Complexity of Grundy Coloring and its Variants

Édouard Bonnet, Florent Foucaud, Eun Jung Kim, and Florian Sikora

Institute for Computer Science and Control, Hungarian Academy of Sciences, Budapest, Hungary (MTA SZTAKI)

August 5, 2015

Grundy colorings

- Order the vertices $v_1, v_2 \ldots v_n$ to *maximize* the number of colors used by the greedy coloring.
- That is, v_i is colored with $c(v_i)$ the first color that does *not* appear in its neighborhood.

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Grundy number $\Gamma(G)$, connected/weak Grundy number

A brief History of Grundy colorings

- 1939: Studied in directed acyclic graphs by Grundy.
- 1979: Formally defined by Kristen and Selkow.
- 1983: Ochromatic number defined by Simmons.
- 1987: Erdös et al. proved that ochromatic number = Grundy number.

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- 2011: Weak Grundy defined by Kierstead and Saoub.
- 2014: Connected Grundy defined by Benevides et al.

Algorithmic motivations

- $\Gamma(G)$ upper bounds the number of colors used by any greedy heuristic for Min Coloring.
- $\Gamma(G) \leqslant C\chi(G)$ on some classes of graphs gives a C-approximation for Min Coloring.
- see Sampaio's PhD thesis for further motivations.

Complexity of computing the Grundy number

$k = \Gamma(G)$ and w denotes the treewidth of the graph
XP algorithm: $n^{f(\kappa)}$; FPT algorithm: $f(\kappa)n^{O(1)}$.

- NP-hard on (co-)bipartite graphs, chordal graphs, line graphs.
- Solvable in $n^{2^{k-1}}$ [Z '06].
- Solvable in $2^{O(kw)}n$ and[1] in $n^{O(w^2)}$ [TP '97].

[1] one can show that $k \leqslant w \log n + 1$

Complexity of computing the Grundy number

Our contribution:

- Solvable in time $O^*(2.443^n)$.
- An $O^*(2^{o(w \log w)})$ algorithm would contradict the ETH (that is, allow to solve SAT in subexponential time).
- Solvable in time $f(k)n^{O(1)}$ in chordal graphs, H-minor free graphs, claw-free graphs.

Complexity of the variants

- Weak Grundy Coloring is NP-complete [GV '97].
- Connected Grundy Coloring is NP-complete on chordal graphs, co-bipartite graphs [B+ '14].

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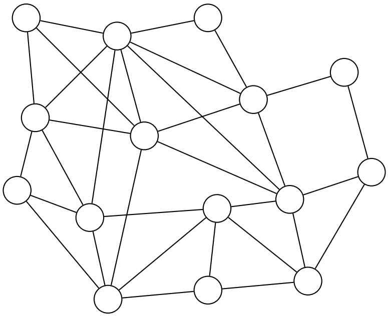
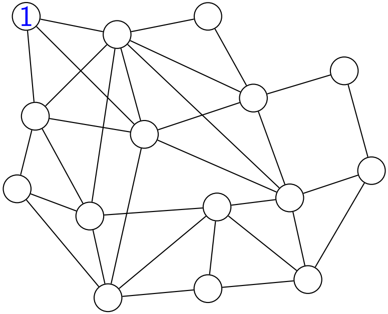
Our contribution:

- Weak Grundy Coloring is solvable in $k^{2^{k-1}}(n+m)n$ (FPT in k).
- Connected Grundy Coloring is NP-complete, even for $k = 7$ colors (not in XP unless P=NP).

Grundy Coloring



Can you achieve color 6?



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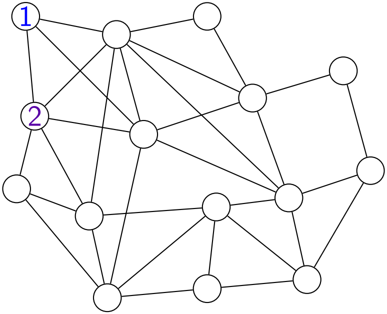

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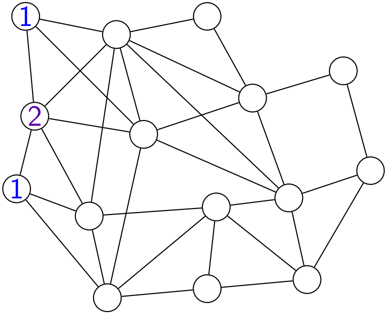

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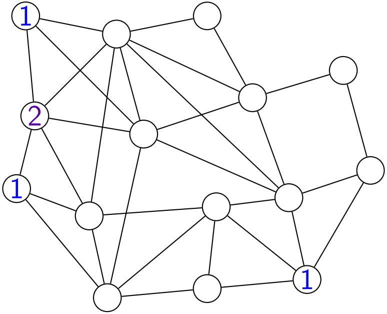

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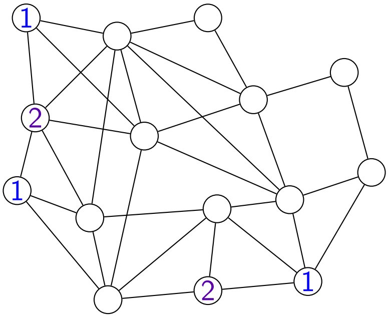

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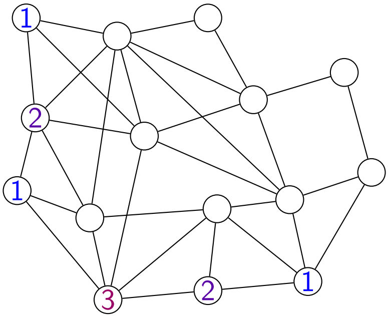

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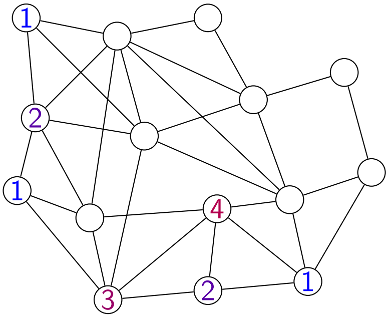

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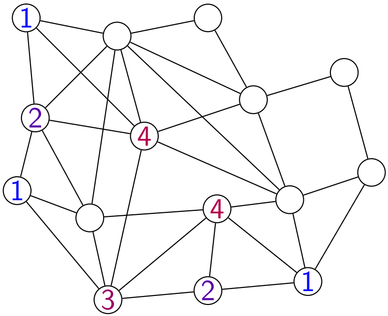

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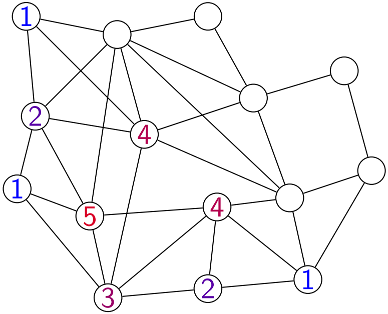

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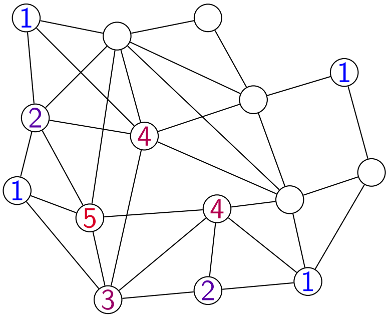

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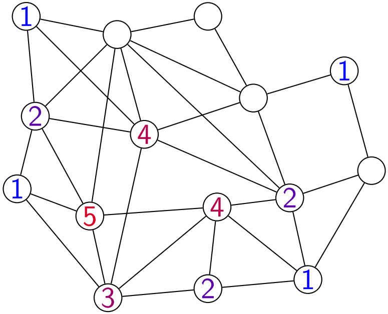

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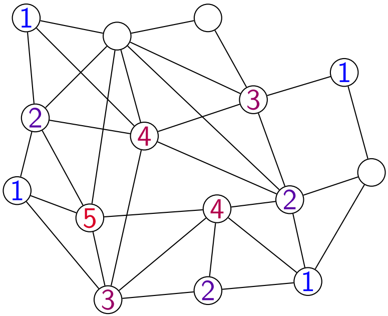

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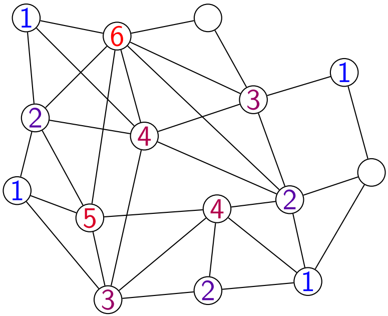

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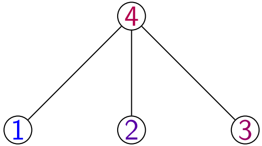

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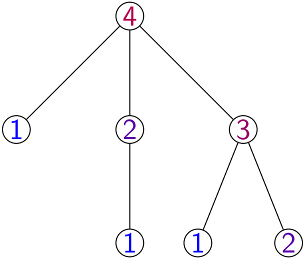

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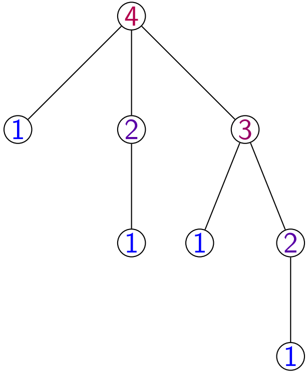

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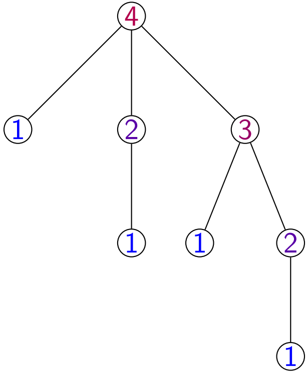

$t_1 = 1$ and $t_k = \Sigma_{1 \leqslant i \leqslant k-1} t_i$.
So, $t_k = 2^{k-1}$.

Theorem (Zaker '06)

The Grundy number can be computed in $O(f(k)n^{2^{k-1}})$.

Exact exponential algorithm

Try all possible ordering of the vertices and check if at least k colors are used by the greedy coloring: $\Theta(n!)$-time.

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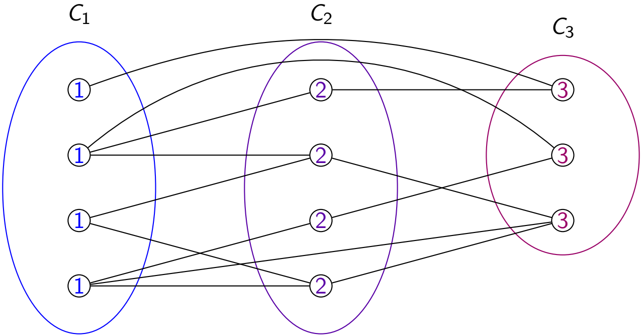
Can we improve on this trivial algorithm?

Solving Grundy Coloring

In a witness:



- Any color class is an independent dominating set in the graph induced by the next classes.

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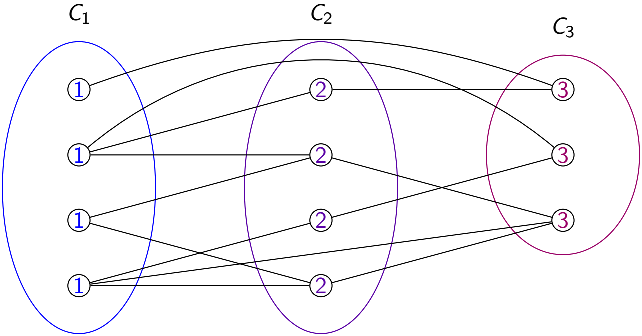

- Any color class is an independent dominating set in the graph induced by the next classes.
- $\Gamma(S) = \max\{\Gamma(S \setminus X), X \text{ ind. dom. set in } G[S]\} + 1$.

Solving Grundy Coloring

- Enumerating the minimal independent dominating sets takes time $O^*(3^{\frac{n}{3}}) = O(1.443^n)$.
- So, filling a cell of the table takes 1.443^i for a subset of size i.
- Hence, the total running time is $\sum_{i=0}^{n} \binom{n}{i} 1.443^i = (1 + 1.443)^n = 2.443^n$.

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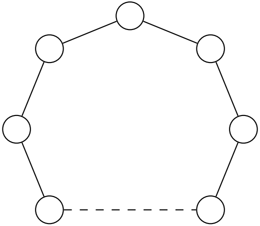
Cannot replace n by the treewidth w:

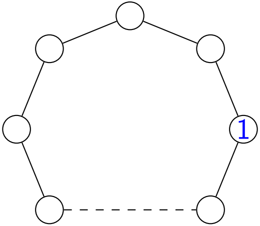
Theorem

Under the ETH, Grundy Coloring cannot be solved in $O^*(c^w)$ for any constant c (not even in $O^*(o(w^w))$).

Connected Grundy Coloring



Connected Grundy number $= 3$, unbounded witness



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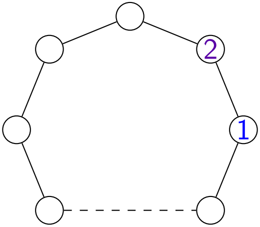

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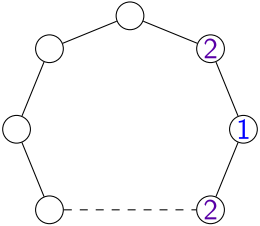

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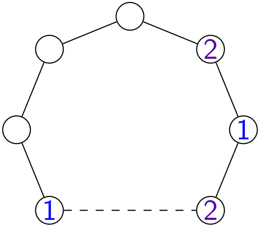

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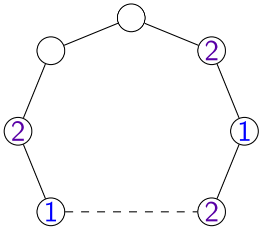

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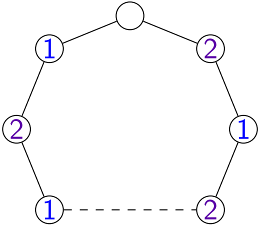

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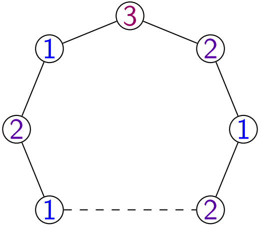

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Theorem (BCDGMSS '14)

Connected Grundy Coloring *is* NP*-complete.*

Theorem

Connected Grundy Coloring *is* NP*-complete even for* $k = 7$*.*

- Reduction from 3SAT-3OCC.

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- We move along a "path" P_1 of *literal* vertices:
  coloring such a vertex by 3 $\equiv$ setting the literal to true.

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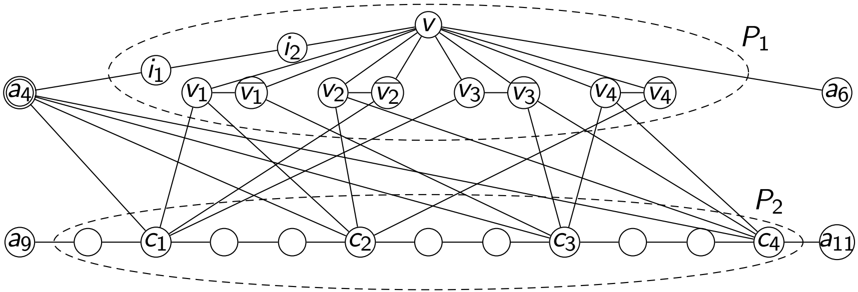
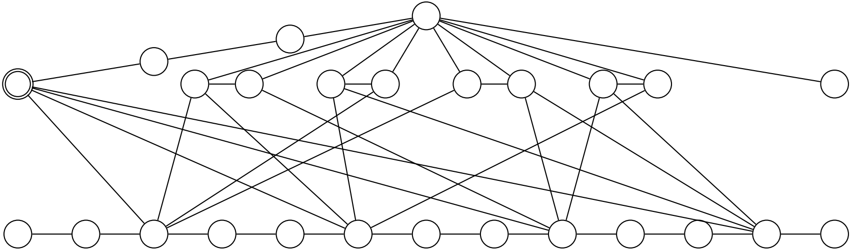
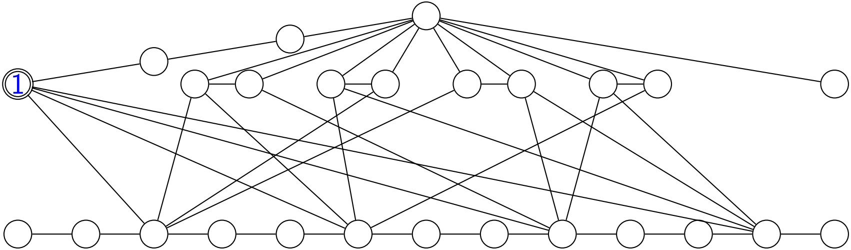
- To achieve color 7, three special neighbors of the c_js should be colored by 1, 2 and 3 respectively.



P_1 and P_2 for the instance
$\{x_1 \vee \neg x_2 \vee x_3\}, \{x_1 \vee x_2 \vee \neg x_4\}, \{\neg x_1 \vee x_3 \vee x_4\}, \{x_2 \vee \neg x_3 \vee x_4\}.$



A connected Grundy coloring setting all the c_js to 4.



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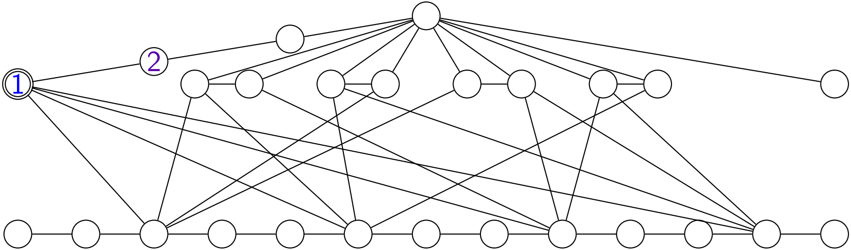

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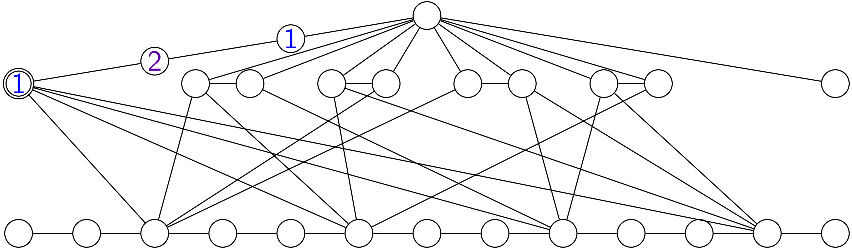

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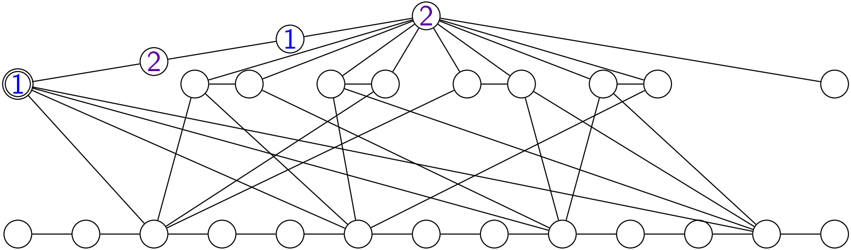

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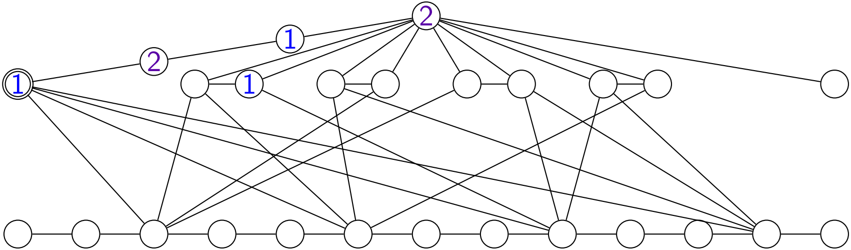

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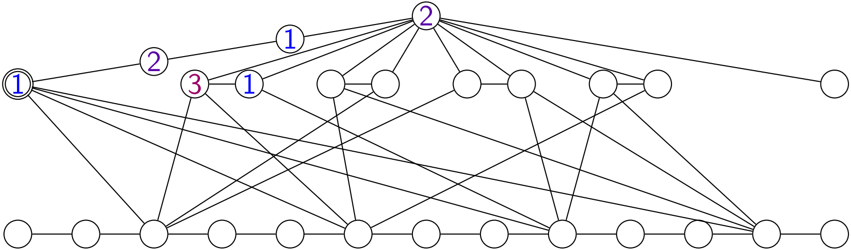

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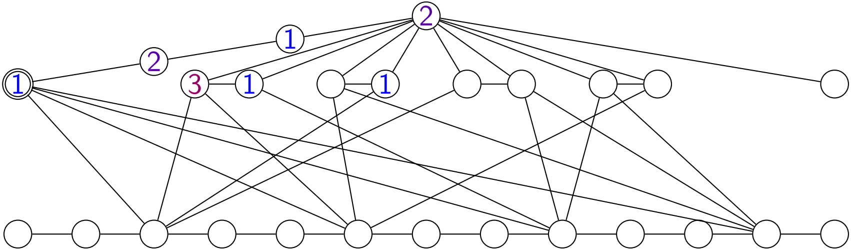

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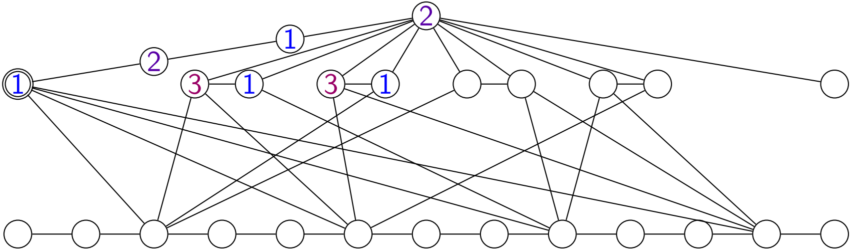

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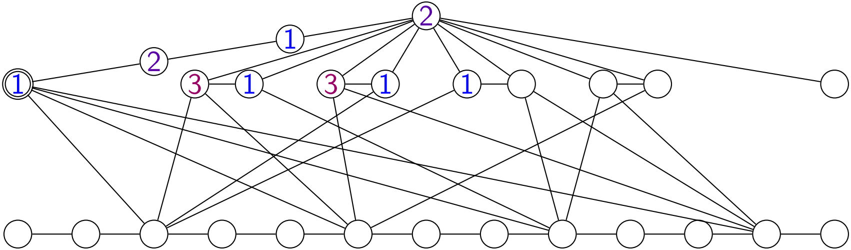

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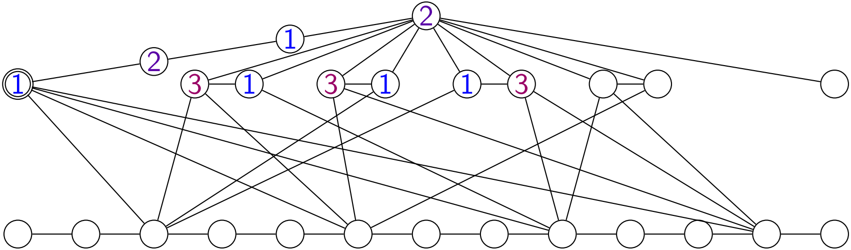

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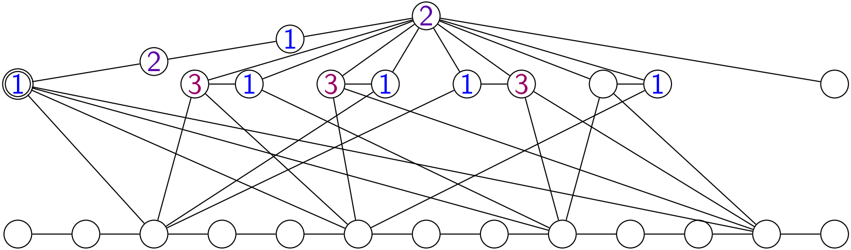

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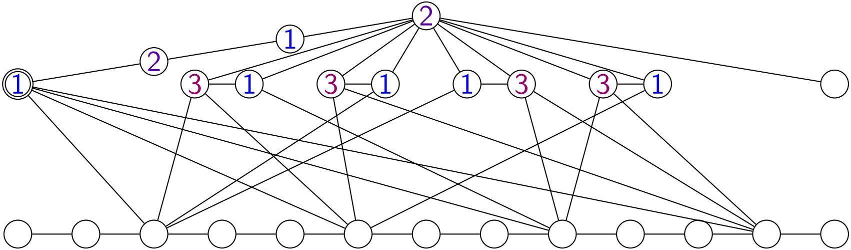

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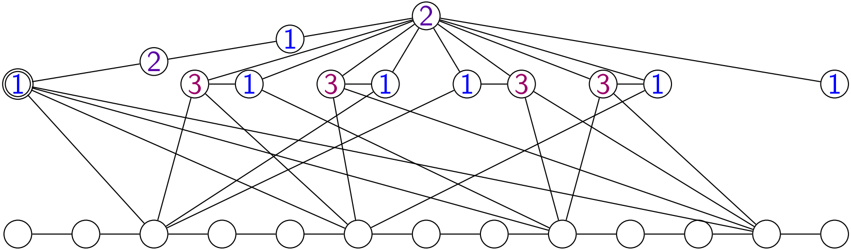

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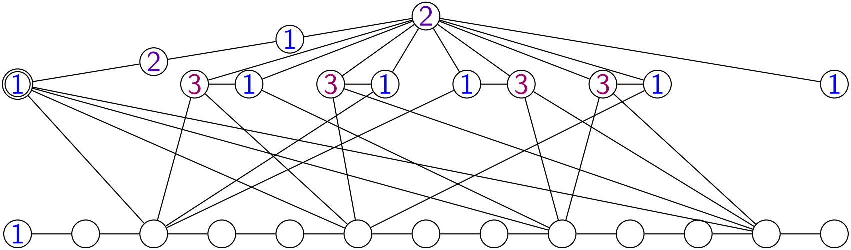

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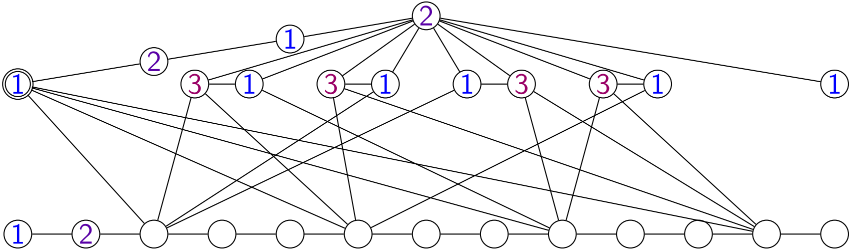

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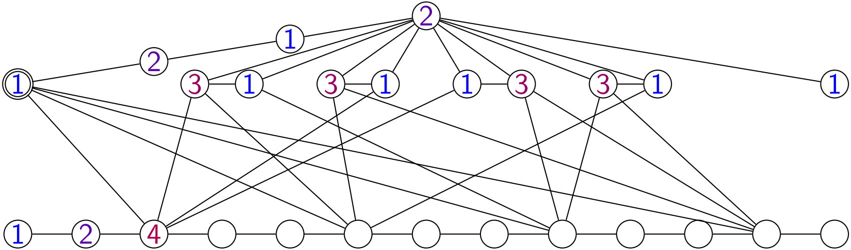

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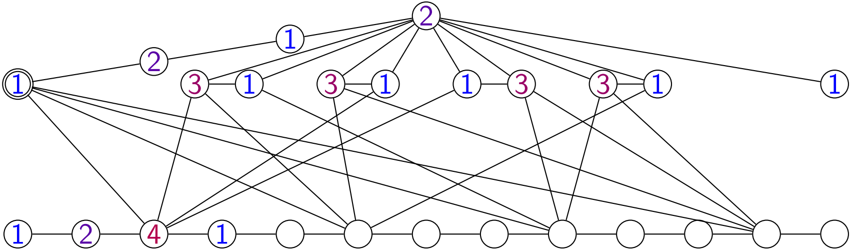

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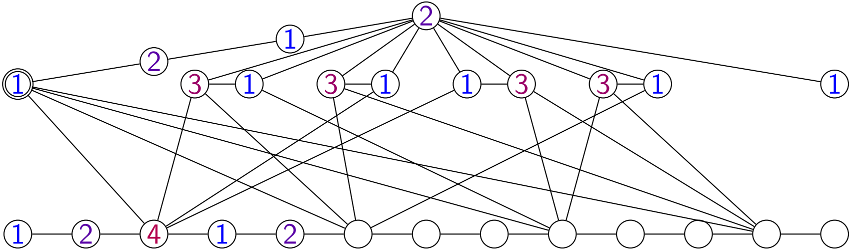

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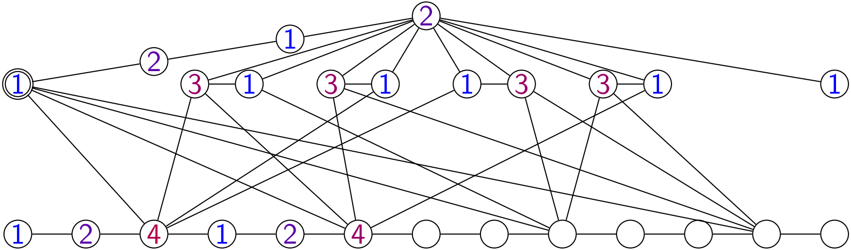

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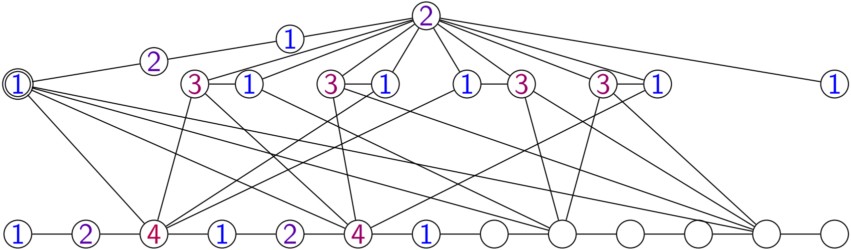

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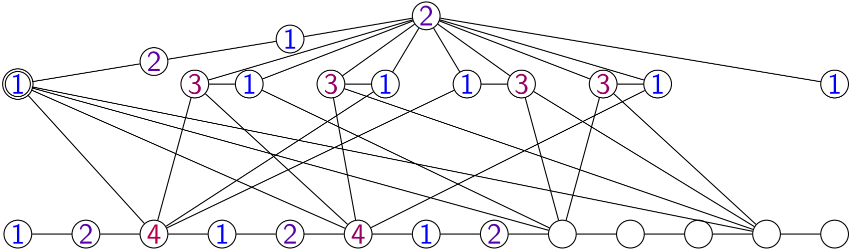

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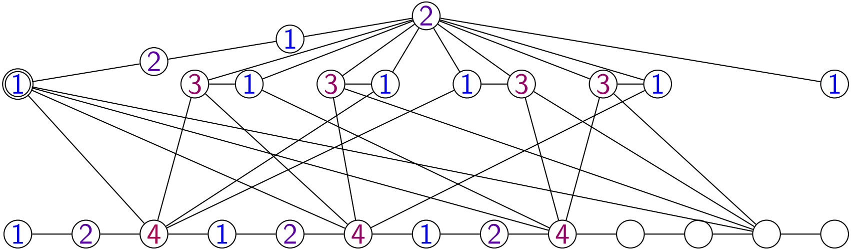

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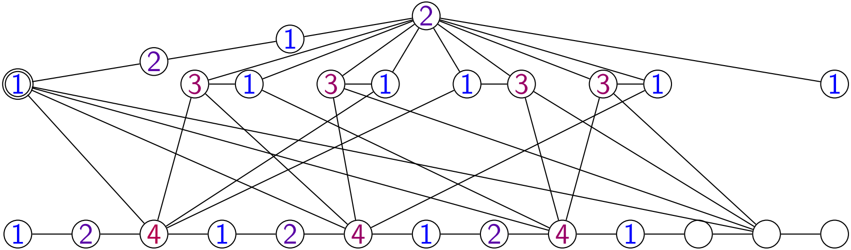

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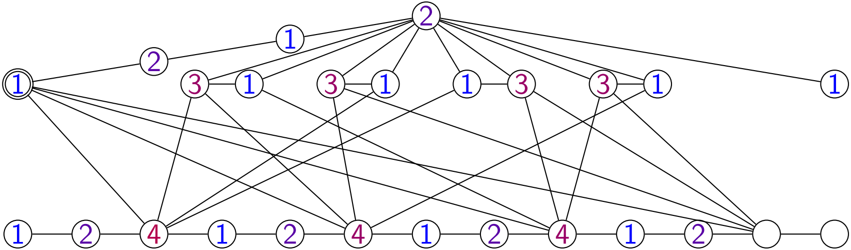

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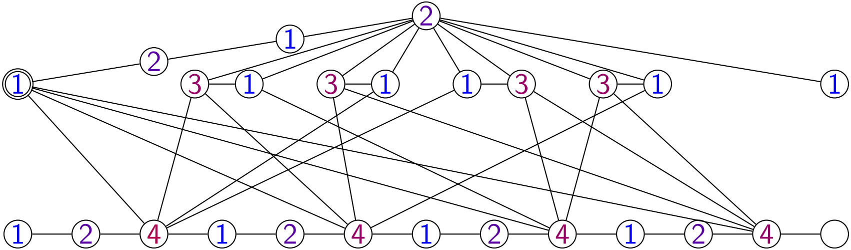

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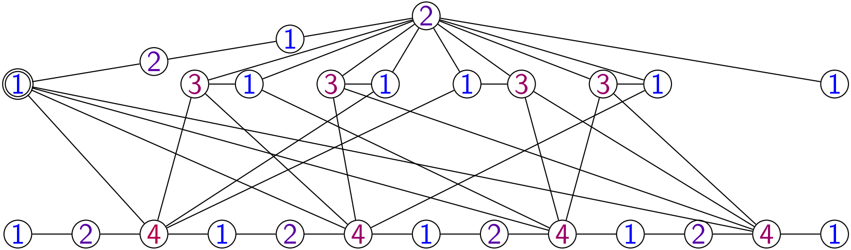

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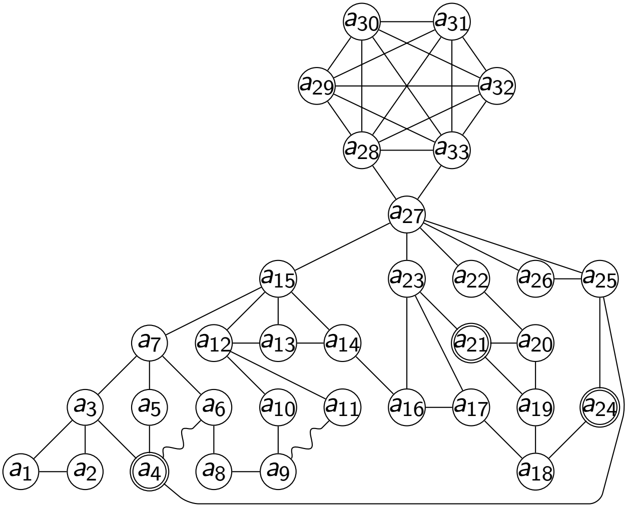
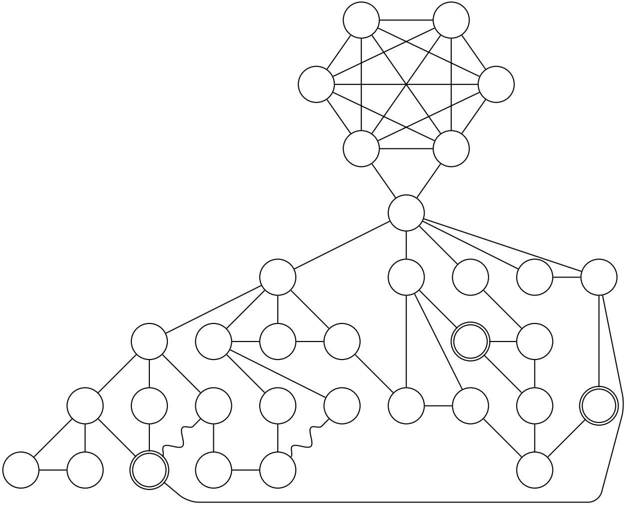
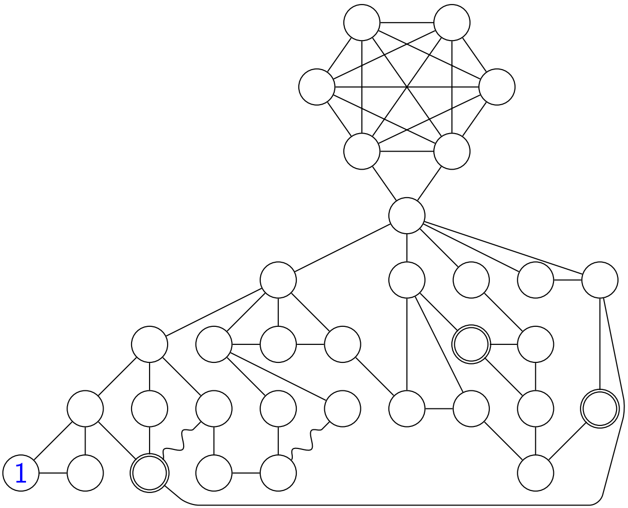


The doubly-circled vertices are linked to all the *clause* vertices c_js.



A connected Grundy coloring achieving color 7.



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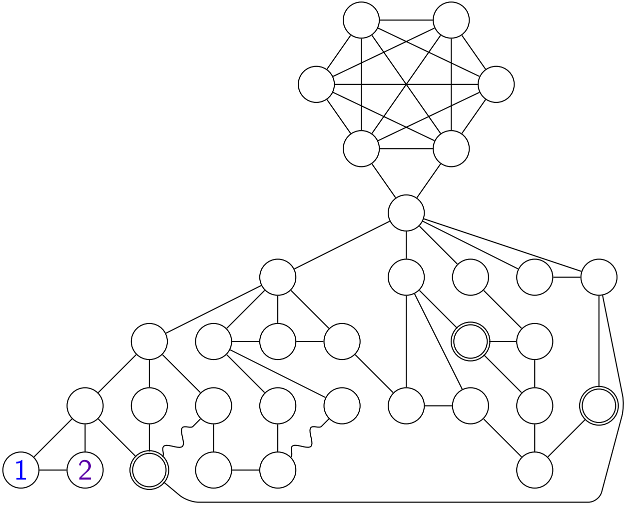

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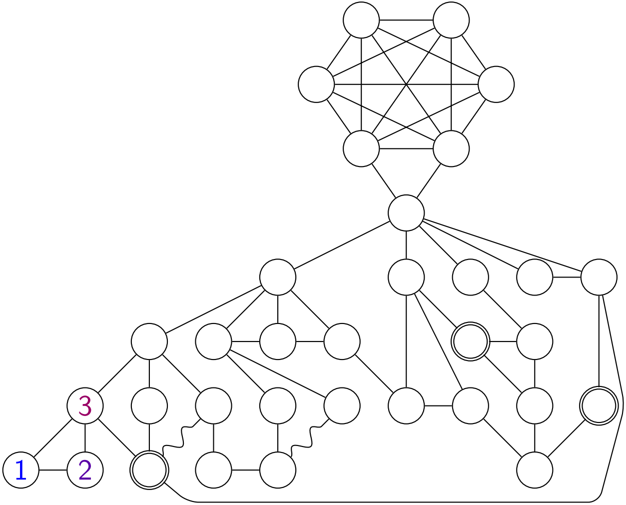

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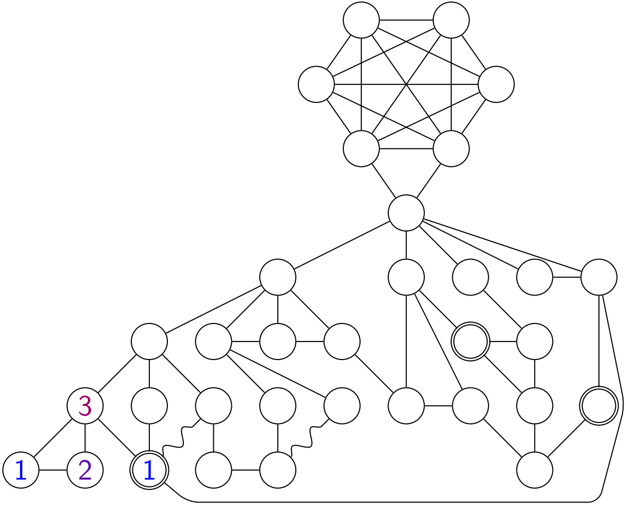

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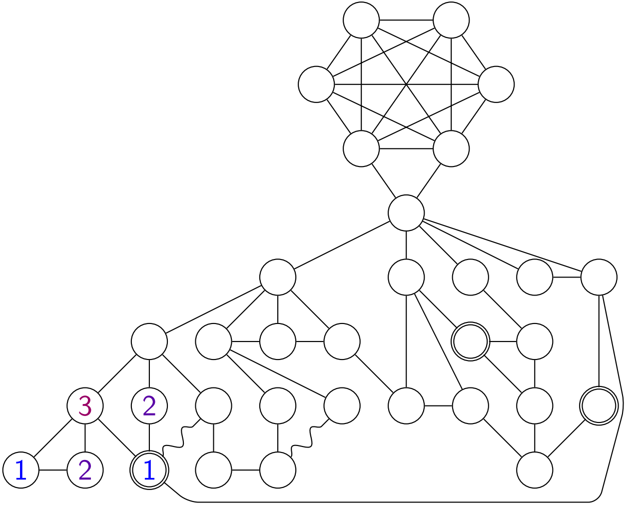

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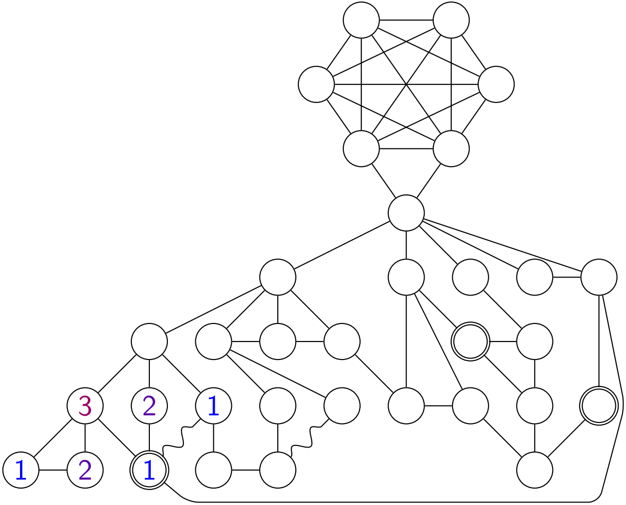

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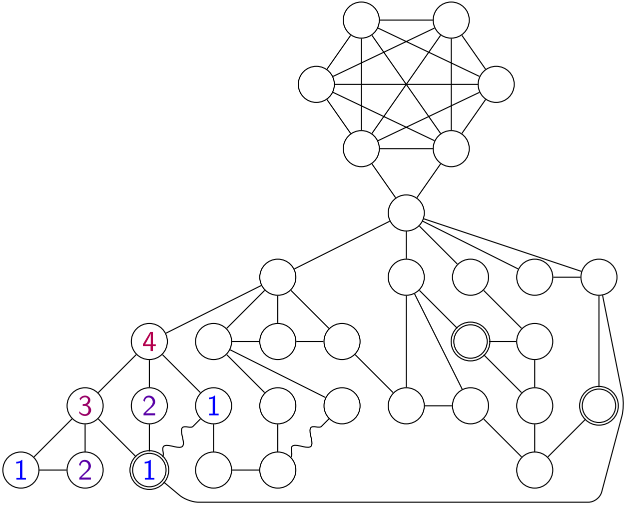

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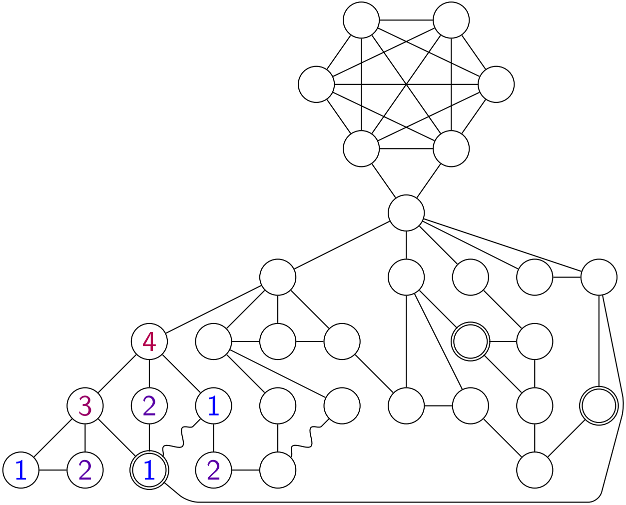

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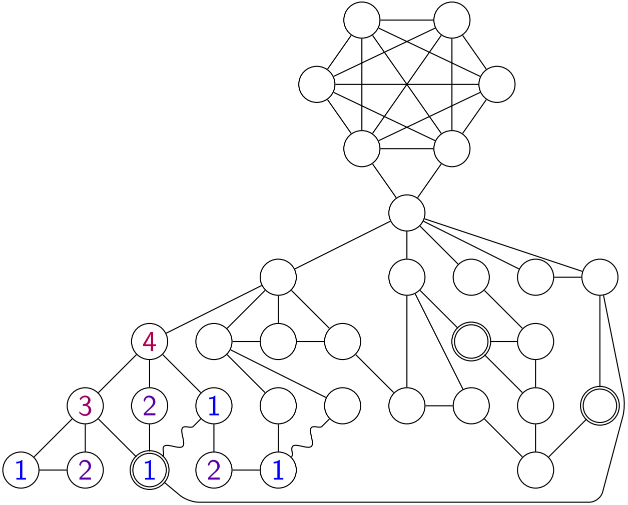

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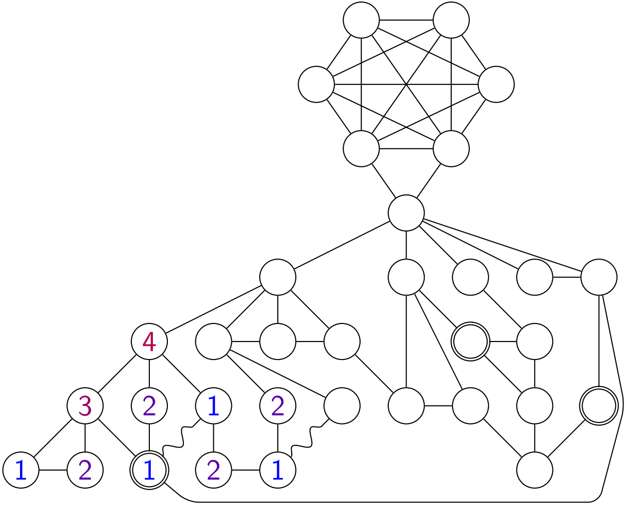

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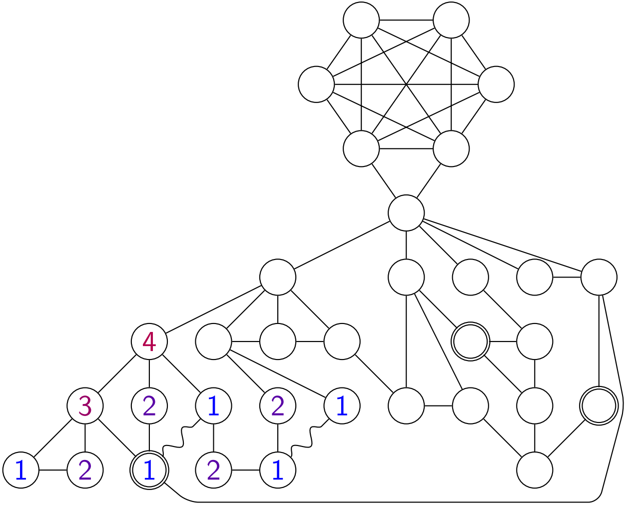

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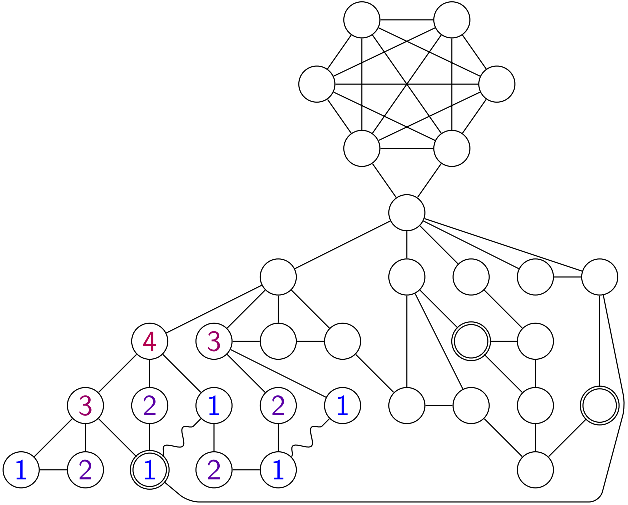

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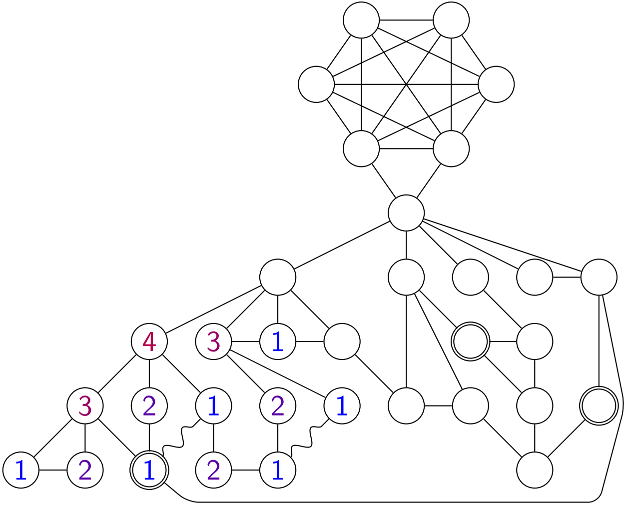

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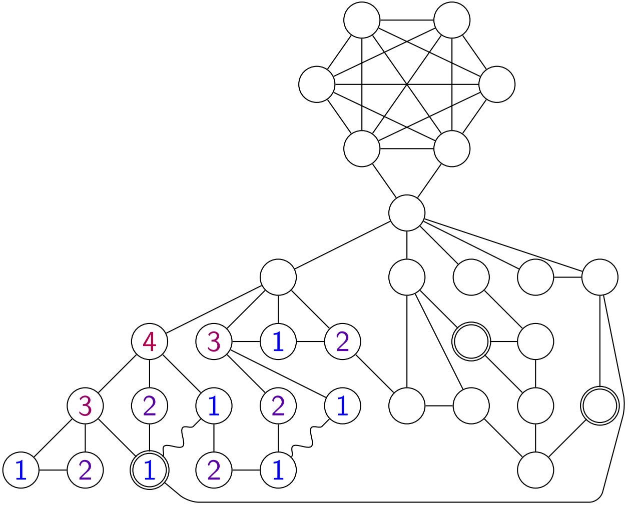

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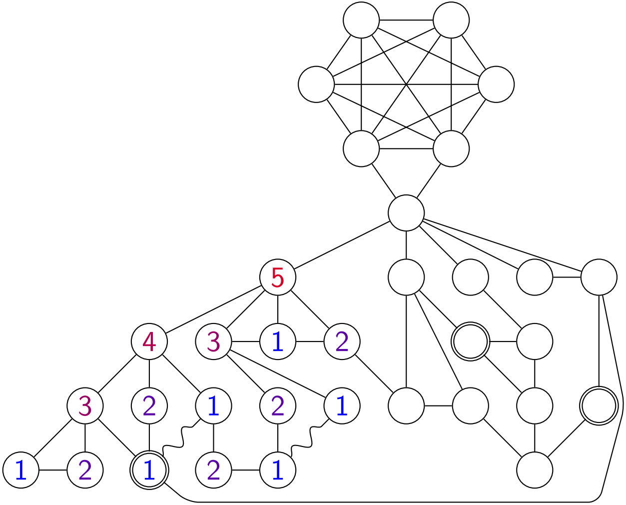

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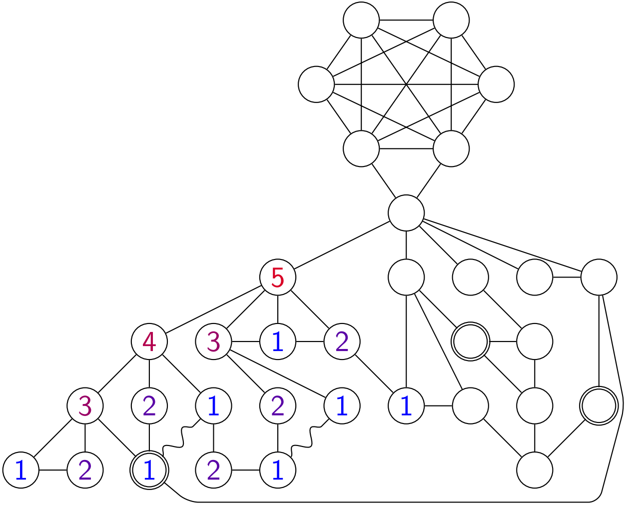

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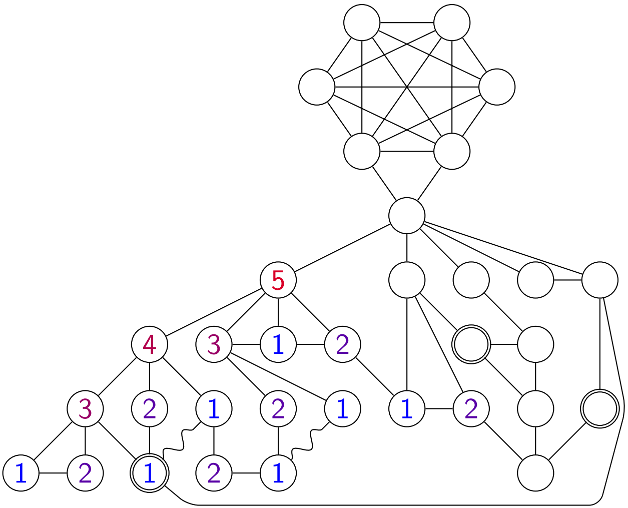

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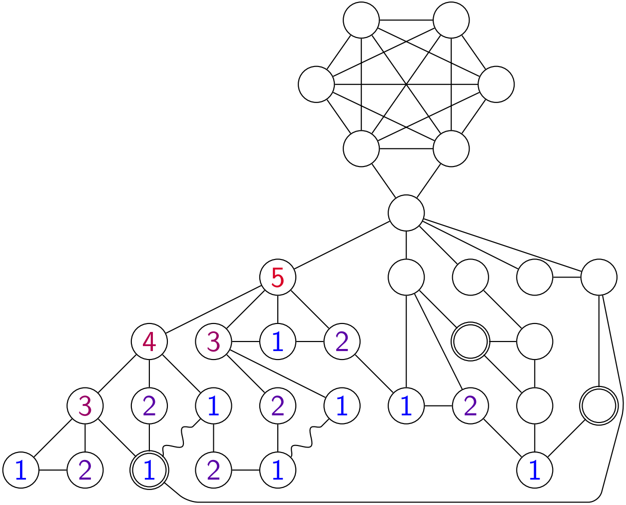

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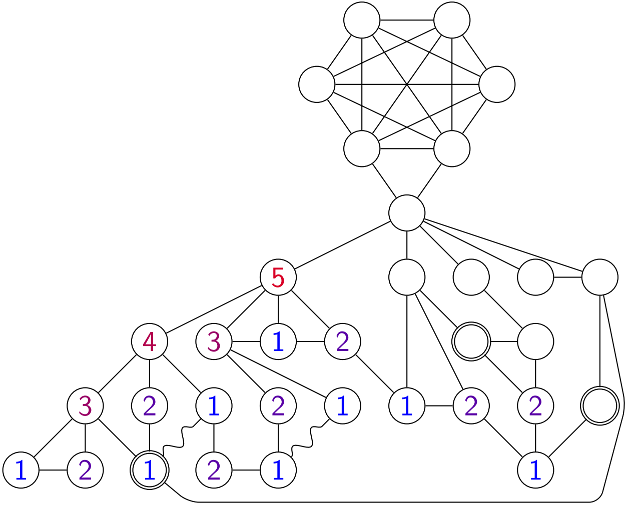

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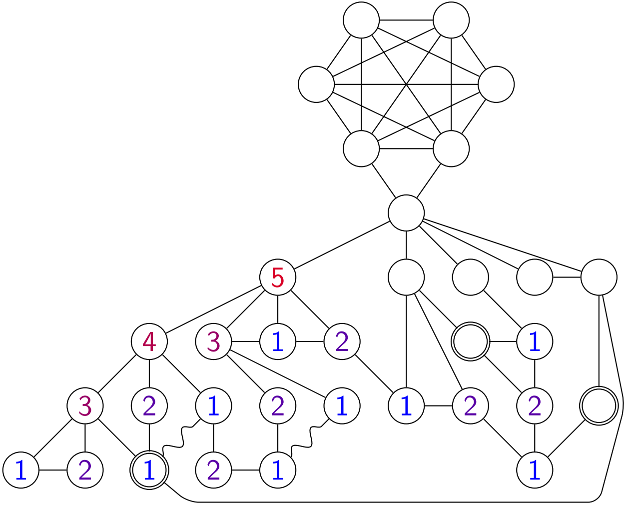

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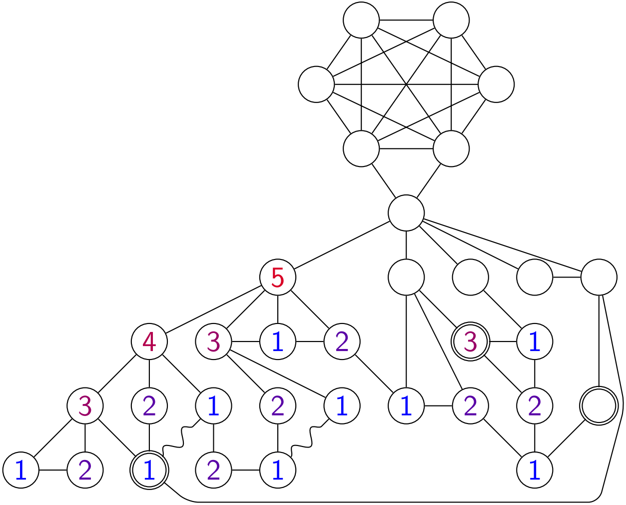

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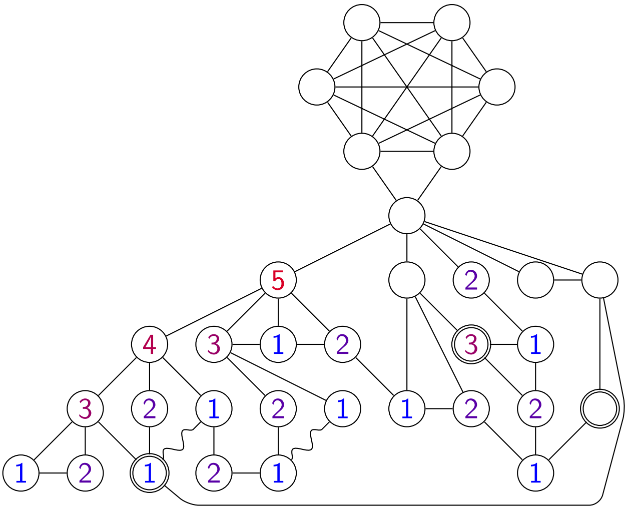

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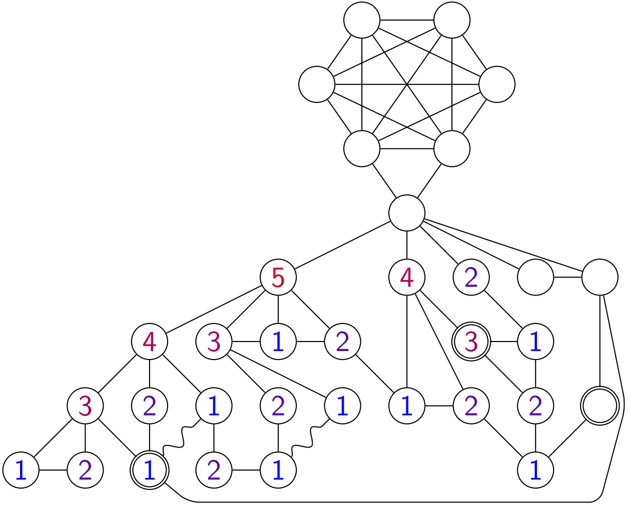

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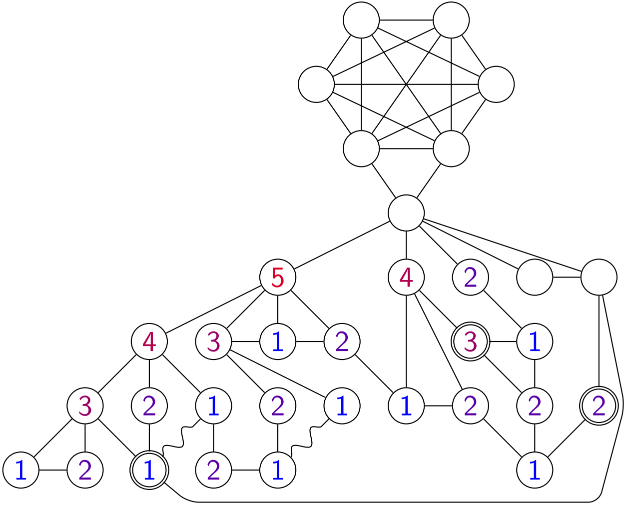

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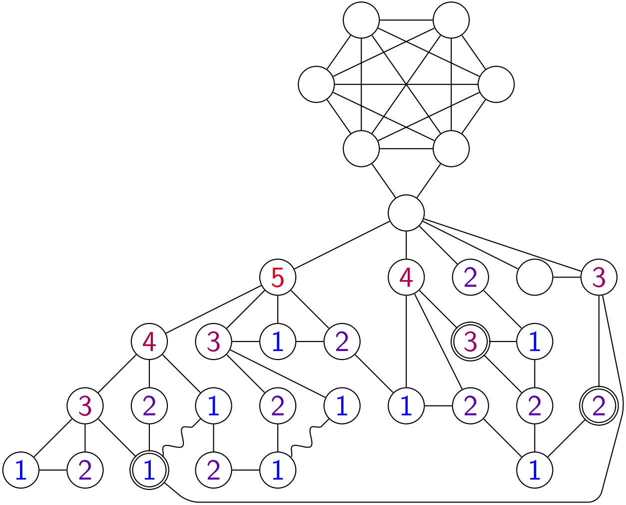

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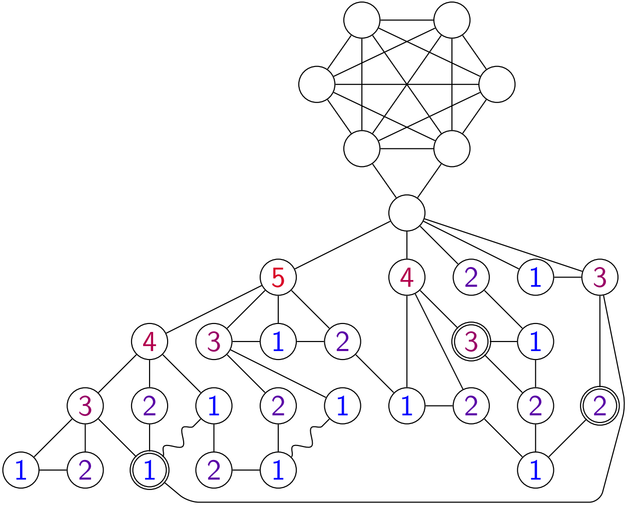

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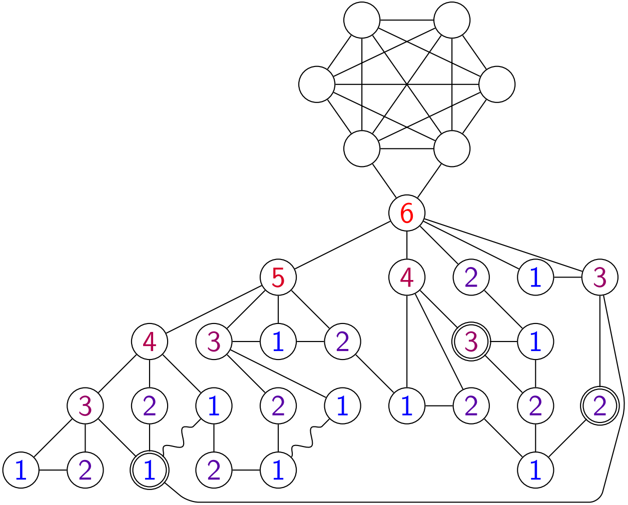

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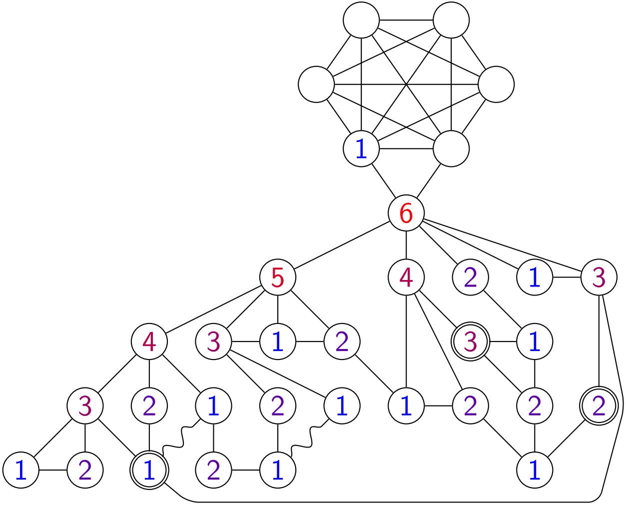

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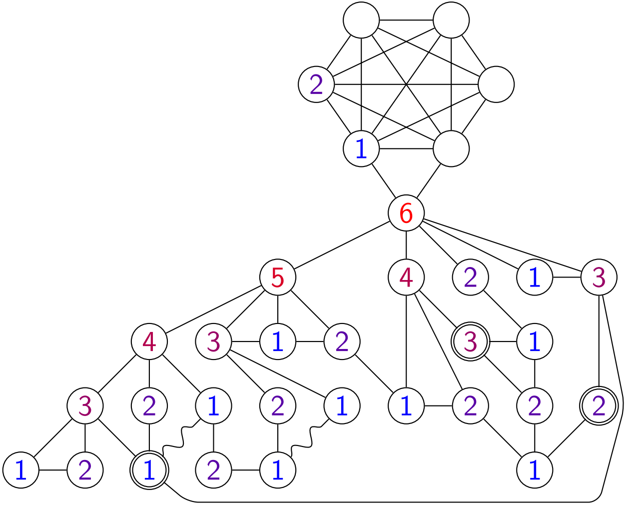

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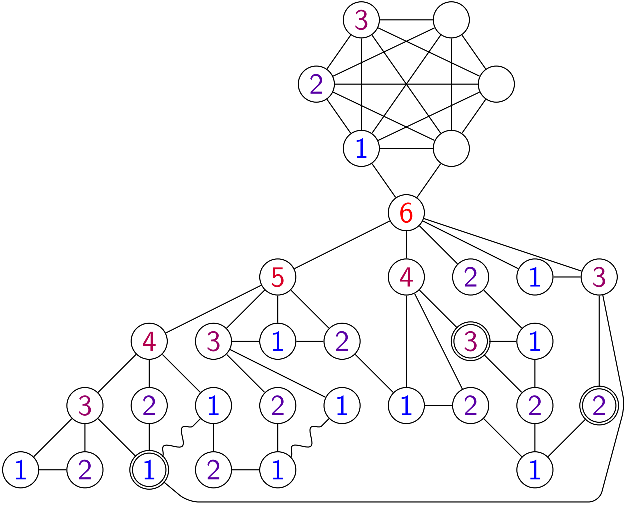

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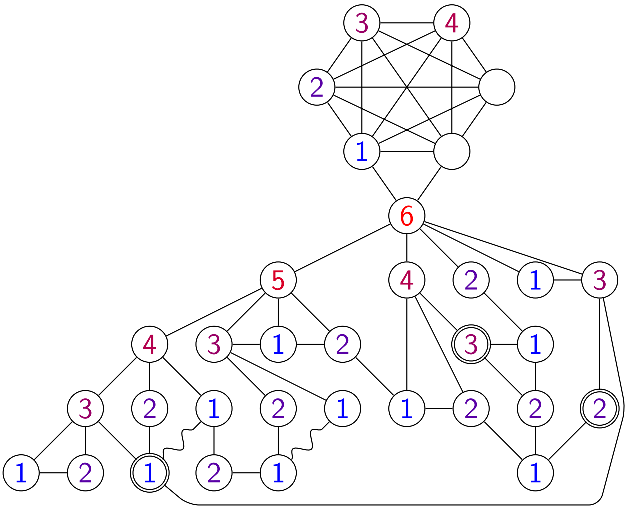

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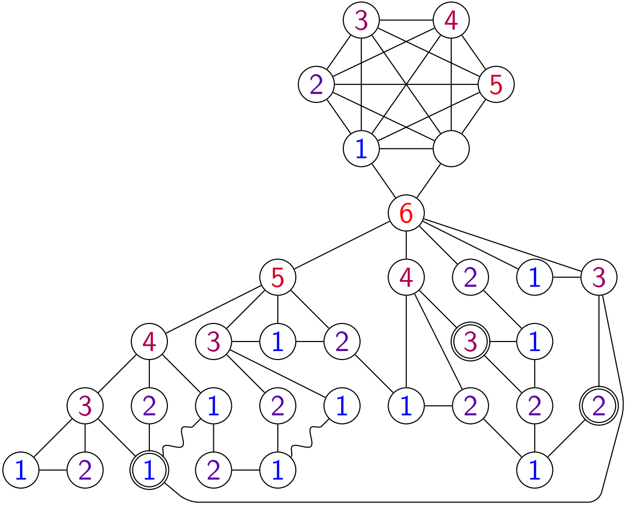

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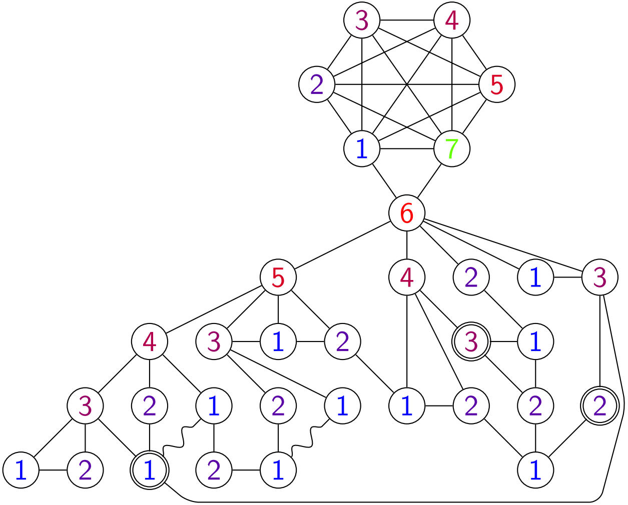

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Weak Grundy Coloring

Color Coding

Theorem

WEAK GRUNDY COLORING *is solvable in* $O^*(k^{2^{k-1}})$.

- Color each vertex uniformly at random between 1 and k.

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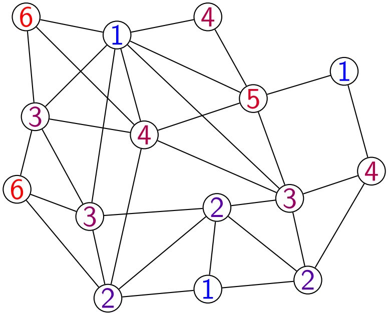
- Solving the instance is easier with this extra information.
- Repeat $100k^{2^{k-1}}$ tries to have a small probability of failure.

Guess #1



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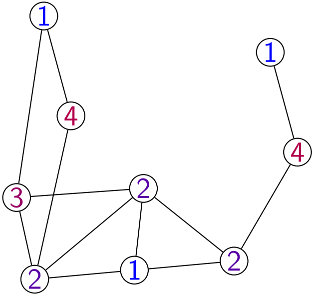

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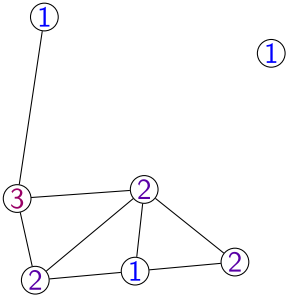
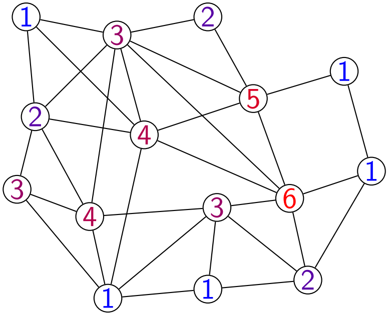


Guess #2



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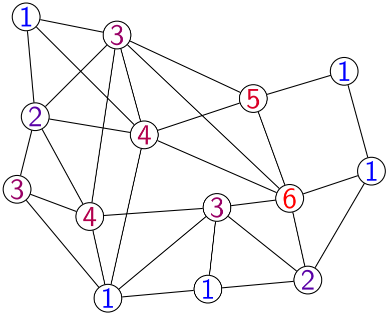

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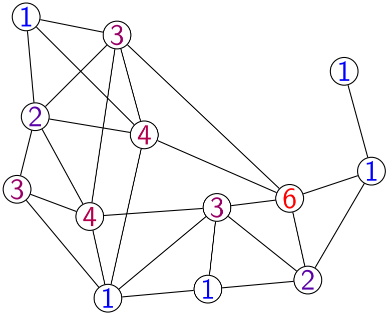

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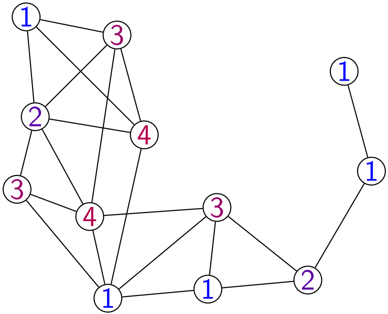
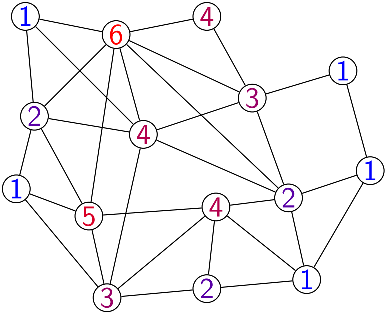


. . . $O(k^{2^k})$ unsuccessful guesses later . . .



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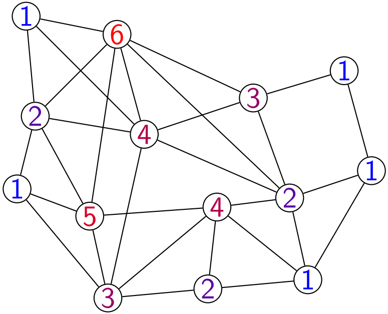

Open Questions

- Is Grundy Coloring solvable in $O(f(k)n^c)$?
- Is Grundy Coloring solvable in $O(f(w)n^c)$?
- Is Grundy Coloring solvable in $O^*(2^n)$?
- What is the complexity of Connected Grundy Coloring for $k = 4$, $k = 5$ (and $k = 6$)?