Fixed-parameter Approximability of Boolean MinCSPs

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Constraint Satisfaction Problem (CSP)

- **Problem:** a set of relations $\{R_i\}_{1 \leqslant i \leqslant \ell}$ over a domain D.
- **Instance:** a set of constraints of the form $R_i(x_{a_1}, \ldots, x_{a_j})$ over variables $x_1, \ldots, x_n \in D$.
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3-SAT:
$D = \{0, 1\}, R_1 = \{0, 1\}^3 \setminus \{(0, 0, 0)\}, \dots, R_8 = \{0, 1\}^3 \setminus \{(1, 1, 1)\}$.

Instance:
$R_4(x_3, x_2, x_4), R_8(x_1, x_3, x_5), R_1(x_1, x_2, x_3), R_1(x_3, x_4, x_5), R_5(x_3, x_2, x_4)$

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Instance:
$x_3 \vee \neg x_2 \vee \neg x_4, \neg x_1 \vee \neg x_3 \vee \neg x_5, x_1 \vee x_2 \vee x_3, x_3 \vee x_4 \vee x_5, \neg x_3 \vee x_2 \vee x_4$

CSP: a very short bio

- 1974: Birth in Montanari's *"Networks of constraints: Fundamental properties and applications to picture processing"*.
- 1978: First dichotomy result by Schaefer.
  For $D = \{0, 1\}$, each CSP is either in P or NP-complete.
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Unified framework propitious to classification which can express many but not all problems in NP.

Boolean CSPs

- The classical complexity is settled.
- Each new kind of classification (approximability, parameterized complexity, counting...) starts with Boolean CSPs.
- Boolean CSPs are already quite expressive.

Optimization variant of Boolean CSPs

The four main settings:

- MaxCSPs: **maximizing** the number of **satisfied** constraints.
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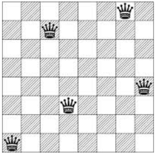
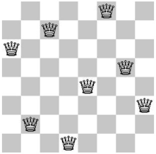
- MinOnes: **minimizing** the number of variables set to 1.
- MaxOnes: **maximizing** the number of variables set to 1.





Approximability of Boolean CSPs

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- Less concise classification for MinCSPs, MinOnes, MaxOnes [Khanna et al. '00].

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Theorem 2.13 (MIN CSP **classification**) *For any constraint set $\mathcal{F}$, the problem (*WEIGHTED*)* MIN CSP($\mathcal{F}$) *is in* PO *or is* APX*-complete or* MIN UNCUT*-complete or* MIN 2CNF DELETION*-complete or* NEAREST CODEWORD*-complete or* MIN HORN DELETION*-complete or or even deciding if the optimum is zero is* NP*-hard. Furthermore,*

(1) *If $\mathcal{F}$ is 0-valid or 1-valid or 2-monotone, then (*WEIGHTED*)* MIN CSP($\mathcal{F}$) *is in* PO.

(2) *Else if $\mathcal{F}$ is IHS-B then (*WEIGHTED*)* MIN CSP($\mathcal{F}$) *is* APX*-complete.*

(3) *Else if $\mathcal{F}$ is width-2 affine then (*WEIGHTED*)* MIN CSP($\mathcal{F}$) *is* MIN UNCUT*-complete.*

(4) *Else if $\mathcal{F}$ is 2CNF then (*WEIGHTED*)* MIN CSP($\mathcal{F}$) *is* MIN 2CNF DELETION*-complete.*

(5) *Else if $\mathcal{F}$ is affine then (*WEIGHTED*)* MIN CSP($\mathcal{F}$) *is* NEAREST CODEWORD*-complete.*

(6) *Else if $\mathcal{F}$ is weakly positive or weakly negative then (*WEIGHTED*)* MIN CSP($\mathcal{F}$) *is* MIN HORN DELETION*-complete.*

(7) *Else deciding if the optimum value of an instance of (*WEIGHTED*)* MIN CSP($\mathcal{F}$) *is zero is* NP*-complete.*

Parameterized Complexity of Boolean CSPs

- Each MaxCSP is FPT (parameter = #satisfied constraints).
- Dichotomy FPT/W[1]-hard of ExactOnes [Marx '05].
- Classification of MaxOnes and ExactOnes w.r.t. parameterized complexity and kernels [Kratsch et al. '10].
- Each MinOnes problem is FPT. Dichotomy poly-kernel or not [Kratsch and Wahlström '10].

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What about the parameterized complexity of MinCSPs?

Probably no strong link approximation/FPT

- MinCSP($\Gamma_{\text{2-SAT}}$) is FPT [Razgon and O'Sullivan '08] but not constant-approximable under UGC [Chawla et al. '06].
- MinCSP($x, \neg x, (a \to b) \wedge (c \to d)$) is W[1]-hard [Marx and Razgon '09] but constant-approximable [Khanna et al. '00].

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FPT **constant-approximation** already appears as more robust.

MinCSPs parameterized approximability

Our goal: For each finite set of relation Γ, tell if MinCSP(Γ) has an **FPT constant-approximation** (FPA) or not.

We establish:

- If Γ is 0-valid, or 1-valid, or bijunctive, or IHS-B, MinCSP(Γ) has an FPA.
- Otherwise, If Γ is affine, MinCSP(Γ) is ODD SET-complete[1].
- Otherwise, MinCSP(Γ) has no FPA unless FPT=W[P].

[1] under A-reductions

Set of constraints

- 0-valid: all satisfied by setting every variable to *false*.
- 1-valid: all satisfied by setting every variable to *true*.
- bijunctive: every constraint is a conjunction of 2-clauses.
- IHS-B: IHS-B$^+$ or IHS-B$^-$.
- IHS-B$^+$: $\neg x$, $x \to y$, $x_1 \vee x_2 \vee \ldots \vee x_k$ with $k \leqslant B$.
- IHS-B$^+$: x, $x \to y$, $\neg x_1 \vee \neg x_2 \vee \ldots \vee \neg x_k$ with $k \leqslant B$.
- affine: $x_1 \oplus x_2 \oplus \ldots \oplus x_k = c$ with $c \in \{0, 1\}$.

Helped by the co-clone lattice

- **Primitive positive:** pp-definition over $\Gamma = \exists\wedge$ whose atomic formulas are in Γ (small lie).
- **Co-clone:** $\langle\Gamma\rangle$ = set of the relations pp-definable over Γ.
- **Base of C:** Γ such that $C = \langle\Gamma\rangle$.
- FPA-ness is preserved from Γ to any finite basis of $\langle\Gamma\rangle$.

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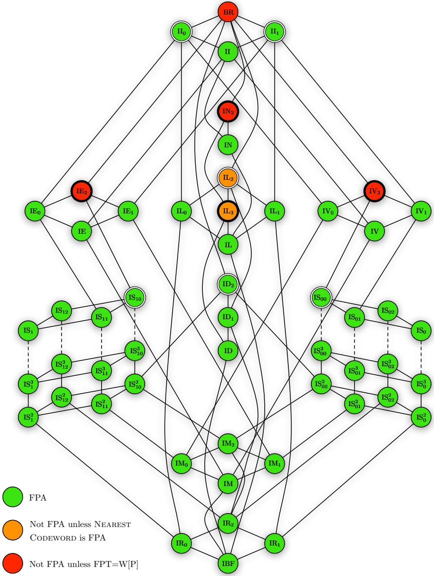
The lattice of Boolean co-clones has good properties:

- countably many co-clones.
- finitely many non-trivial maximal (minimal) co-clones.

Our results on the lattice of co-clones



Tractability results

- 0-valid and 1-valid CSPs are in P.
- Bijunctive: Treat a conjunction as separate constraints + MinCSP($\Gamma_{\text{2-SAT}}$) is FPT.
- IHS-B: **(2)** *Else if $\mathcal{F}$ is IHS-B then (*WEIGHTED*)* MIN CSP($\mathcal{F}$) *is* APX*-complete.*
  There is a $B + 1$-approximation.

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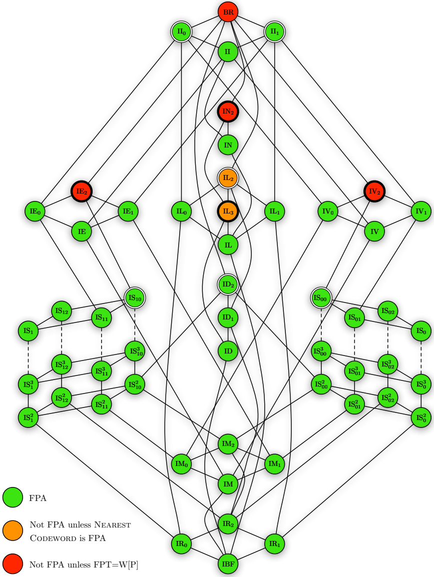

Orange results

Cycle of A-reductions involving Odd Set and the two problems MinCSP($\mathrm{EVEN}^4, \neg x, x$) and MinCSP($\mathrm{EVEN}^4, x \oplus y$).

Theorem

Odd Set *has no FPA unless* k-Densest Subgraph *has an FPT approximation scheme.*

Theorem

Odd Set *has no FPA under LPC+ETH.*

ETH: 3-SAT has no subexponential algorithm.
LPC: 3-SAT $\in$ PCP($\log \phi + O(1), O(1)$).

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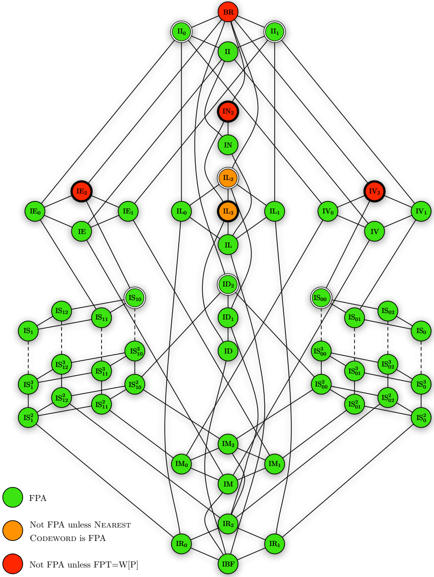

Hardness results

Theorem (Marx '10)

MONOTONE CIRCUIT SAT *has no FPA unless FPT=W[P].*

A-reduction to Horn-SAT and dual-Horn-SAT.

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An FPA for MinCSP(NAE^3) would imply the polynomiality of CSP(NAE^3) by fixing k to 0.

Thank you for your attention!