The Complexity of Grundy Coloring and its Variants

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September 25, 2014

Warm Up

Connected Grundy Coloring

Weak Grundy Coloring

Grundy Colorings

- Order the vertices $v_1, v_2 \ldots v_n$ to maximize the number of colors used by the greedy coloring.
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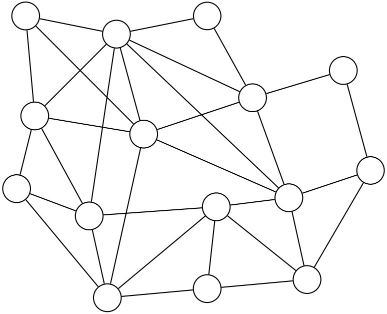
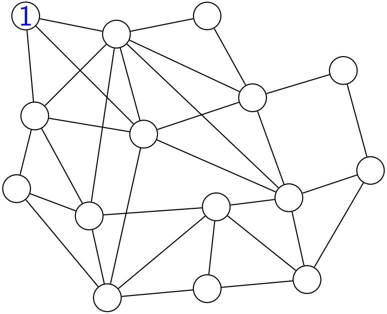
- Connected version: $\forall i, G[v_1 \cup \ldots \cup v_i]$ is connected.
- Weak version: v_i can be colored with any color in $\{1, \ldots, c(v_i)\}$.



Can you achieve color 6?



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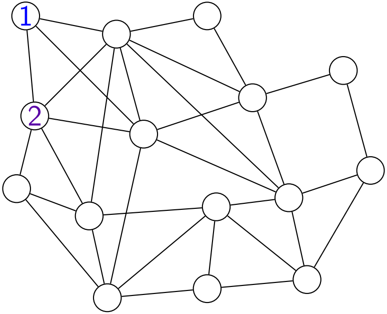

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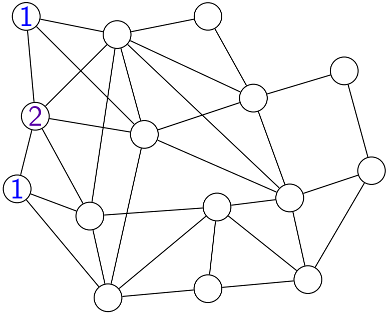

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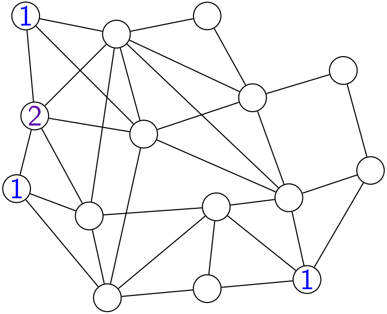

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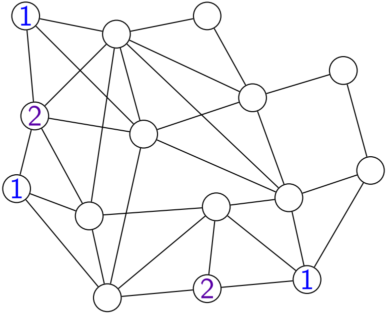

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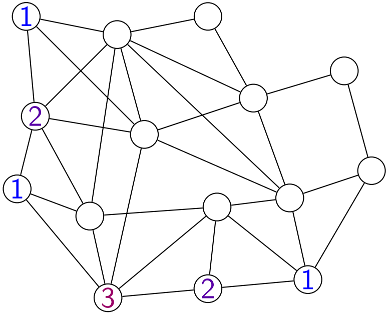

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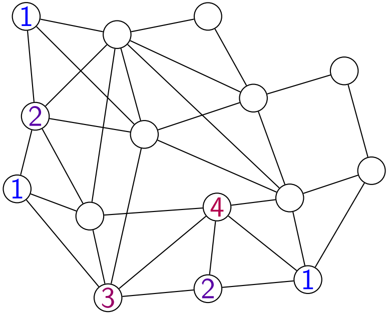

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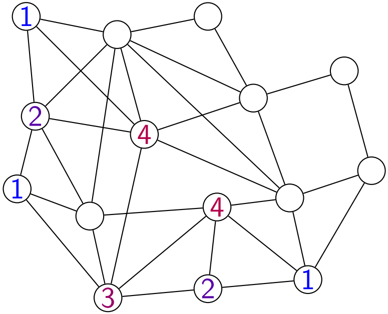

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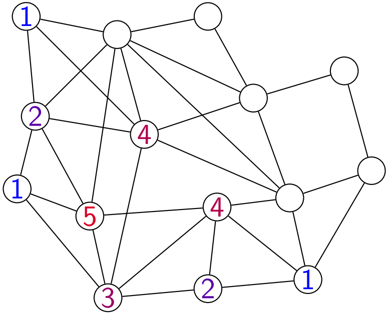

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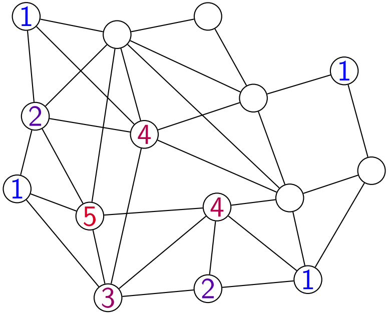

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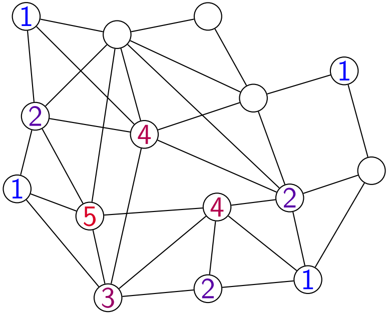

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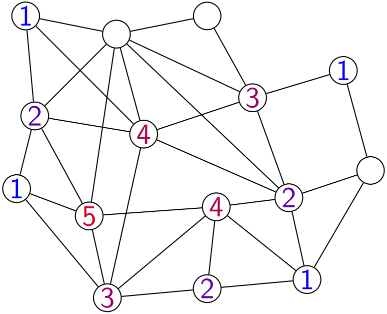

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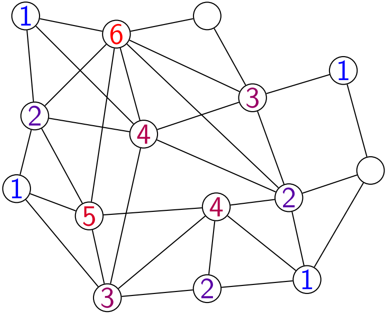

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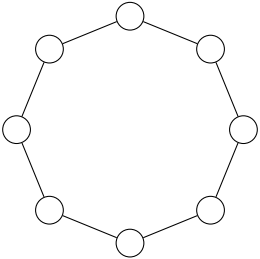
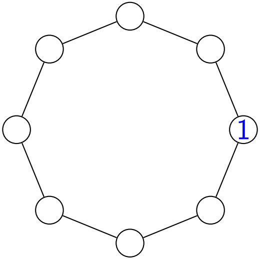


Grundy number $= 3$



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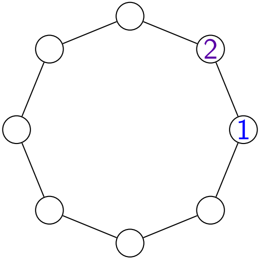

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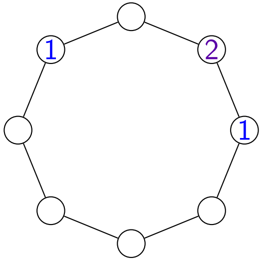

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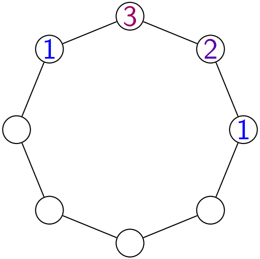

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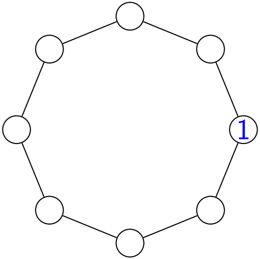

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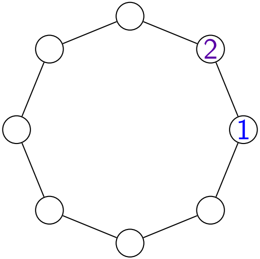

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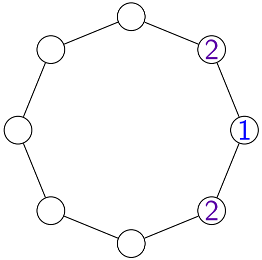

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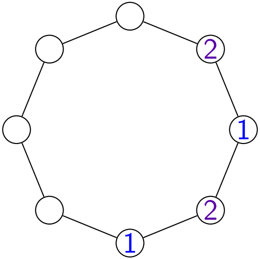

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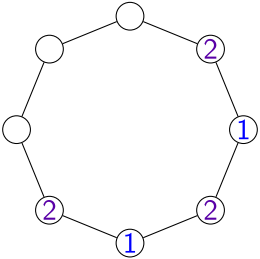

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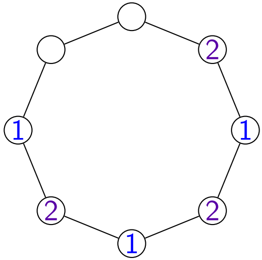

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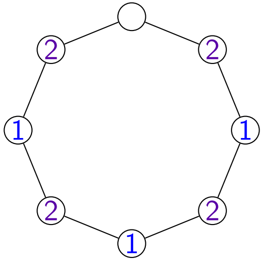

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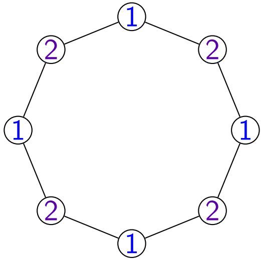

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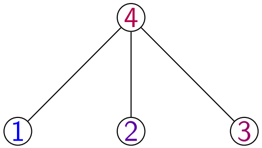

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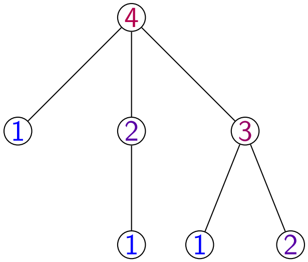

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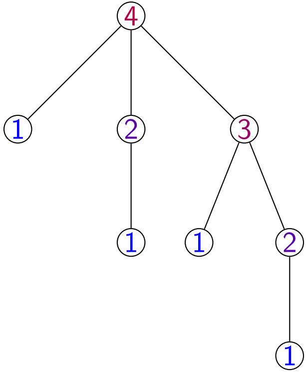

$t_1 = 1$ and $t_k = \Sigma_{1 \leqslant i \leqslant k-1} t_i$.
So, $t_k = 2^{k-1}$.

Theorem (Zaker '05)

The Grundy number can be computed in $O(f(k)n^{2^{k-1}})$.

XP algorithm: $O(f(k)n^{g(k)})$

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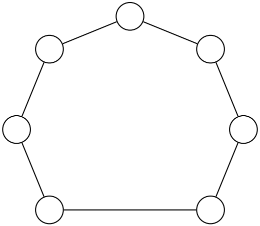
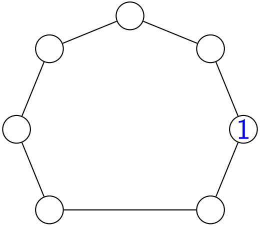
Can we do the same for the connected Grundy number?



Connected Grundy number = 3, unbounded witness



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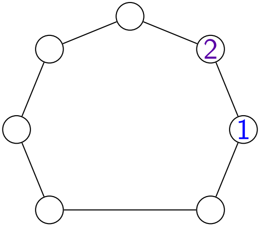

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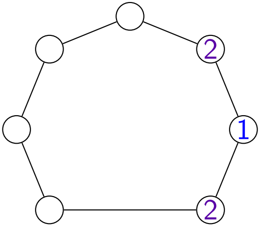

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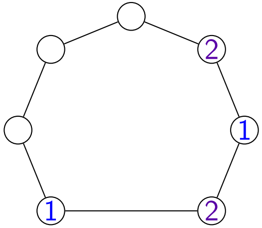

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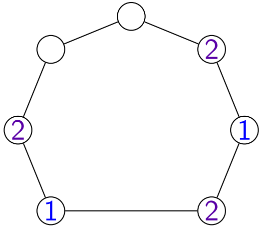

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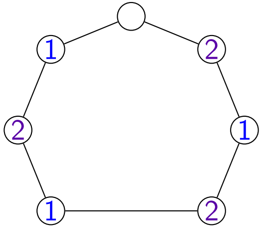

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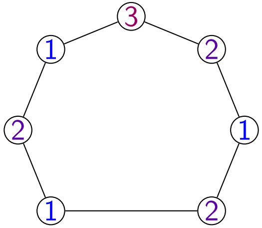

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Theorem (BCDGMSS '14)

Connected Grundy Coloring *is* NP*-complete.*

Theorem

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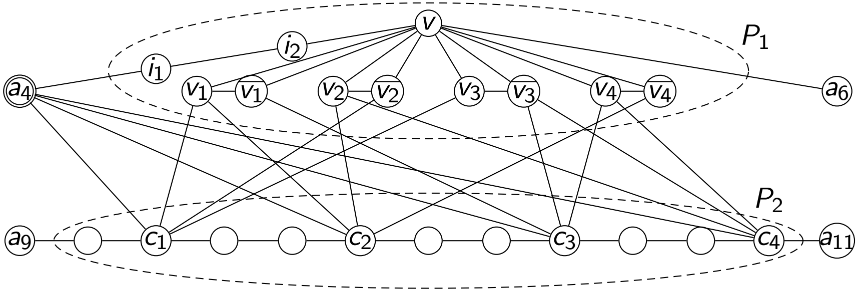
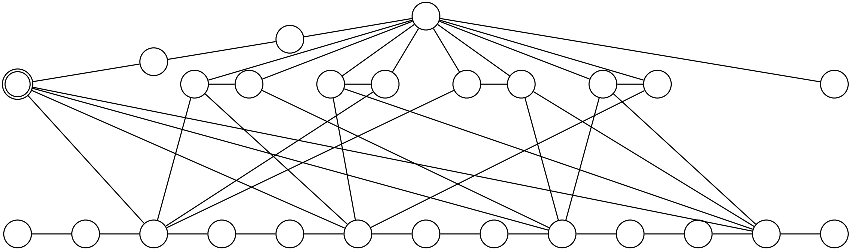
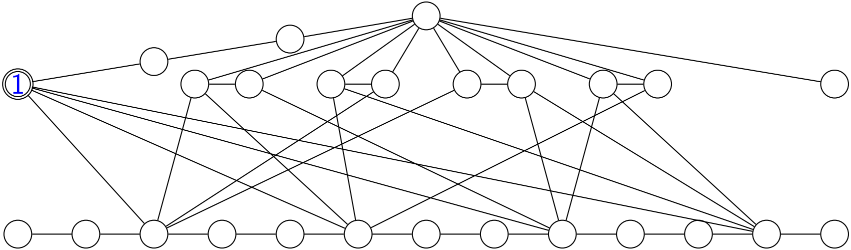
- To achieve color 7, three special neighbors of the c_js should be colored by 1, 2 and 3 respectively.



P_1 and P_2 for the instance
$\{x_1 \vee \neg x_2 \vee x_3\}, \{x_1 \vee x_2 \vee \neg x_4\}, \{\neg x_1 \vee x_3 \vee x_4\}, \{x_2 \vee \neg x_3 \vee x_4\}.$



A connected Grundy coloring setting all the c_js to 4.



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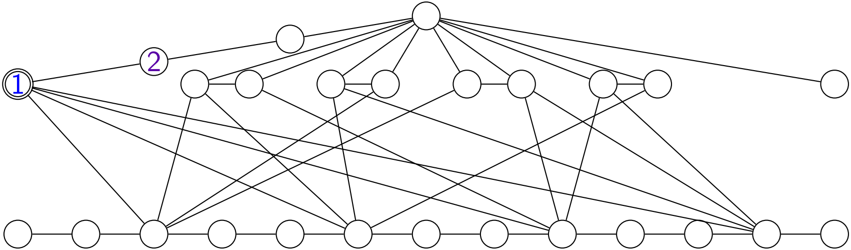

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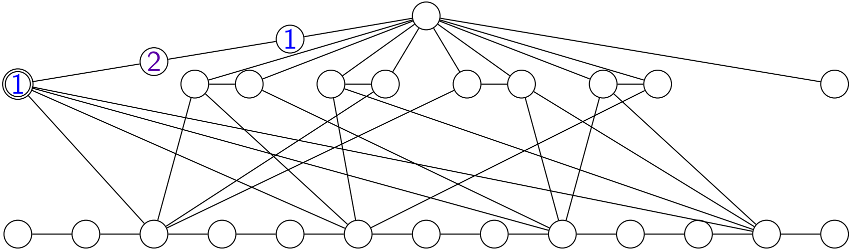

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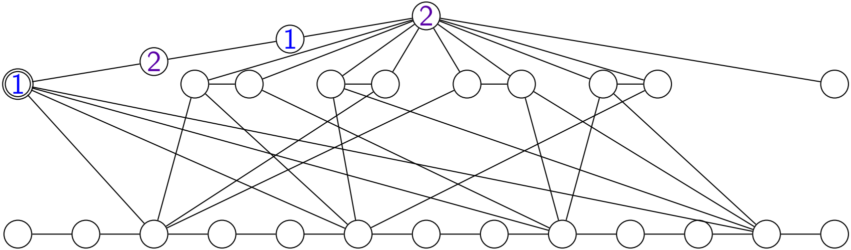

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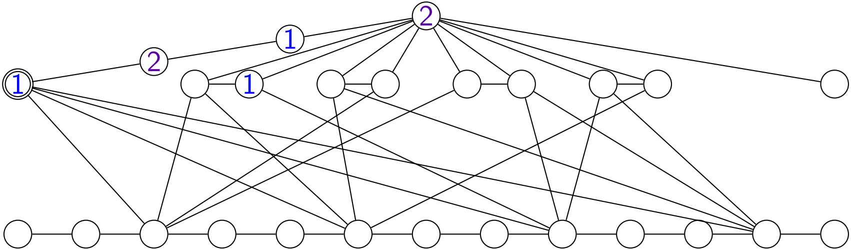

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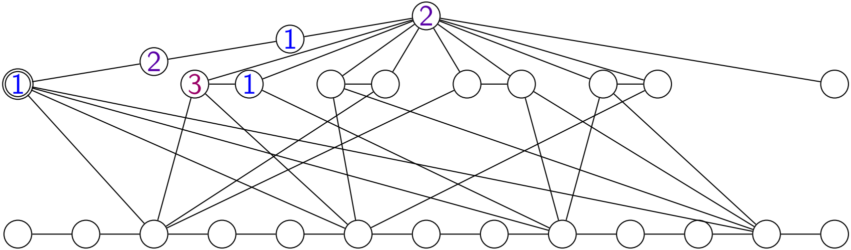

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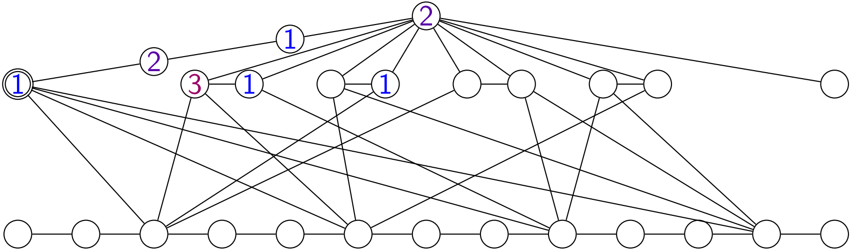

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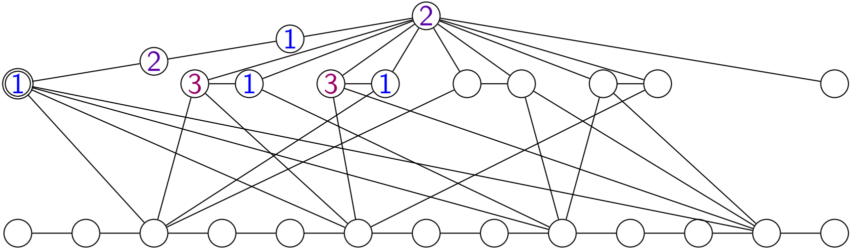

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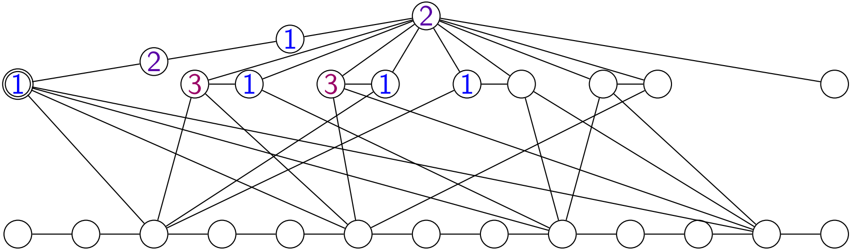

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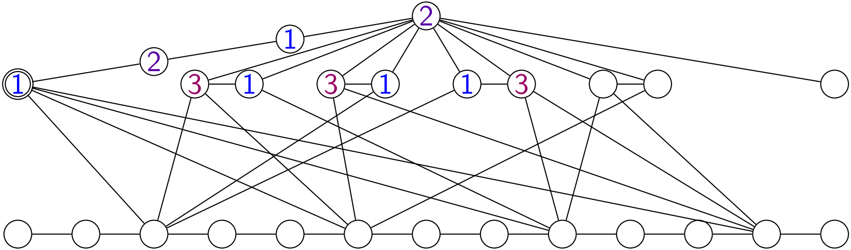

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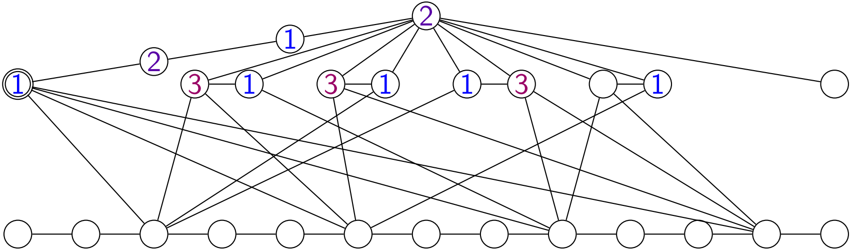

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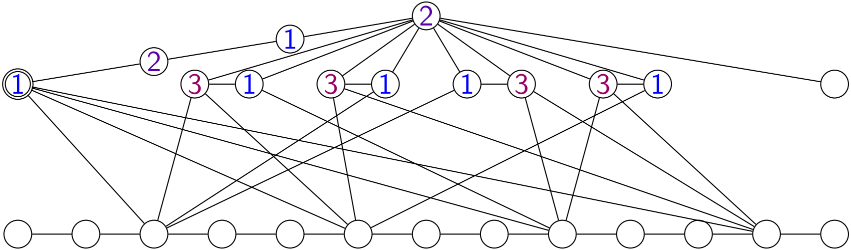

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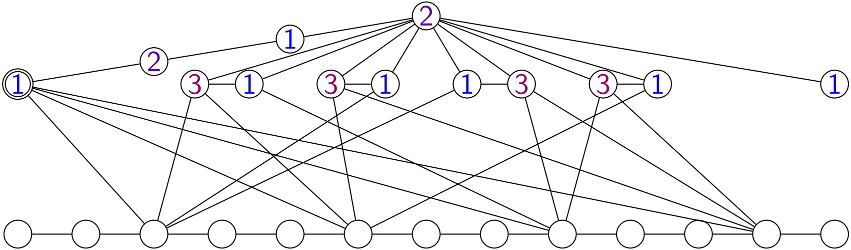

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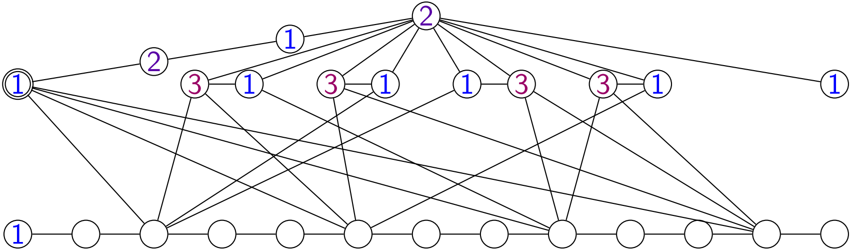

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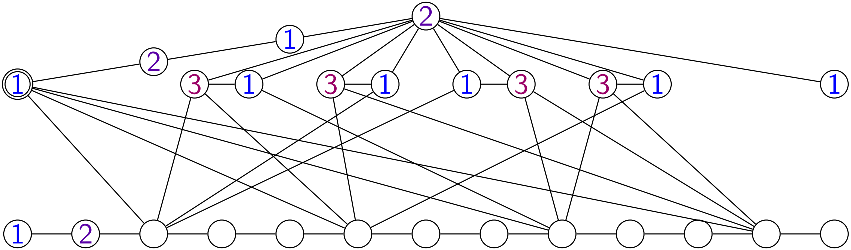

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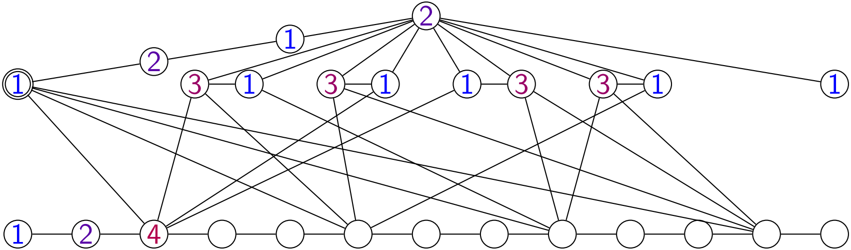

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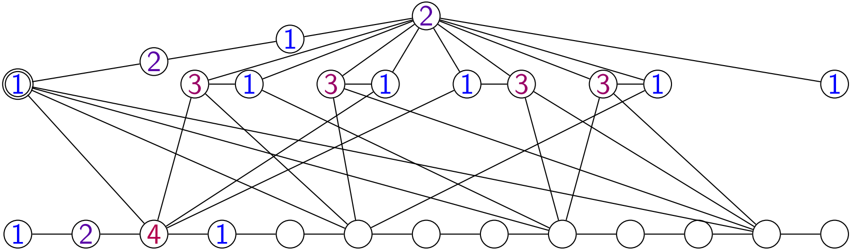

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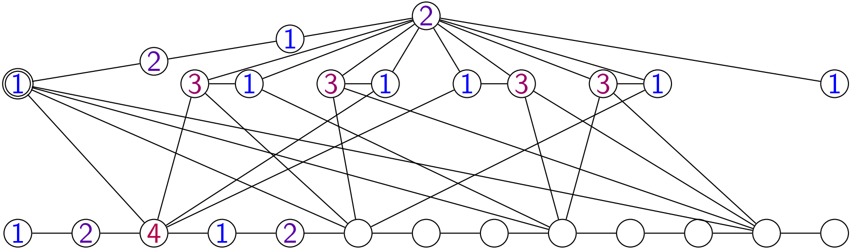

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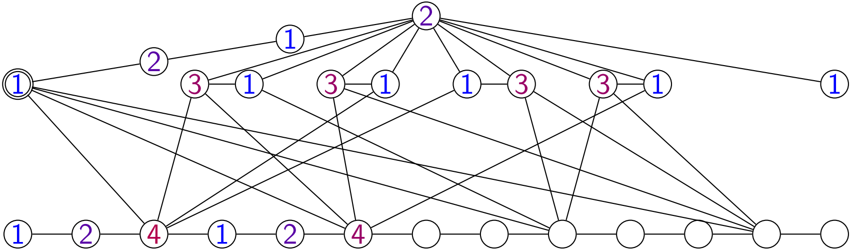

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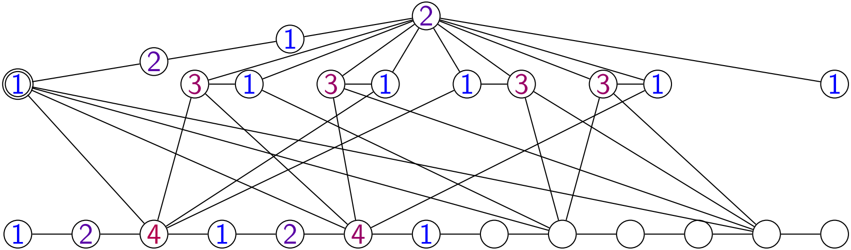

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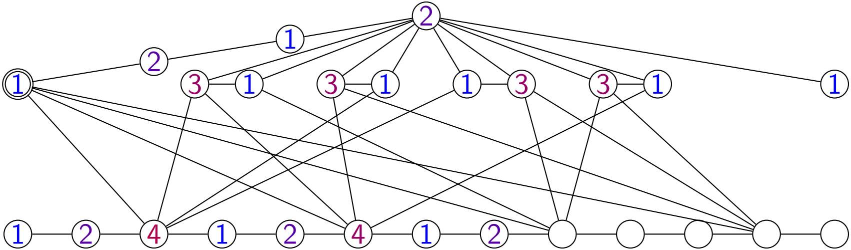

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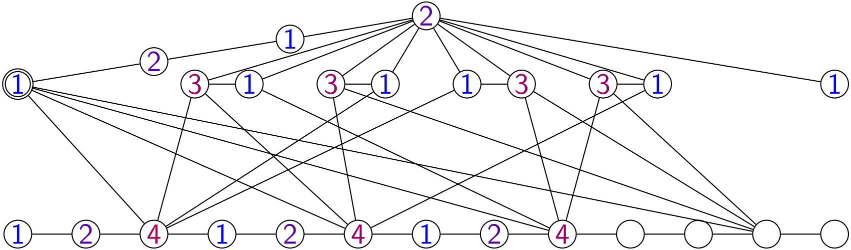

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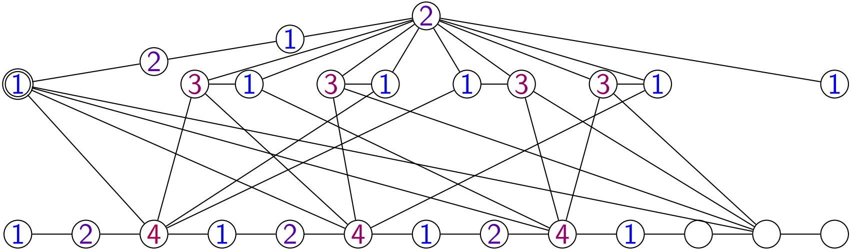

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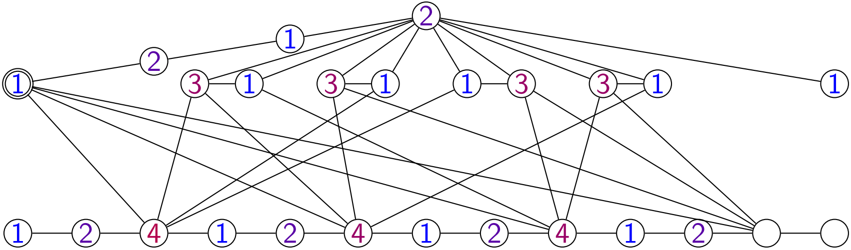

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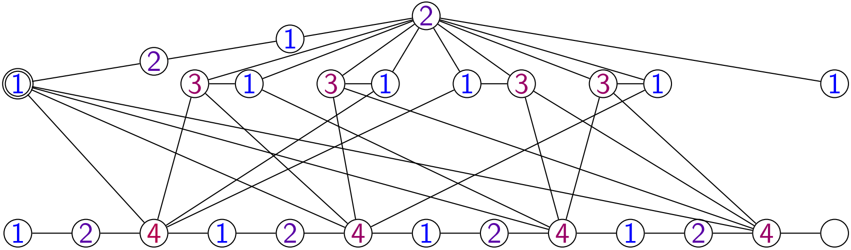

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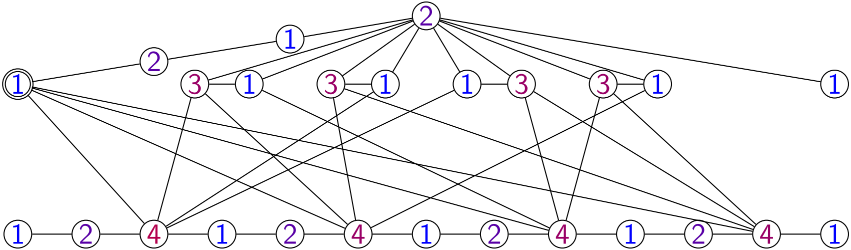

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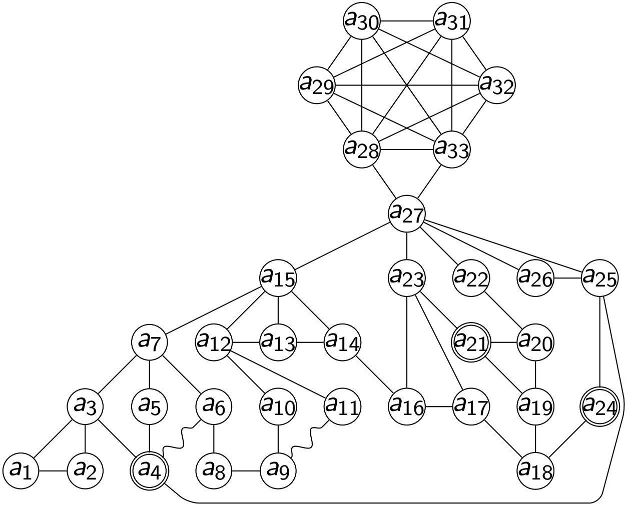
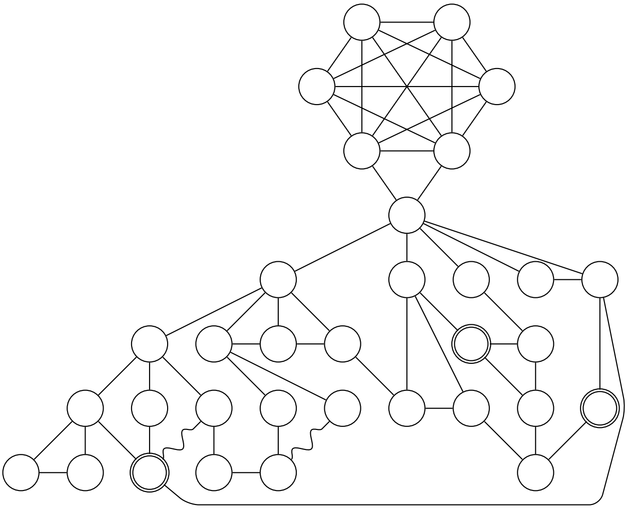
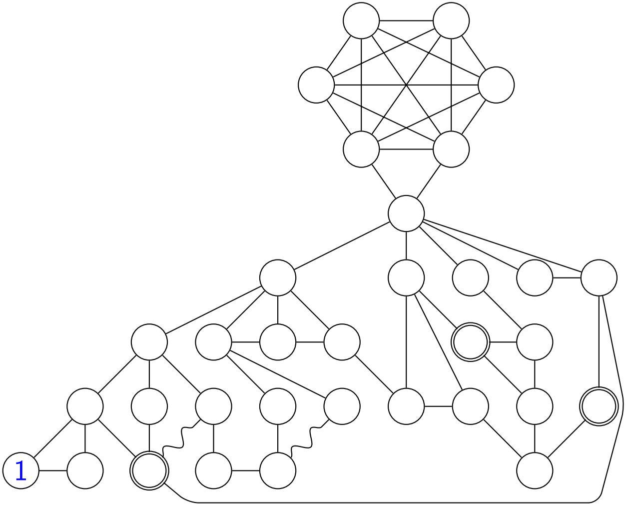


The doubly-circled vertices are linked to all the *clause* vertices c_js.



A connected Grundy coloring achieving color 7.



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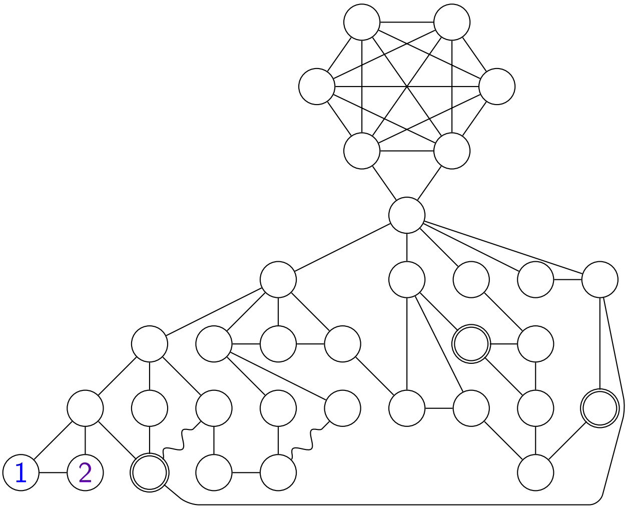

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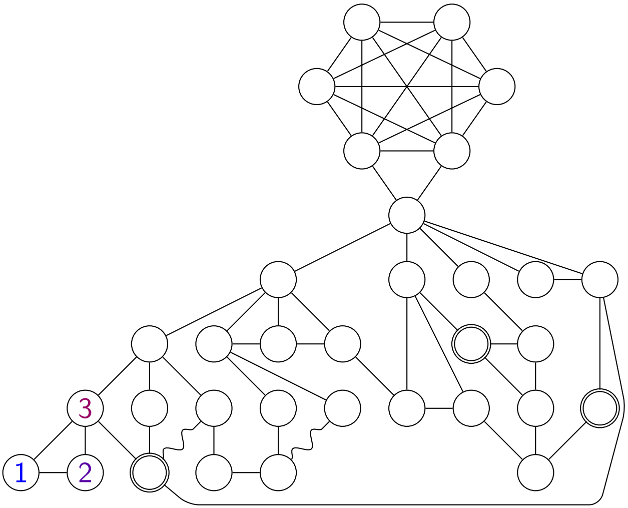

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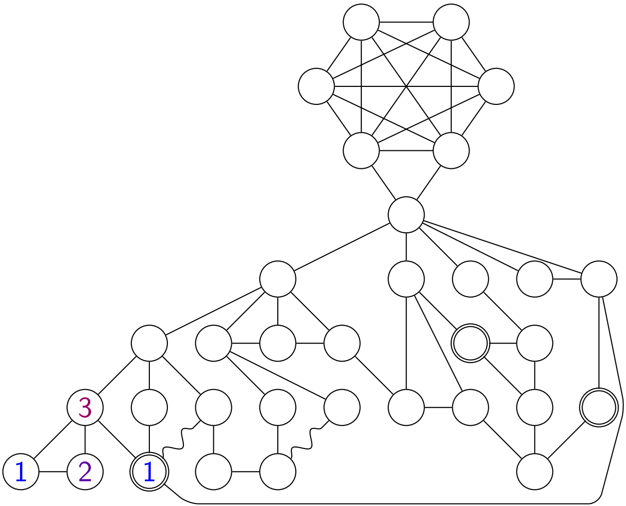

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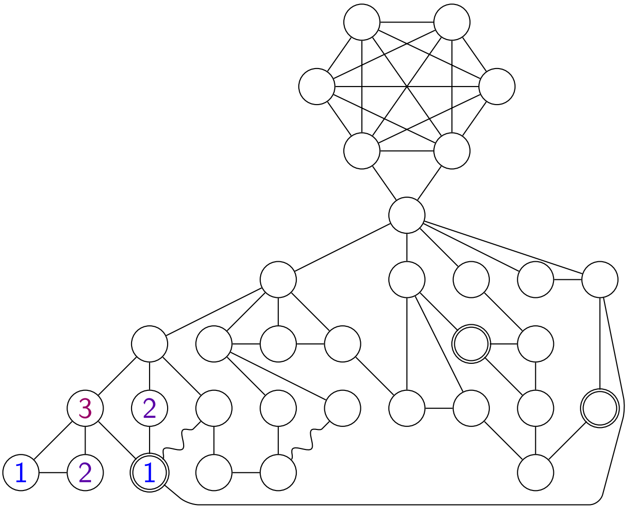

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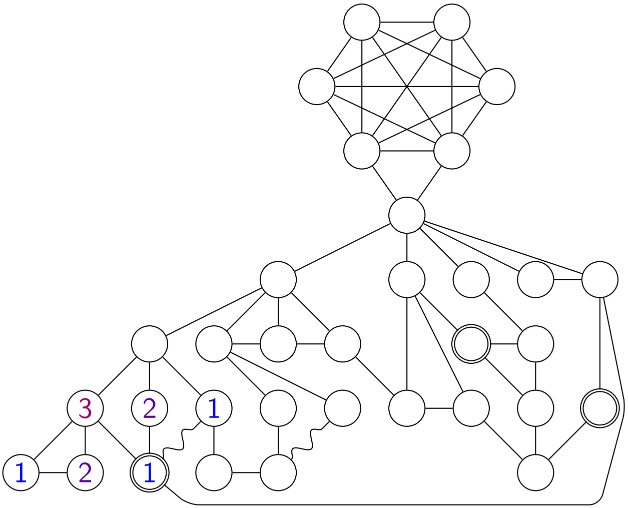

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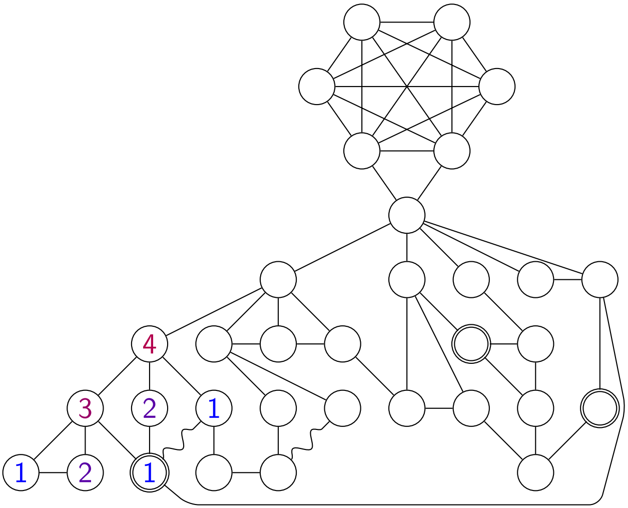

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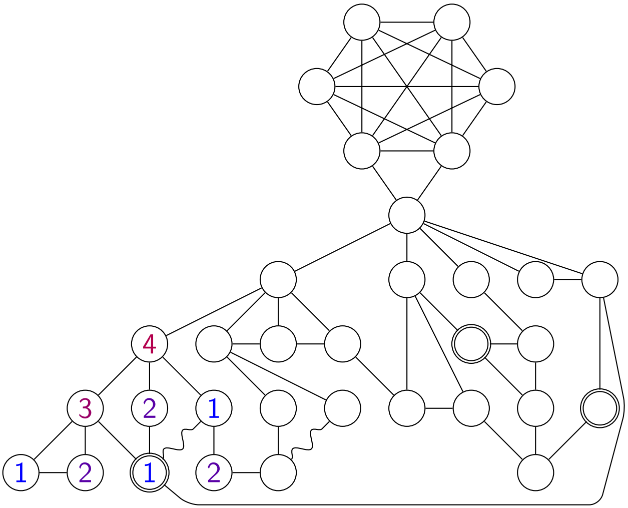

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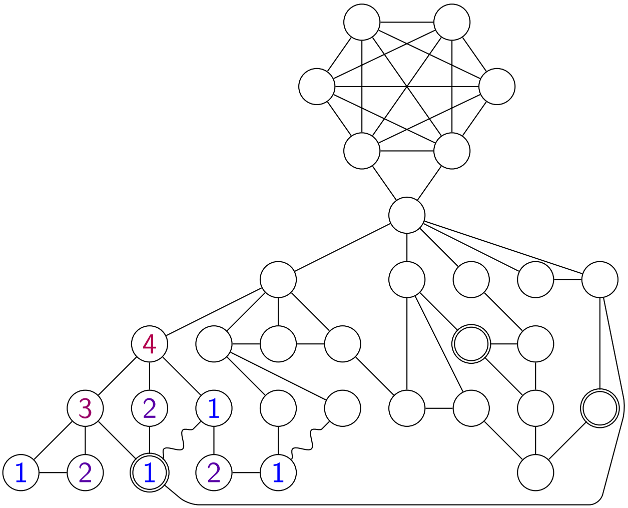

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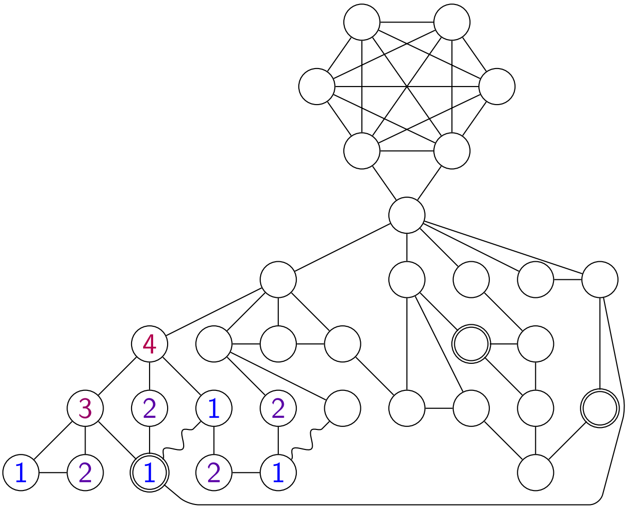

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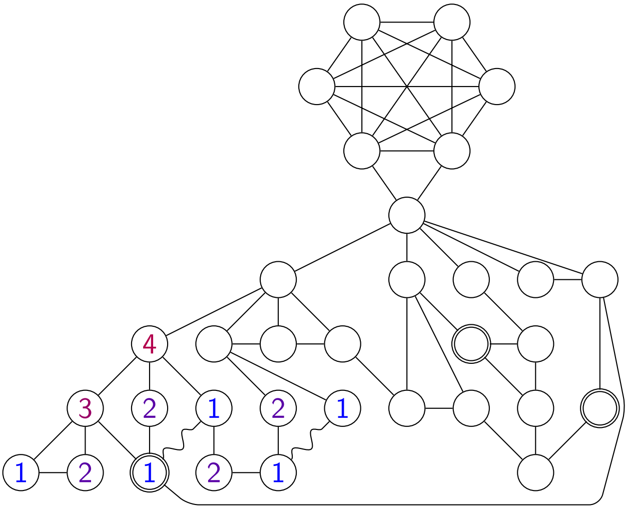

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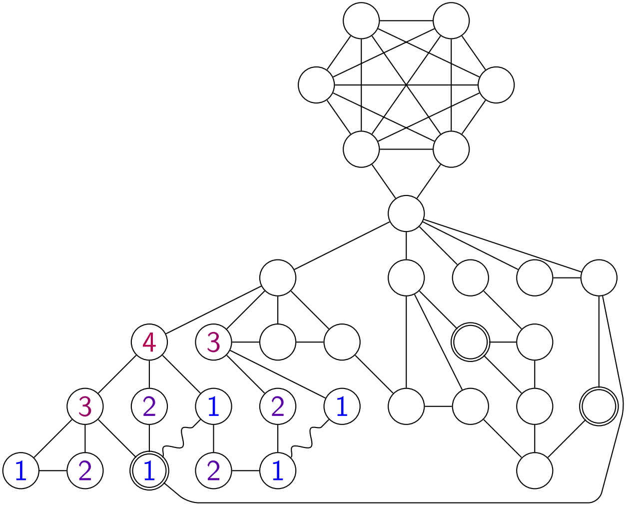

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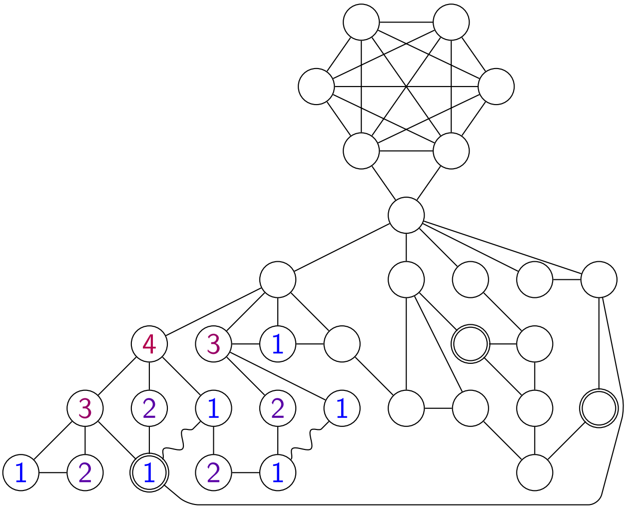

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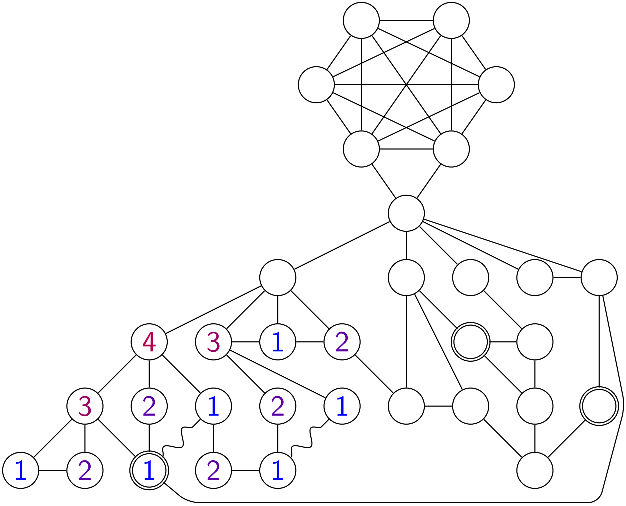

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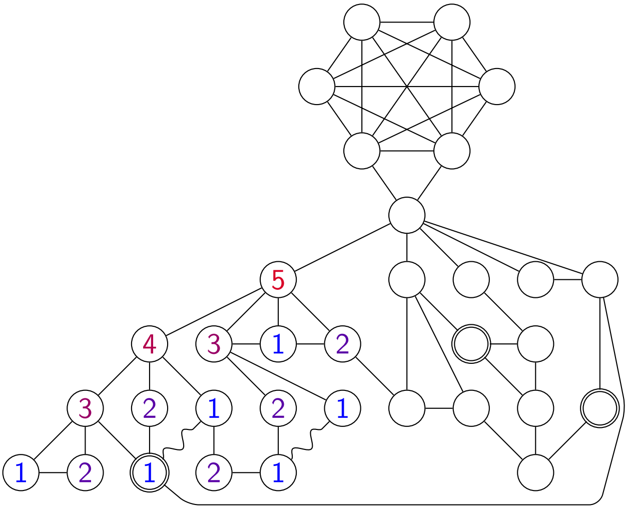

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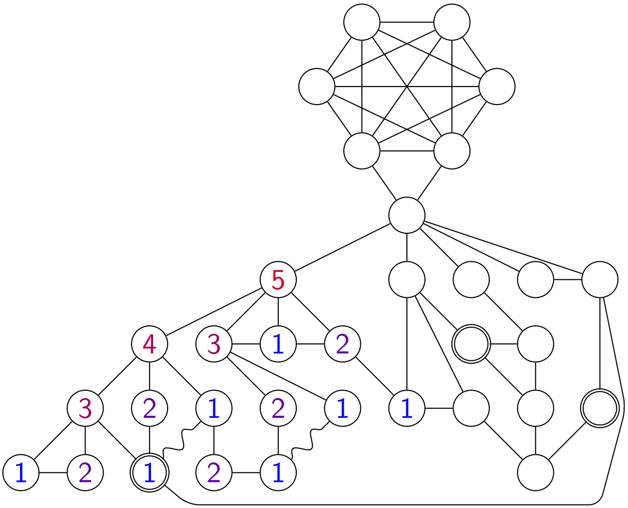

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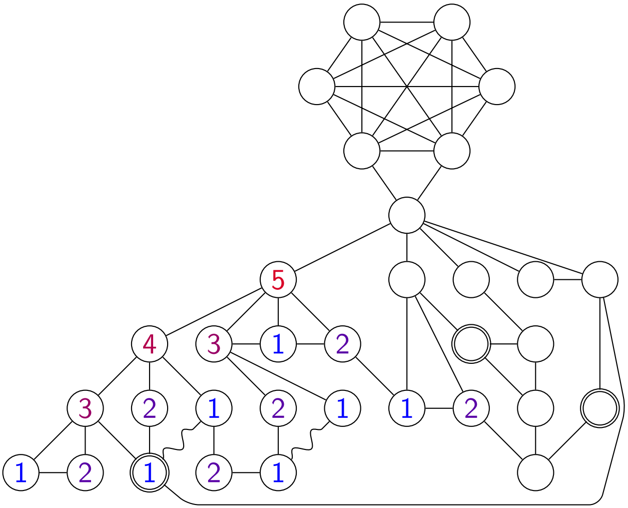

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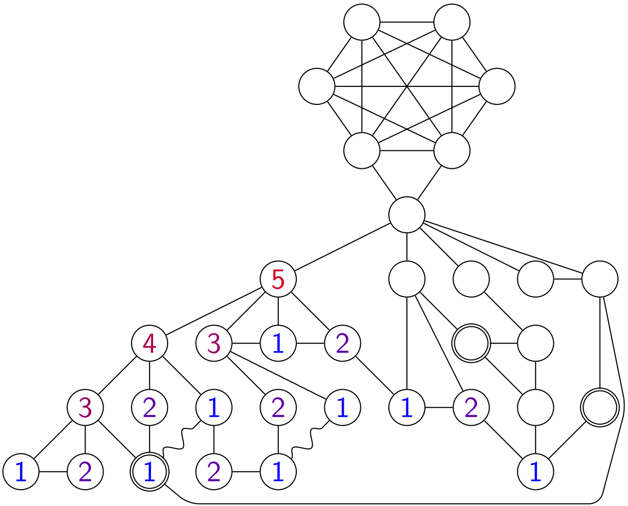

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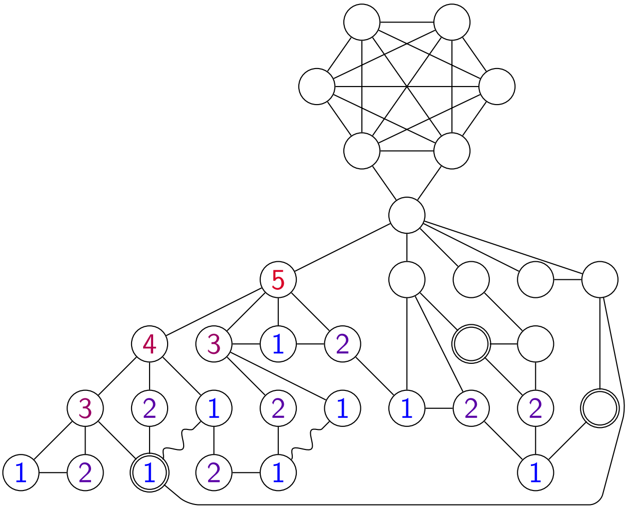

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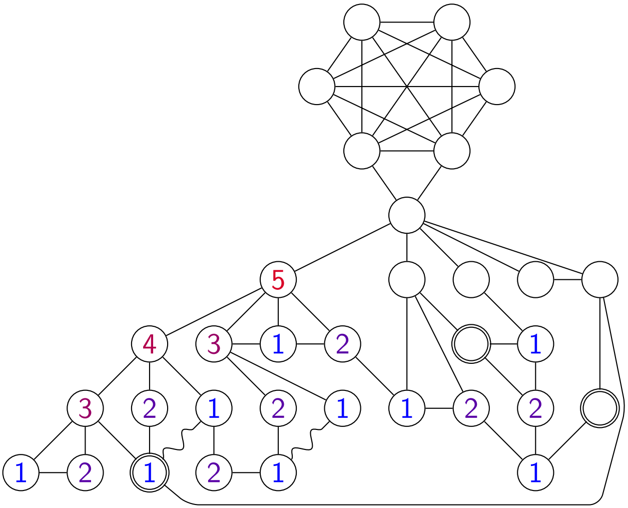

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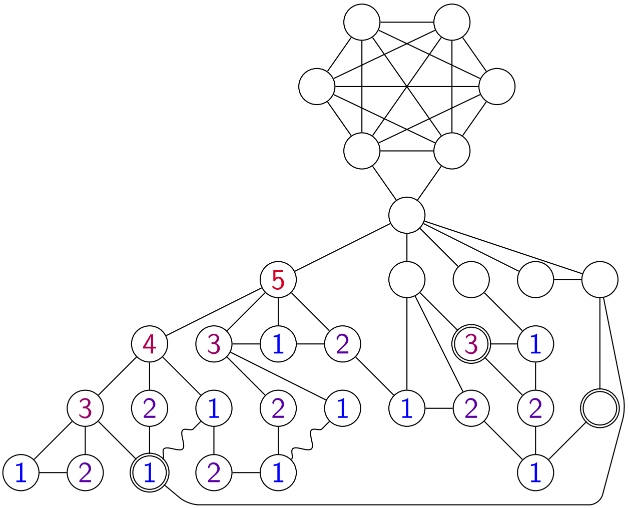

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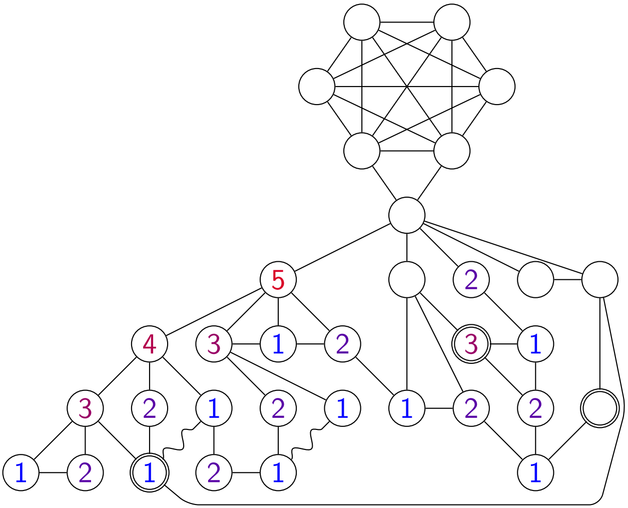

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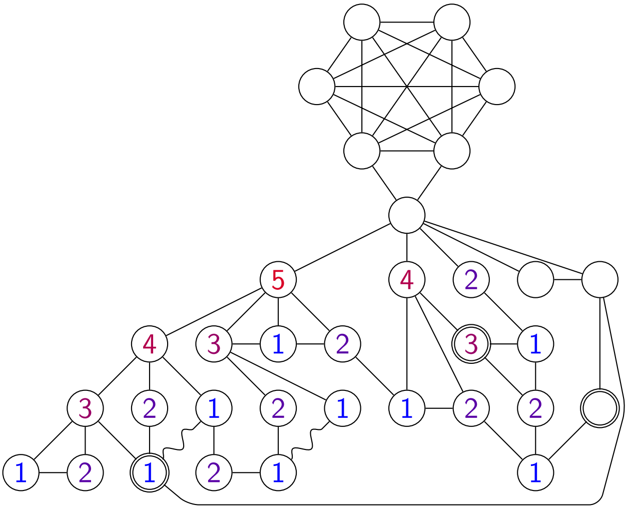

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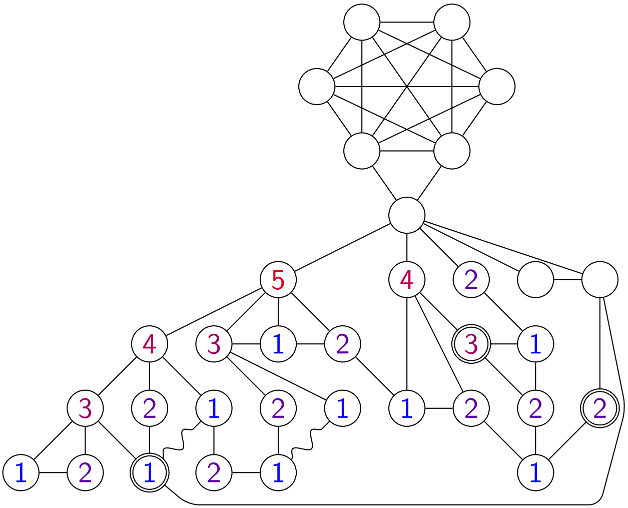

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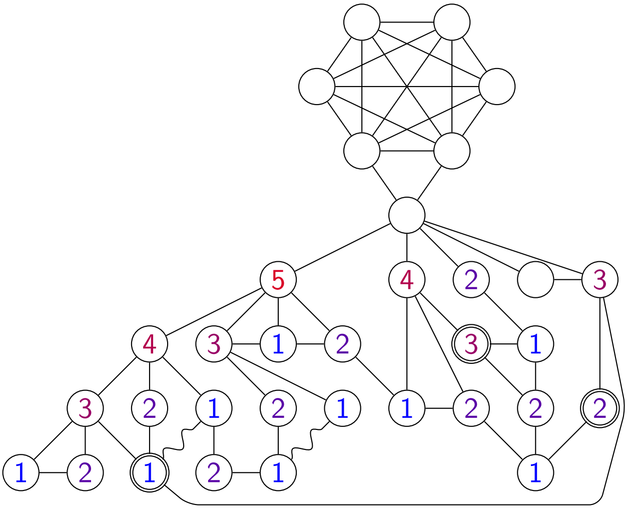

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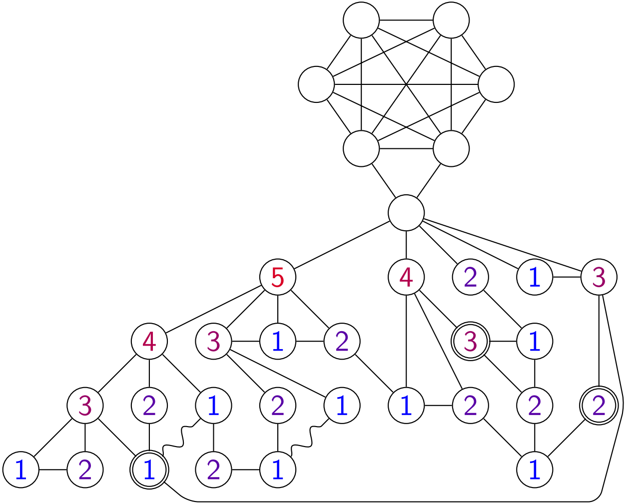

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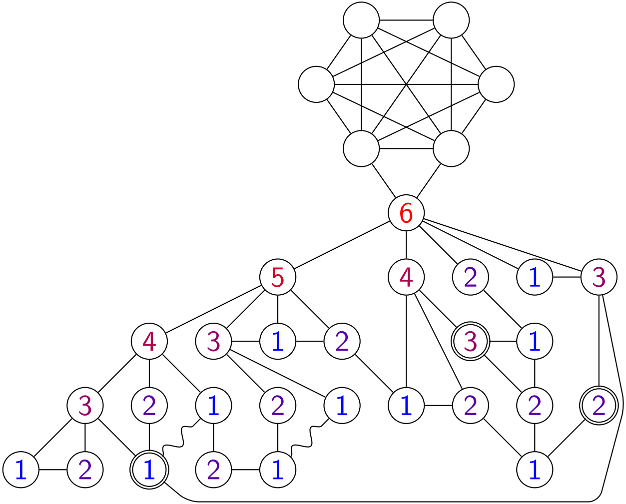

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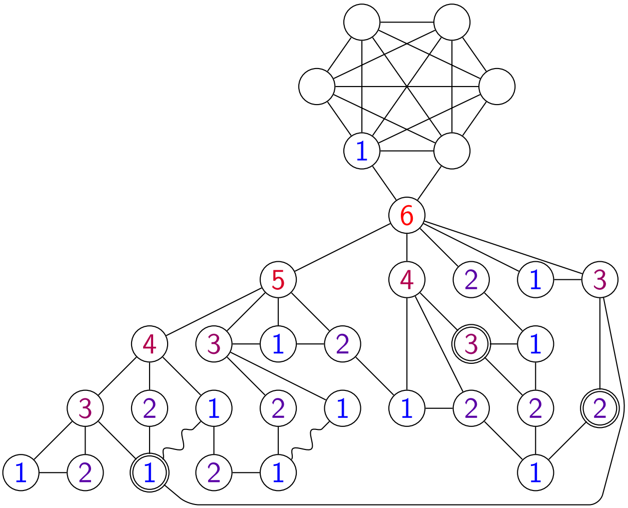

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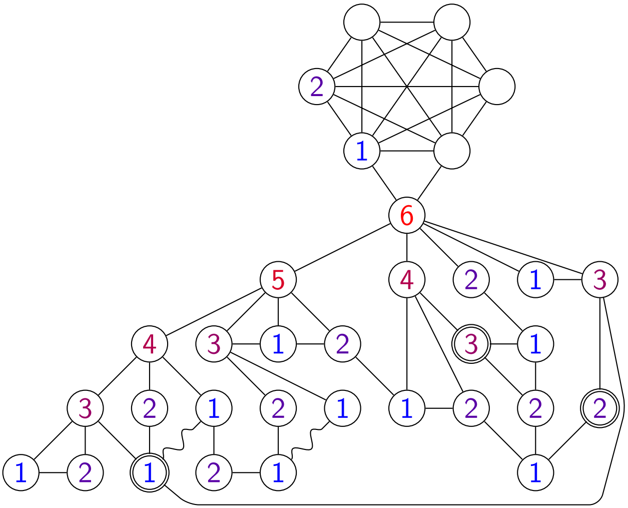

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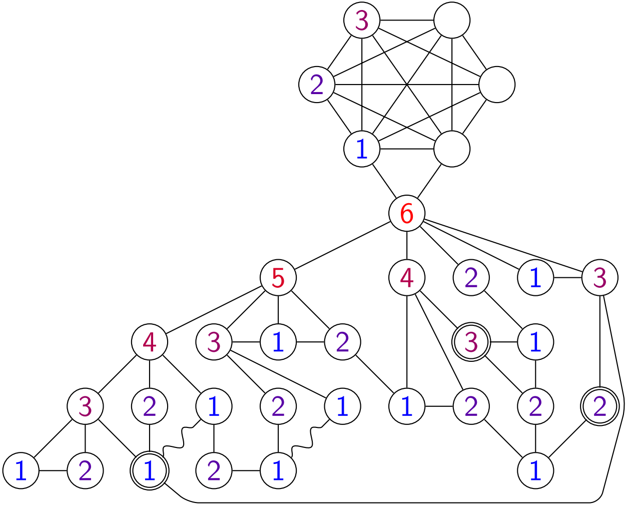

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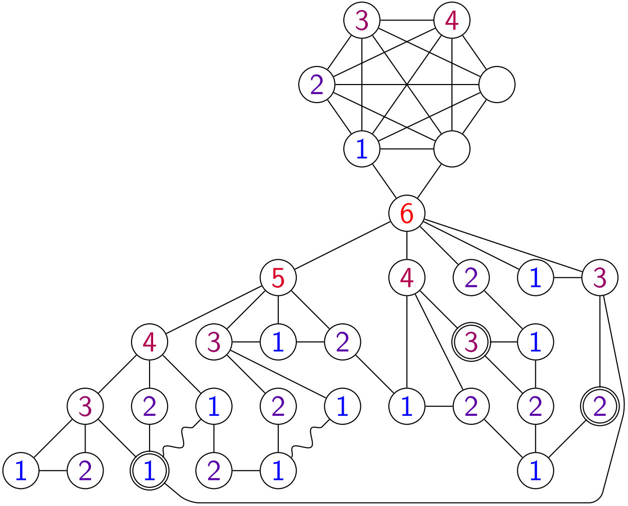

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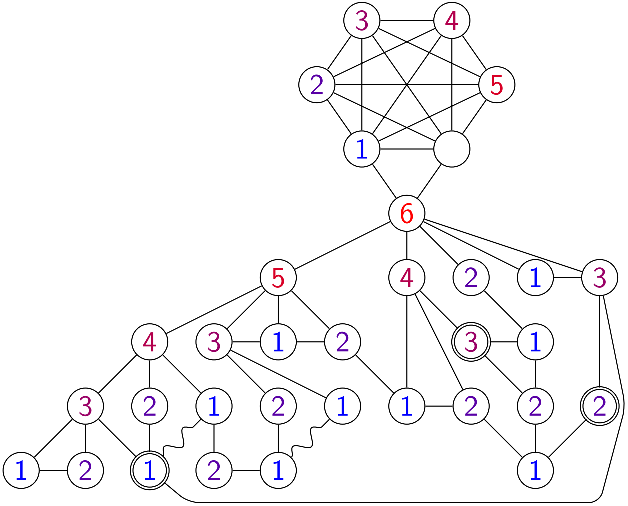

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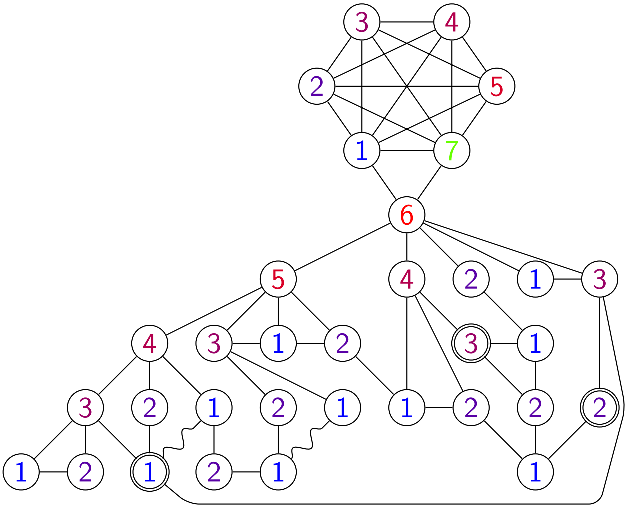

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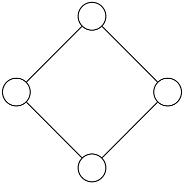

Weak Grundy number $= 3$, (connected) Grundy number $= 2$.

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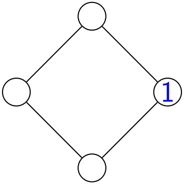

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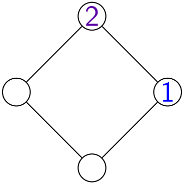

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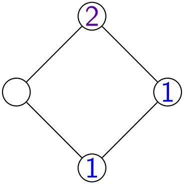

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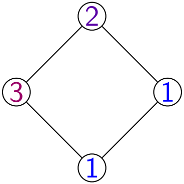

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- Solving the instance is *easier* with this extra information.

Now, can you propose an algorithm in $O(f(k)n^c)$ for finding a path of length k in a graph?

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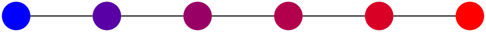

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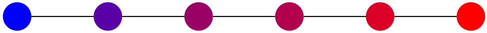

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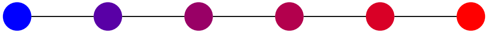


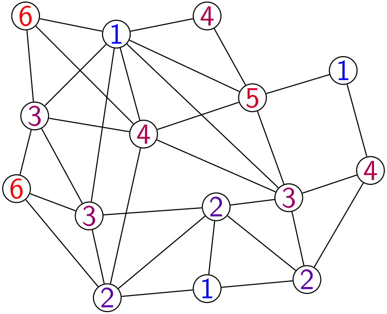
- How many times do you repeat the random coloring?
- How do find a path of length k in colored instances?

Theorem

WEAK GRUNDY COLORING *is in* FPT.

FPT algorithm: $O(f(k)n^c)$.

Guess #1



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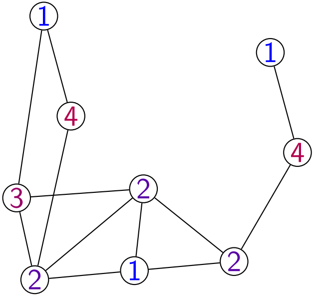

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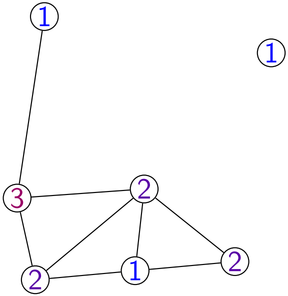
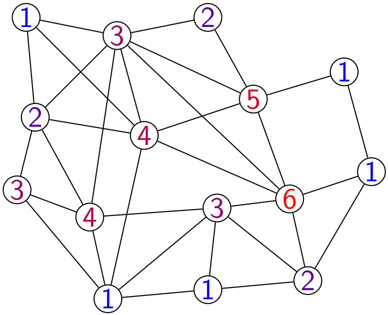


Guess #2



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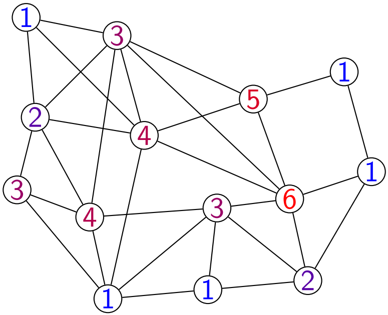

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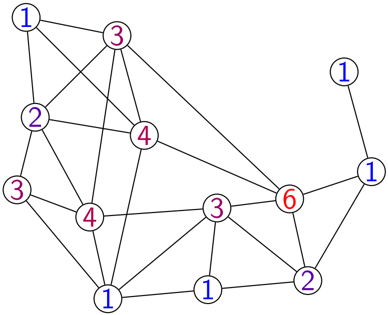

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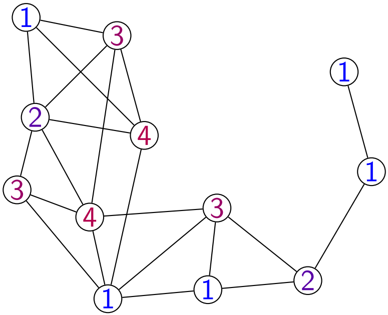

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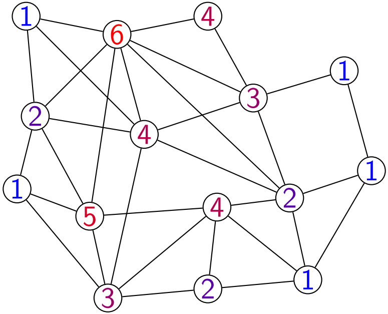

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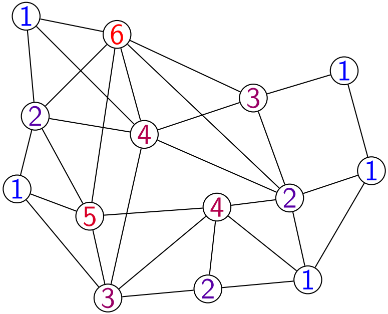

Open Problems

- Is Grundy Coloring solvable in $O(f(k)n^c)$?
- What is the complexity of Connected Grundy Coloring for $k = 4$, $k = 5$ and $k = 6$?
- What is the complexity of Weak Connected Grundy Coloring for k constant?