Grundy Coloring & friends, Half-Graphs, Bicliques

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ENS Lyon, LIP

STACS 2020, Montpellier, March 11th

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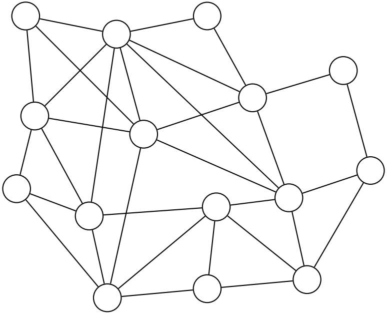
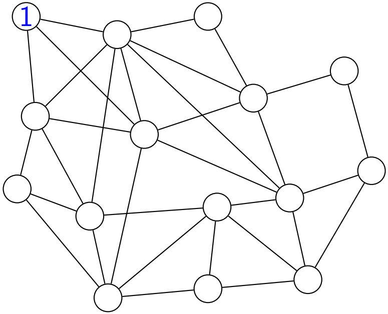


Do you see a Grundy coloring reaching color 6?



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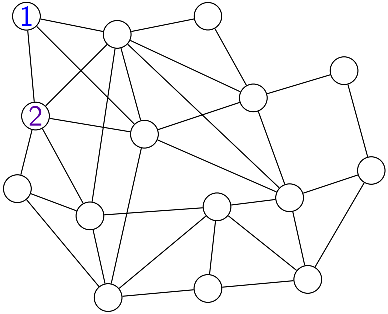

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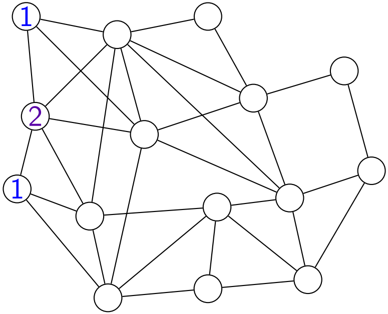

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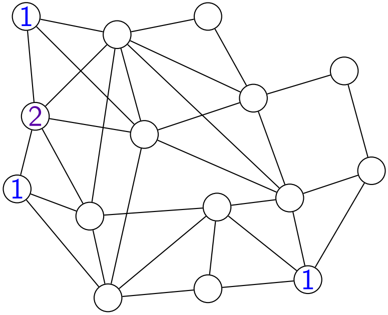

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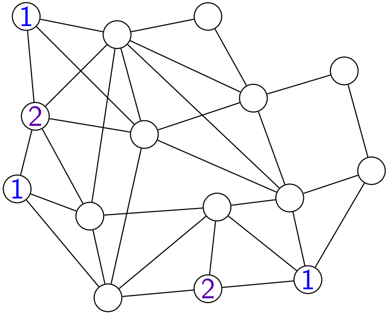

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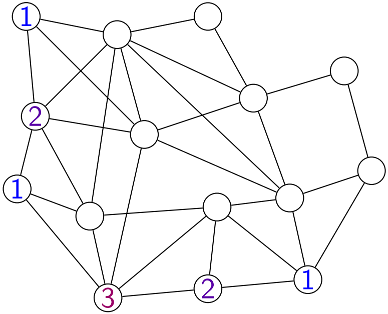

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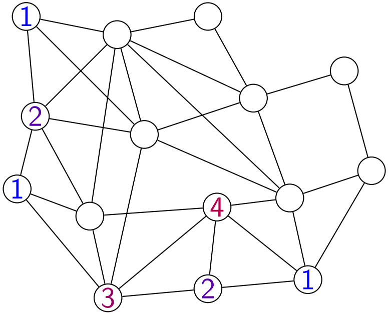

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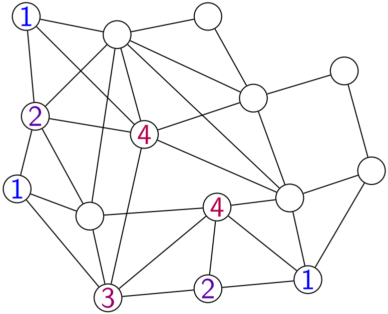

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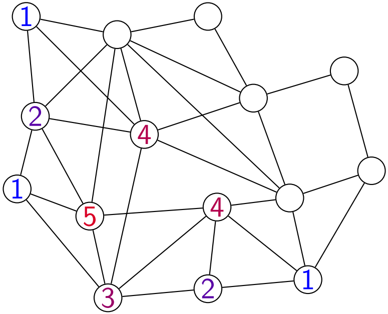

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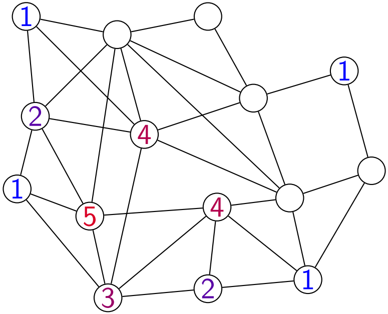

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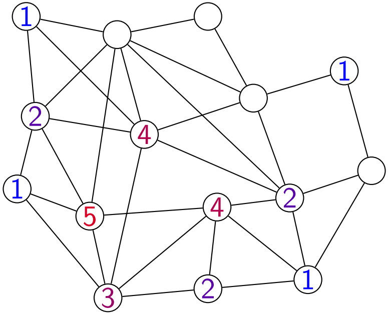

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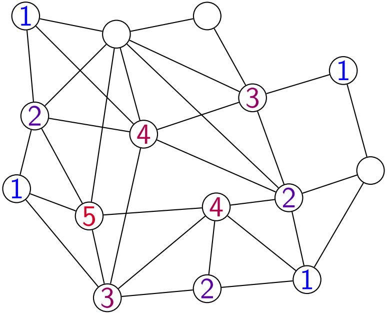

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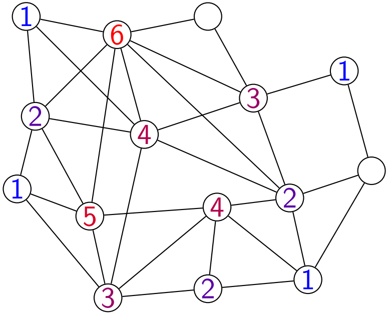

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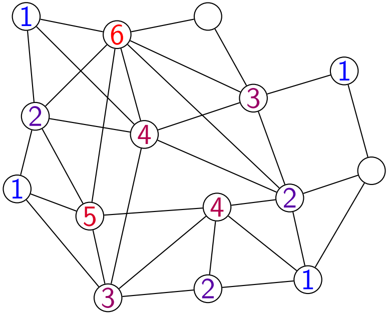
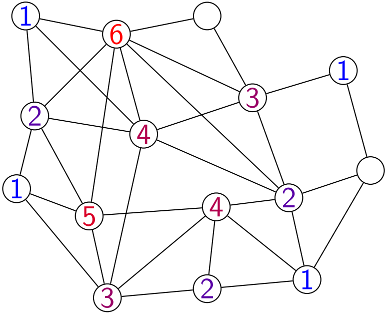


Witness = induced subgraph having the same Grundy number.



k-witness = induced subgraph having Grundy number at least k.

A brief History of Grundy colorings

- 1939: Studied in directed acyclic graphs by Grundy.
- 1979: Kristen and Selkow defines the Grundy number $\Gamma(G)$.
- 1983: Simmons defines the ochromatic[1] number $\chi^o(G)$.
- 1987: Erdős et al. prove that $\chi^o(G) = \Gamma(G)$.

[1] maximum number of colors used among all vertex-orderings, with the following rules (1) proper coloring, and (2) minimizing the number of colors for that ordering.

Algorithmic motivations

- $\Gamma(G)$ upper bounds the number of colors used by any greedy heuristic for Min Coloring.
- $\Gamma(G) \leqslant C\chi(G)$ on some classes of graphs gives a C-approximation for Min Coloring.
- Online coloring.
- see Sampaio's and Gastineau's PhD theses for further motivations.

How many vertices (at most) do we need to achieve color k?

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④

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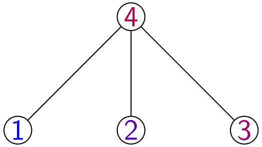

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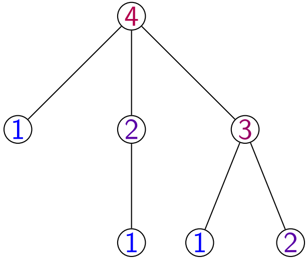

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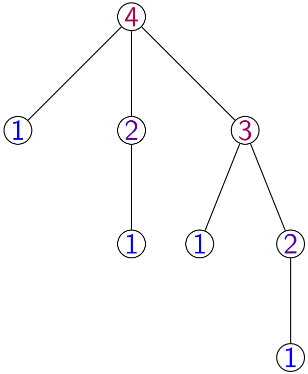

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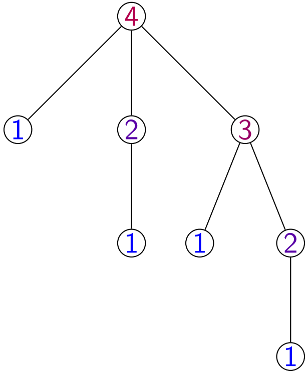

A minimal witness is of size at most 2^{k-1}.

Theorem (Zaker '06)

The Grundy number can be computed in $f(k)n^{2^{k-1}}$.

Main question

Can $\Gamma(G) = k$ be decided in $f(k)n^{O(1)}$? At least in $f(k)n^{k^{O(1)}}$?

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No and no!

Theorem

Under the ETH, GRUNDY COLORING *is not solvable in* $f(k)n^{2^{o(k)}}$.

Also shows W[1]-hardness

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Main ingredients:

- Constant-length "paths" of half-graphs (propagation).
- Truncated binomial trees (verification gadget).

We will now see what those things are

First thoughts on the parameterized reduction

- Reduce from a hard parameterized problem (e.g. k-Multicolored Clique) and keep the parameter small.

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We need to encode **choices** and have a way to **propagate** them

Let's be more specific

Starting point: k-Multicolored Subgraph Isomorphism;
Given a k-colored graph $G = (V_1 \cup \ldots \cup V_k, E)$ and a cubic graph $H = ([k], F)$, is there a $i \in [k] \mapsto v_i \in V_i$ s.t. $ij \in F \Rightarrow v_i v_j \in E$?

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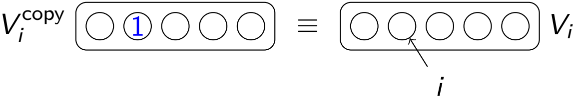
Choice encoding: An independent set of size $|V_i|$ where "coloring 1 the p-th vertex $\equiv$ mapping i to the p-th vertex of V_i"



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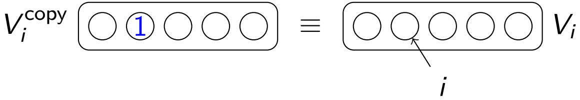
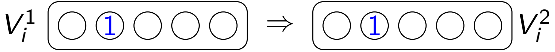
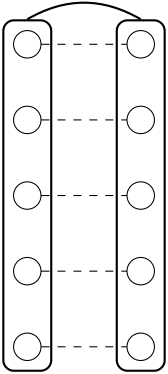


Propagation: Coloring 1 the p-th vertex in one copy forces it in the next copies

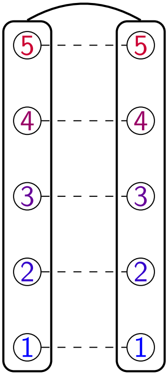


Anti-Matching: biclique minus a perfect matching



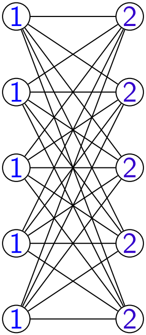
Would be ideal for propagation

Anti-Matching: biclique minus a perfect matching



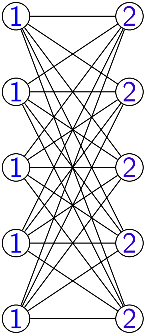
But unbounded Grundy number

Biclique



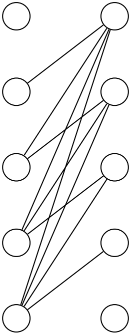
Grundy number 2

Biclique



But do not propagate anything

Simple Half-Graph



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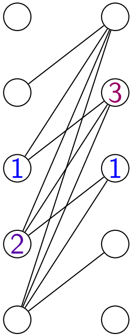
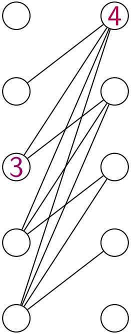


has Grundy number 3

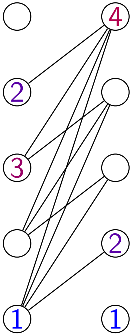
Simple Half-Graph



has Grundy number 3

A 4 would need to be supported by a 3 on the other side

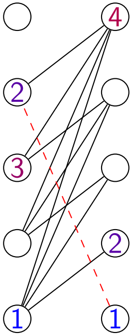
Simple Half-Graph



has Grundy number 3

This would imply on both sides a 2 supported by a 1

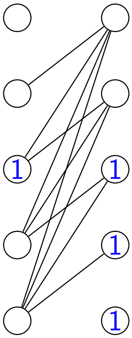
Simple Half-Graph



has Grundy number 3

Impossible due to $2K_2$-freeness

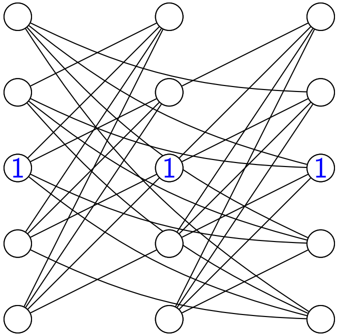
Simple Half-Graph



has Grundy number 3

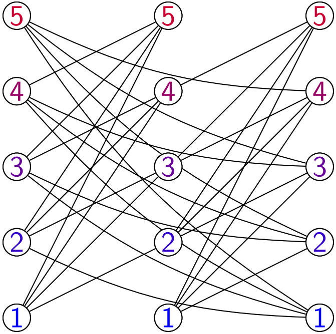
Only *half*-propagation

Cycles of Half-Graphs



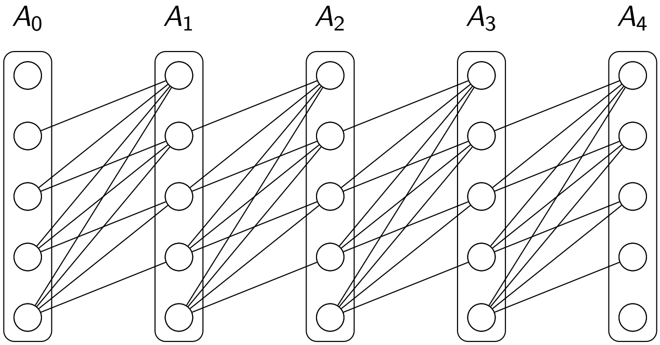
Would yield *full* propagation

Cycles of Half-Graphs



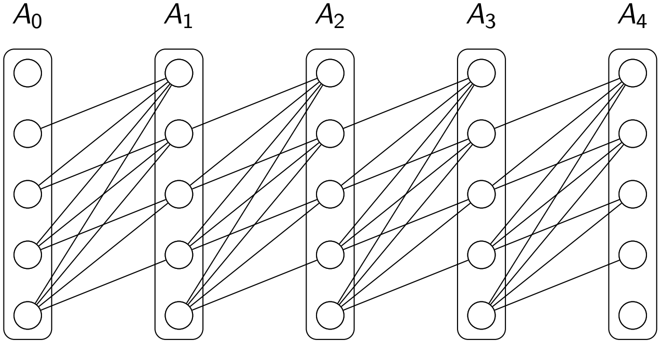
But unbounded Grundy again

Length-ℓ Path of Half-Graphs



Grundy number bounded by 4^{ℓ}, proof by induction on ℓ

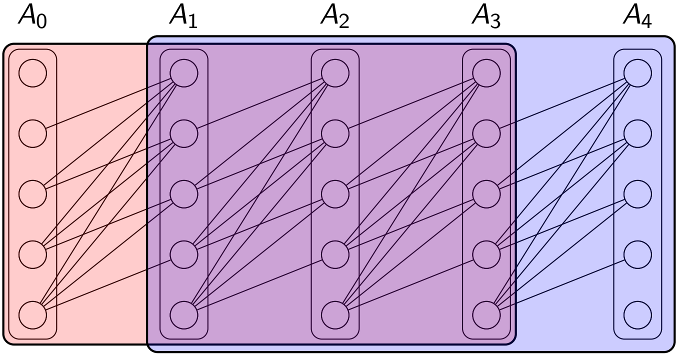
Length-ℓ Path of Half-Graphs



Grundy number bounded by 4^{ℓ}, proof by induction on ℓ

Let H be a minimal colored witness for $\Gamma(A_0 \cup \ldots \cup A_\ell) = \Gamma(G)$

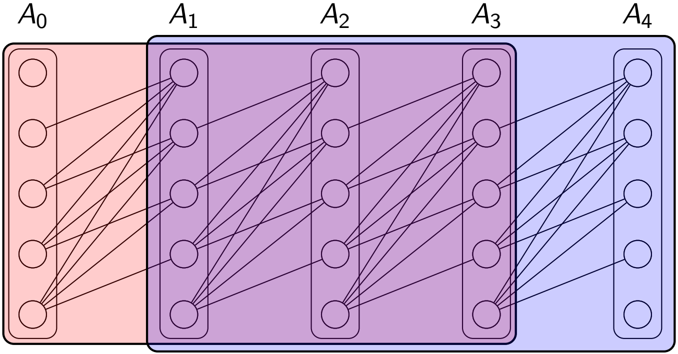
Length-ℓ Path of Half-Graphs



Grundy number bounded by 4^ℓ, proof by induction on ℓ

By hypothesis, A_0 and A_ℓ contain at least $\Gamma(G) - 4^{\ell-1}$ colors of H

Length-ℓ Path of Half-Graphs



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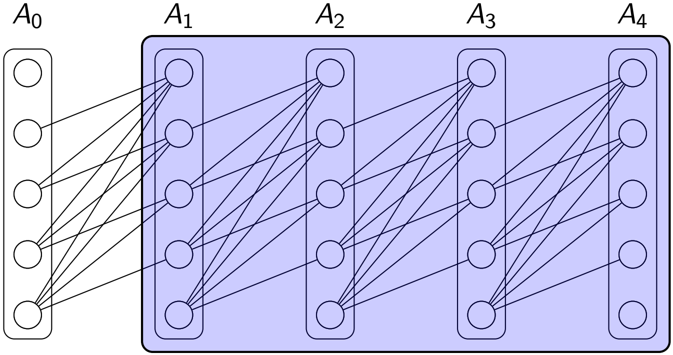
Hence A_0 and A_ℓ share at least $\Gamma(G) - 2 \cdot 4^{\ell-1} > 4^{\ell-1}$ colors

Length-ℓ Path of Half-Graphs



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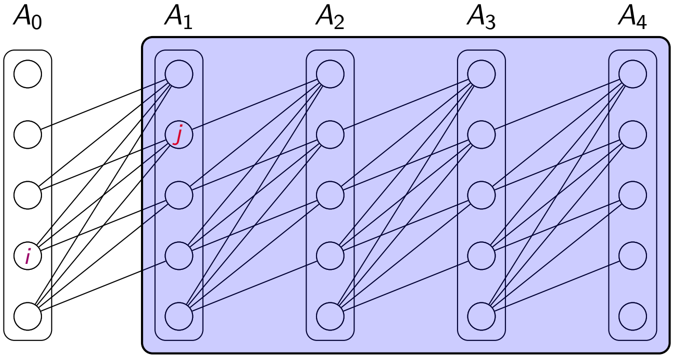
These color classes restricted to $G[A_1 \cup \ldots \cup A_\ell]$ form a witness

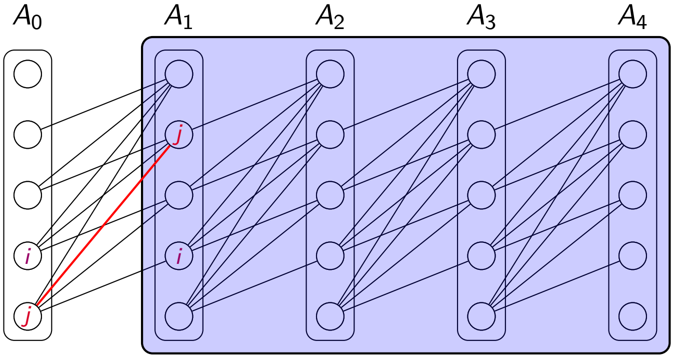
Length-ℓ Path of Half-Graphs



Grundy number bounded by 4^ℓ, proof by induction on ℓ

Indeed if a color j in A_1 is only supported by an $i < j$ in A_0

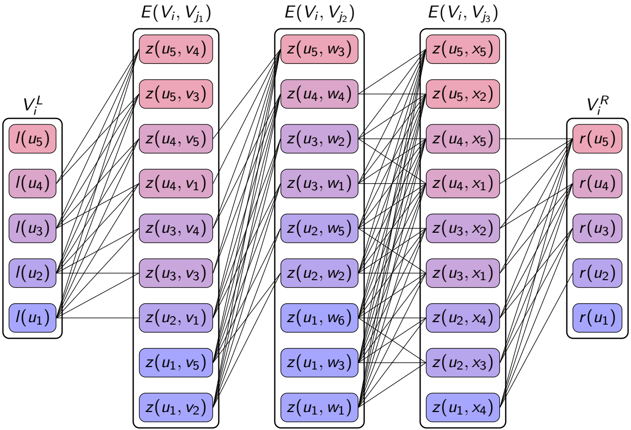
Length-ℓ Path of Half-Graphs



Grundy number bounded by 4^ℓ, proof by induction on ℓ

Then the j appearing in A_0 could not be supported by an i

Encoding V_i with a length-4 Path of Half-Graphs



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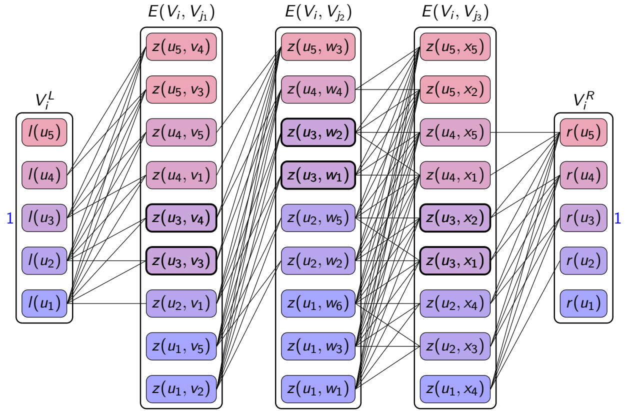
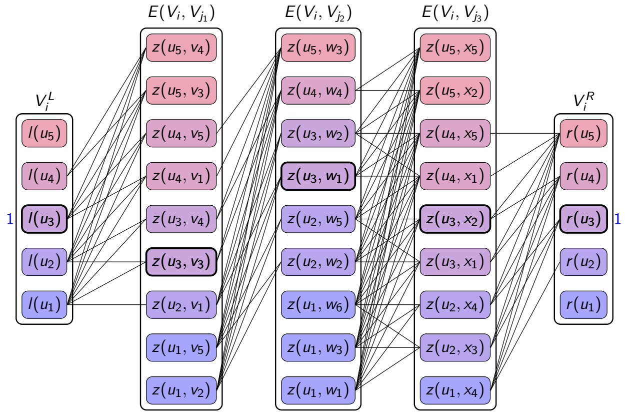


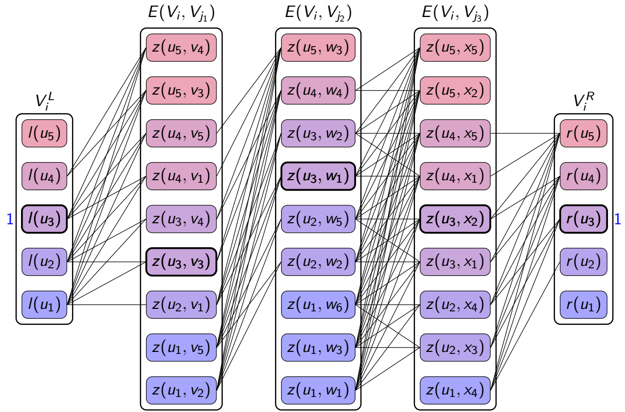
1 forced at a pair $l(u_a), r(u_a) \Rightarrow$ 1 only at "edges" incident to u_a

Encoding V_i with a length-4 Path of Half-Graphs



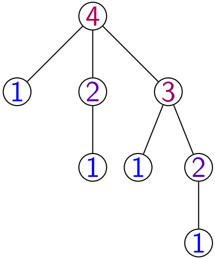
Only useful 1 those with other enpoints selected in their color class

Encoding V_i with a length-4 Path of Half-Graphs



Need gadget which activates iff some pairs are colored 1

Back to binomial trees $T_q = v(T_1, T_2, \dots, T_{q-1})$



A *unique* optimum Grundy Coloring

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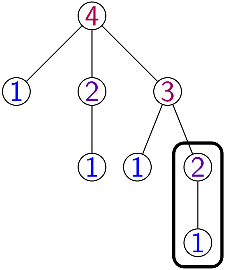

Dominant subtree: largest among its siblings

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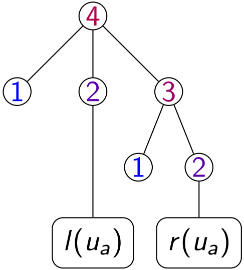
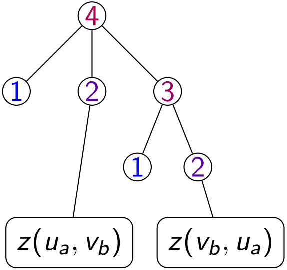


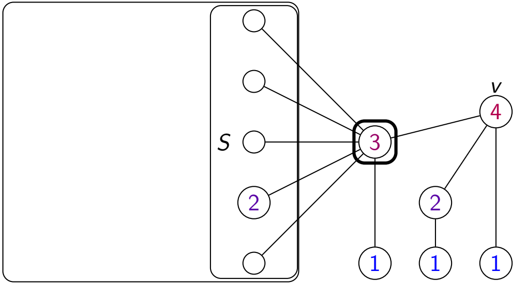
For each $l(u_a), r(u_a)$, copy tree, remove dominant 1, link to the 2

Back to binomial trees $T_q = v(T_1, T_2, \ldots, T_{q-1})$



Same principle for the edge check with $z(u_a, v_b)$ and $z(v_b, u_a)$

The binomial tree with missing dominant subtrees



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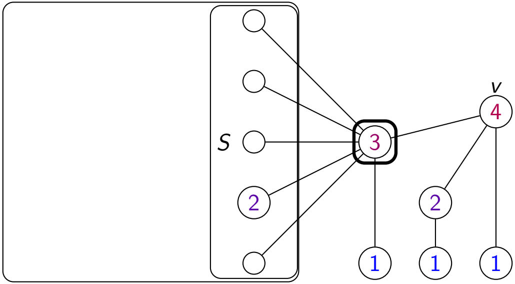
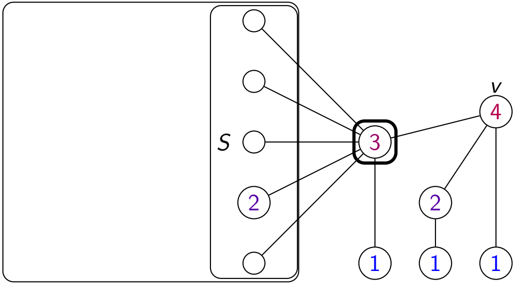


v can get color 4 iff its 3 finds a 2 in S

The binomial tree with missing dominant subtrees



v can get color 4 iff its 3 finds a 2 in S

Remove $k + 3k/2$ dominant 4 in the binomial tree $T_{\log k+10}$ and link each parent 5 to the roots of a V_i or $E(V_i, V_j)$

Wrapping up

Deciding if the Grundy number is $\log k + O(1)$ is as hard as k-MULTICOLORED SUBGRAPH ISOMORPHISM for cubic patterns.

Theorem (Marx '10)

Under the ETH, k-MULTICOLORED SUBGRAPH ISOMORPHISM with cubic patterns cannot be solved in $f(k)n^{o(k/\log k)}$.

Thus,

Theorem

Under the ETH, GRUNDY cannot be solved in $f(k)n^{2^{o(k)}}$.

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Thus,

Theorem

Under the ETH, Grundy cannot be solved in $f(k)n^{2^{o(k)}}$.

Nothing better than trying all colorings of all subsets of size 2^{k-1}

Summary of our results

- GRUNDY COLORING is W[1]-hard and unlikely solvable in $f(k)n^{2^{o(k)}}$
- b-CHROMATIC CORE is W[1]-hard:
  simple half-graphs + GRID TILING + ad-hoc tricks
- PARTIAL GRUNDY COLORING and b-CHROMATIC CORE are FPT in $K_{t,t}$-free graphs

Lemma

Every $K_{t,t}$-free graph with a large number of large-degree vertices admits $kK_{1,k}$ as an induced sugraph.

Open Questions

- Is Partial Grundy FPT?
- Are Grundy and b-Chromatic Core FPT in $H_{t,t}$-free graphs?
- Is Grundy FPT in $K_{t,t}$-free graphs?
- In general, can some of the FPT algorithms in $K_{t,t}$-free graphs be lifted to $H_{t,t}$-free graphs?

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Thank you for your attention!