EPTAS for Max Clique on Disks and Unit Balls

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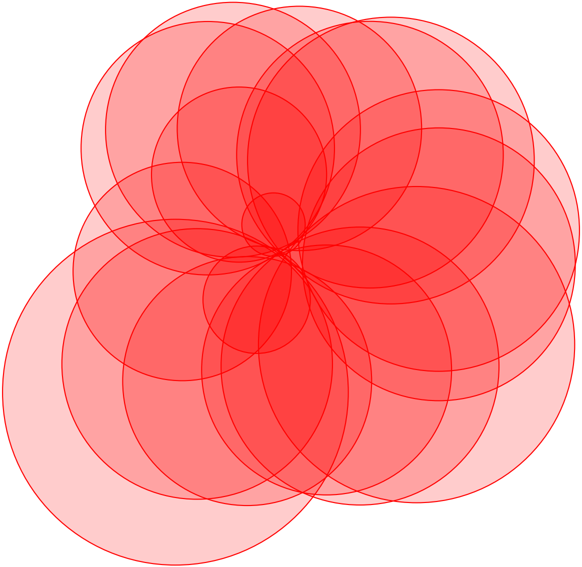
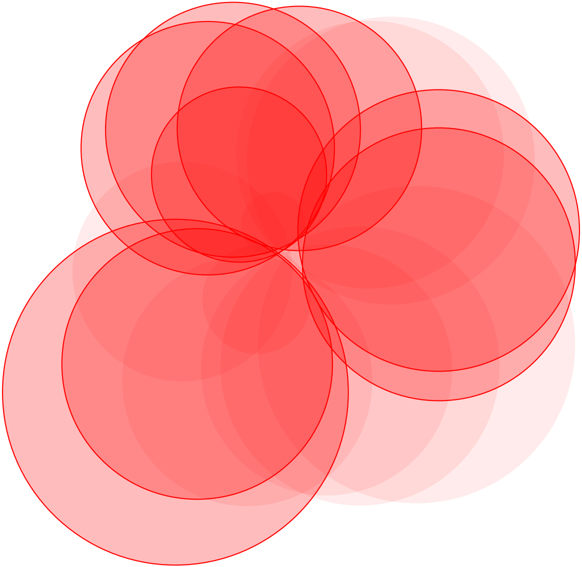
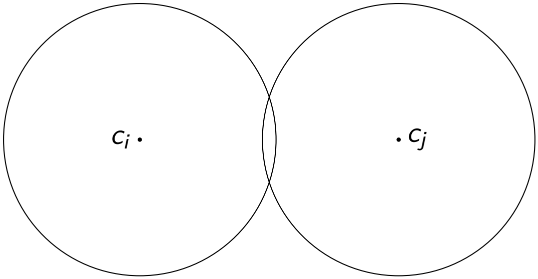


Find a largest collection of disks that pairwise intersect



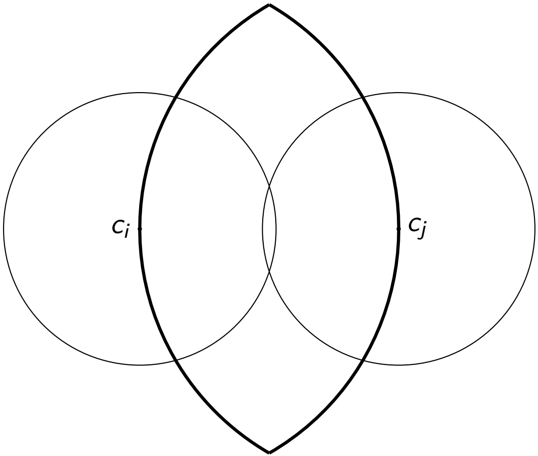
Example of a solution

Polynomial algorithm on unit disks by Clark, Colbourn, and Johnson, 1990.



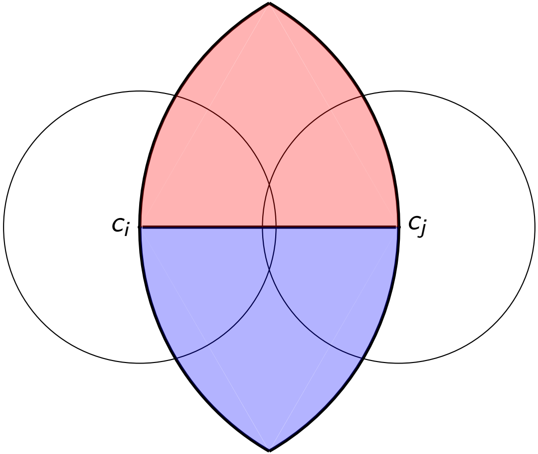
Guess two farthest disks in an optimum solution S.

Polynomial algorithm on unit disks by Clark, Colbourn, and Johnson, 1990.



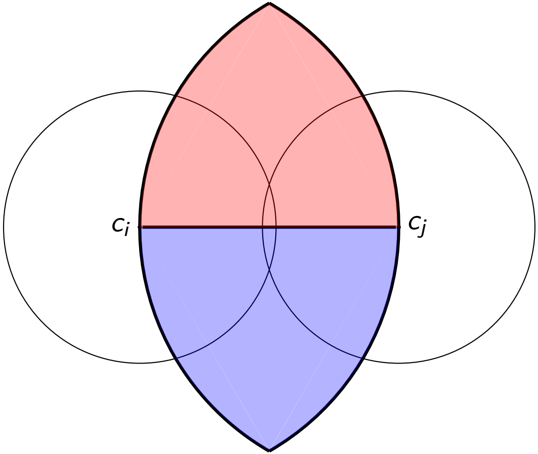
Hence, all the centers of S lie inside the bold digon.

Polynomial algorithm on unit disks by Clark, Colbourn, and Johnson, 1990.



Two disks centered in the same-color region intersect.

Polynomial algorithm on unit disks by Clark, Colbourn, and Johnson, 1990.



We solve MAX CLIQUE in a co-bipartite graph.

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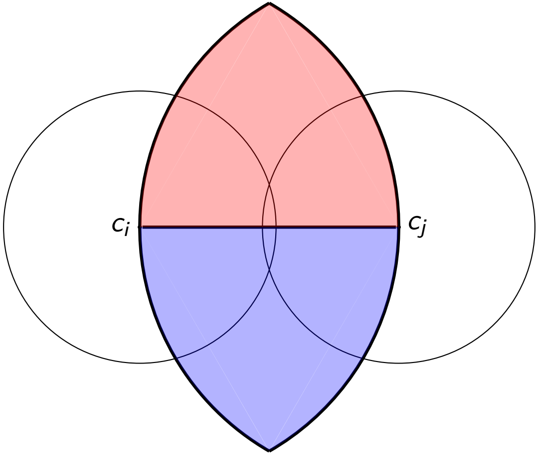


We solve Max Independent Set in a bipartite graph.

Disk graphs

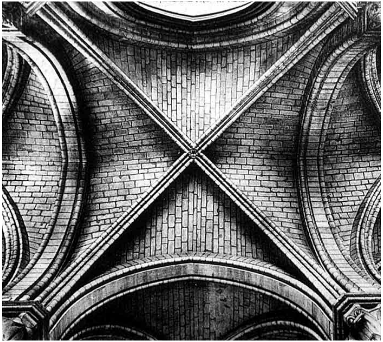
Unweighted problems

3-Colourability [?]	NP-complete	[+]Details
Clique [?]	Unknown to ISGCI	[+]Details
Clique cover [?]	NP-complete	[+]Details
Colourability [?]	NP-complete	[+]Details
Domination [?]	NP-complete	[+]Details
Feedback vertex set [?]	NP-complete	[+]Details
Graph isomorphism [?]	Unknown to ISGCI	[+]Details
Hamiltonian cycle [?]	NP-complete	[+]Details
Hamiltonian path [?]	NP-complete	[+]Details
Independent dominating set [?]	NP-complete	[+]Details
Independent set [?]	NP-complete	[+]Details
Maximum bisection [?]	NP-complete	[+]Details
Maximum cut [?]	NP-complete	[+]Details
Minimum bisection [?]	NP-complete	[+]Details
Monopolarity [?]	NP-complete	[+]Details
Polarity [?]	NP-complete	[+]Details
Recognition [?]	NP-hard	[+]Details

Inherits the NP-hardness of planar graphs.

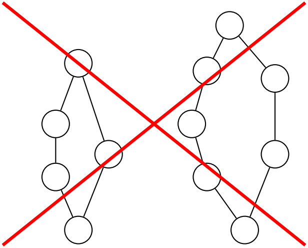
So what is known for MAX CLIQUE on disk graphs?

- Polynomial-time 2-approximation
- Polynomial-time 2.553-approximation for unit balls
- NP-hardness only known in dimension $\log n$



Theorem (BGKRzS '18)

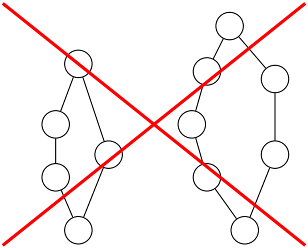
The disjoint union of two odd cycles is not a complement of disk graph



iocp = induced odd cycle packing number

iocp $\leqslant 1$

Can we solve Max Independent Set more efficiently if there are no two vertex-disjoint odd cycles as an induced subgraph?



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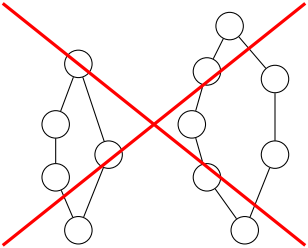


Theorem (BGKRzS '18)

QPTAS and $2^{\tilde{O}(n^{2/3})}$*-algorithm for* MIS *for iocp* $\leqslant 1$ *.*

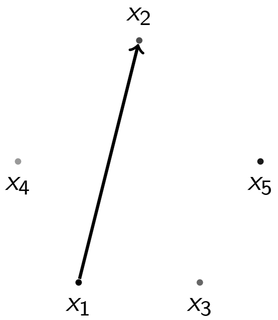
Corollary

QPTAS and $2^{\tilde{O}(n^{2/3})}$*-algorithm for* MAX CLIQUE *on disk graphs.*

What about unit ball graphs?

$x_1, \ldots, x_s \in \mathbb{R}^3$ are the *consecutive* centers an $\overline{\text{odd-cycle}}$.

Move continuously a vector with the following steps:



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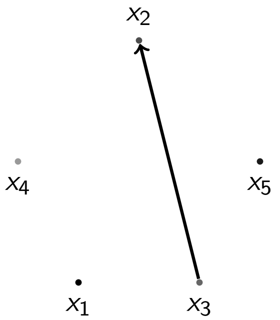

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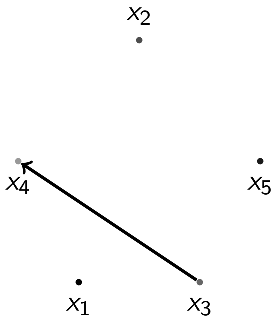

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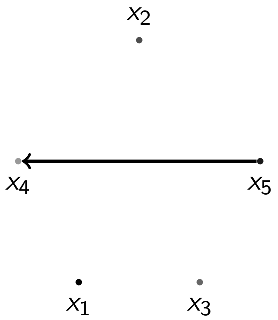

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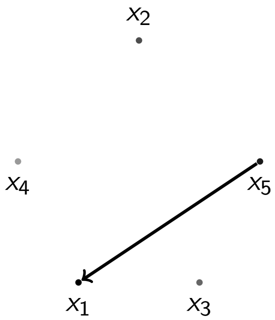

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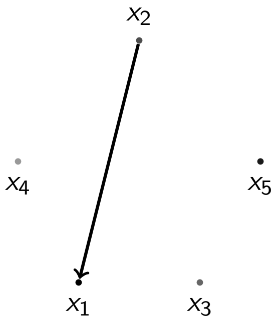

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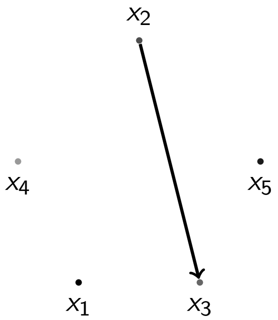

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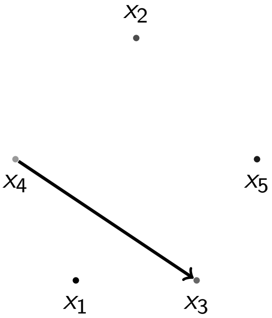

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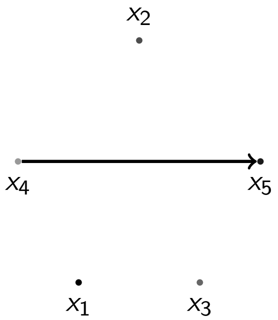

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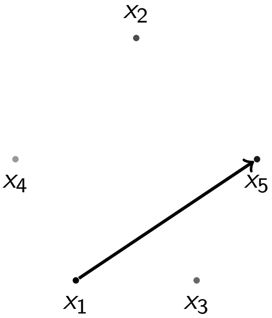

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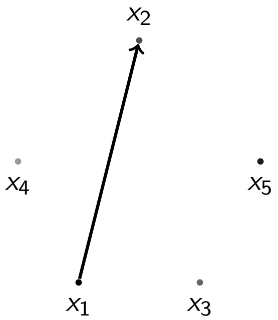

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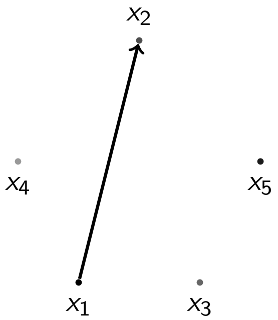
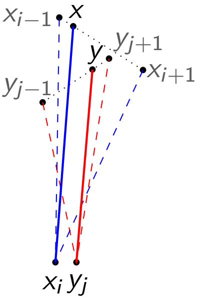
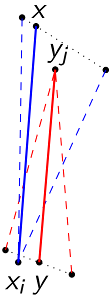


The curve drawn by the directions on the unit sphere is antipodal

What would happen with the complement of two odd cycles?

As two antipodal curves intersect, we have one of the following:





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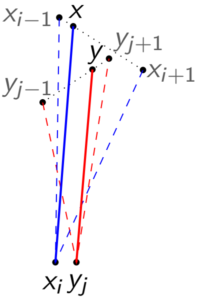

Left case: $d(x_i, y) + d(y_j, x) > d(x_i, x) + d(y_j, y)$

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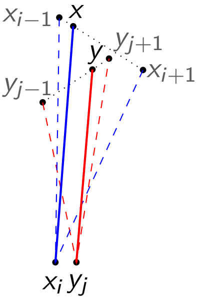

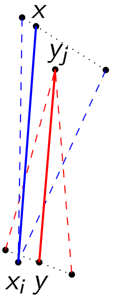

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$$d(y_j, x_{i\pm1}) \leqslant 2 < d(x_i, x_{i\pm1})$$

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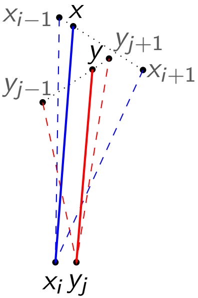

Left case: $d(x_i, y) + d(y_j, x) > d(x_i, x) + d(y_j, y)$

Hence, $d(y_j, x) < d(x_i, x)$ (and similarly $d(x_i, y) < d(y_j, y)$)

EPTAS for MIS on graphs with constant VCdim and iocp

VC dim of $\mathcal{S}$ = maximum size of a set with all intersections with $\mathcal{S}$.
VCdim(G) = VC dimension of the neighborhood set-system.
$\alpha(G)$ = size of a maximum independent set in G.

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Theorem

Max Independent Set can be $1+\varepsilon$-approximated in time $2^{\tilde{O}(1/\varepsilon^3)} n^{O(1)}$ on graphs G with

- *VCdim(G) = $O(1)$,*
- *$\alpha(G) = \Omega(n)$, and*
- *iocp(G) $\leqslant 1$.*

ε-nets

Classic result of Haussler and Welzl in VC dimension theory

Theorem (ε-nets)

A set-system $(\mathcal{S}, U)$ with VC dimension d and only sets of size at least $\varepsilon|U|$ has a hitting set of size $O(\frac{d}{\varepsilon} \log \frac{1}{\varepsilon})$.

Furthermore, any sample of size $\frac{10d}{\varepsilon} \log \frac{1}{\varepsilon}$ is a hitting set w.h.p.

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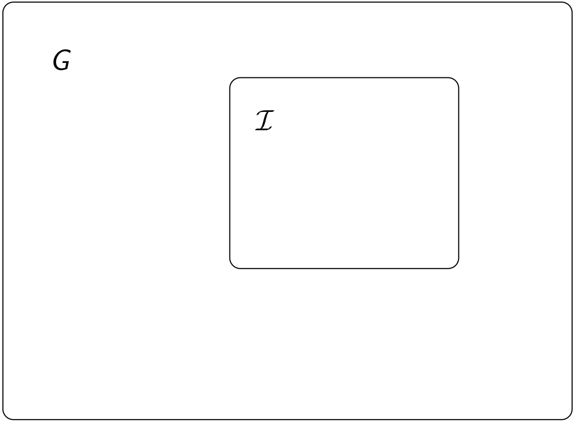
Furthermore, any sample of size $\frac{10d}{\varepsilon} \log \frac{1}{\varepsilon}$ is a hitting set w.h.p.

We will apply that result to the set-system
$(\{N(u) \cap \mathcal{I} \mid u \in V(G), |N(u) \cap \mathcal{I}| \geqslant \varepsilon^3 |\mathcal{I}|\}, \mathcal{I})$.
In words, the large neighborhoods over I.

First step: sampling

$\mathcal{I}$ is a fixed maximum independent set.
We can assume that $|\mathcal{I}| = \Theta(n)$.



We pick randomly S of $\tilde{O}(1/\varepsilon^3)$ vertices.

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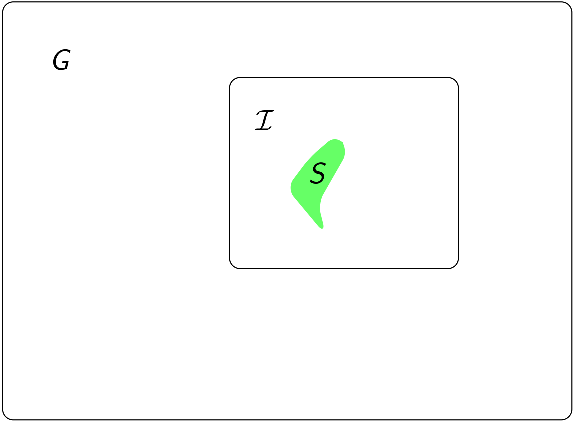

With probability $f(\varepsilon) > 0$, $S \subseteq \mathcal{I}$.

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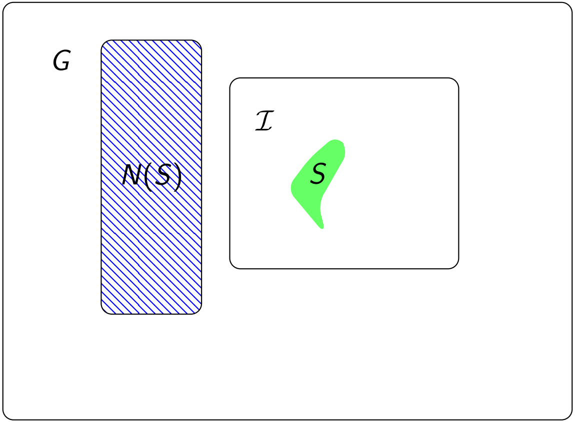

We delete the neighborhood of S.

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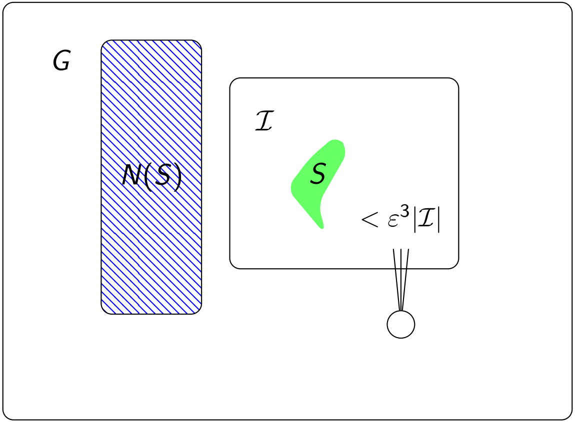

The remaining vertices have few vertices in $\mathcal{I}$.

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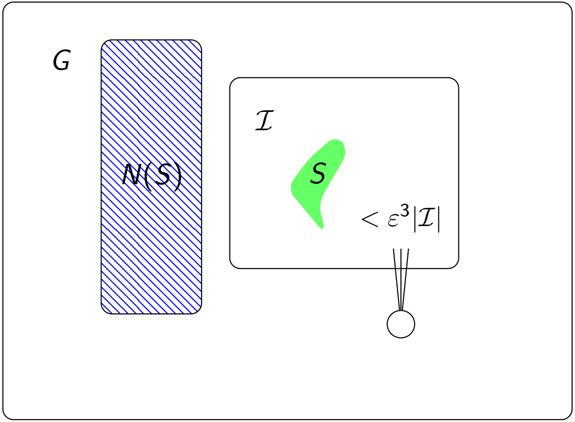

This is due to the theorem of ε-nets.

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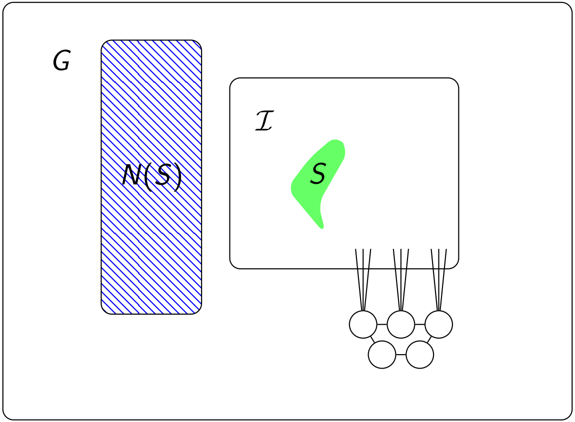

We compute a shortest odd cycle C.

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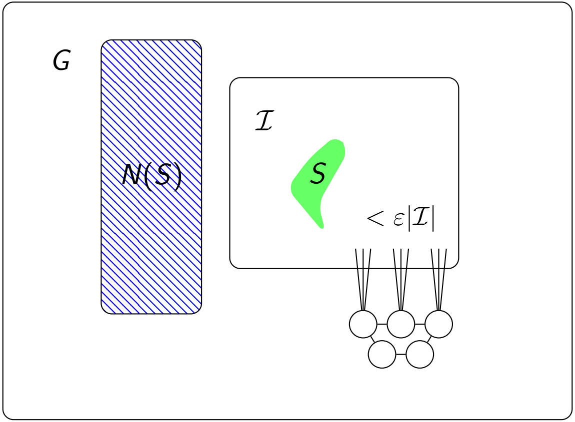

If $|C| \leqslant 1/\varepsilon^2$, we delete its neighborhood.

First step: sampling

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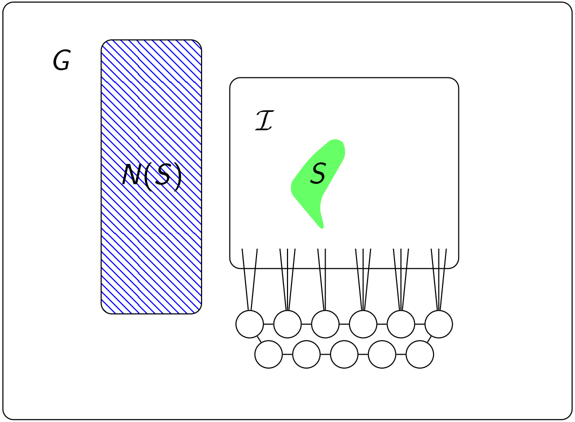
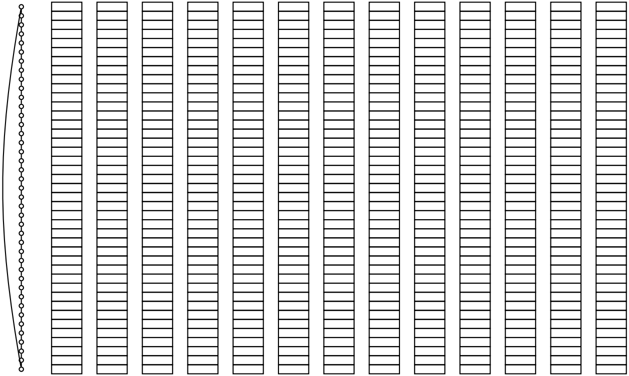


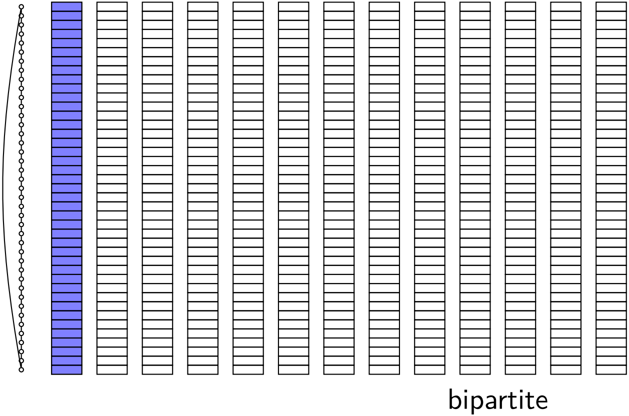
So, we might assume that $|C| > 1/\varepsilon^2$.

Second step: find a small odd cycle transversal



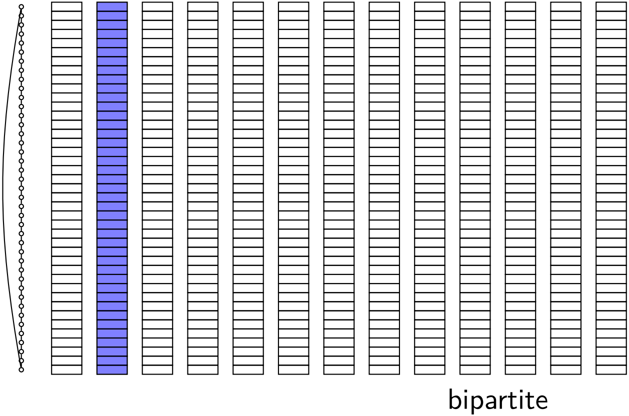
In column, the successive neighborhood of C, *layers*.
Rows indicate the closest neighbor on C, *strata*.

Second step: find a small odd cycle transversal



Deleting a j-th neighborhood of C, leaves a bipartite to the right.

Second step: find a small odd cycle transversal



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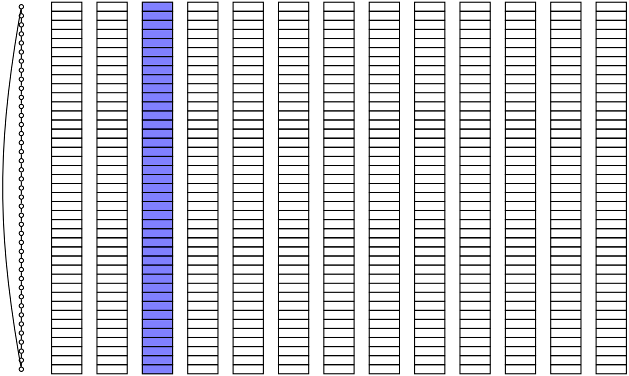

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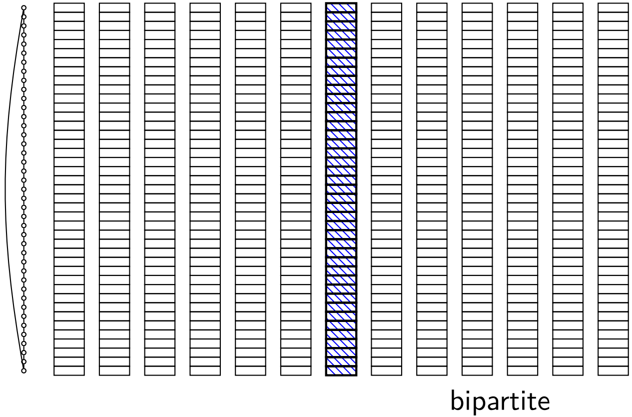
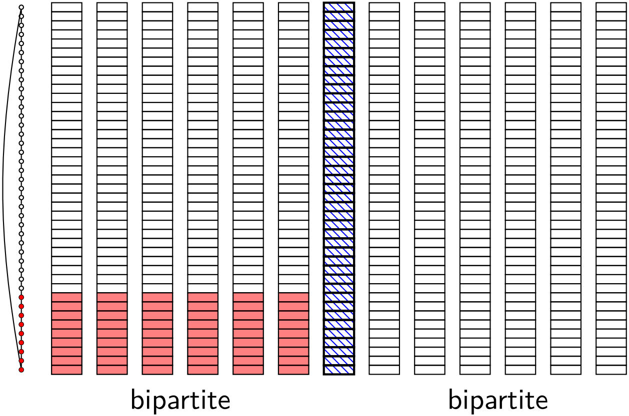


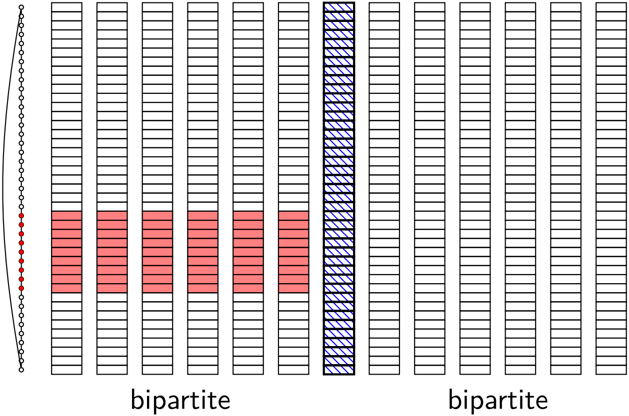
We delete the lightest of the $\approx 1/\varepsilon$ first layers.

Second step: find a small odd cycle transversal



$\approx 1/\varepsilon$ consecutive strata form an odd cycle transversal.

Second step: find a small odd cycle transversal



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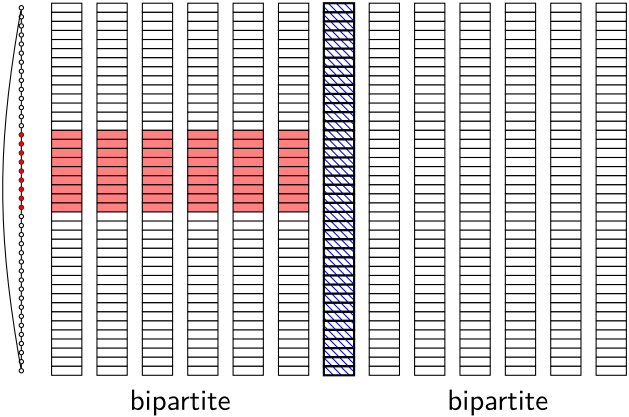

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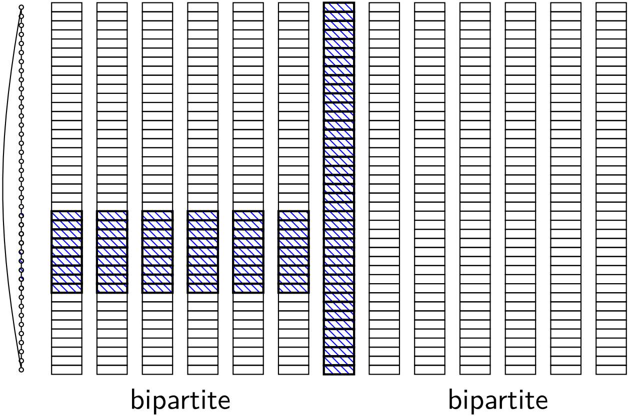

We remove the lightest block of strata.

Other shapes?

2-subdivisions: graphs where each edge is subdivided exactly twice
co-2-subdivisions: complements of 2-subdivisions

Lemma

For some $\alpha > 1$, $\textsc{Max Clique}$ on co-2-subdivisions is not α-approximable algorithm in $2^{n^{0.99}}$, unless the ETH fails.

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Lemma

For some $\alpha > 1$, $\textsc{Max Clique}$ *on co-2-subdivisions is not* α*-approximable algorithm in* $2^{n^{0.99}}$*, unless the ETH fails.*

Theorem

Intersection graphs of the following objects contain all the co-2-subdivisions:

- *(filled) triangles*
- *(filled) ellipses arbitrarily close to unit disks*
- *balls (3-dimensional disks), even with arbitrarily close radii*
- *unit 4-dimensional disks*

hence, they inherit the same lower bound

Open questions

- Is Max Clique NP-hard on disk and unit ball graphs?
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Thank you for your attention!