The Complexity of Grundy Coloring and its Variants

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Grundy coloring

The worst way of reasonably coloring a graph.

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- Order the vertices $v_1, v_2, \ldots, v_n$ to ***maximize*** #colors used by the greedy coloring.
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- Weak version: v_i can be colored with any color $\leqslant c(v_i)$.

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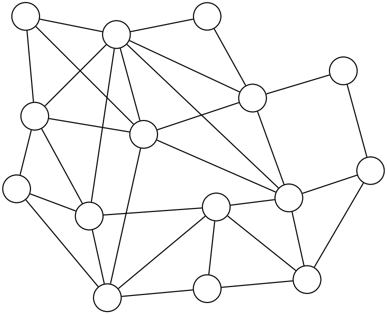
Grundy number $\Gamma(G)$, connected/weak Grundy number



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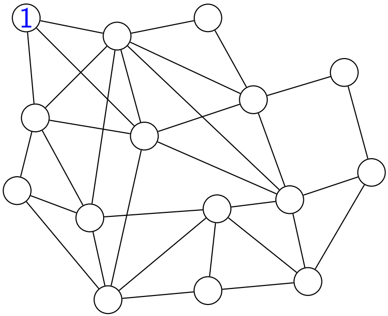

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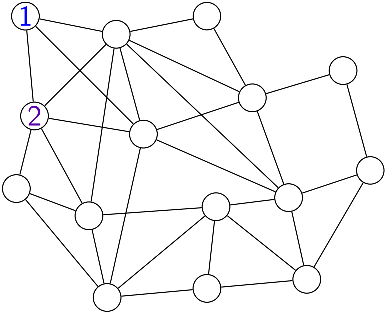

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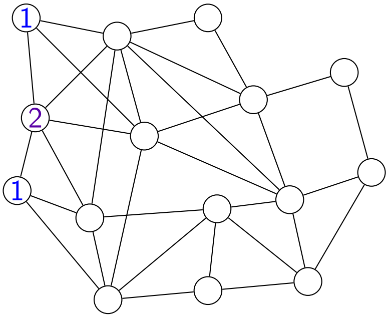

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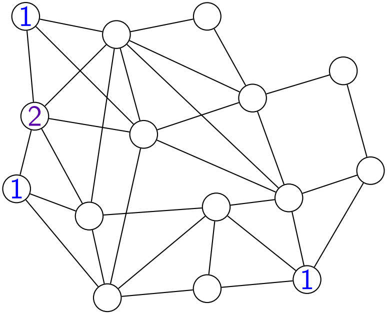

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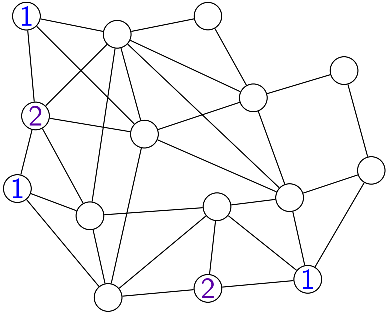

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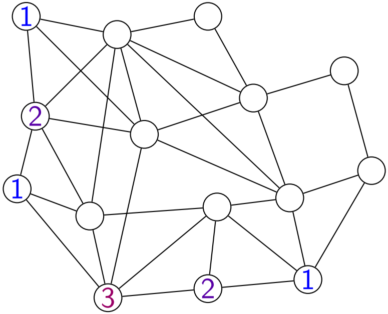

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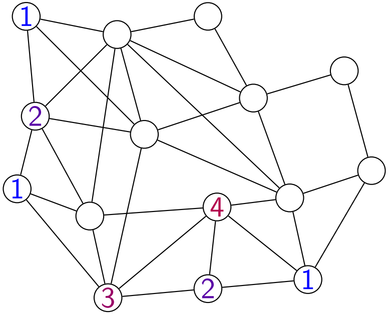

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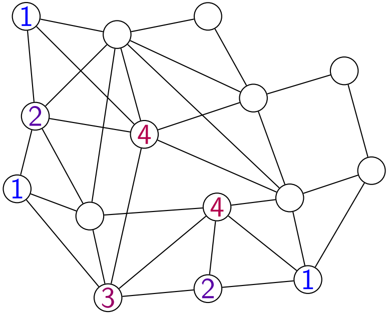

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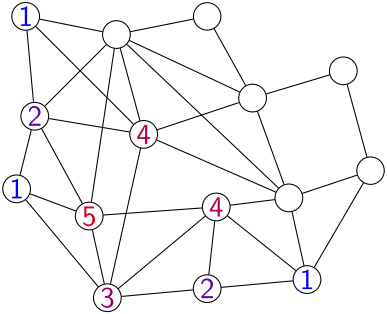

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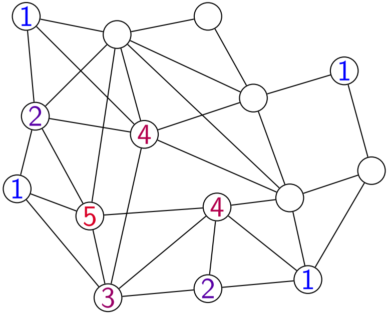

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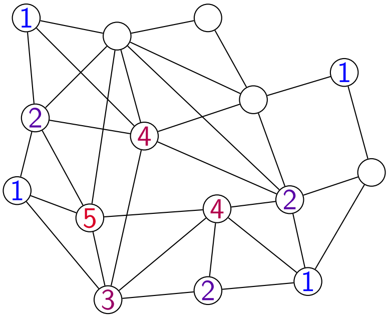

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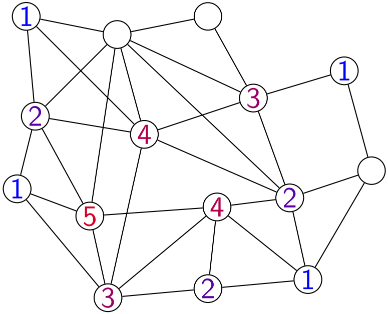

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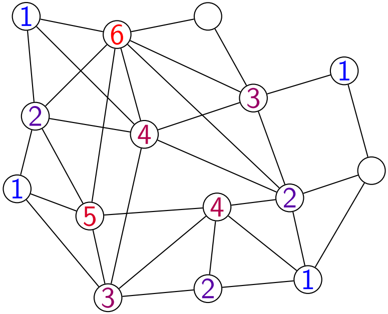

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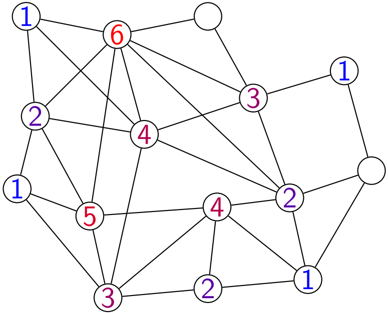
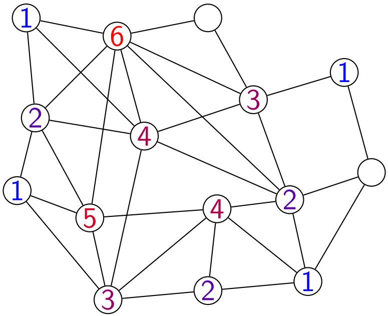
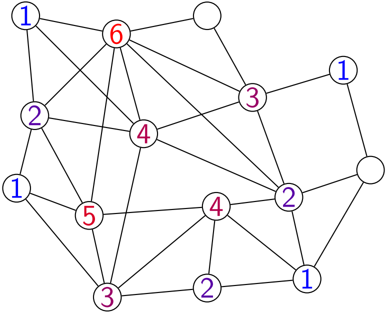


Was it a weak Grundy coloring?



Was it a connected Grundy coloring?



(Minimal) witness = (minimal) induced subgraph having the same X Grundy number, where $X \in \{\ ^{-1}, \text{weak}, \text{connected}\}$.

A brief History of Grundy colorings

- 1939: Studied in directed acyclic graphs by Grundy.
- 1979: Formally defined by Kristen and Selkow.
- 1983: Ochromatic number defined by Simmons.
- 1987: Erdös et al. proved that ochromatic number = Grundy number.

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- 1983: Ochromatic number defined by Simmons.
- 1987: Erdös et al. proved that ochromatic number = Grundy number.
- 2011: Weak Grundy defined by Kierstead and Saoub.
- 2014: Connected Grundy defined by Benevides et al.

Algorithmic motivations

- $\Gamma(G)$ upper bounds the number of colors used by any greedy heuristic for Min Coloring.
- $\Gamma(G) \leqslant C\chi(G)$ on some classes of graphs gives a C-approximation for Min Coloring.
- Online coloring.
- see Sampaio's PhD thesis for further motivations.

More questionable motivations

- (Weak) Grundy Coloring is to (Independent) Dominating Set what Coloring is to Independent Set.
- Is Sudoku more interesting than Grundy Coloring?
- Idea for commercialization: rename color i to 2^i and set the goal to 2048.

Complexity of computing the Grundy number

$k = \Gamma(G)$ and w denotes the treewidth of the graph
XP algorithm: $n^{f(\kappa)}$; FPT algorithm: $f(\kappa)n^{O(1)}$.

- NP-hard on (co-)bipartite, chordal, line, claw-free graphs.
- Solvable in $n^{2^{k-1}}$ [Z '06].
- Solvable in $2^{O(kw)}n$ and[1] in $n^{O(w^2)}$ [TP '97].

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parameter	XP	FPT
k	$n^{2^{k-1}}$	?
w	$n^{O(w^2)}$	?
$k + w$	-	$2^{O(kw)}n$

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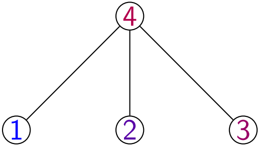

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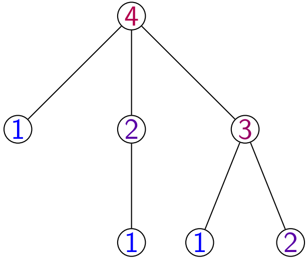

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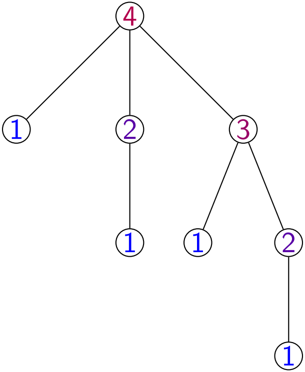

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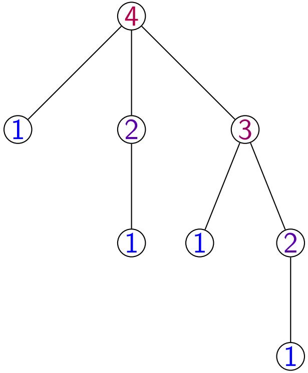

A minimal witness is of size at most 2^{k-1}.
(and its vertices are at distance $\leqslant k$ of the vertex colored by k)

Theorem (Zaker '06)

The Grundy number can be computed in $O(f(k)n^{2^{k-1}})$.

Complexity of computing the Grundy number

Outline [BFKS '15]:

- Solvable in time $O^*(2.443^n)$.
- FPT in various classes such as H-minor free graphs, chordal graphs, claw-free graphs.
- An $O^*(2^{o(w \log w)})$ algorithm would contradict the ETH.

Complexity of the variants

- Weak Grundy Coloring is NP-complete [GV '97].
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Outline [BKFS '15]:

- Weak Grundy Coloring is solvable in $O^*(2^{2^{O(k)}})$ but not in $2^{2^{o(k)}}2^{o(n+m)}$ under the ETH.
- Connected Grundy Coloring is NP-complete for $k = 7$.

Grundy Coloring

Exact exponential algorithm

Try all possible orderings and run the corresponding greedy coloring: $O^*(n!)$.

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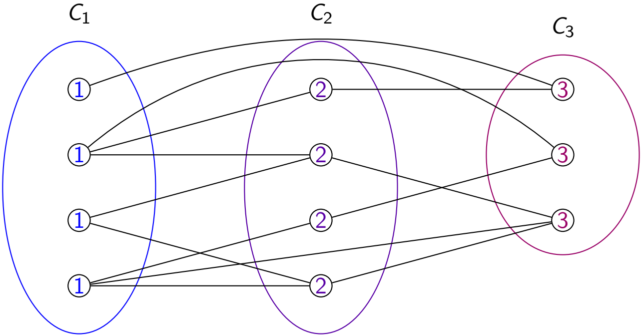
Can we improve on this trivial algorithm?

Solving Grundy Coloring

In a minimal witness:



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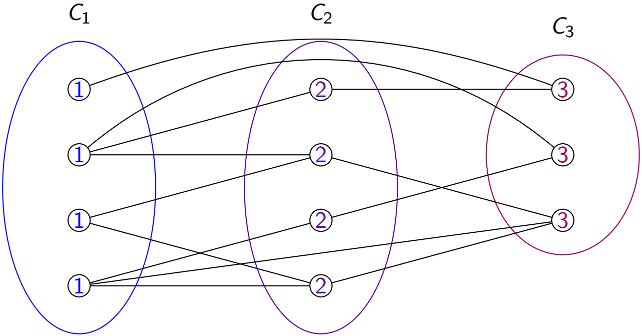

- C_i is an independent dominating set in $G[\bigcup_{i \leqslant j \leqslant k} C_j]$.
- $\Gamma(S) = \max\{\Gamma(S \setminus X), X \text{ ind. dom. set in } G[S]\} + 1$.

Solving Grundy Coloring

- Enumerating the independent dominating sets takes time $O^*(3^{\frac{n}{3}}) = O(1.443^n)$.
- So, filling a cell of the table takes 1.443^i for a subset of size i.
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$O^*(2^n)$ aglorithms? Polynomial space? Connected Grundy?

On H-minor free graphs

INDUCED SUBGRAPH ISOMORPHISM: B induced subgraph of A?

Theorem (FG '01)

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Why does it imply an FPT algorithm for Grundy Coloring?

Minimal witnesses have size at most 2^{k-1}.
So, there are less than $k2^{2^{2k}}$ graphs B to try.

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Besides, $\omega(G) \leqslant \Gamma(G)$.
Therefore, $\mathrm{tw}(G) \leqslant \Gamma(G) - 1$
$\rightsquigarrow$ run FPT algorithm in $2^{O(\mathrm{tw}(G)\Gamma(G))} = 2^{O(\Gamma(G)^2)}$.

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In a claw-free graph, if $d(v) = \Delta(G)$, then $\chi(G[N(v)]) \geqslant \frac{\Delta(G)}{2}$.
Besides, $\Gamma(G) \geqslant \chi(G) \geqslant \chi(G[N(v)])$ holds in any graph.

Weak Grundy Coloring

Color Coding

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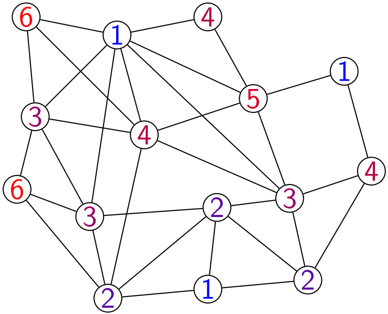
- Solving the instance is easier with this extra information.
- Repeat $100k^{2^{k-1}}$ tries to have a small probability of failure.

Guess #1



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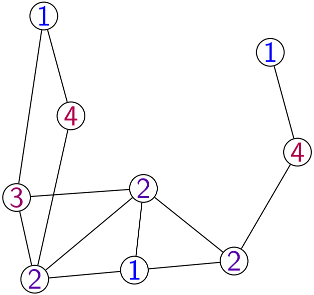

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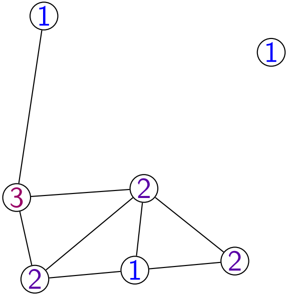
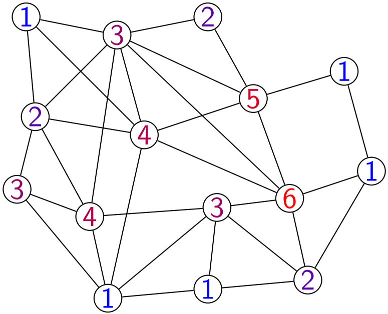


Guess #2



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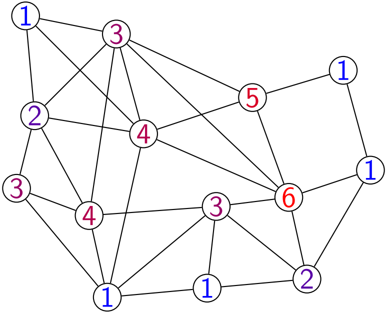

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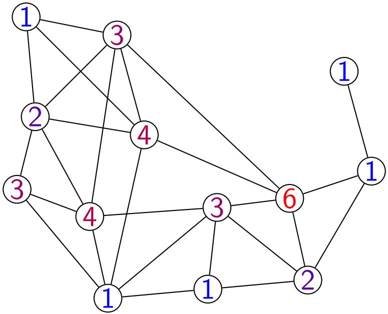

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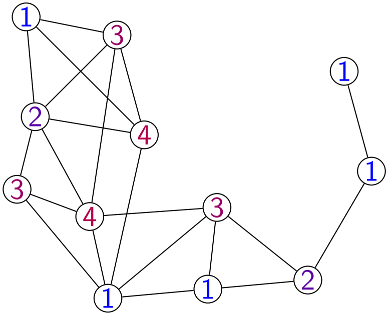

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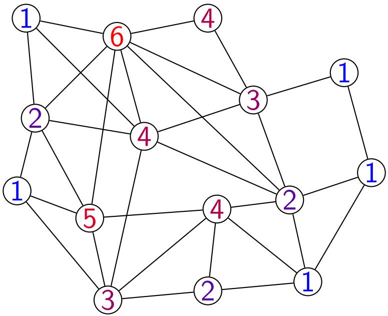

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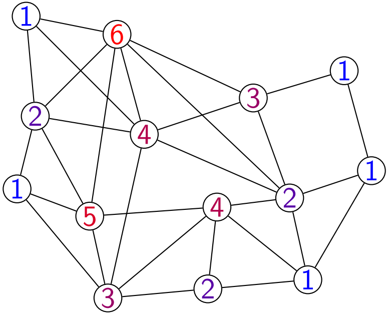
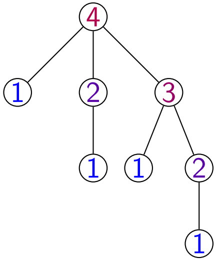


Back to binomial trees $T_k = v(T_1, T_2, \ldots, T_{k-1})$



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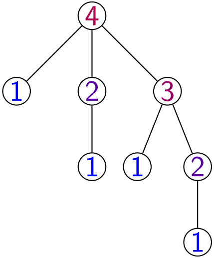

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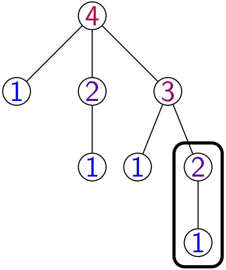
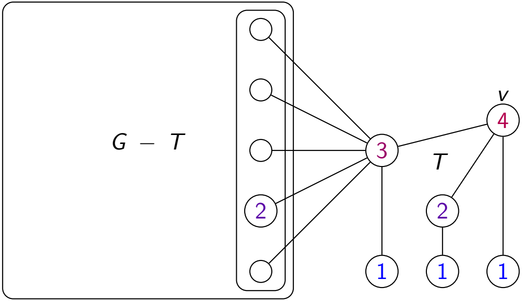


A *unique* optimal (weak) Grundy Coloring.
Dominant subtree: largest among its siblings.

The binomial tree with missing dominant subtrees



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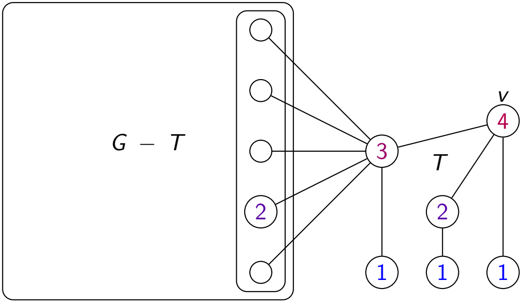

User guide: show that *only* v can get color 4, with degree-based considerations.

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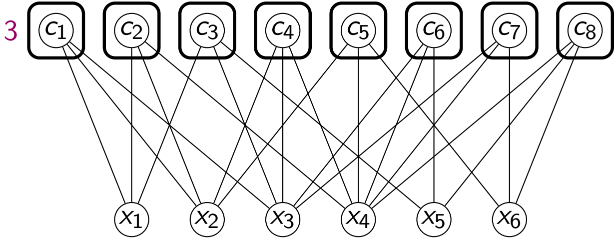

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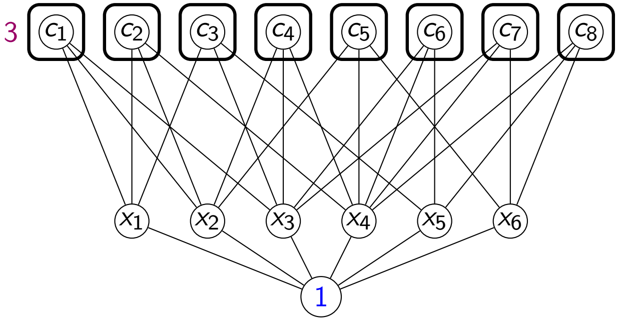

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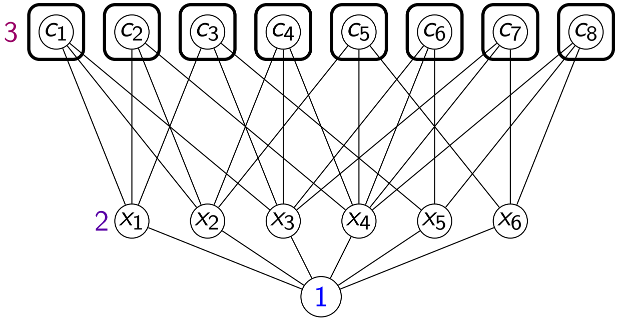

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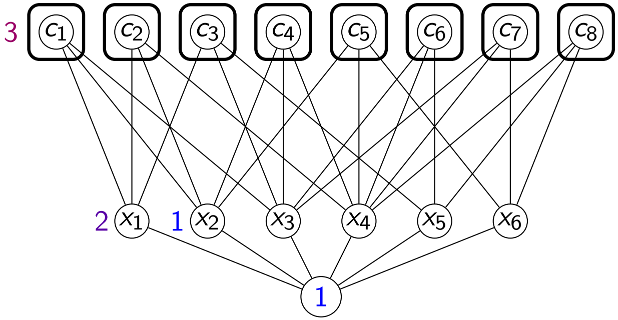

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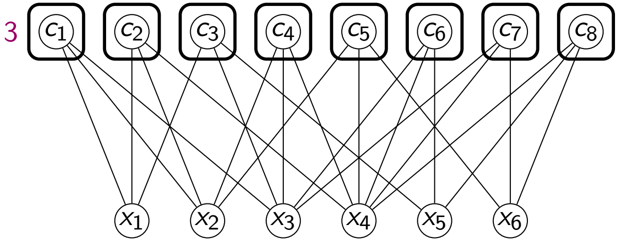

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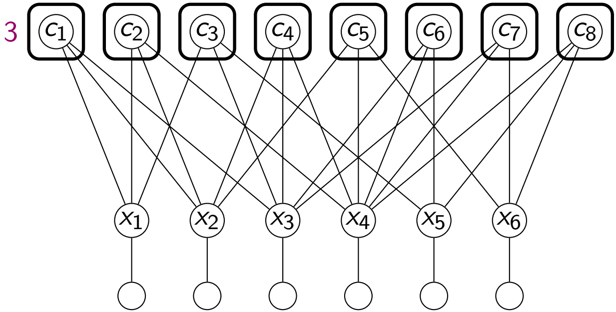

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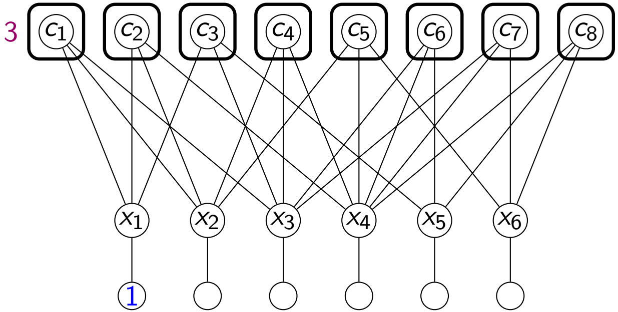

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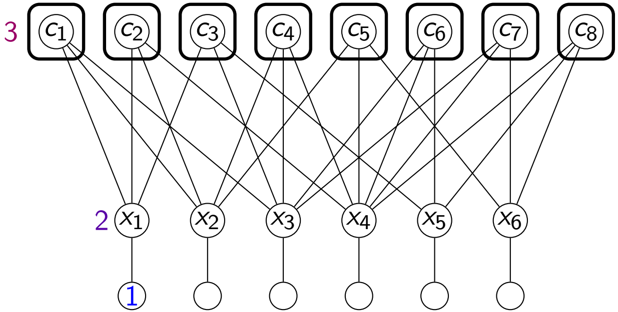

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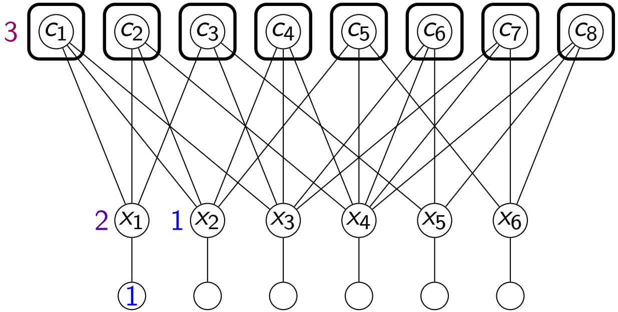

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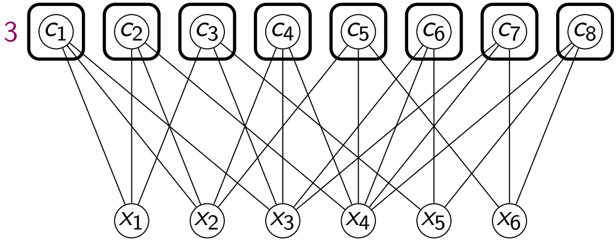

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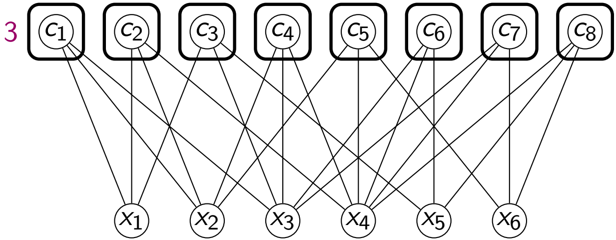

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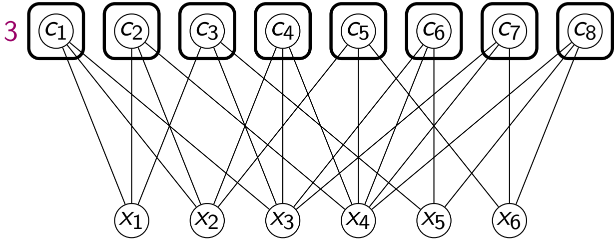

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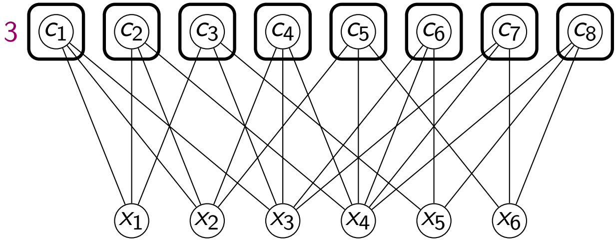

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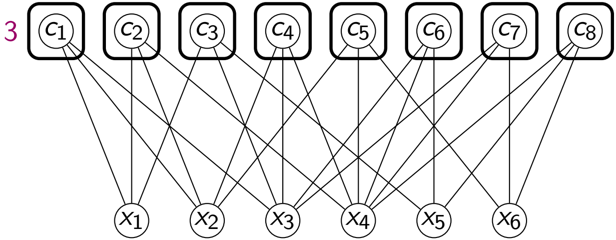

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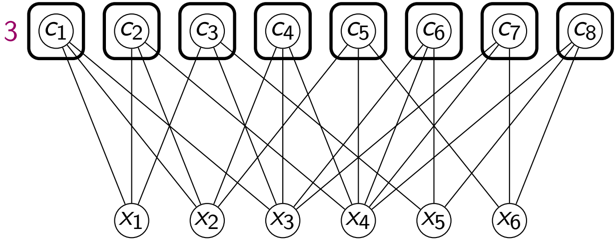

Solving (Weak) Grundy in $2^{2^{o(k)}} 2^{o(n+m)}$ would disprove the ETH.

Back to Grundy Coloring parameterized by treewidth

Binomial tree gadgetry + tricks for fine-grained lower bounds:

Theorem

Solving Grundy Coloring in $O^*(2^{o(w \log w)})$ *would disprove the ETH.*

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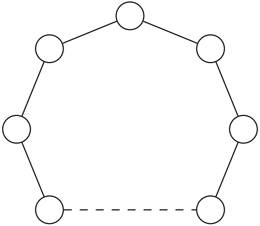
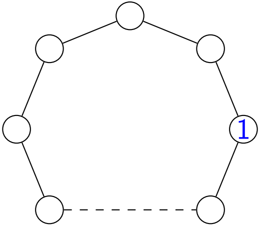
Might become tight if the $k + w$ algorithm is improved to $O^*(k^w) = O^*(w^w(\log n)^w) = O^*(w^w w^{2w}) = O^*(w^{O(w)})$.

Connected Grundy Coloring



Connected Grundy number = 3, unbounded witness



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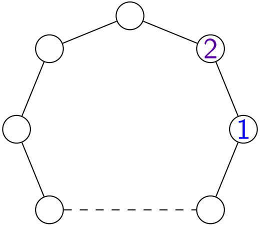

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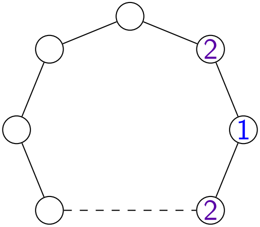

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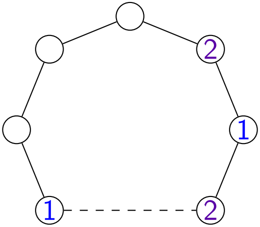

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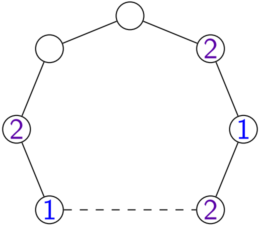

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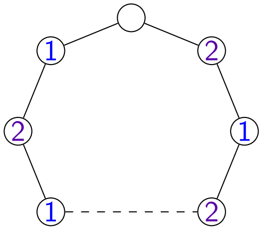

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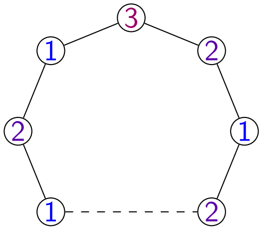

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paraNP-hardness of Connected Grundy Coloring

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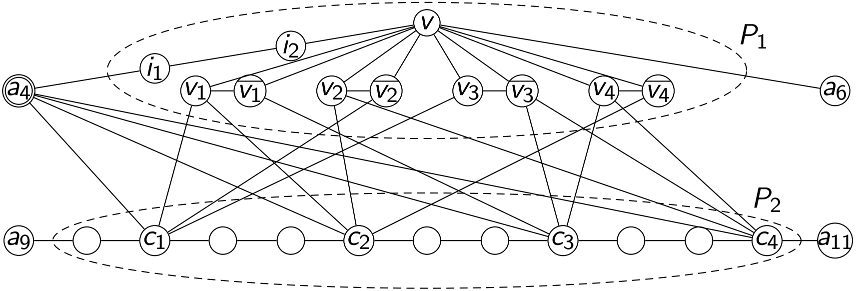
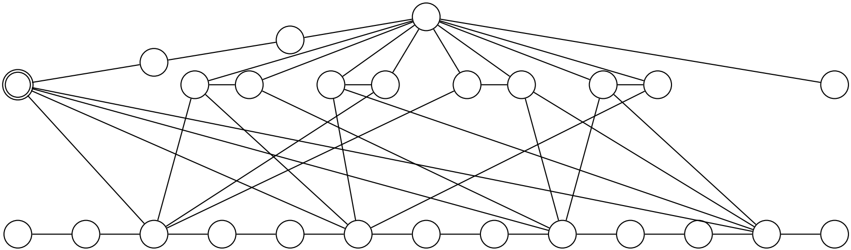
- To achieve color 7, three special neighbors of the c_js should be colored by 1, 2 and 3 respectively.

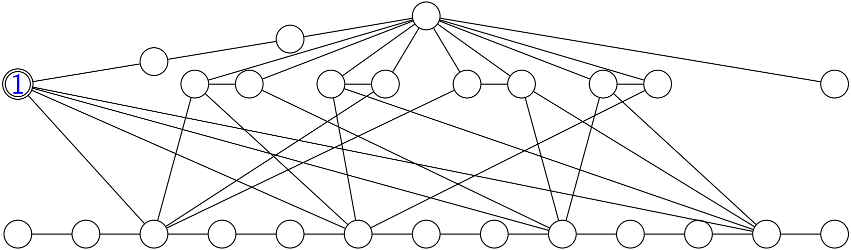


P_1 and P_2 for the instance

$\{x_1 \vee \neg x_2 \vee x_3\}, \{x_1 \vee x_2 \vee \neg x_4\}, \{\neg x_1 \vee x_3 \vee x_4\}, \{x_2 \vee \neg x_3 \vee x_4\}.$



A connected Grundy coloring setting all the c_js to 4.



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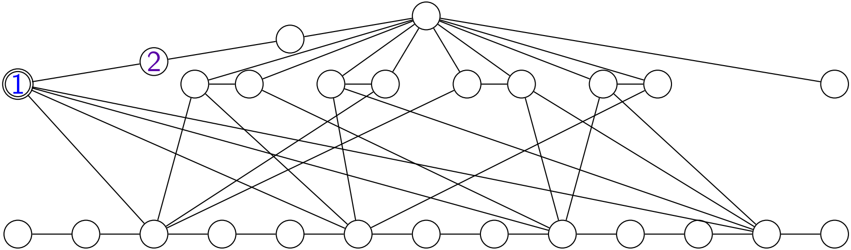

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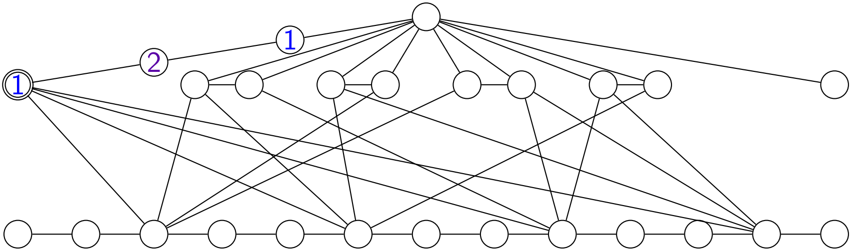

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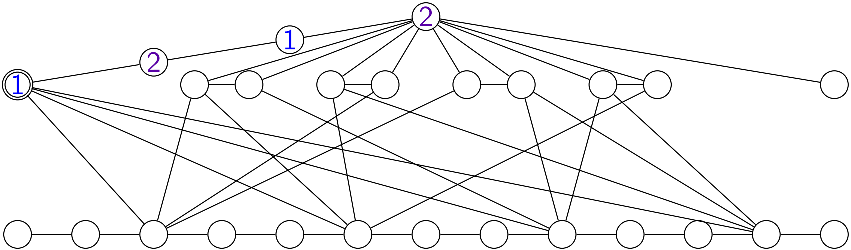

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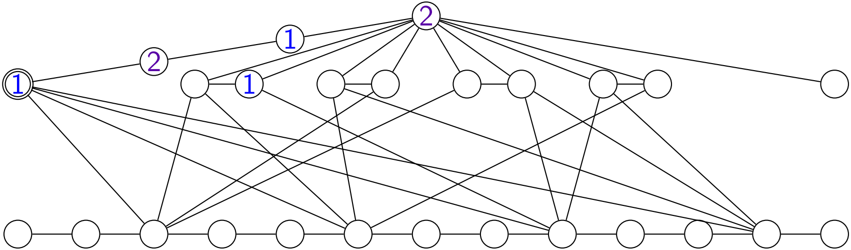

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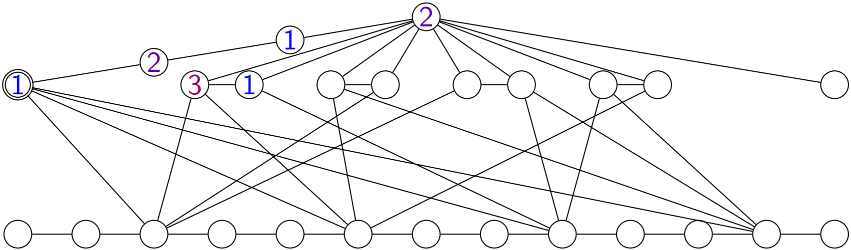

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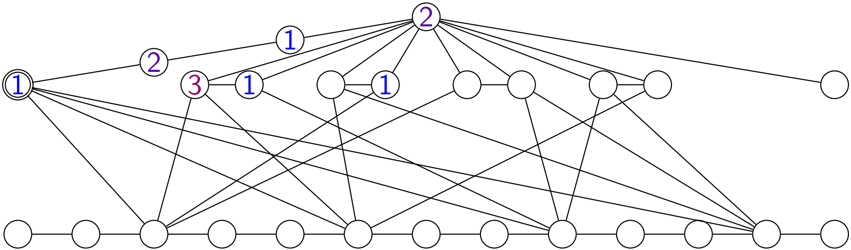

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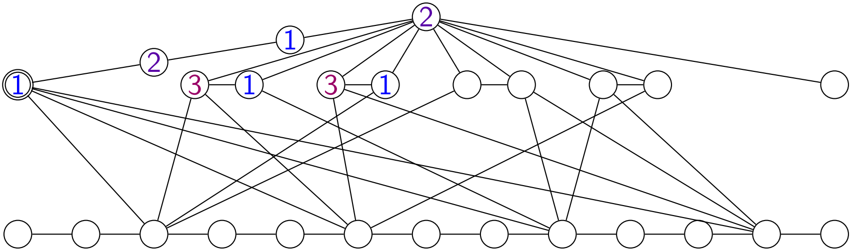

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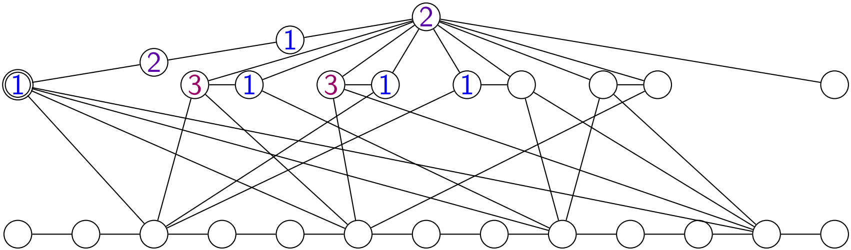

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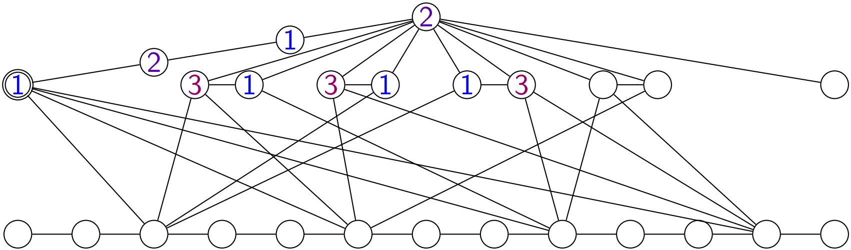

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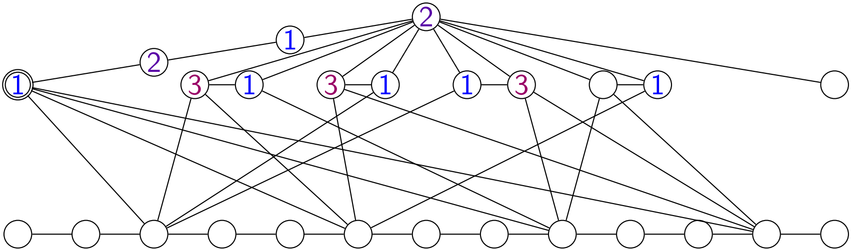

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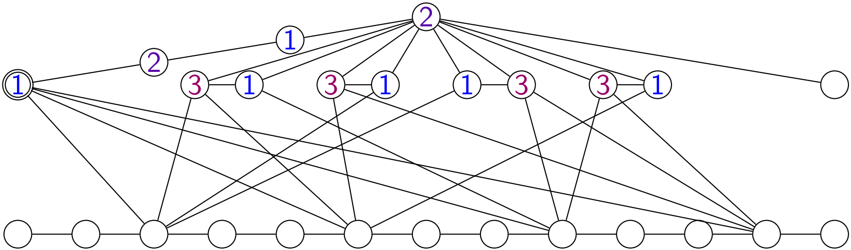

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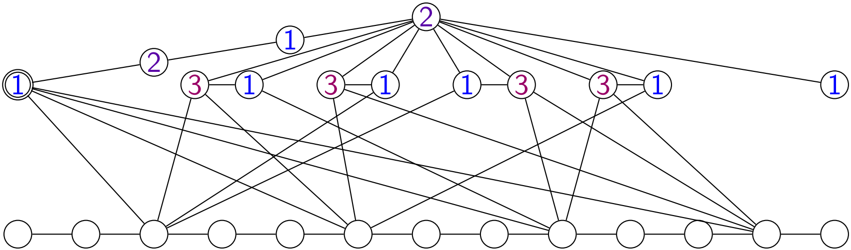

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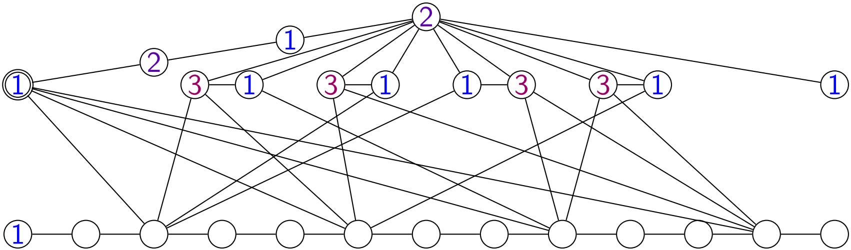

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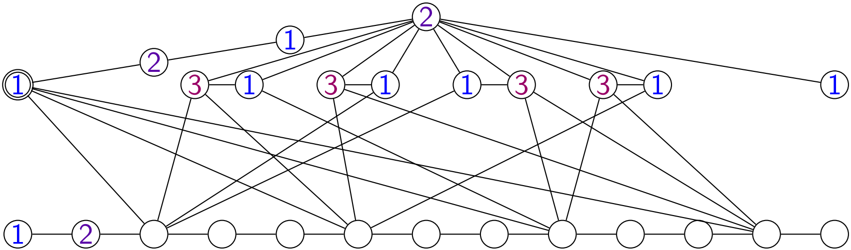

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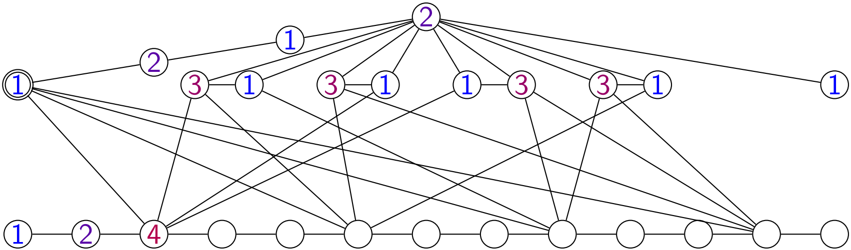

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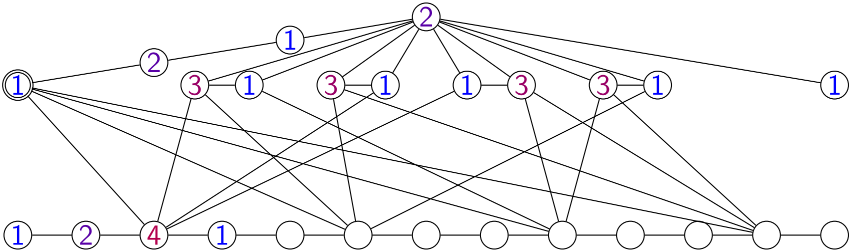

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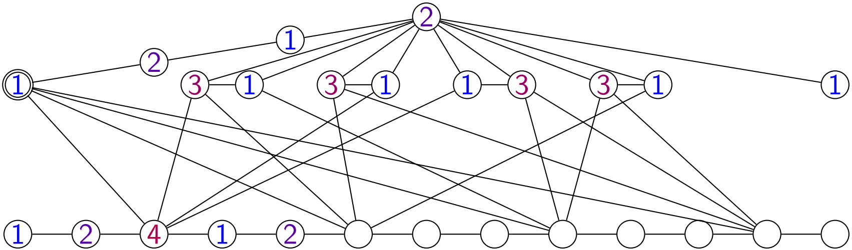

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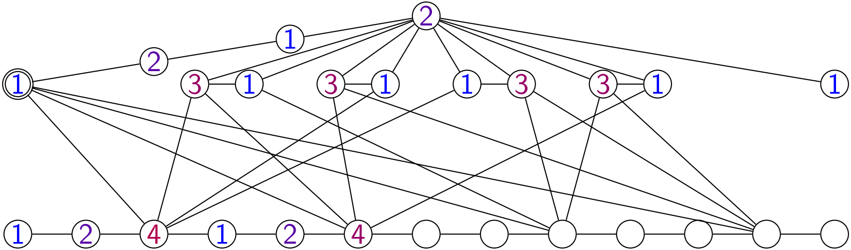

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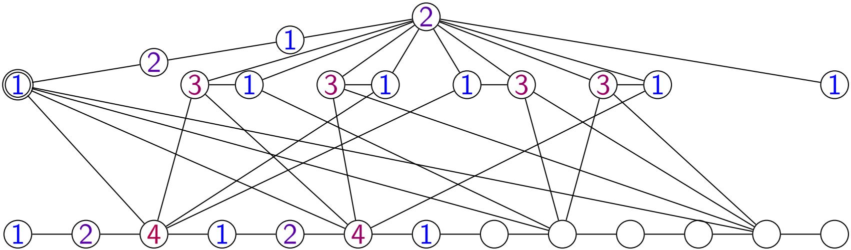

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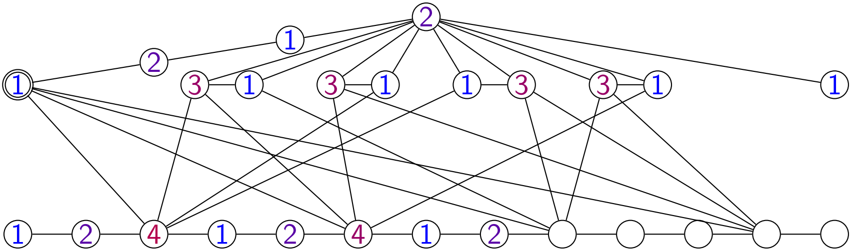

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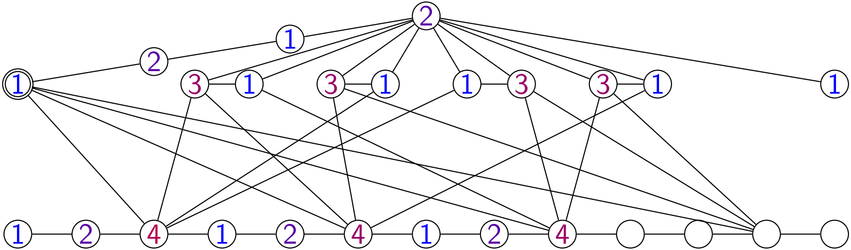

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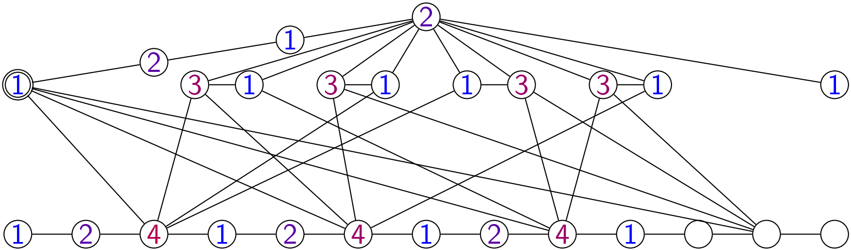

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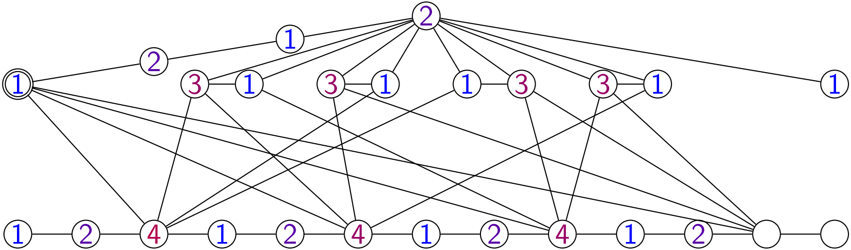

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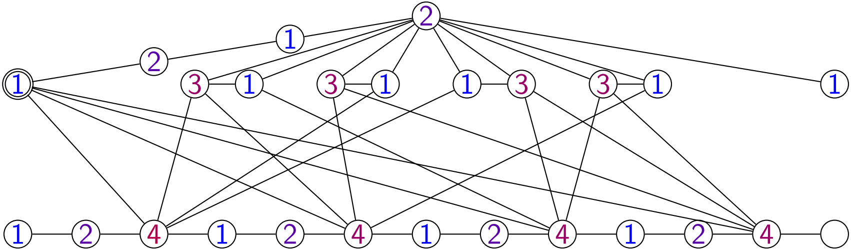
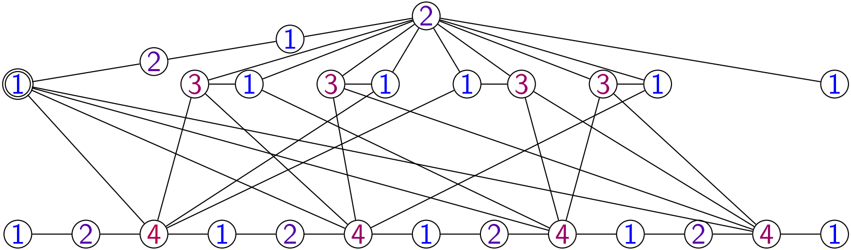
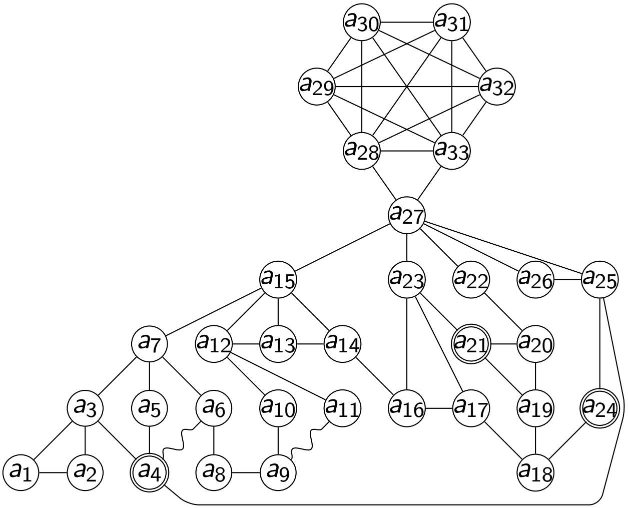


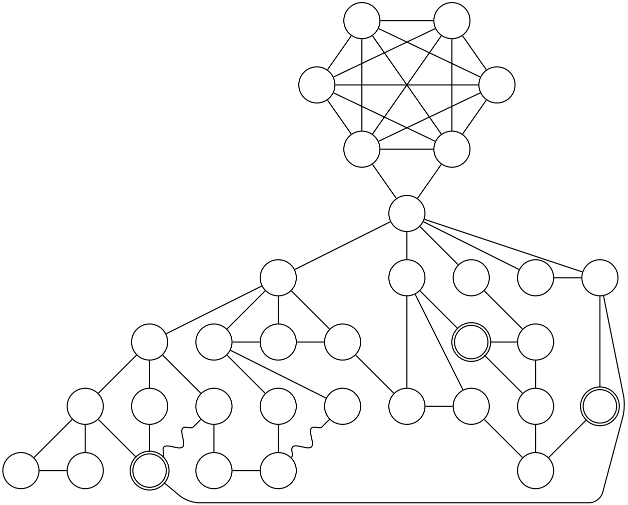
A connected Grundy coloring setting all the c_js to 4.



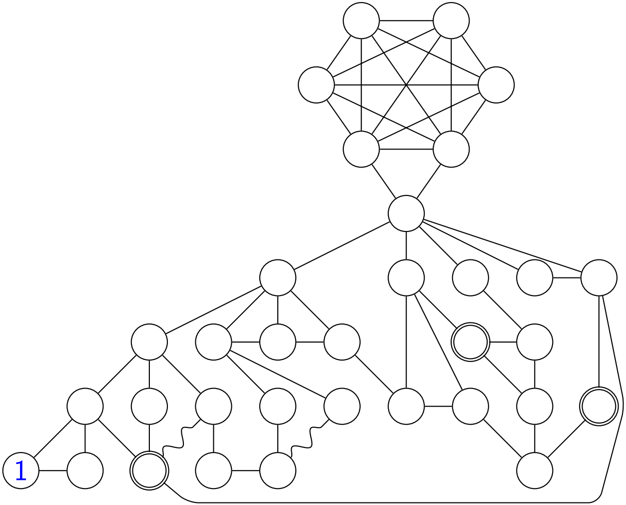
A connected Grundy coloring setting all the c_js to 4.



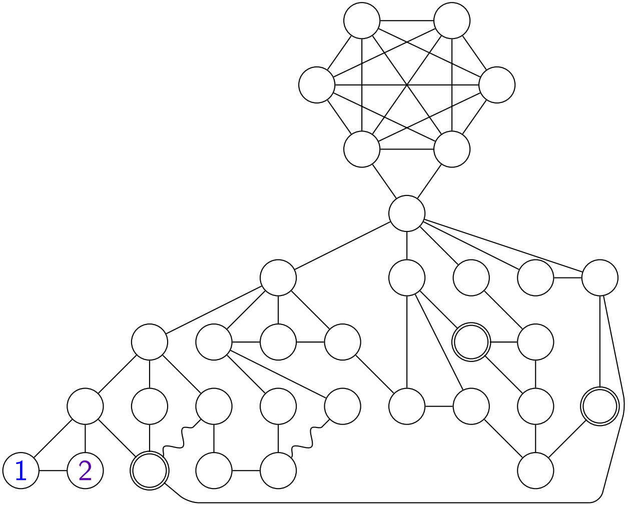
The doubly circled vertices are linked to all the *clause* vertices c_js.



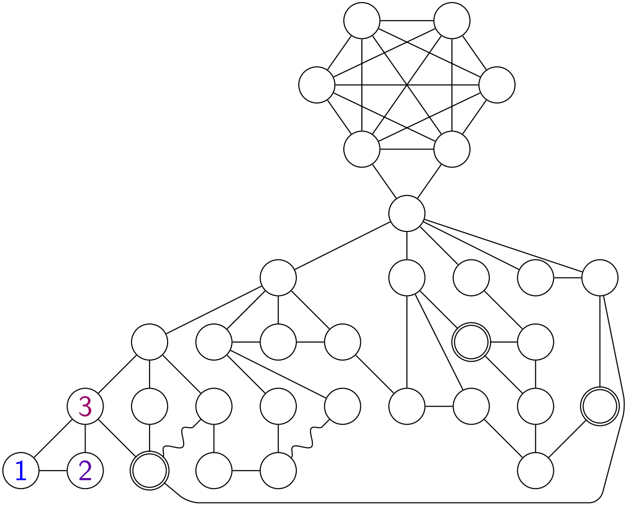
A connected Grundy coloring achieving color 7.



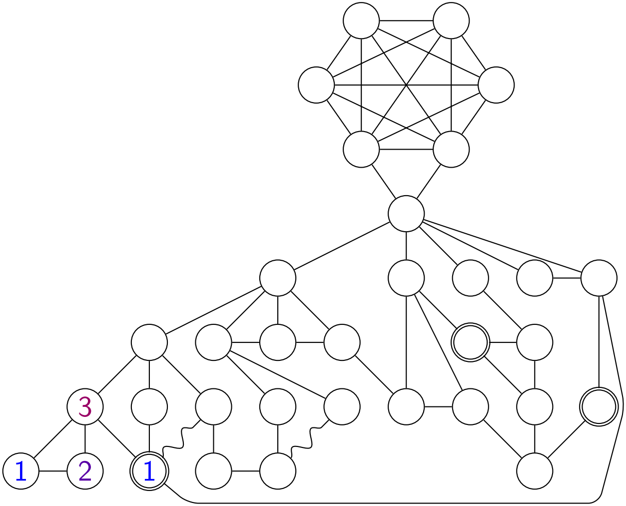
A connected Grundy coloring achieving color 7.



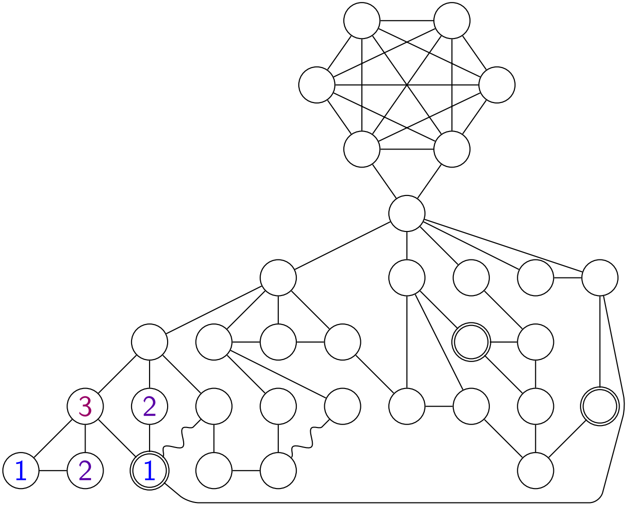
A connected Grundy coloring achieving color 7.



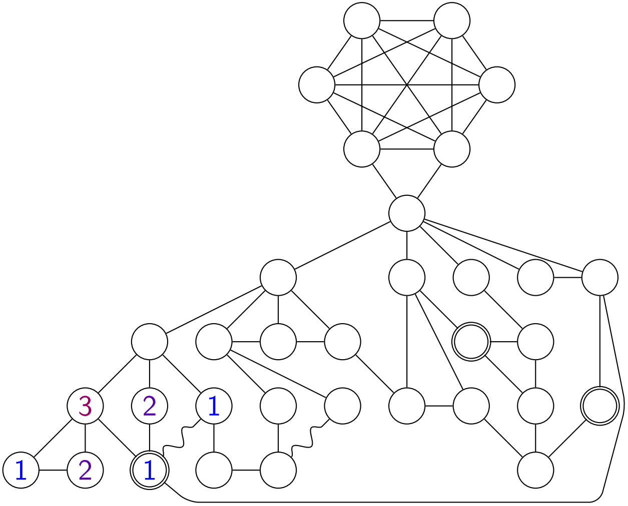
A connected Grundy coloring achieving color 7.



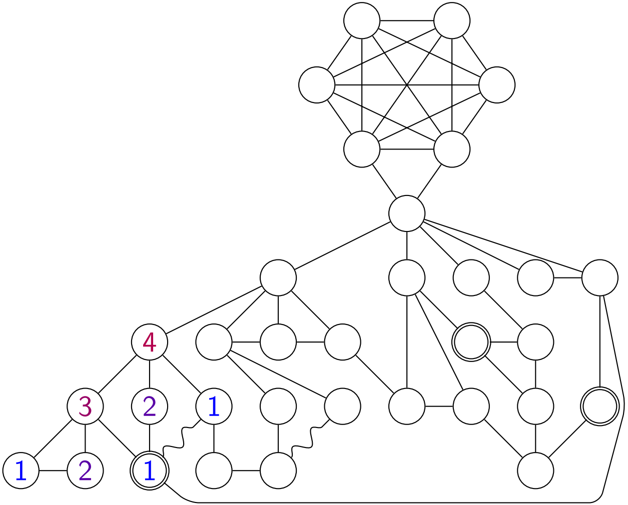
A connected Grundy coloring achieving color 7.



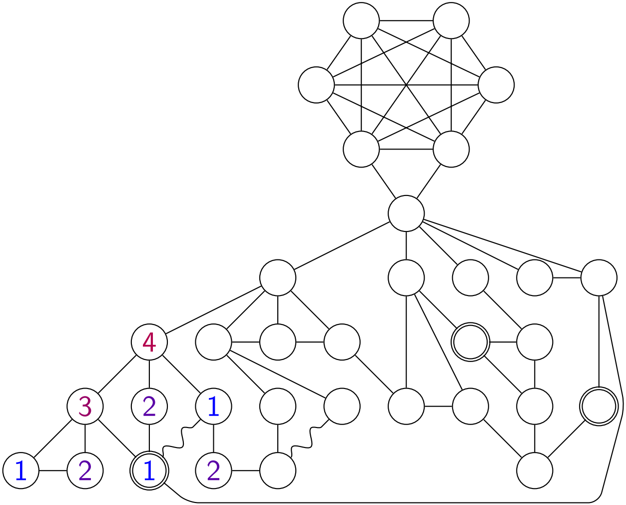
A connected Grundy coloring achieving color 7.



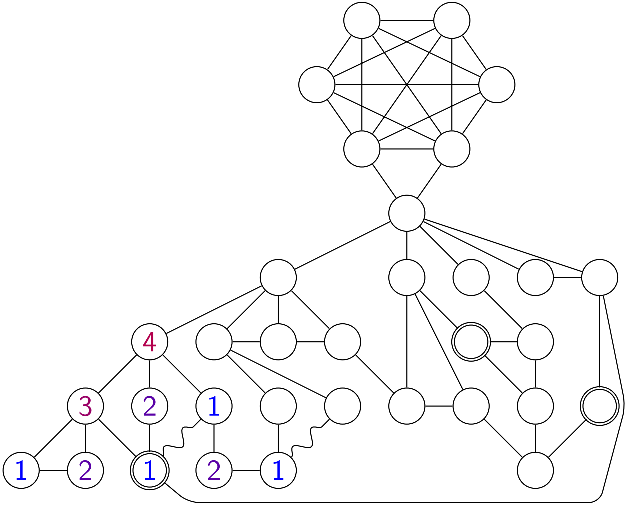
A connected Grundy coloring achieving color 7.



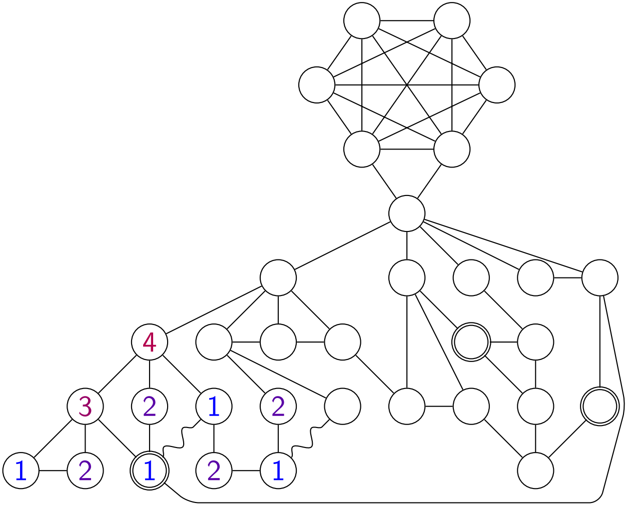
A connected Grundy coloring achieving color 7.



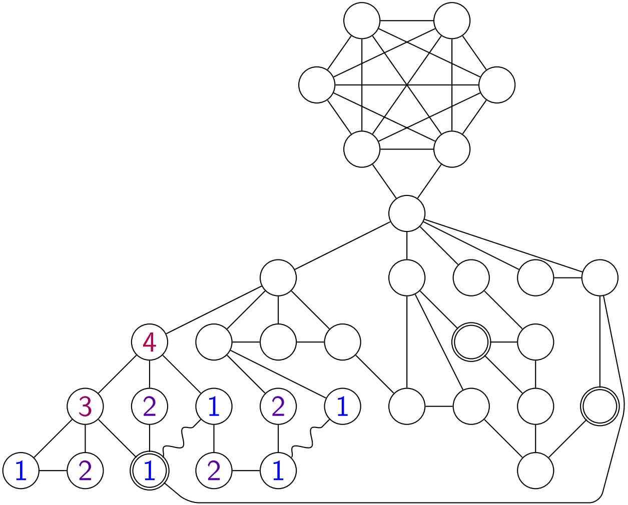
A connected Grundy coloring achieving color 7.



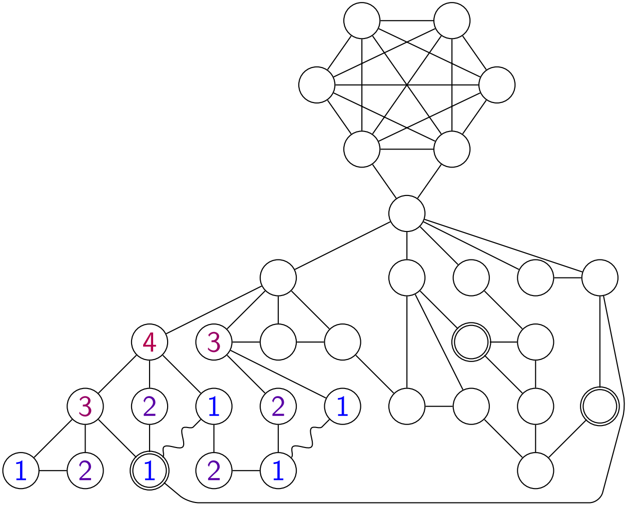
A connected Grundy coloring achieving color 7.



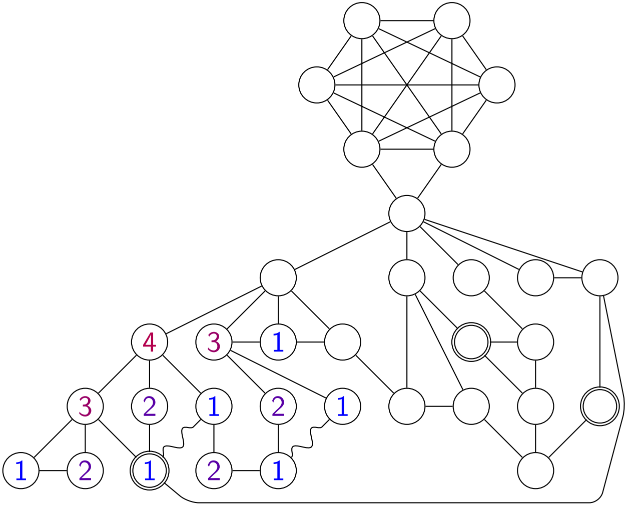
A connected Grundy coloring achieving color 7.



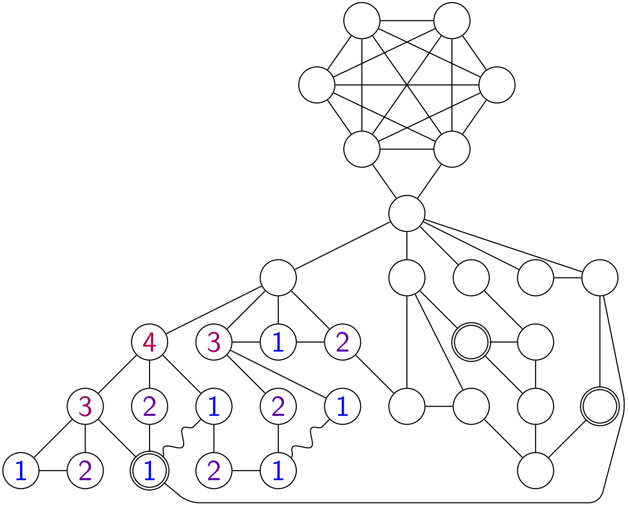
A connected Grundy coloring achieving color 7.



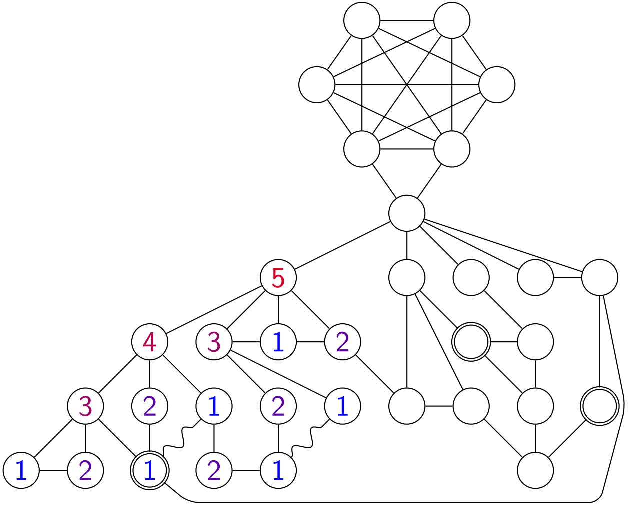
A connected Grundy coloring achieving color 7.



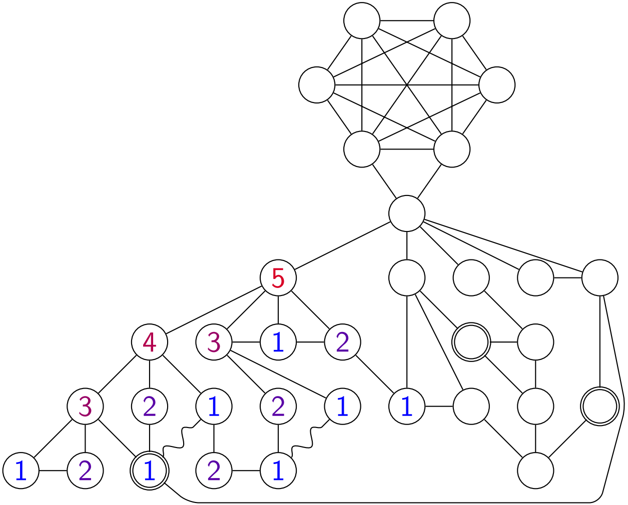
A connected Grundy coloring achieving color 7.



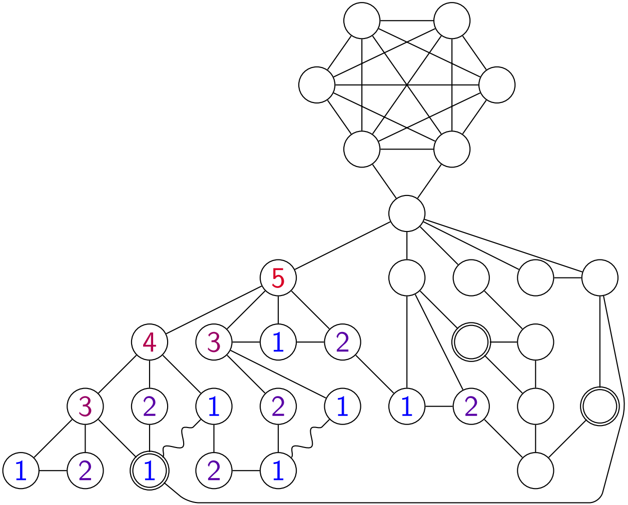
A connected Grundy coloring achieving color 7.



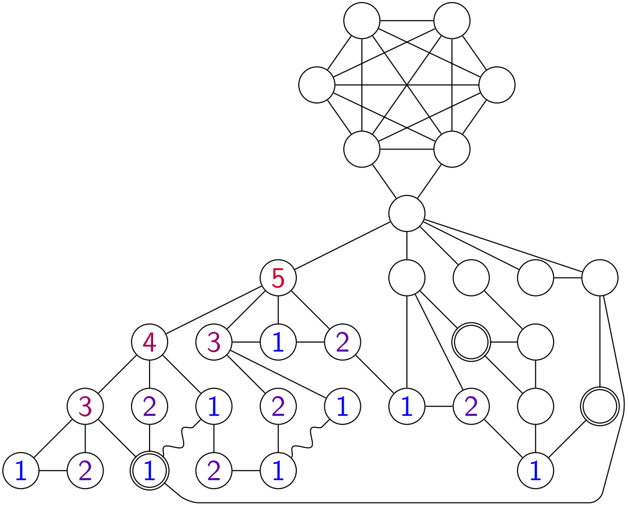
A connected Grundy coloring achieving color 7.



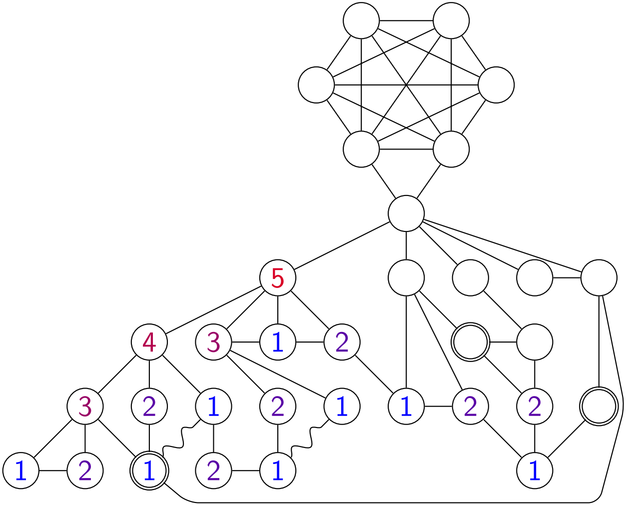
A connected Grundy coloring achieving color 7.



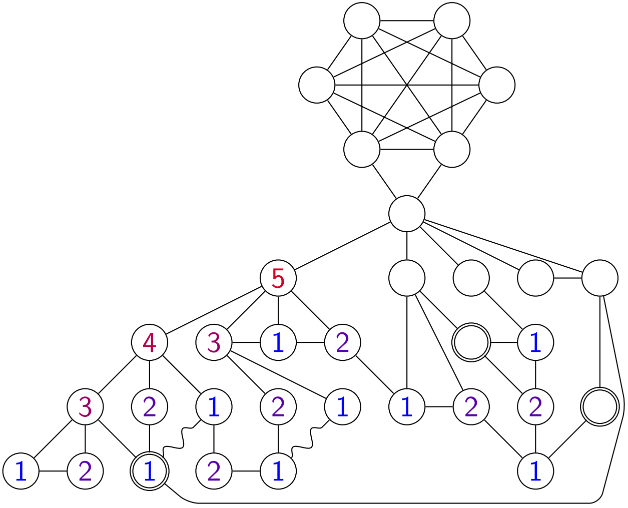
A connected Grundy coloring achieving color 7.



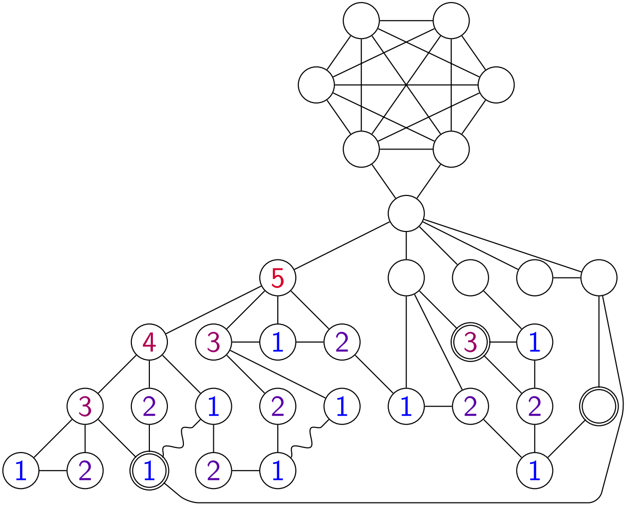
A connected Grundy coloring achieving color 7.



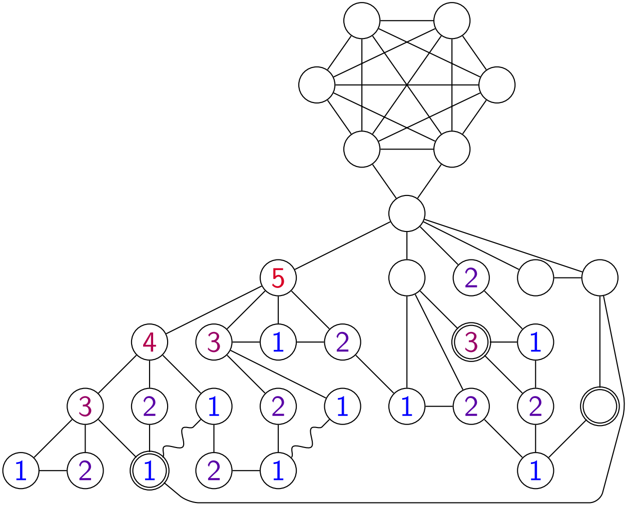
A connected Grundy coloring achieving color 7.



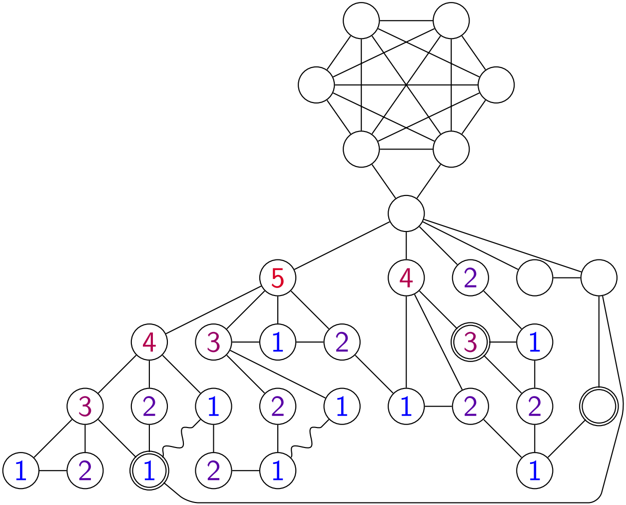
A connected Grundy coloring achieving color 7.



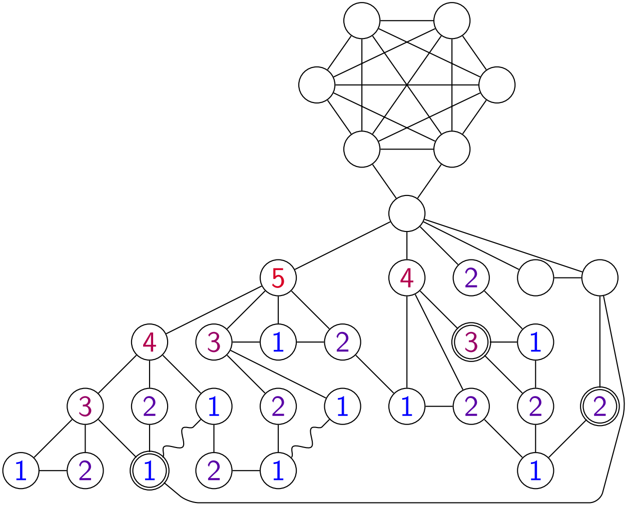
A connected Grundy coloring achieving color 7.



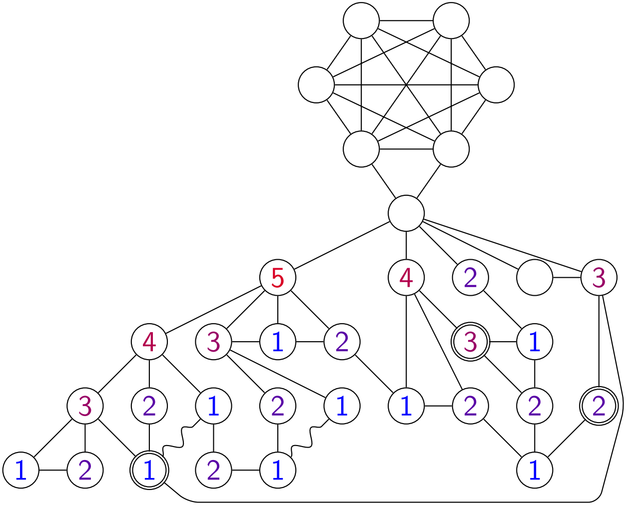
A connected Grundy coloring achieving color 7.



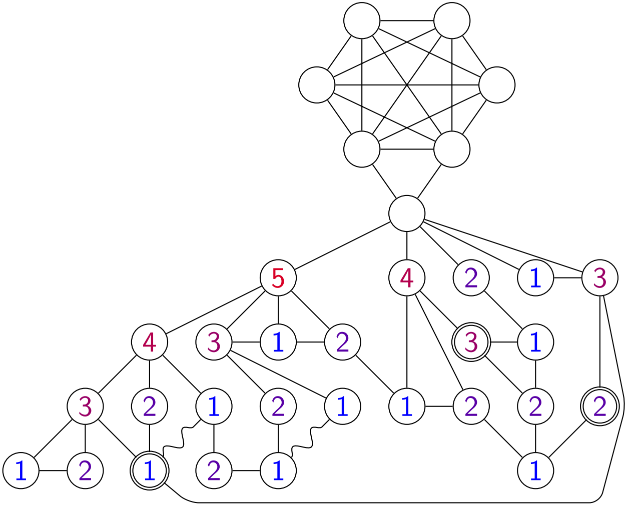
A connected Grundy coloring achieving color 7.



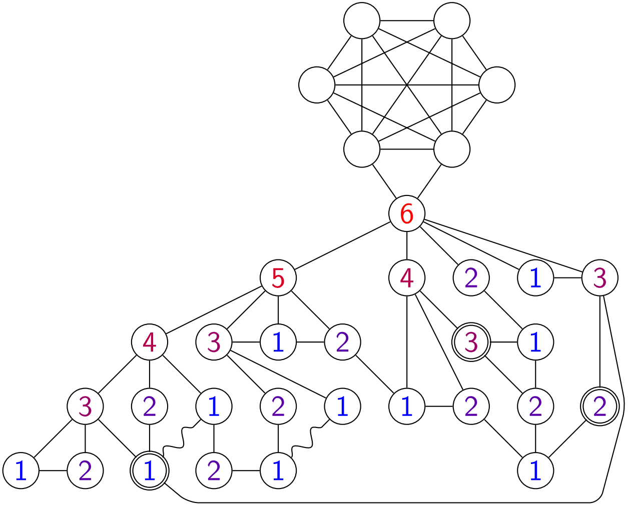
A connected Grundy coloring achieving color 7.



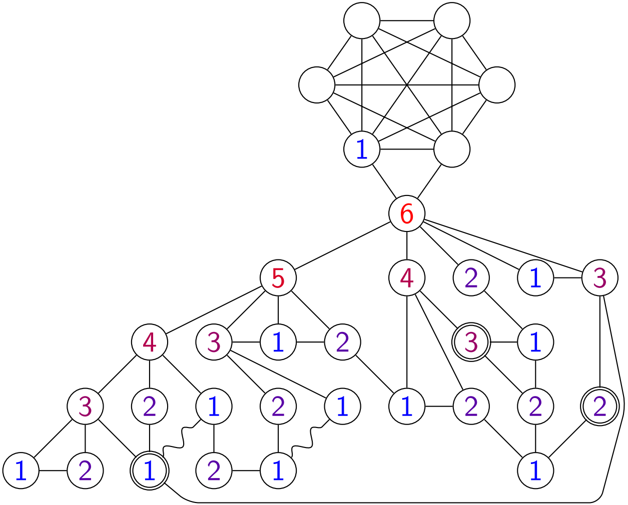
A connected Grundy coloring achieving color 7.



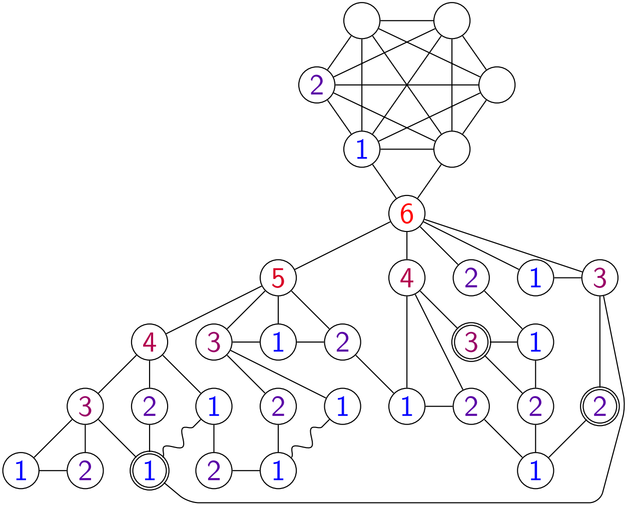
A connected Grundy coloring achieving color 7.



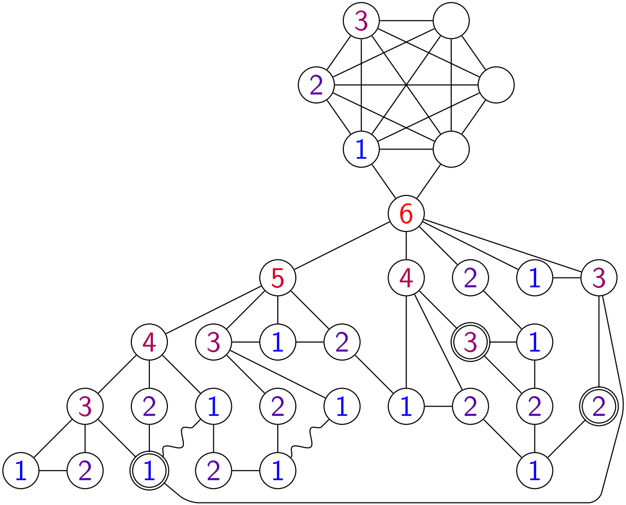
A connected Grundy coloring achieving color 7.



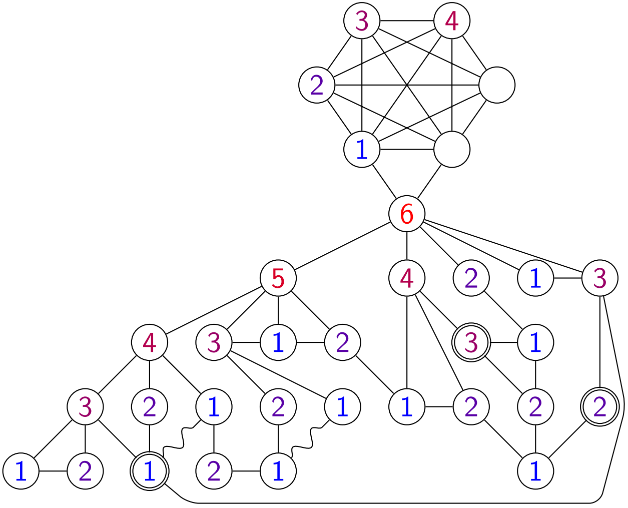
A connected Grundy coloring achieving color 7.



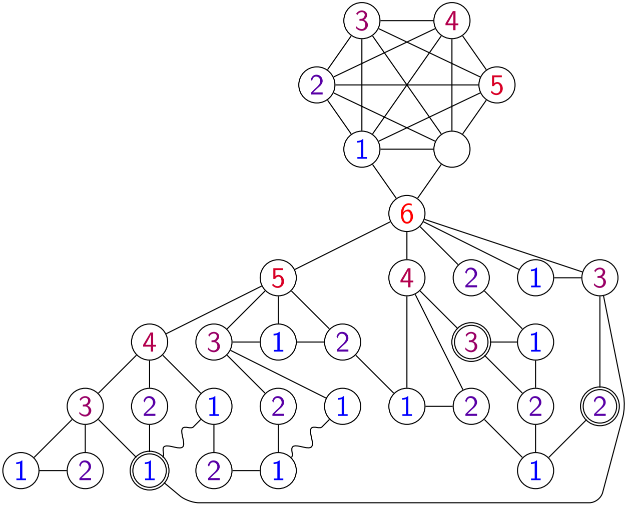
A connected Grundy coloring achieving color 7.



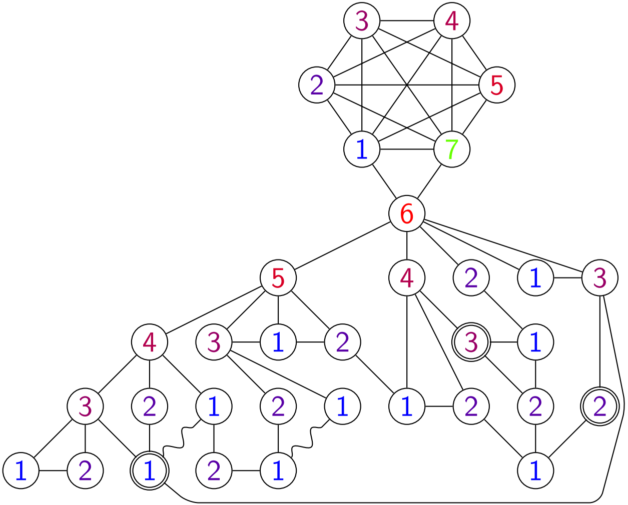
A connected Grundy coloring achieving color 7.



A connected Grundy coloring achieving color 7.



A connected Grundy coloring achieving color 7.



A connected Grundy coloring achieving color 7.

Open Questions

- Is Grundy Coloring FPT in the highest color k?
- Is Grundy Coloring FPT in the treewidth w?
- Is (Weak) Grundy Coloring solvable in $O^*(2^n)$?
- Is Connected Grundy Coloring solvable in $O^*(c^n)$?