Twin-Width and Contraction Sequences

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ENS de Lyon, LIP

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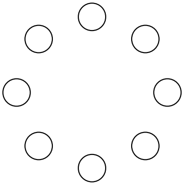
The modeling power of graphs



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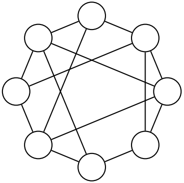

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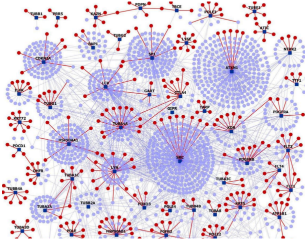

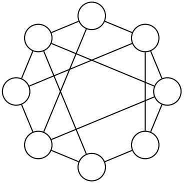

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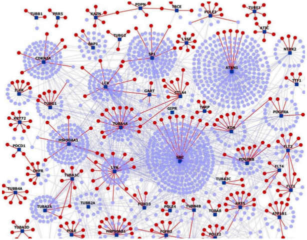

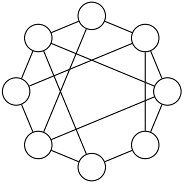

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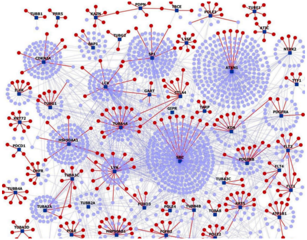
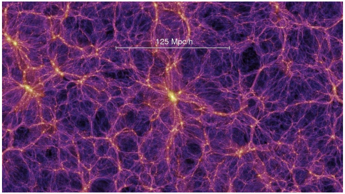
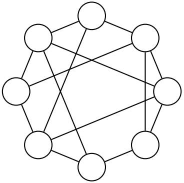
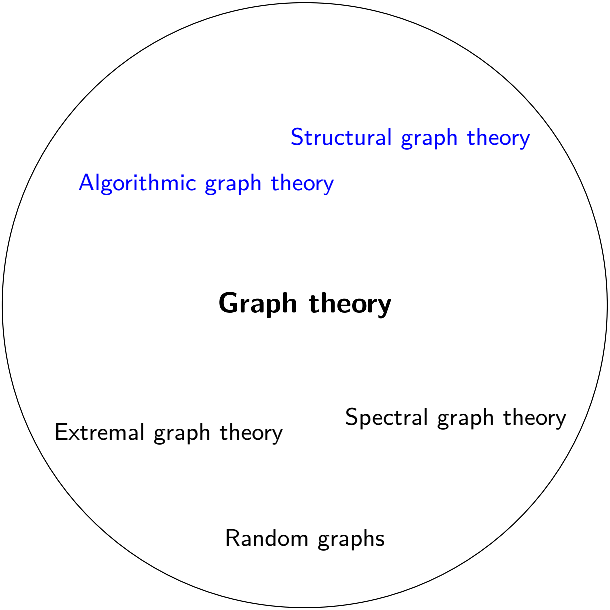


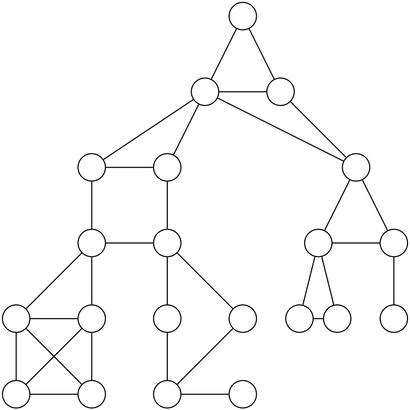




Graph theory and its interactions

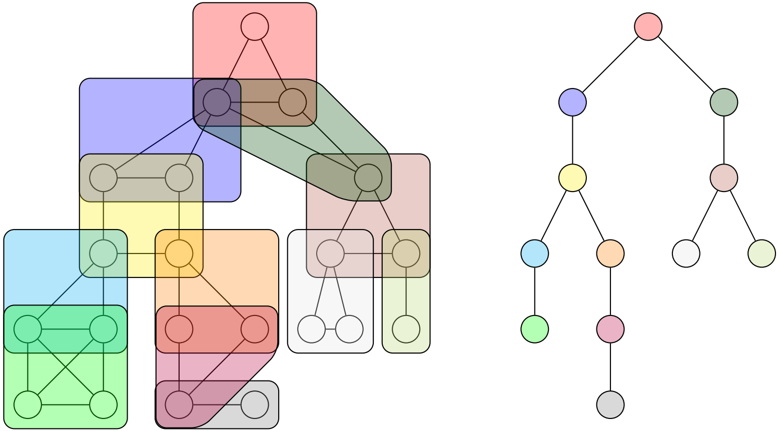


Tree-decomposition



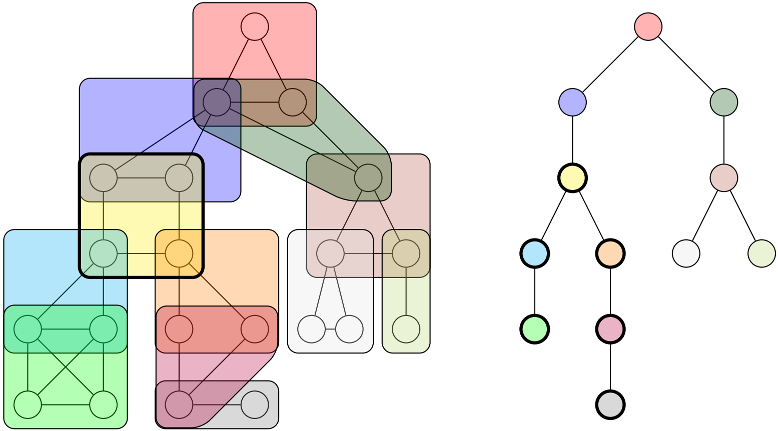
Tree-decomposition

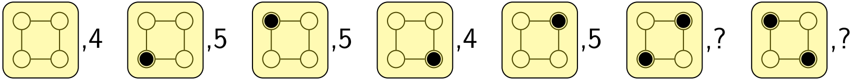
Cover by bags mapping to a tree s.t. each vertex lies in a subtree



Tree-decomposition: solving Max Independent Set

For each trace in each bag, keep a best solution in what is below





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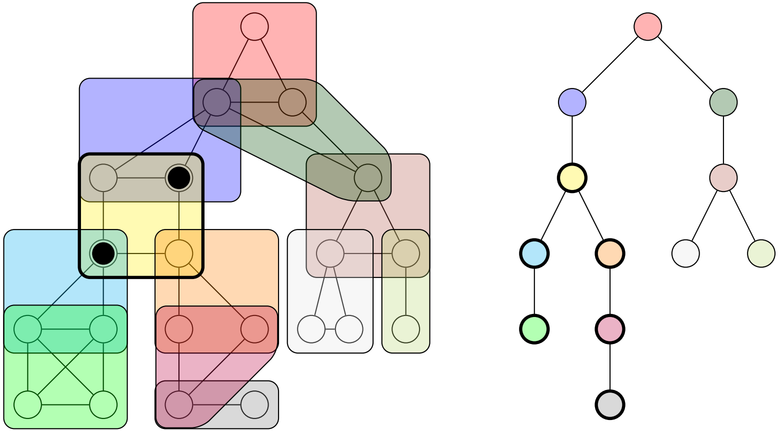

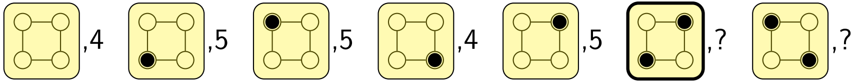

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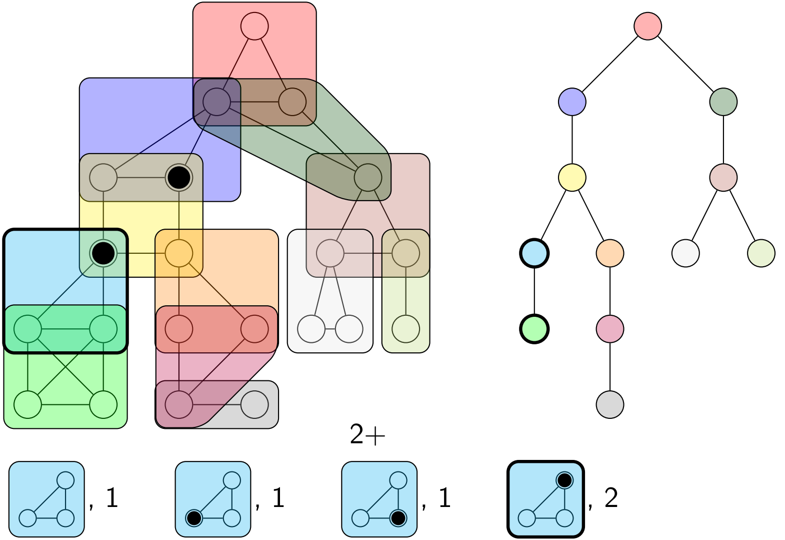

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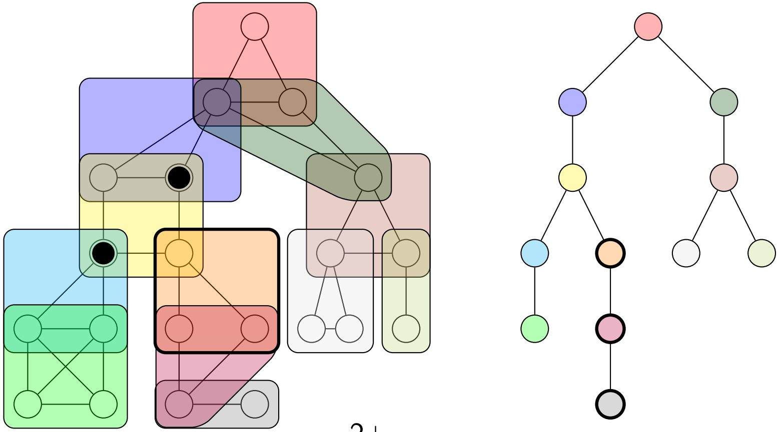
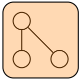


2+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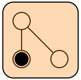
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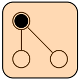
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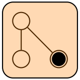
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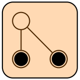
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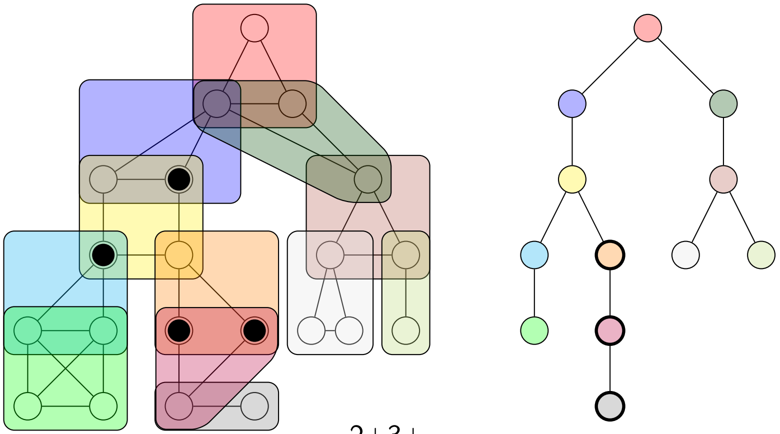
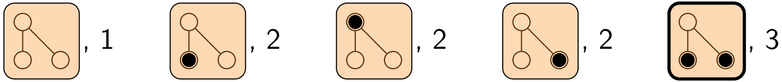


2+3+



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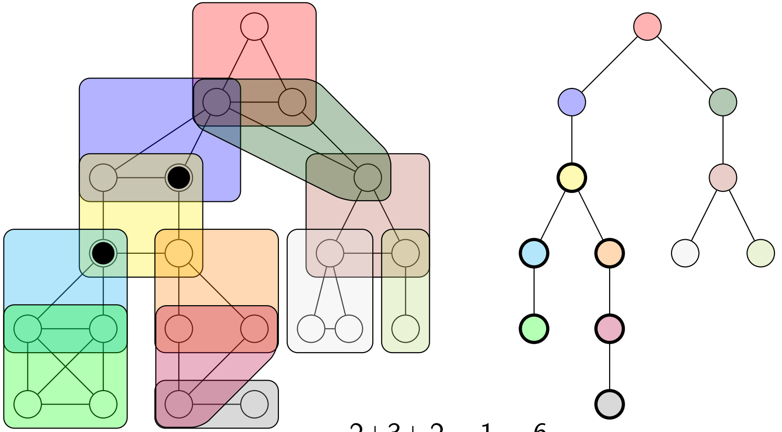
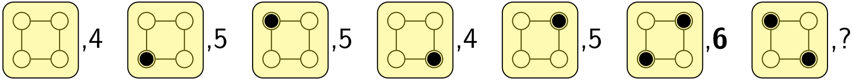


$2+3+2-1=6$



Treewidth

Minimum largest bag size over all tree decompositions minus 1

- rediscovered several times in the 70's and 80's...
- made central by *Graph Minors* and algorithmic graph theory
- previous slide: $2^{O(\mathrm{tw})}n$ time for MAX INDEPENDENT SET
- generalized by Courcelle's theorem
- inspired other tree-based width parameters: clique-width, rank-width, mim-width, etc.

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Computing a tree decomposition?

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Computing a tree decomposition? NP-hard but various algorithms

width $2\mathrm{tw}+1$ in $2^{O(\mathrm{tw})}n$

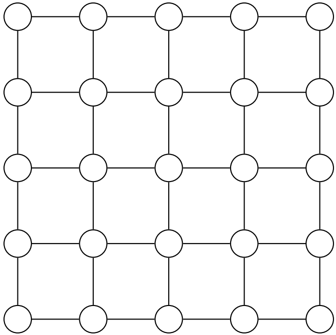
width $5\mathrm{tw}+4$ in $2^{6.76\mathrm{tw}}n\log n$

width tw in $2^{O(\mathrm{tw}^2)}n$

width $\mathrm{tw}\sqrt{\log \mathrm{tw}}$ in $n^{O(1)}$

width tw in 1.74^n

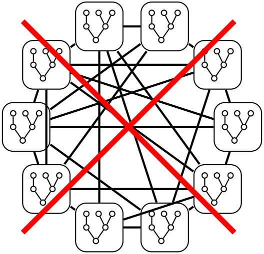
Low treewidth is very restrictive



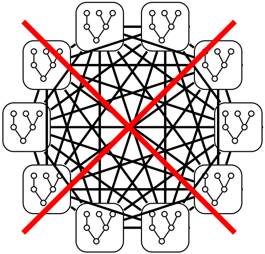
Grids have unbounded treewidth, clique-width, rank-width

Sparsity theory

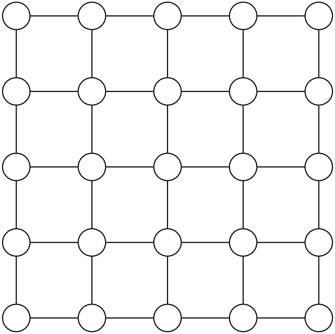
Bounded expansion: only sparse graph by shallow tree contraction

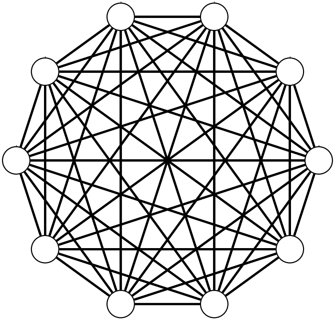


Nowhere denseness: no large clique by shallow tree contraction



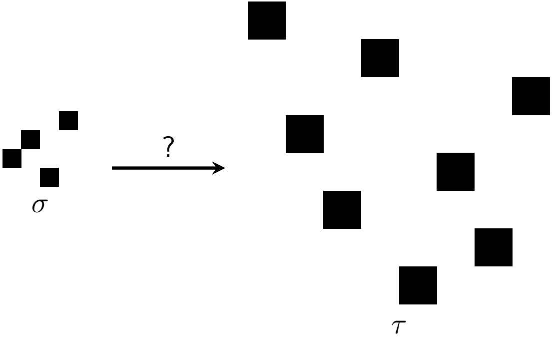
Going beyond sparsity and bounded clique-width?





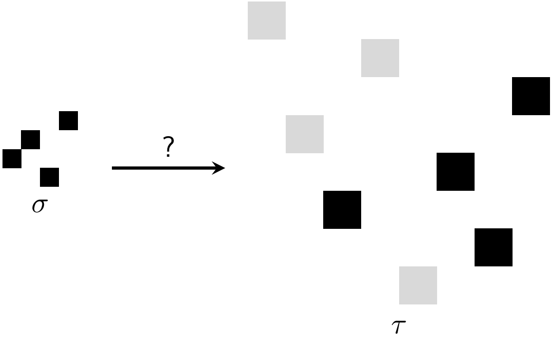
Conciliating the grid and the clique

The genesis of twin-width: Permutation Pattern



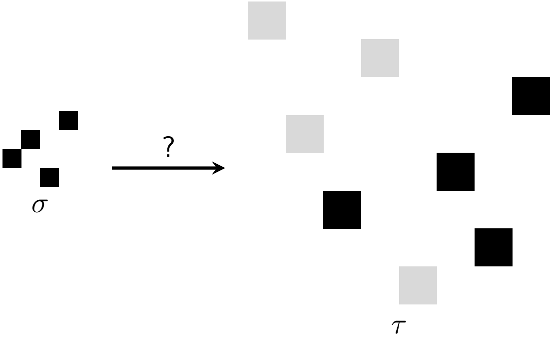
Is 3124 in 57362841?

The genesis of twin-width: Permutation Pattern



Is 3124 in 5**736**2**8**41? Yes

The genesis of twin-width: Permutation Pattern



Theorem (Guillemot, Marx '14)

Permutation Pattern *can be solved in time* $f(|\sigma|)|\tau|$.

Guillemot and Marx's win-win algorithm

Is σ in τ?

Theorem (Marcus, Tardos '04)

$\forall t$, $\exists c_t$ $\forall$ $n \times n$ 0,1-matrix with $\geqslant c_t n$ 1-entries has a t-grid minor.

$$\text{4-grid minor} \quad \left[\begin{array}{cc|cc|cc|cc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
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$\geqslant c_{|\sigma|} n$ 1-entries: answer YES from the $|\sigma|$-grid minor, or
$< c_{|\sigma|} n$ 1-entries: merge of two "similar" rectangles of 1s

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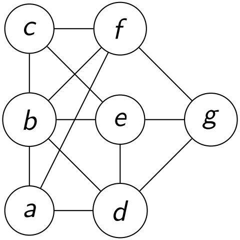
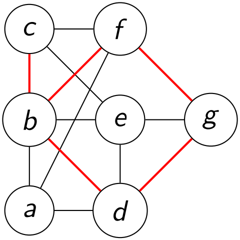
If the latter always holds: exploitable "decomposition" of τ

Graphs



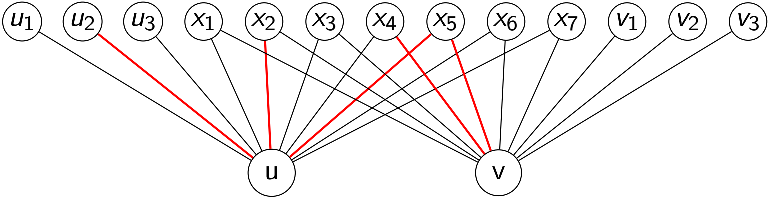
Two outcomes between a pair of vertices:
edge or non-edge

Trigraphs



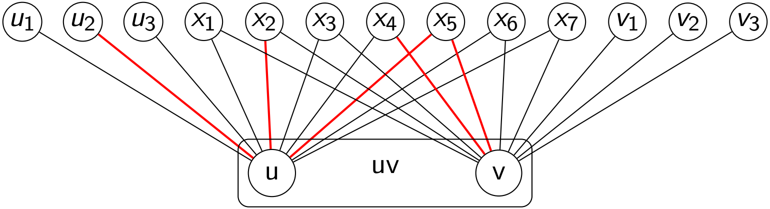
Three outcomes between a pair of vertices:
edge, or non-edge, or red edge (error edge)

Contractions in trigraphs



Identification of two non-necessarily adjacent vertices

Contractions in trigraphs



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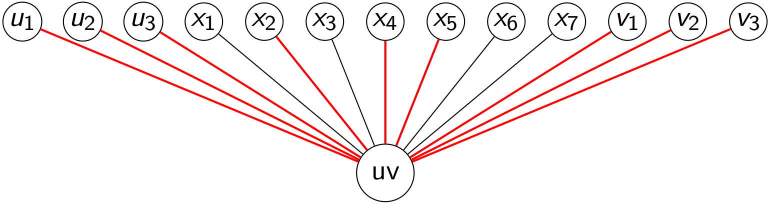
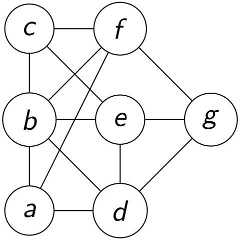


edges to $N(u) \triangle N(v)$ turn red, for $N(u) \cap N(v)$ red is absorbing

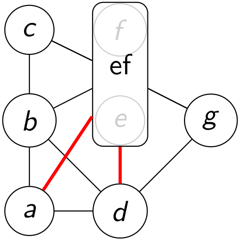
Contraction sequence



A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that G_i is obtained by performing one contraction in G_{i+1}.

$\mathcal{R}(G_i)$: red graph of G_i, obtained by removing its black edges

Contraction sequence



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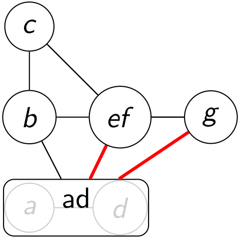

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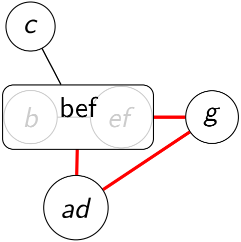

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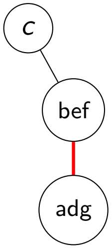

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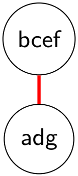

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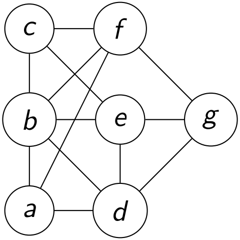
Twin-width

tww(G): Least integer d such that G admits a contraction sequence where all red graphs have *maximum degree* at most d.



Maximum red degree $= 0$

overall maximum red degree = 0

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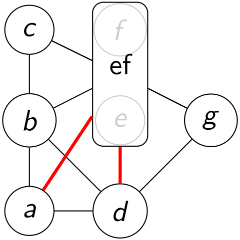

Maximum red degree $= 2$

overall maximum red degree = 2

Twin-width

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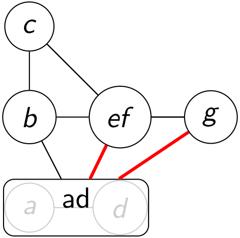

Maximum red degree $= 2$

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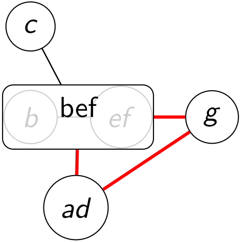

Maximum red degree = 2

overall maximum red degree = 2

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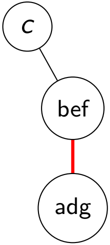
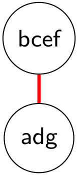


Maximum red degree $= 1$

overall maximum red degree = 2

Twin-width

tww(G): Least integer d such that G admits a contraction sequence where all red graphs have *maximum degree* at most d.



Maximum red degree $= 1$

overall maximum red degree = 2

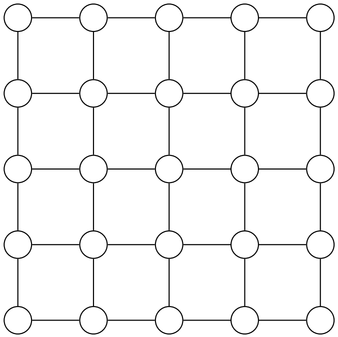
Twin-width

tww(G): Least integer d such that G admits a contraction sequence where all red graphs have *maximum degree* at most d.

Maximum red degree $= 0$

overall maximum red degree = 2

Grids have twin-width at most 4



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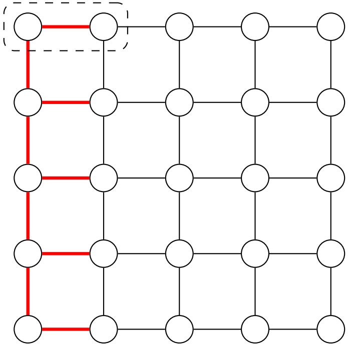

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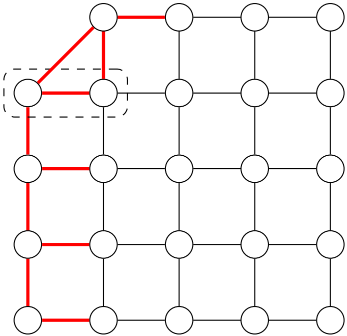

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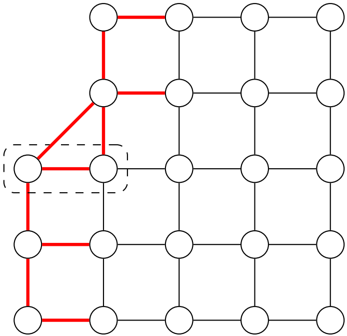

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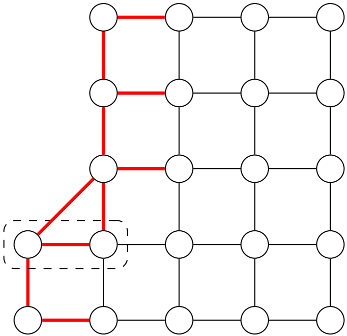

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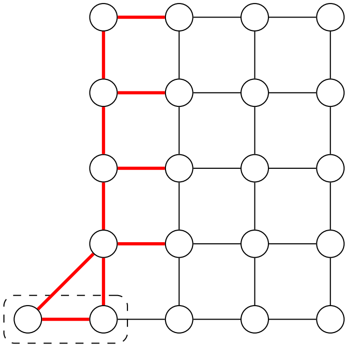

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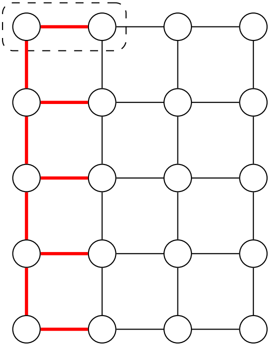
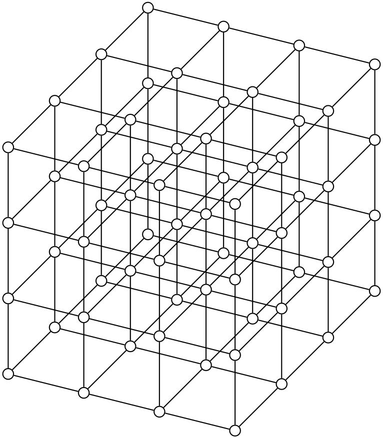
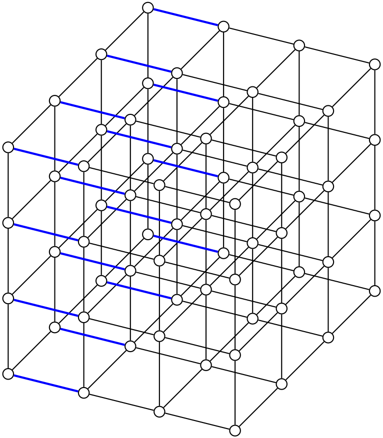


4-sequence for 2-dimensional grids

3-dimensional grids

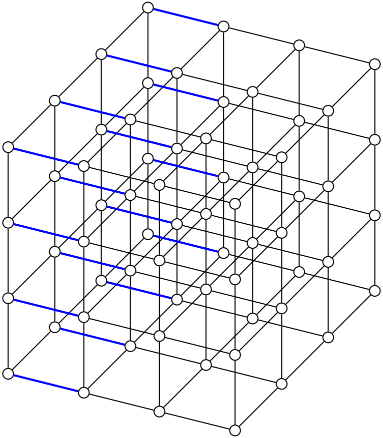


3-dimensional grids



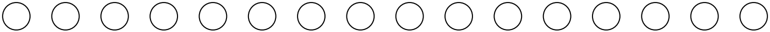
Contract the blue edges

3-dimensional grids



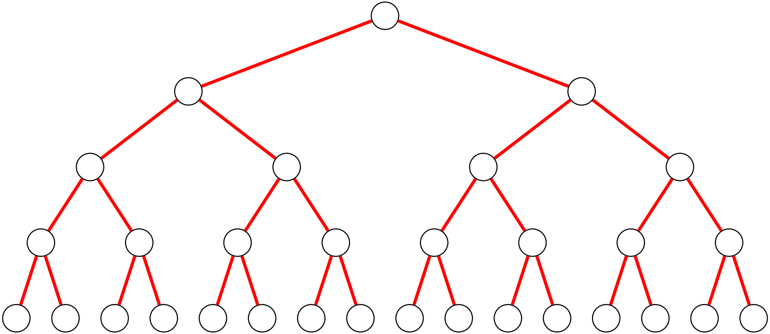
The d-dimensional grid has twin-width $\Theta(d)$

$(\geqslant 2\log n)$-subdivisions have twin-width at most 4



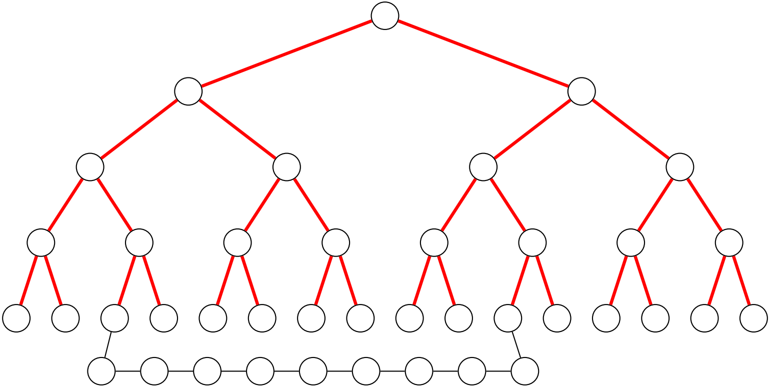
Consider the *branching* vertices

($\geqslant 2\log n$)-subdivisions have twin-width at most 4



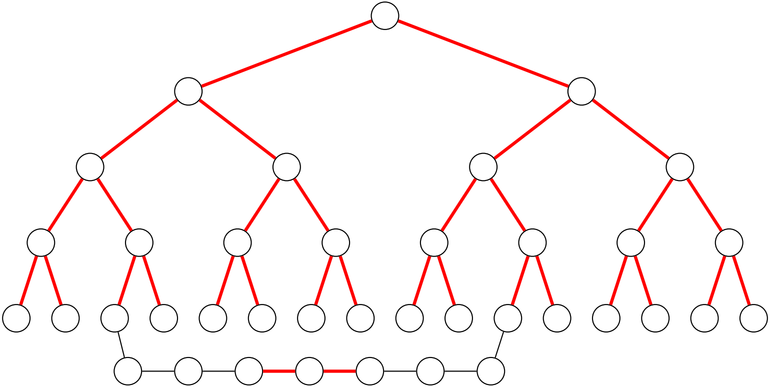
and make them leaves of a red full binary tree

($\geqslant 2 \log n$)-subdivisions have twin-width at most 4



Take any subdivided edge

($\geqslant 2\log n$)-subdivisions have twin-width at most 4



Shorten it to the length of the path in the red tree

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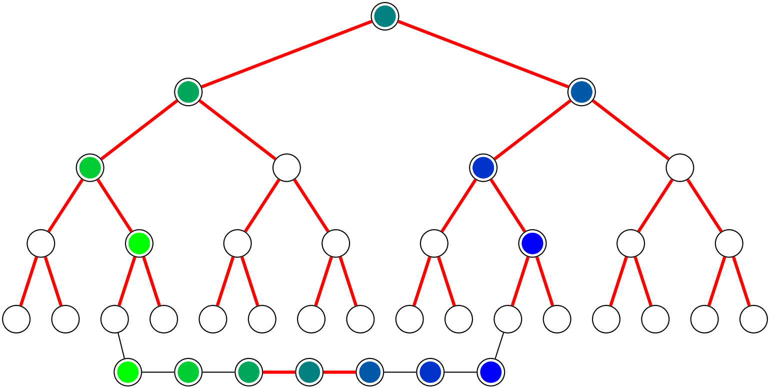
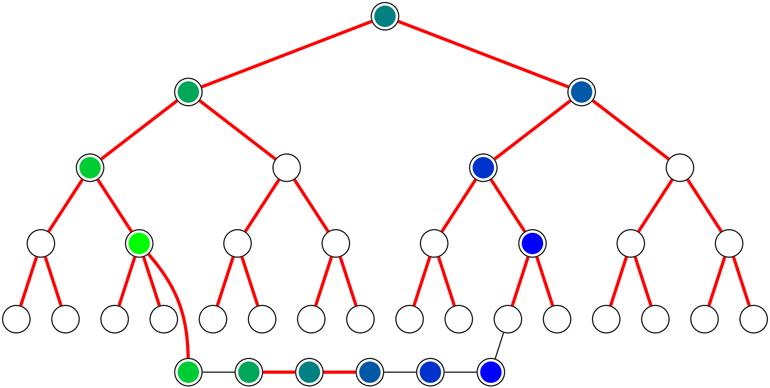


Zip the subdivided edge onto the tree

$(\geqslant 2 \log n)$-subdivisions have twin-width at most 4



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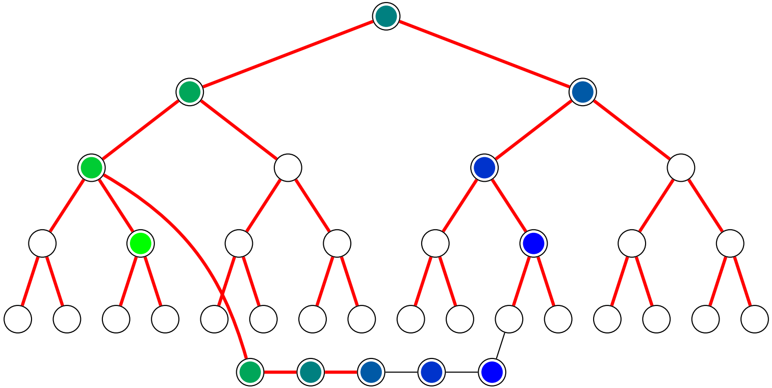

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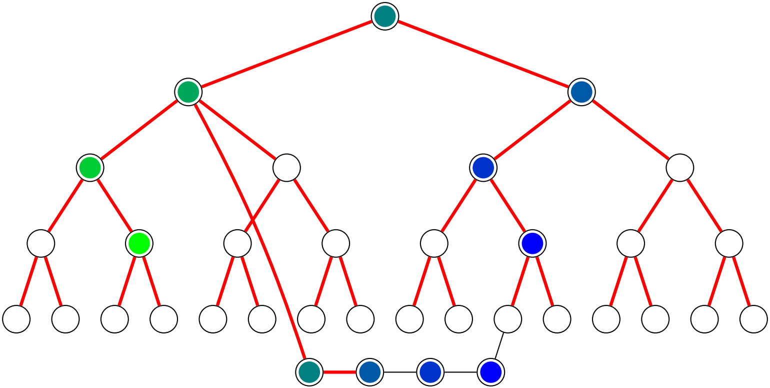

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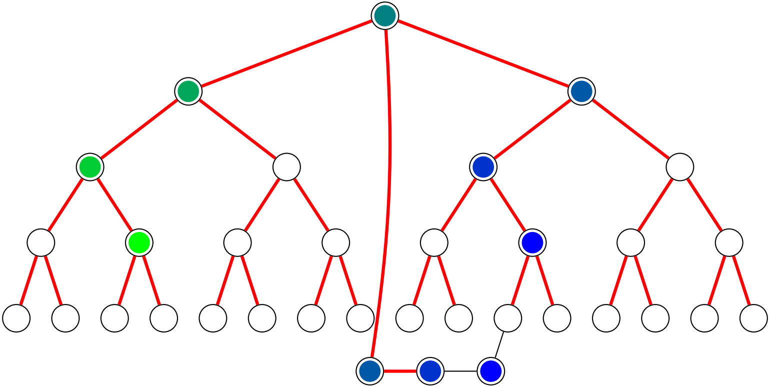

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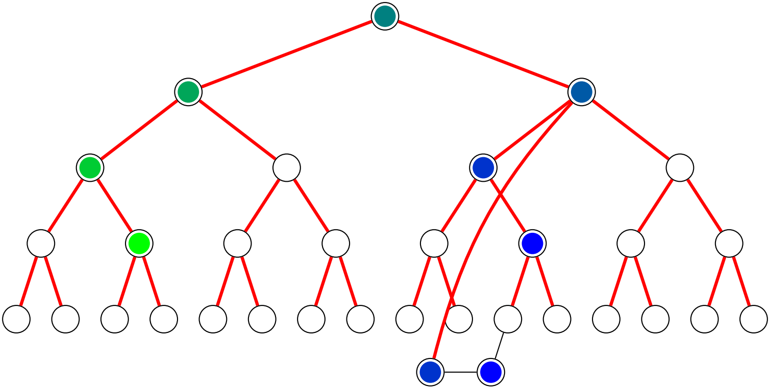

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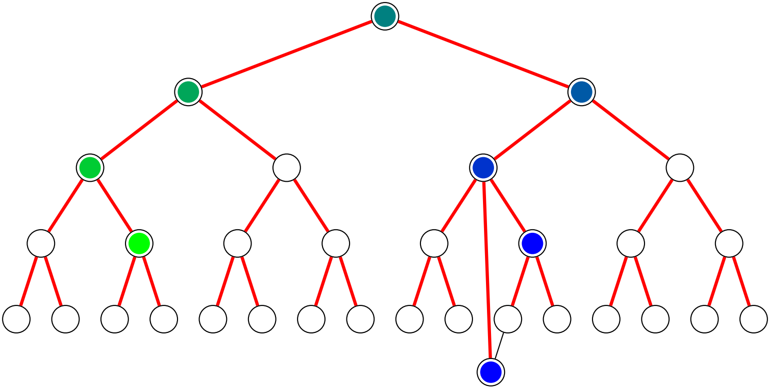

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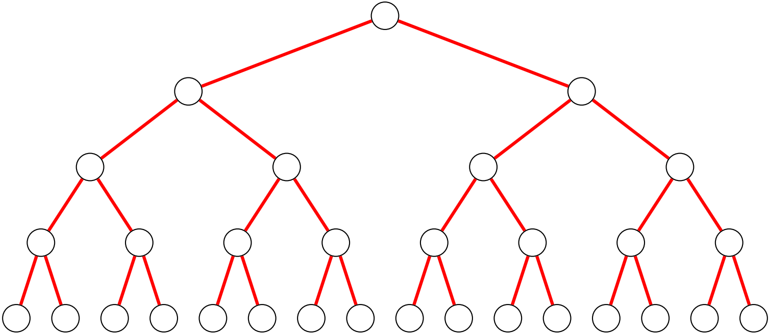

Take another subdivided edge and repeat

Mixed minor

Mixed cell: at least two distinct rows and two distinct columns

$$\left[\begin{array}{cc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

3-mixed minor

Mixed minor

Mixed cell: at least two distinct rows and two distinct columns

$$
\left[\begin{array}{cc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
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0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]
$$

3-mixed minor

A matrix is said t-**mixed free** if it does not have a t-mixed minor

Twin-width and mixed freeness

Theorem (B., Kim, Thomassé, Watrigant '20)

*If G admits **a** t-mixed free adjacency matrix, then* $tww(G) = 2^{2^{O(t)}}$.

Twin-width and mixed freeness

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If $\exists \prec$ *s.t.* $Adj_{\prec}(G)$ *is* t*-mixed free, then* $tww(G) = 2^{2^{O(t)}}$*.*

Twin-width and mixed freeness

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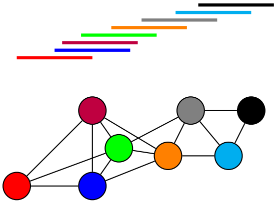
If $\exists \prec$ s.t. $Adj_{\prec}(G)$ is t-mixed free, then $tww(G) = 2^{2^{O(t)}}$.

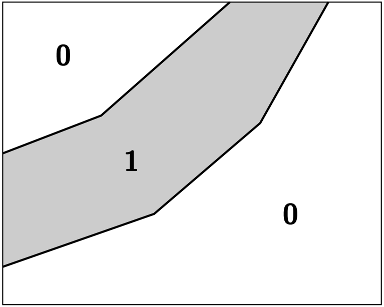
Now to bound the twin-width of a class $\mathcal{C}$:
1) Find a *good* vertex-ordering procedure
2) Argue that, in this order, a t-mixed minor would contradict the structure of $\mathcal{C}$

Unit interval graphs

Intersection graph of unit segments on the real line

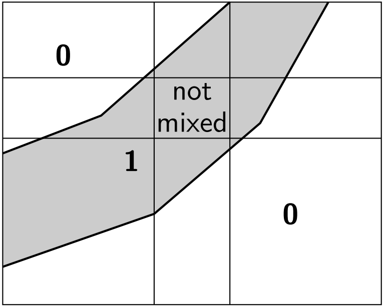


Unit interval graphs



order by left endpoints

Unit interval graphs



No 3-by-3 grid has all 9 cells crossed by two non-decreasing curves

Classes known to have effective bounded twin-width

- Bounded rank-width, and even, boolean-width graphs,
- every hereditary proper subclass of permutation graphs,
- posets of bounded antichain size (seen as digraphs),
- unit interval graphs,
- K_t-minor free graphs,
- map graphs,
- subgraphs of d-dimensional grids,
- K_t-free unit d-dimensional ball graphs,
- $\Omega(\log n)$-subdivisions of all the n-vertex graphs,
- cubic expanders defined by iterative random 2-lifts from K_4,
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Can we solve problems faster, given an $O(1)$-sequence?

k-Independent Set

Algorithms in time $f(k)|V(G)|^{o(k)}$ are unlikely in general

Theorem (B., Geniet, Kim, Thomassé, Watrigant '21)

k-Independent Set can be solved in time $O(d^{2k}k^2|V(G)|)$ given a d-sequence $G = G_n, \ldots, G_1$.

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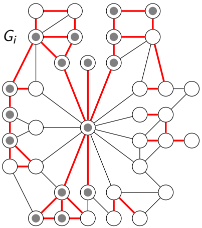
Main idea: **For every $D \in \binom{V(G_i)}{\leqslant k}$ such that $\mathcal{R}(G_i)[D]$ is connected, store a largest independent set in $G[\bigcup D]$ intersecting every vertex of D, for i going to n down to 1.**



k-Independent Set: First observations

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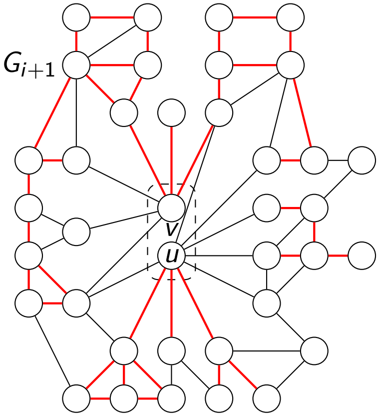
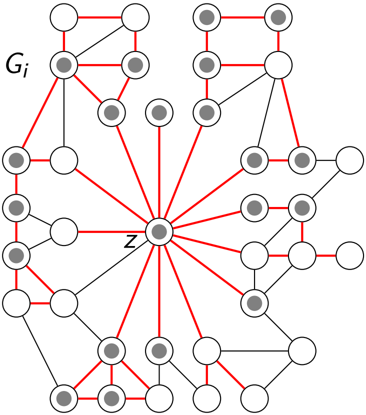
Running time: As $\mathcal{R}(G_i)$ has maximum degree at most d, it has at most $d^{2k}i$ connected sets on up to k vertices.

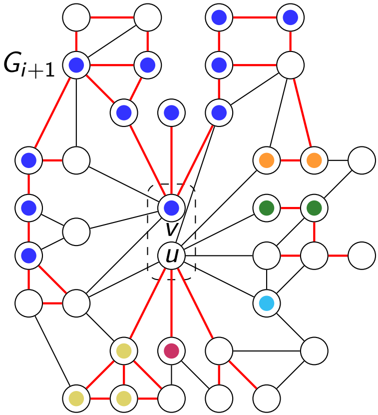
k-Independent Set: Update of partial solutions

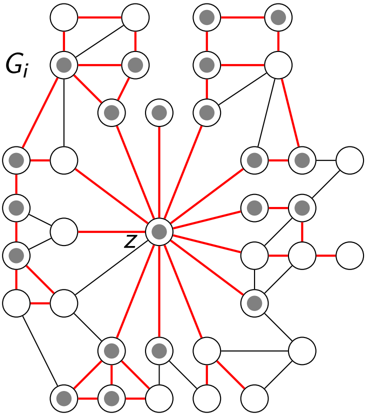




Best partial solution inhabiting •?

k-Independent Set: Update of partial solutions





3 unions of red connected subgraphs to consider in G_{i+1}
with u, or v, or both

Model checking

Graph FO/MSO Model Checking **Parameter:** $|\varphi|$
Input: A graph G and a first-order/monadic second-order sentence $\varphi \in FO/MSO(\{E\})$
Question: $G \models \varphi$?

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$$\varphi = \exists x_1 \exists x_2 \cdots \exists x_k \forall x \bigvee_{1 \leqslant i \leqslant k} x = x_i \vee \bigvee_{1 \leqslant i \leqslant k} E(x, x_i) \vee E(x_i, x)$$

$G \models \varphi$? $\Leftrightarrow$

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$G \models \varphi$? $\Leftrightarrow$ k-DOMINATING SET

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$G \models \varphi$? $\Leftrightarrow$ 3-Coloring

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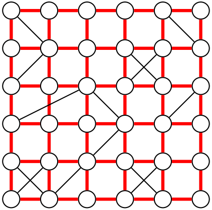
$G \models \varphi$? $\Leftrightarrow$ 3-COLORING

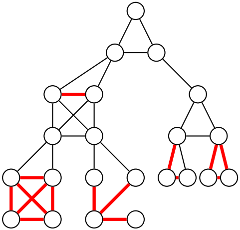
When can we solve MODEL CHECKING in time $f(\varphi)\lvert V(G)\rvert^{O(1)}$?

Reduced parameters and component twin-width

$$p^{\downarrow}(G) = \min\{\max_{i \in [n]} p(\mathcal{R}(G_i)) : G_n, \ldots, G_1 \text{ sequence of } G\}$$

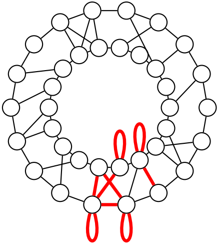


reduced maximum degree = twin-width



reduced component size $\equiv$ cliquewidth

= component twin-width



reduced #edges $\equiv$ linear cliquewidth

Model checking on graphs of bounded twin-width

Recast of the Courcelle–Makowsky–Rotics theorem:

Theorem (B., Kim, Reinald, Thomassé '22)

MSO model checking can be solved in time $f(|\varphi|, d) \cdot |V(G)|$ on graphs G given with a d-sequence of component twin-width.

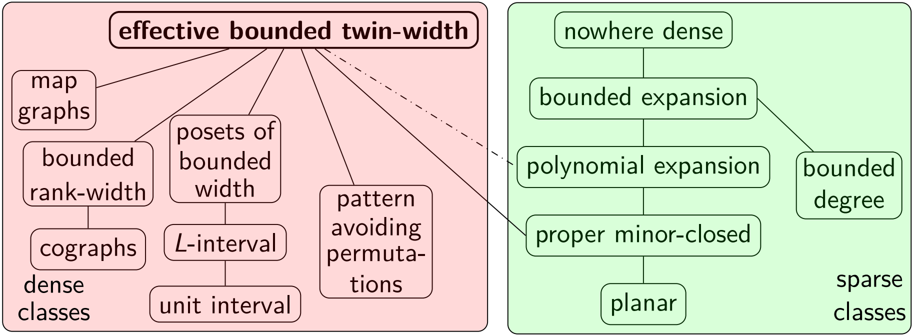
Generalization of the k-Independent Set algorithm:

Theorem (B., Kim, Thomassé, Watrigant '20)

FO model checking can be solved in time $f(|\varphi|, d) \cdot |V(G)|$ on graphs G given with a d-sequence.

Gaifman's locality + MSO model checking algorithm

Classes for which FO model checking is FPT



Theorem (Bergé, B., Déprés '22)

Deciding if the twin-width of a graph is at most 4 is NP-complete.

First-order interpretations and transductions

FO interpretation: redefine the edges by a first-order formula
$\varphi(x, y) = \neg E(x, y)$ (complement)
$\varphi(x, y) = E(x, y) \vee \exists z E(x, z) \wedge E(z, y)$ (square)

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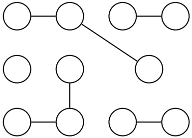
FO transduction: color by $O(1)$ unary relations, interpret, delete



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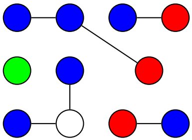

First-order interpretations and transductions

FO interpretation: redefine the edges by a first-order formula
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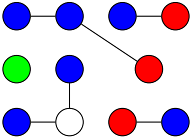

$$\varphi(x, y) = E(x, y) \vee (G(x) \wedge B(y) \wedge \neg\exists z R(z) \wedge E(y, z))$$
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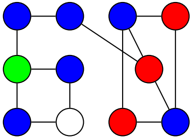

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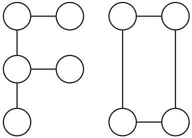
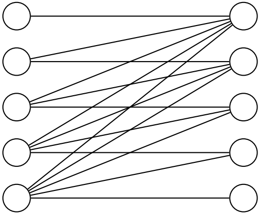


Stable and dependent (for hereditary classes)

Due to [Baldwin, Shelah '85; Braunfeld, Laskowski '22]

Stable class: no transduction of the class contains all ladders
Dependent class: no transduction of the class contains all graphs

ladder



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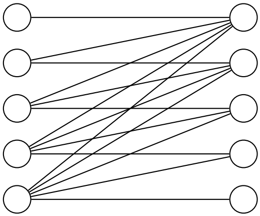

Bounded-degree graphs $\to$ stable
Unit interval graphs $\to$ dependent but not stable
Interval graphs $\to$ independent

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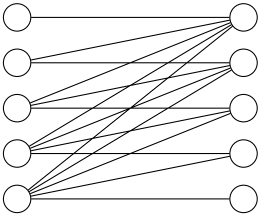
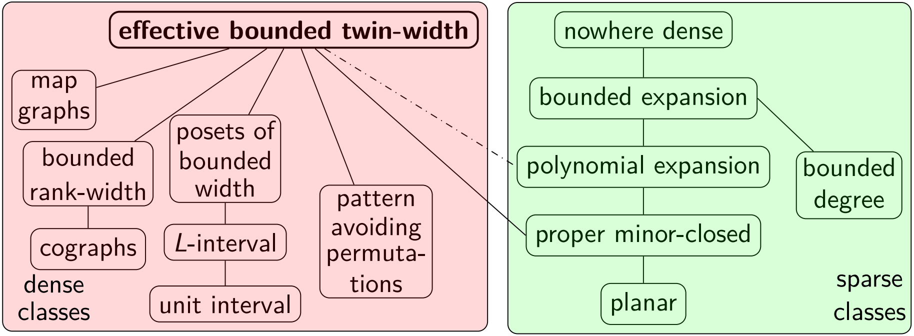


Bounded-degree graphs $\rightarrow$ stable
Unit interval graphs $\rightarrow$ dependent but not stable
Interval graphs $\rightarrow$ independent

Bounded twin-width graphs $\rightarrow$ dependent but not stable

Classes for which FO model checking is FPT



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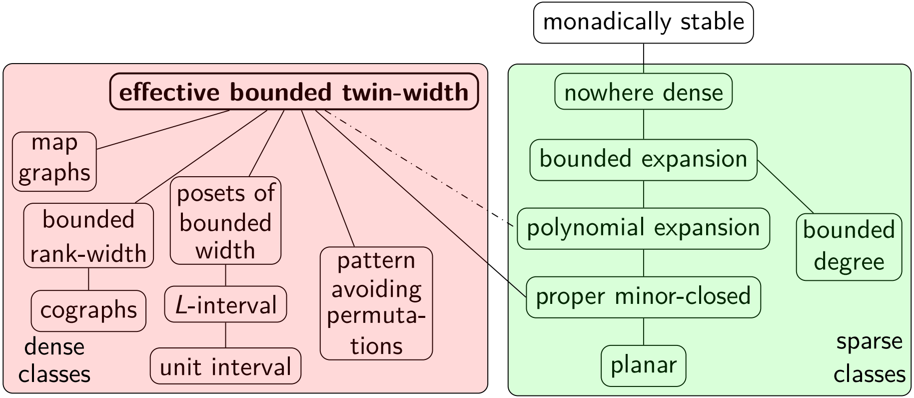

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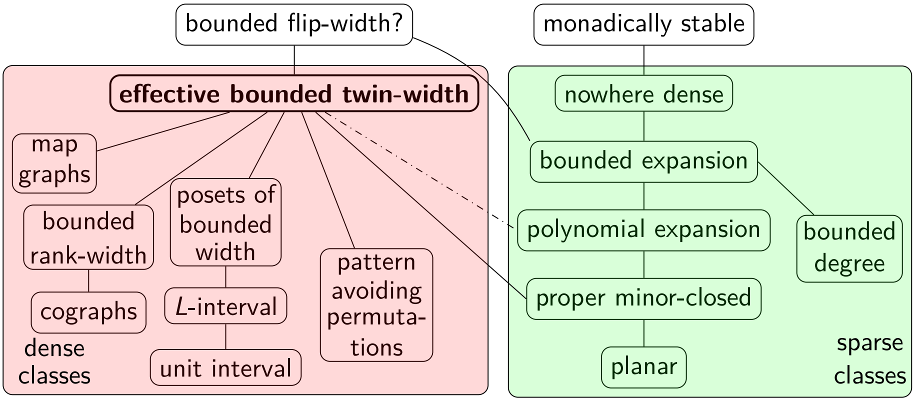

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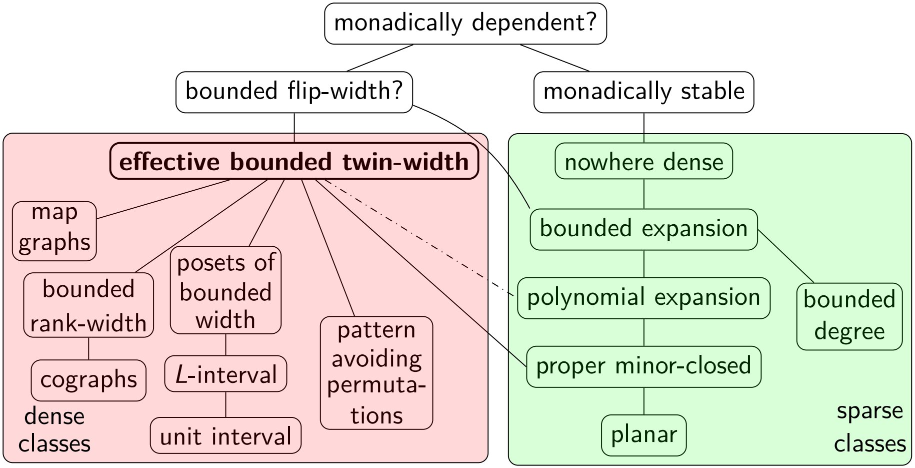

First-order transductions and twin-width

Theorem (B., Kim, Thomassé, Watrigant '20)

For every class $\mathcal{C}$ *with bounded twin-width and transduction* T*, the class* $\mathsf{T}(\mathcal{C})$ *has bounded twin-width.*

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Theorem (B., Bourneuf, Geniet, Thomassé '24)

There is a fixed proper permutation class $\mathcal{P}$ such that a class has bounded twin-width if and only if it is the transduction of $\mathcal{P}$.

The lens of contraction sequences

Class of bounded	FO transduction of	constr. on red graphs	efficient MC
linear rank-width	linear order	bd #edges	MSO
rank-width	tree order	bd component	**MSO**
twin-width	**proper perm. class**	bd degree	**FO**

Equivalences for **ordered** graphs

Theorem (B., Giocanti, Ossona de Mendez, Toruńczyk, Thomassé, Simon '21)

Let $\mathcal{C}$ be a hereditary class of ordered graphs. TFAE:

(*i*) *$\mathcal{C}$ has bounded twin-width.*

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Bounded twin-width is the structural characterization of tame ordered binary structures

Open questions

- Algorithm to compute/approximate twin-width in general
- Fully classify classes with tractable FO model checking
- Constructions of subcubic unbounded twin-width graphs
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Thank you for your attention!