ABC \times AD_Y\text{: } i = \text{ind}(a, b, c, d), j = \text{ind}(a, a', b, d), k = \text{ind}(a, a', d)$$
$$(a, b, c) \to (a, b, i, j, k) \to (a, d, i, j, k)_Y \to (a', d, i, j, k)_X \to (a', d, i', j', k')_Y$$

No orthogonal quadruple in $S \Rightarrow \forall u, v,\ d(u,v) \leqslant 4$



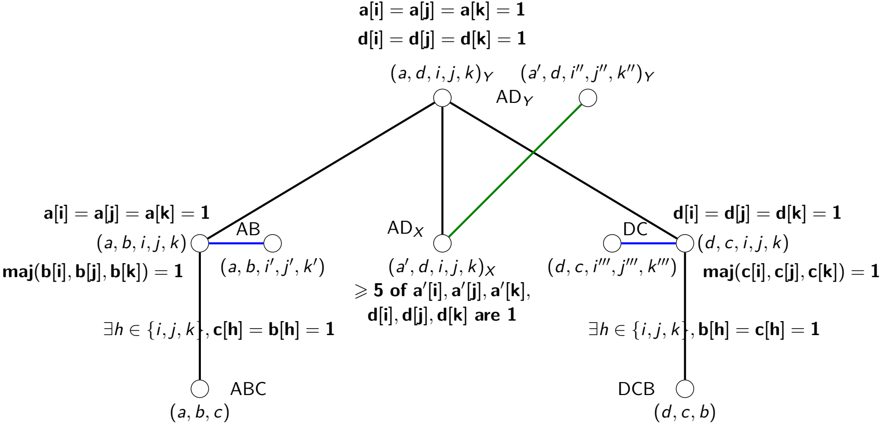
$$\mathrm{AB} \times \mathrm{DC}:\ i = \mathrm{ind}(a, b, c, d)$$
$$(a, b, i', j', k') \to (a, b, i, i, i) \to (a, d, i, i, i)_Y \to (d, c, i, i, i) \to (d, c, i'', j'', k'')$$

a, b, c, d orthogonal $\Rightarrow d((a, b, c), (d, c, b)) = 7$



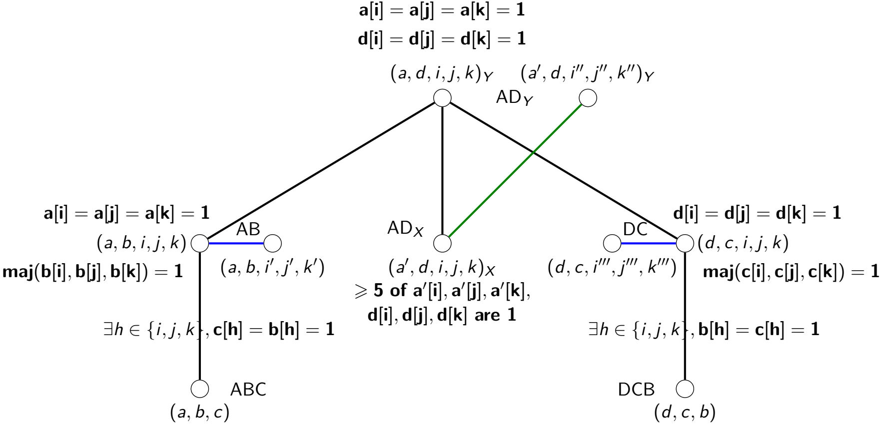
No path of length at most 6 via u or v

a, b, c, d orthogonal $\Rightarrow d((a,b,c),(d,c,b)) = 7$



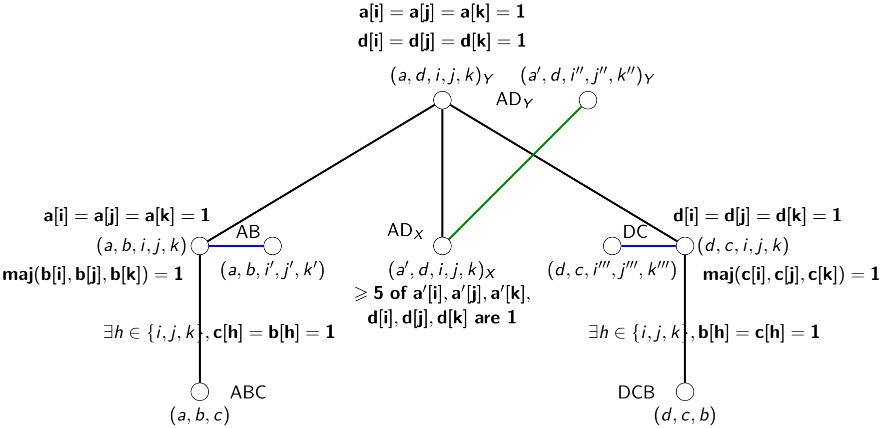
Case (a): no skew edge used

a, b, c, d orthogonal $\Rightarrow d((a,b,c),(d,c,b)) = 7$



Implies $(a, b, i, j, k) \in AB$ and $(d, c, i, j, k) \in DC$ on the path,
thus $\exists h \in \{i, j, k\}$ $a[h] = b[h] = c[h] = d[h] = 1$, a contradiction

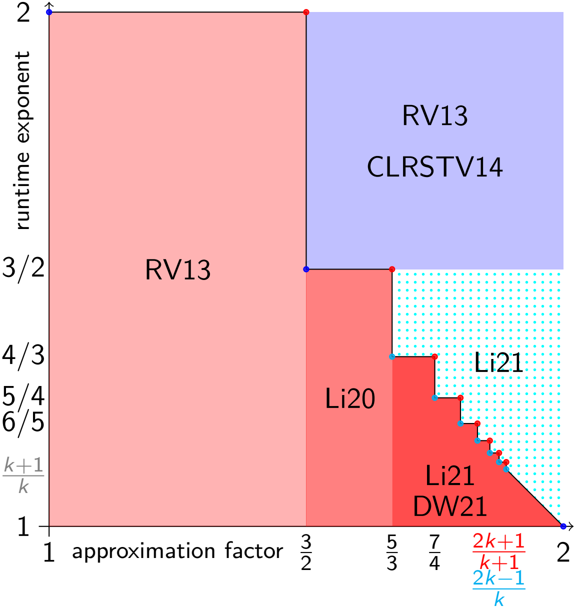
a, b, c, d orthogonal $\Rightarrow d((a,b,c),(d,c,b)) = 7$



Case (b): at least one skew edge used

$(a, b, c) \rightarrow (a, b, i, j, k) \rightarrow (a, d, i, j, k)_Y$ (or symmetric), hence contradiction

Recent developments for directed unweighted DIAMETER



Recent developments for undirected unweighted DIAMETER