Algorithms based on contraction sequences

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based on joint works with Colin Geniet, Eun Jung Kim, Amadeus Reinald, Stéphan Thomassé, and Rémi Watrigant

ENS Lyon, LIP

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Unification and generalization via contraction sequences

Fixed-parameter tractable ($f(k)n^{O(1)}$) algorithms for

- Hamiltonian Cycle parameterized by treewidth k:
  Courcelle's theorem or progdyn on tree-decompositions

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  Bova et al., Gajarský et al.
- k-INDUCED SUBGRAPH ISOMORPHISM on unit intervals:
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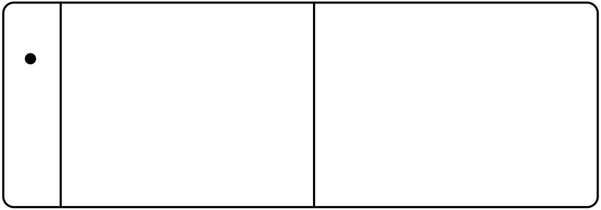
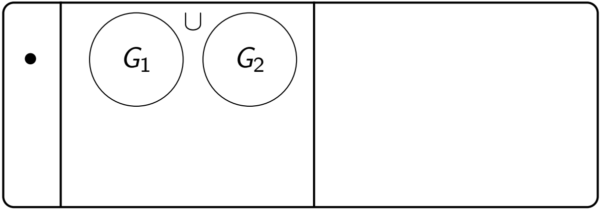
$\boxed{\text{All these results are in fact part of the same framework}}$

Cographs



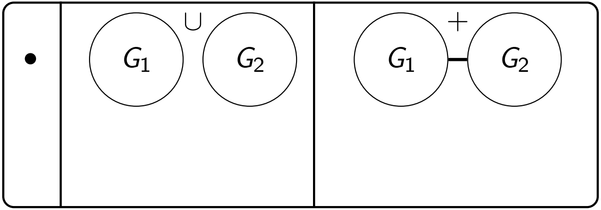
A single vertex is a cograph,

Cographs



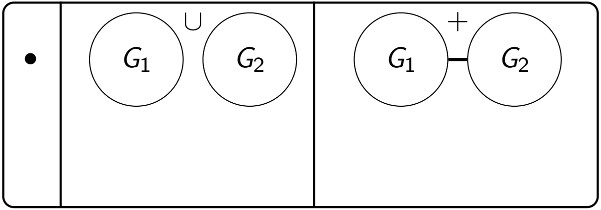
as well as the union of two cographs,

Cographs

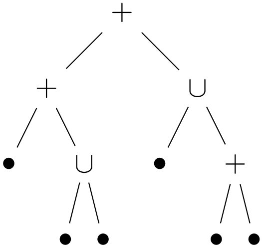


and the complete join of two cographs.

Cographs



Many NP-hard problems are polytime solvable on cographs



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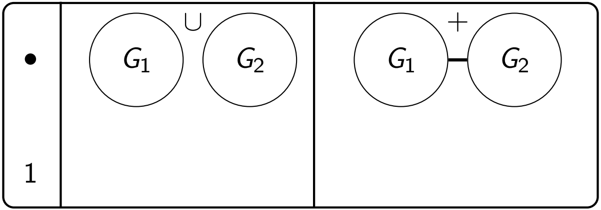
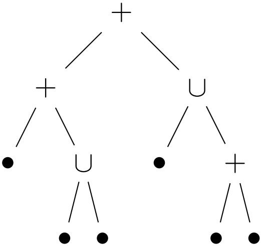


For instance the independence number $\alpha(G)$ is polytime



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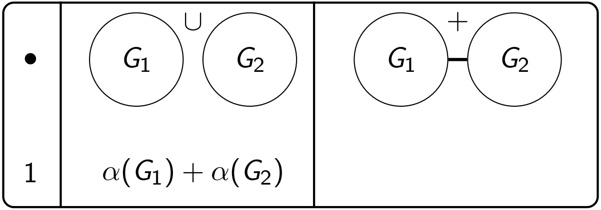
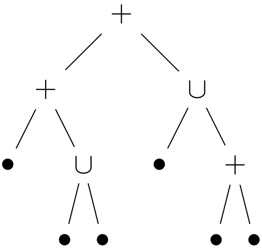


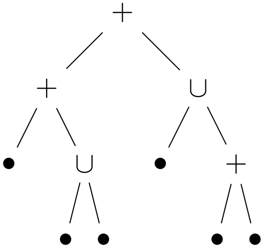
In case of a disjoint union: combine the solutions



Cographs

$\bullet$	$G_1 \cup G_2$	$G_1 + G_2$
1	$\alpha(G_1) + \alpha(G_2)$	$\max\{\alpha(G_1), \alpha(G_2)\}$

In case of a complete join: pick the larger one



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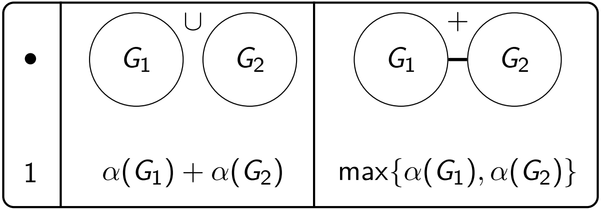

•
G_1 ∪ G_2
G_1 + G_2
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$\alpha(G_1) + \alpha(G_2)$
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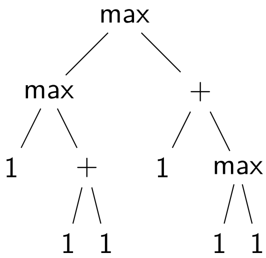

max
max
+
1
+
1
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Another cograph definition

Every induced subgraph has two twins

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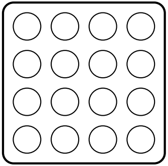
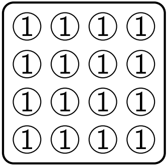


Is there another algorithmic scheme based on this definition?

Another cograph definition

Every induced subgraph has two twins



We store in each vertex its inner max independent set

Another cograph definition

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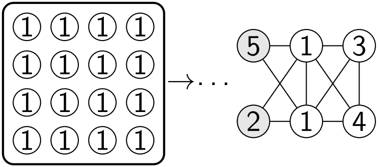
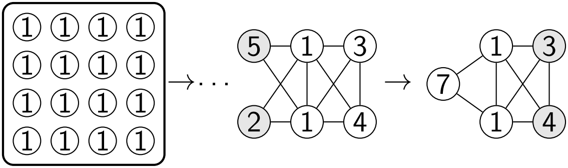


We can find a pair of false/true twins

Another cograph definition

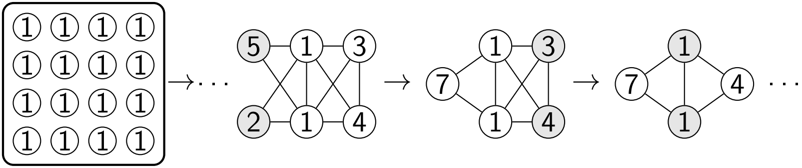
Every induced subgraph has two twins



Sum them if they are false twins

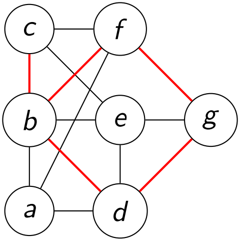
Another cograph definition

Every induced subgraph has two twins



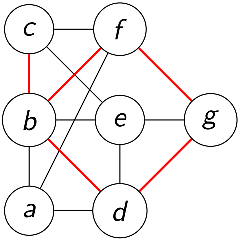
Max them if they are true twins

Trigraphs



Three outcomes between a pair of vertices:
edge, or non-edge, or red edge (error edge)

Trigraphs



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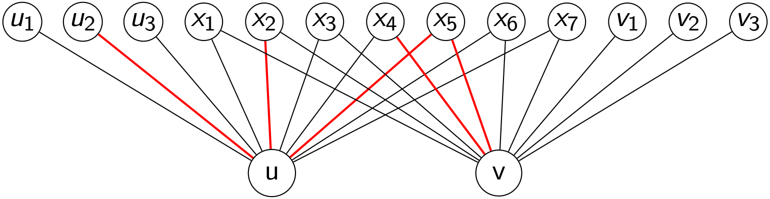
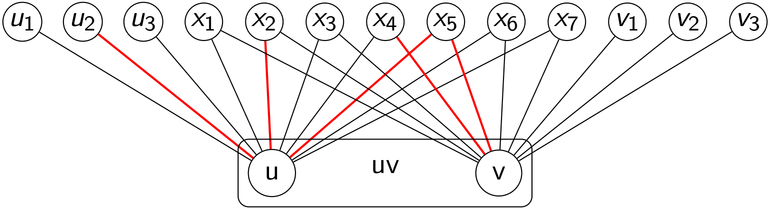
Red graph: trigraph minus its black edges

Contractions in trigraphs



Identification of two non-necessarily adjacent vertices

Contractions in trigraphs



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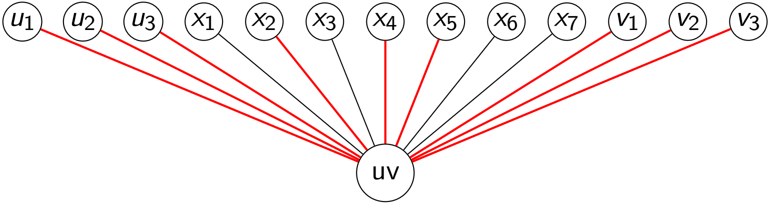
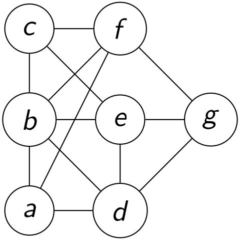


edges to $N(u) \triangle N(v)$ turn red, for $N(u) \cap N(v)$ red is absorbing

Contraction sequence



A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that
G_i is obtained by performing one contraction in G_{i+1}.

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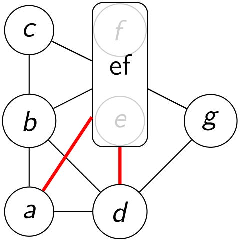

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partition viewpoint: $G_i \leftrightsquigarrow (G, \mathcal{P}_i)$, vertex $\leftrightsquigarrow$ part
$G\langle S \rangle = G[\bigcup \text{vertices of } G \text{ contracted into a vertex of } S]$

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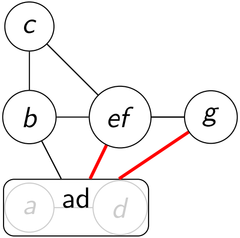

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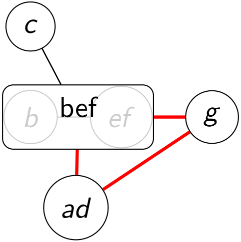

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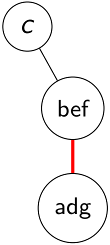

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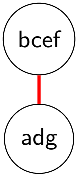

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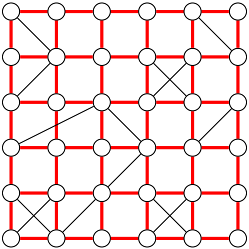
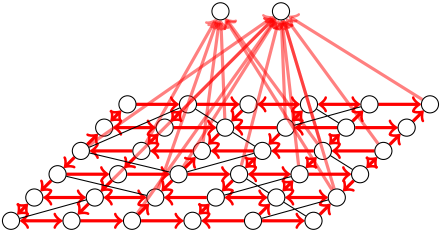
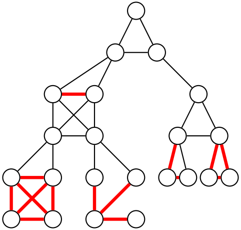
Different conditions imposed in the sequence of red graphs



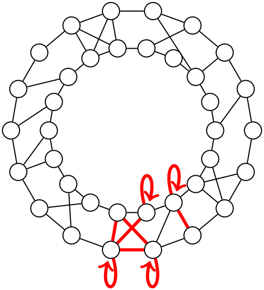
bd degree: defines bd twin-width



bd outdegree: defines bd oriented twin-width

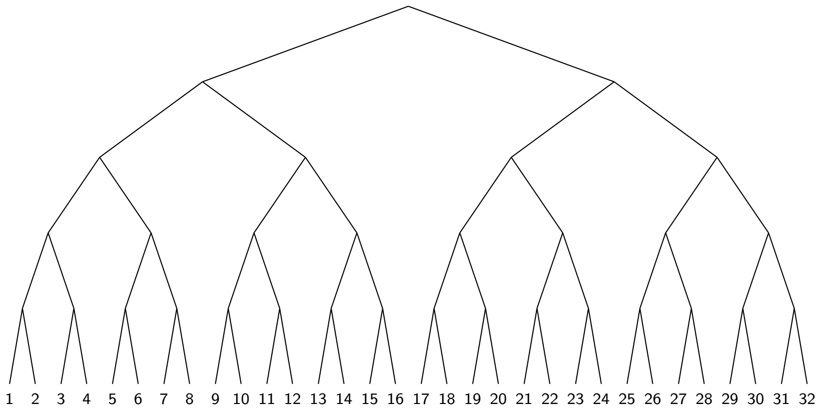


bd component: redefines bd cliquewidth



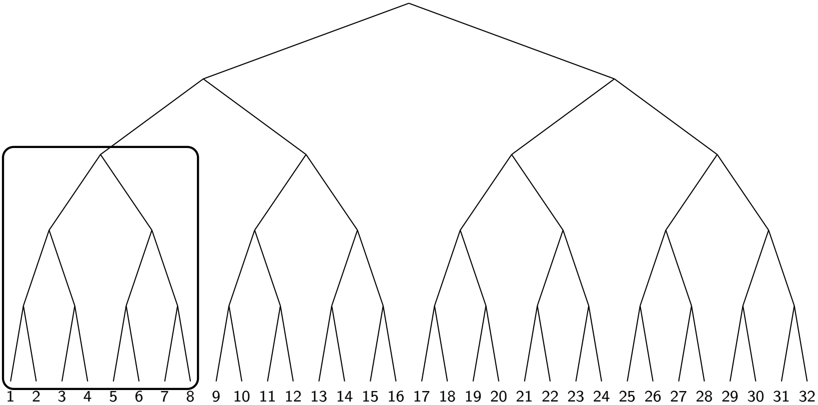
bd #edges: redefines bd linear cliquewidth

Bd boolean-width ⇒ bd component twin-width



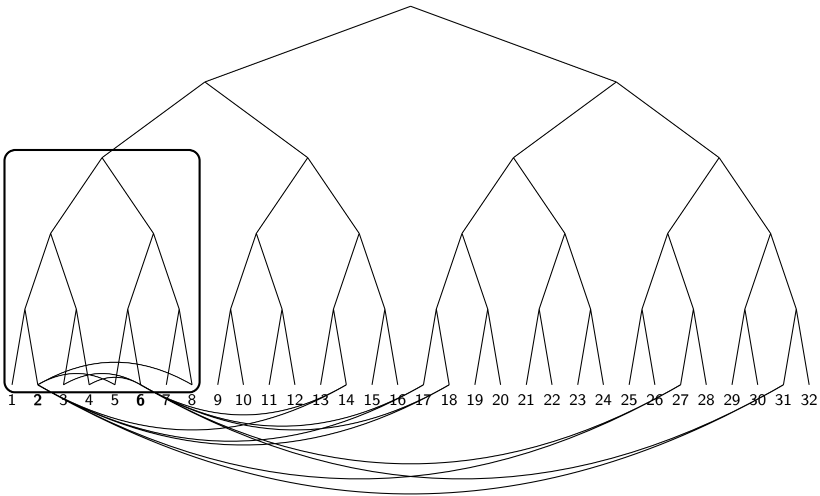
Bd boolean-width: binary tree layout s.t. every edge cut in the tree induces a bipartition with bd # distinct neighborhoods

Bd boolean-width $\Rightarrow$ bd component twin-width



There is a subtree on $\ell \in [d+1, 2d]$ leaves, where d bounds the number of single-vertex neighborhoods in a bipartition

Bd boolean-width ⇒ bd component twin-width



Two vertices have the same neighborhood outside of this subtree

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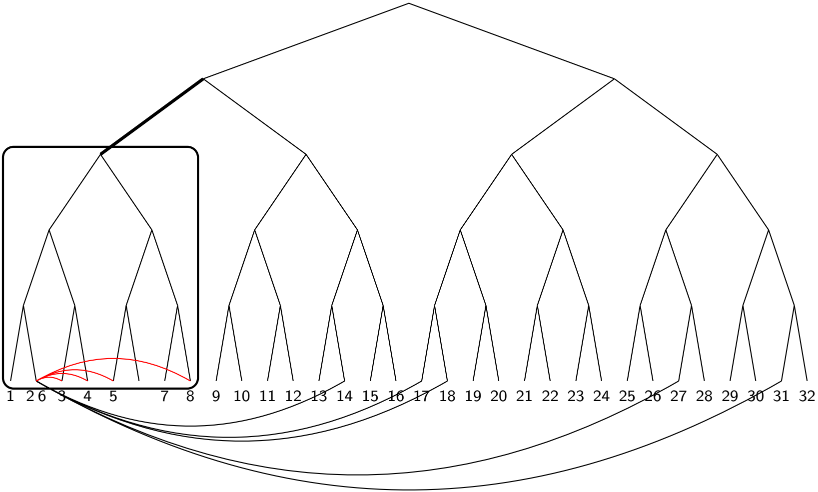

Contracting them preserves the upper bound at $2d$
on the size of red connected components

Component twin-width and boolean-width are tied

Theorem (B., Kim, Reinald, Thomassé '22)

A class has bounded component twin-width iff it has bounded boolean-width/cliquewidth/rank-width.

Proof.

We just saw one direction.

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When red components merge, their subtree gets a same parent. □

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Theorem (B., Kim, Reinald, Thomassé '22)

A class has bounded total twin-width iff it has bounded linear boolean-width/cliquewidth/rank-width.

Is it easier to design algorithms via this characterization?

Solve 3-Coloring on a graph G with a contraction sequence s.t. all red graphs have components of size at most d

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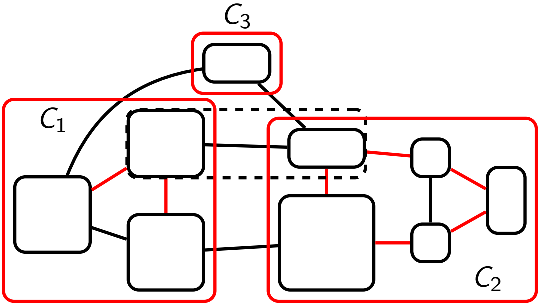

For every red component C keep every profile $V(C) \to 2^{\{1,2,3\}} \setminus \{\emptyset\}$ realizable by a proper 3-coloring of $G\langle C \rangle$

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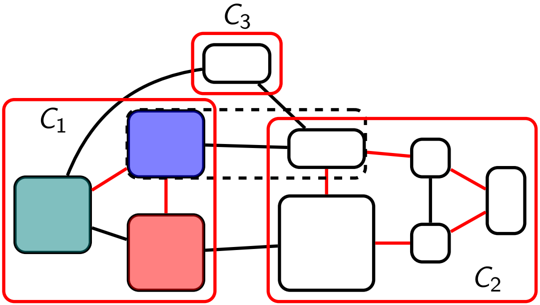
Solve 3-COLORING on a graph G with a contraction sequence s.t. all red graphs have components of size at most d



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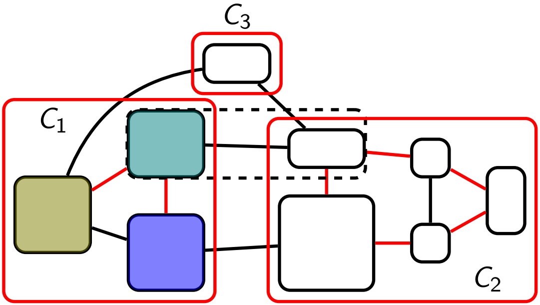

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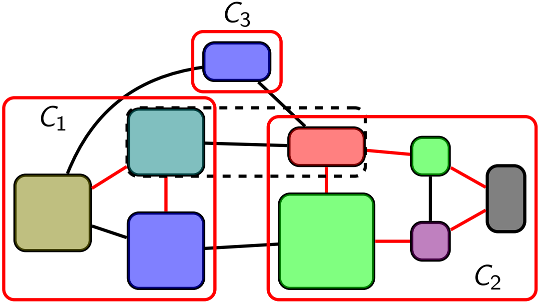

Some tuples of the at most $d+1$ profiles corresponding to merging red components are compatible

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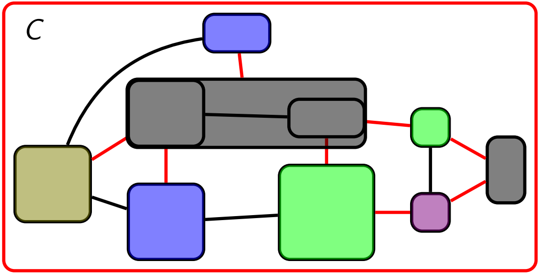

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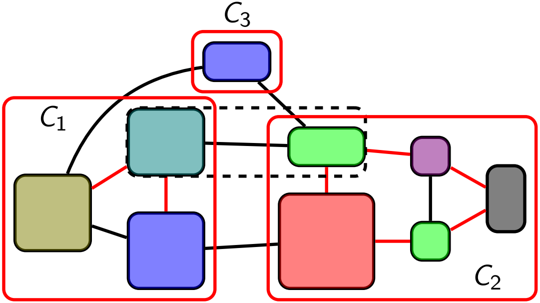

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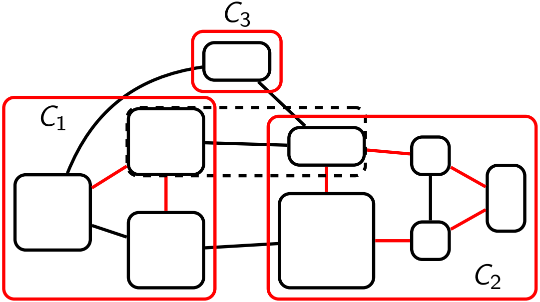

Initialization: time $3n$
Update: time $7^d d^2$
Total: time $7^d d^2 n$
End: still a profile on the single vertex *containing* the whole graph?

Formulas, sentences, and model checking

Graph FO/MSO Model Checking	**Parameter:** $\lvert\varphi\rvert$

Input: A graph G and a first-order/monadic second-order sentence $\varphi \in FO/MSO(\{E\})$

Question: $G \models \varphi$?

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Example:

$$\varphi = \exists x_1 \exists x_2 \cdots \exists x_k \forall x \bigvee_{1 \leqslant i \leqslant k} x = x_i \vee \bigvee_{1 \leqslant i \leqslant k} E(x, x_i) \vee E(x_i, x)$$

$G \models \varphi$? $\Leftrightarrow$

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Example:

$$\varphi = \exists x_1 \exists x_2 \cdots \exists x_k \forall x \bigvee_{1 \leqslant i \leqslant k} x = x_i \vee \bigvee_{1 \leqslant i \leqslant k} E(x, x_i) \vee E(x_i, x)$$

$G \models \varphi$? $\Leftrightarrow$ k-DOMINATING SET

Formulas, sentences, and model checking

GRAPH FO/MSO MODEL CHECKING	**Parameter:** $\lvert\varphi\rvert$
Input: A graph G and a first-order/monadic second-order sentence $\varphi \in FO/MSO(\{E\})$	
Question: $G \models \varphi$?	

Example:

$$\varphi = \exists x_1 \exists x_2 \cdots \exists x_k \bigwedge_{1 \leqslant i < j \leqslant k} \neg(x_i = x_j) \wedge \neg E(x_i, x_j) \wedge \neg E(x_j, x_i)$$

$G \models \varphi$? $\Leftrightarrow$

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$G \models \varphi$? $\Leftrightarrow$ k-Independent Set

Formulas, sentences, and model checking

Graph FO/MSO Model Checking	**Parameter:** $\lvert\varphi\rvert$
Input: A graph G and a first-order/monadic second-order sentence $\varphi \in FO/MSO(\{E\})$	
Question: $G \models \varphi$?	

Example:

$$\varphi = \exists X_1 \exists X_2 \exists X_3 (\forall x \bigvee_{1 \leqslant i \leqslant 3} X_i(x)) \wedge \forall x \forall y \bigwedge_{1 \leqslant i \leqslant 3} (X_i(x) \wedge X_i(y) \rightarrow \neg E(x, y))$$

$G \models \varphi? \Leftrightarrow$

Formulas, sentences, and model checking

Graph FO/MSO Model Checking	**Parameter:** $\|\varphi\|$

Input: A graph G and a first-order/monadic second-order sentence $\varphi \in FO/MSO(\{E\})$
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$$\varphi = \exists X_1 \exists X_2 \exists X_3 (\forall x \bigvee_{1 \leqslant i \leqslant 3} X_i(x)) \wedge \forall x \forall y \bigwedge_{1 \leqslant i \leqslant 3} (X_i(x) \wedge X_i(y) \to \neg E(x, y))$$

$G \models \varphi$? $\Leftrightarrow$ 3-Coloring

Courcelle's theorems

We will reprove with contraction sequences:

Theorem (Courcelle, Makowsky, Rotics '00)

MSO model checking can be solved in time $f(|\varphi|, d) \cdot |V(G)|$ given a witness that the clique-width/component twin-width of the input G is at most d.

generalizes

Theorem (Courcelle '90)

MSO model checking can be solved in time $f(|\varphi|, t) \cdot |V(G)|$ on graphs G of treewidth at most t.

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generalizes

Theorem (Courcelle '90)

MSO model checking can be solved in time $f(|\varphi|, t) \cdot |V(G)|$ on graphs G of treewidth at most t.

- as the incidence graph preserves bounded treewidth, possible edge-set quantification
- linear FPT approximation for treewidth
- (polynomial) FPT approximation for clique-width

Rank-k m-types

Sets of non-equivalent formulas/sentences of quantifier rank at most k satisfied by a fixed structure:

$$\mathrm{tp}_k^{\mathcal{L}}(\mathscr{A}, \vec{a} \in A^m) = \{\varphi(\vec{x}) \in \mathcal{L}[k] : \mathscr{A} \models \varphi(\vec{a})\},$$

$$\mathrm{tp}_k^{\mathcal{L}}(\mathscr{A}) = \{\varphi \in \mathcal{L}[k] : \mathscr{A} \models \varphi\}.$$

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Theorem (folklore)

For $\mathcal{L} \in \{FO, MSO\}$, the number of rank-$k$ m-types is bounded by a function of k and m only.

Proof.

"$\mathcal{L}[k+1]$ are Boolean combinations of $\exists x \mathcal{L}[k]$." $\square$

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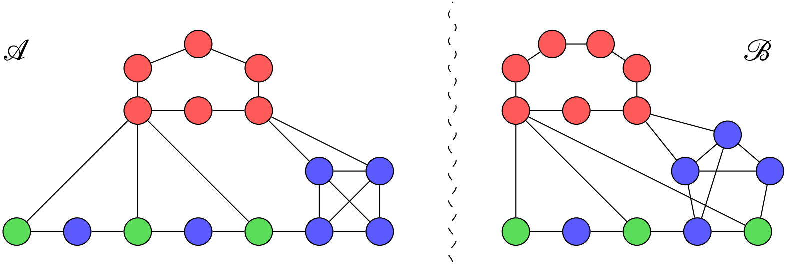
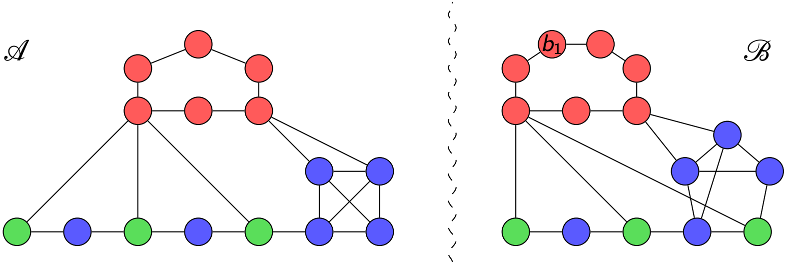
Rank-k types partition the graphs into $g(k)$ classes.
Efficient Model Checking = quickly finding the class of the input.

FO Ehrenfeucht-Fraissé game



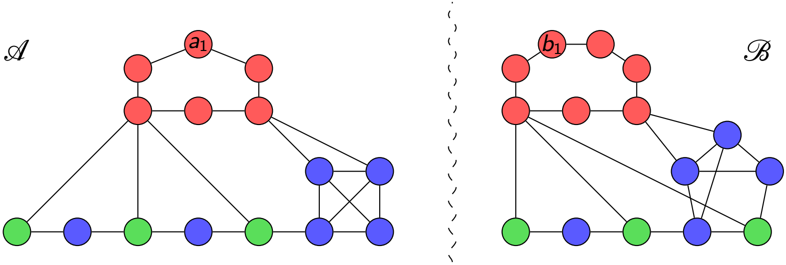
2-player game on two σ-structures $\mathscr{A}, \mathscr{B}$ (for us, colored graphs)

FO Ehrenfeucht-Fraissé game



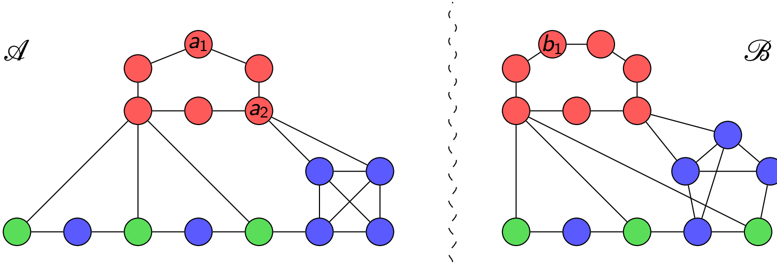
At each round, Spoiler picks a structure ($\mathscr{B}$) and a vertex therein

FO Ehrenfeucht-Fraissé game



Duplicator answers with a vertex in the other structure

FO Ehrenfeucht-Fraissé game



After q rounds, Duplicator wishes that $a_i \mapsto b_i$ is an isomorphism between $\mathscr{A}[a_1, \ldots, a_k]$ and $\mathscr{B}[b_1, \ldots, b_k]$

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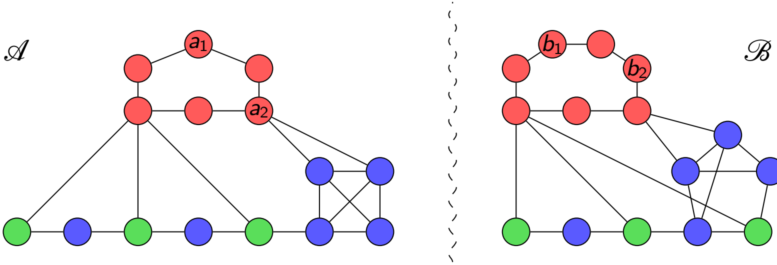

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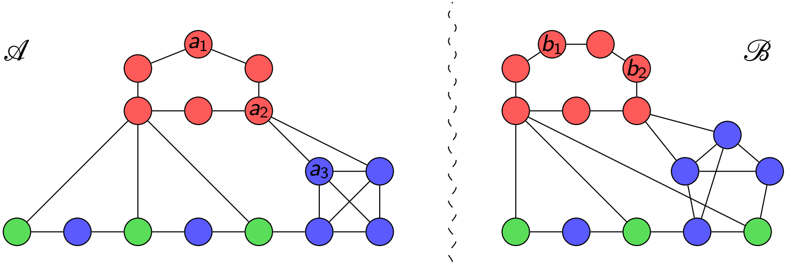
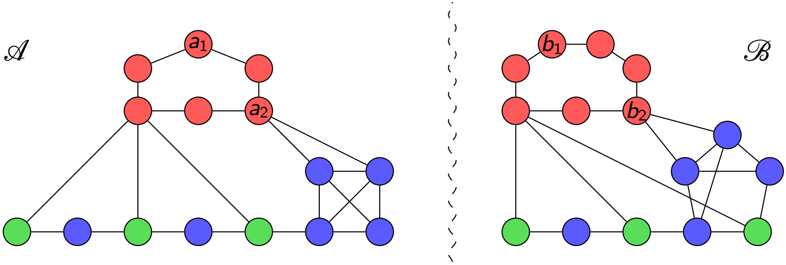


When no longer possible, Spoiler wins

FO Ehrenfeucht-Fraissé game



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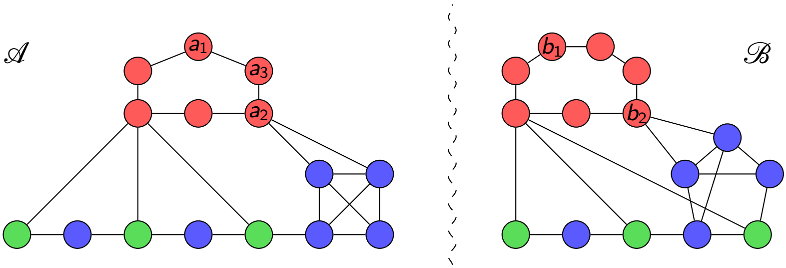
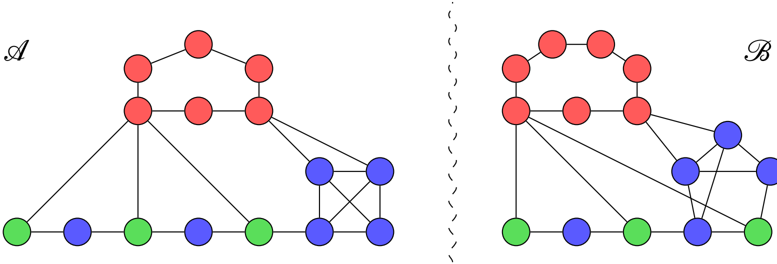


If Duplicator can survive k rounds, we write $\mathscr{A} \equiv_k^{\mathsf{FO}} \mathscr{B}$

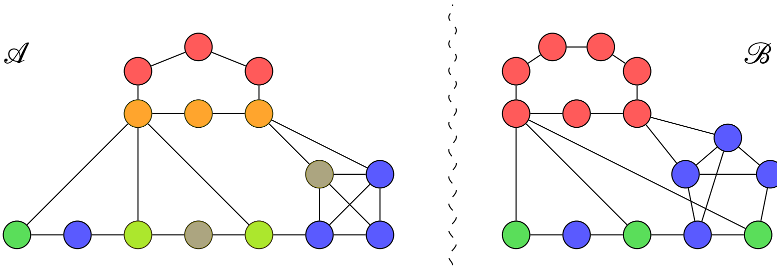
Here $\mathscr{A} \equiv_2^{\mathsf{FO}} \mathscr{B}$ and $\mathscr{A} \not\equiv_3^{\mathsf{FO}} \mathscr{B}$

MSO Ehrenfeucht-Fraissé game



Same game but Spoiler can now play set moves

MSO Ehrenfeucht-Fraissé game



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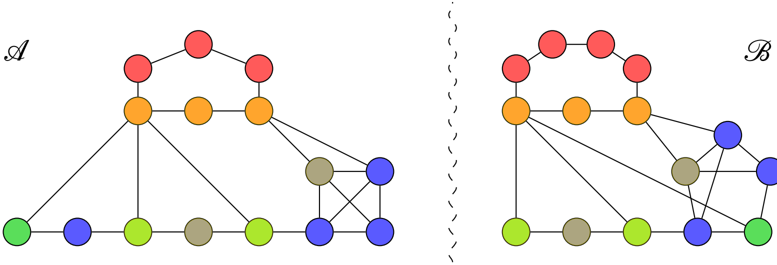
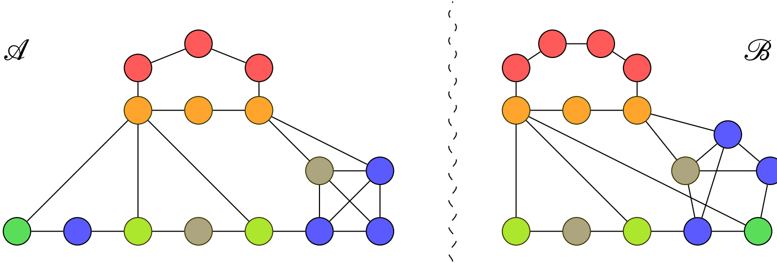


To which Duplicator answers a set in the other structure

MSO Ehrenfeucht-Fraissé game



Again we write $\mathscr{A} \equiv_k^{\mathsf{MSO}} \mathscr{B}$ if Duplicator can survive k rounds

k-round EF games capture rank-k types

Theorem (Ehrenfeucht-Fraissé)

For every σ-structures $\mathscr{A}, \mathscr{B}$ and logic $\mathcal{L} \in \{FO, MSO\}$,

$$\mathscr{A} \equiv_k^{\mathcal{L}} \mathscr{B} \text{ if and only if } tp_k^{\mathcal{L}}(\mathscr{A}) = tp_k^{\mathcal{L}}(\mathscr{B}).$$

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Proof.

Induction on k.

($\Rightarrow$) $\mathcal{L}[k+1]$ formulas are Boolean combinations of $\exists x \varphi$ or $\exists X \varphi$ where $\varphi \in \mathcal{L}[k]$. Use the answer of Duplicator to $x = a$ or $X = A$.

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($\Leftarrow$) If $\mathrm{tp}_{k+1}^{\mathcal{L}}(\mathcal{A}) = \mathrm{tp}_{k+1}^{\mathcal{L}}(\mathcal{B})$, then the type $\mathrm{tp}_k^{\mathcal{L}}(\mathcal{A}, a)$ is equal to some $\mathrm{tp}_k^{\mathcal{L}}(\mathcal{B}, b)$. Move a can be answered by playing b. $\square$

MSO model checking for component twin-width d

Partitioned sentences: sentences on $(E, U_1, \ldots, U_d)$-structures, interpreted as a graph vertex partitioned in d parts

Maintain for every red component C of every trigraph G_i

$$\text{tp}_k^{\text{MSO}}(G, \mathcal{P}_i, C) = \{\varphi \in \text{MSO}_{E,U_1,\ldots,U_d}(k) : (G\langle C\rangle, \mathcal{P}_i\langle C\rangle) \models \varphi\}.$$

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For each $v \in V(G)$, $\mathrm{tp}_k(G, \mathcal{P}_n, \{v\}) =$ type of K_1
$\mathrm{tp}_k(G, \mathcal{P}_1, \{V(G)\}) =$ type of G

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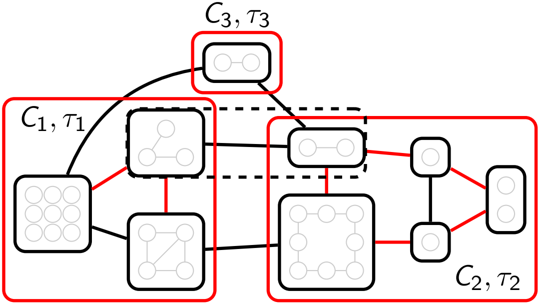

$\tau = \mathrm{tp}_k^{\mathrm{MSO}}(G, \mathcal{P}_i, C)$ based on the $\tau_j = \mathrm{tp}_k^{\mathrm{MSO}}(G, \mathcal{P}_{i+1}, C_j)$?

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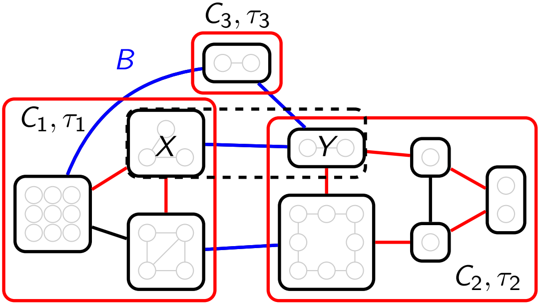
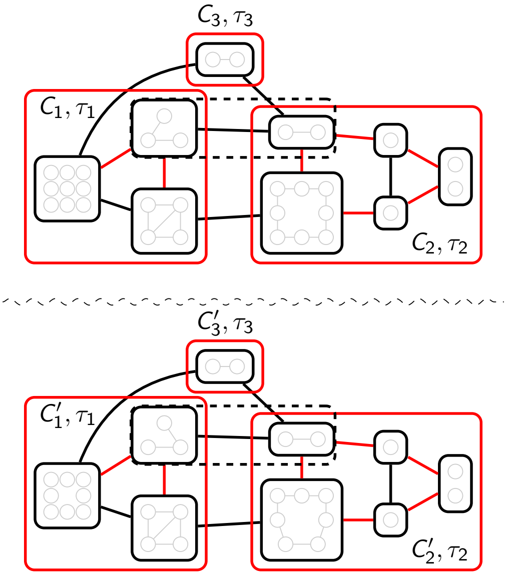


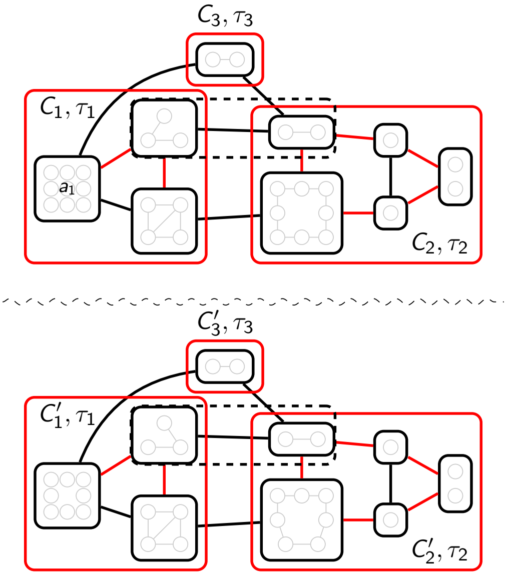
C arises from $C_1, \ldots, C_{d'}$: $\tau = F(\tau_1, \ldots, \tau_{d'}, B, X, Y)$

Showing $\tau = F(\tau_1, \ldots, \tau_{d'}, B, X, Y)$ via MSO EF game



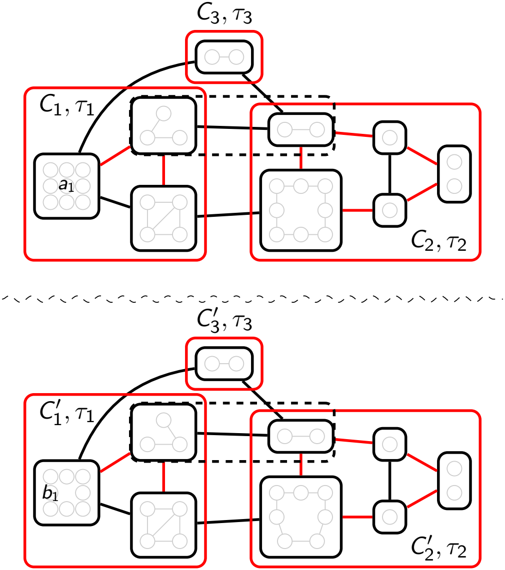
Duplicator combines her strategies in the red components

Showing $\tau = F(\tau_1, \ldots, \tau_{d'}, B, X, Y)$ via MSO EF game



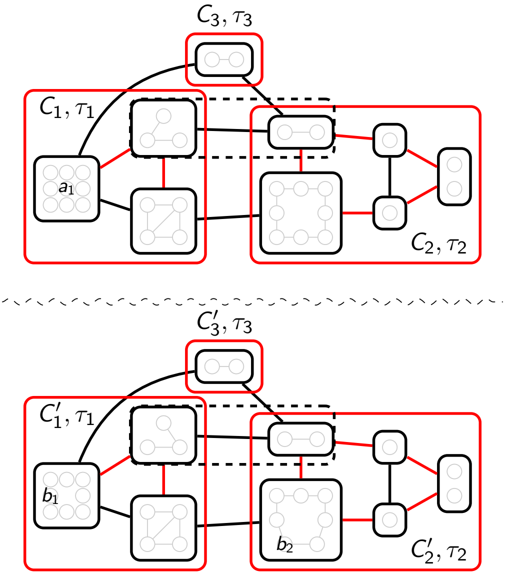
If Spoiler plays a vertex in the component of type τ_1,

Showing $\tau = F(\tau_1, \ldots, \tau_{d'}, B, X, Y)$ via MSO EF game



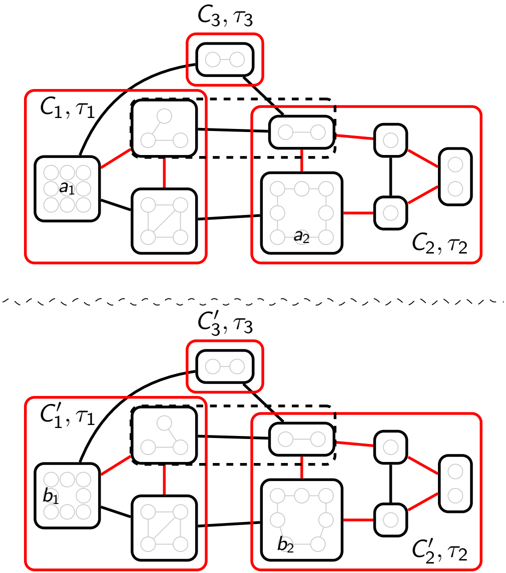
Duplicator answers the corresponding winning move

Showing $\tau = F(\tau_1, \ldots, \tau_{d'}, B, X, Y)$ via MSO EF game



Same in the component of type τ_2

Showing $\tau = F(\tau_1, \ldots, \tau_{d'}, B, X, Y)$ via MSO EF game



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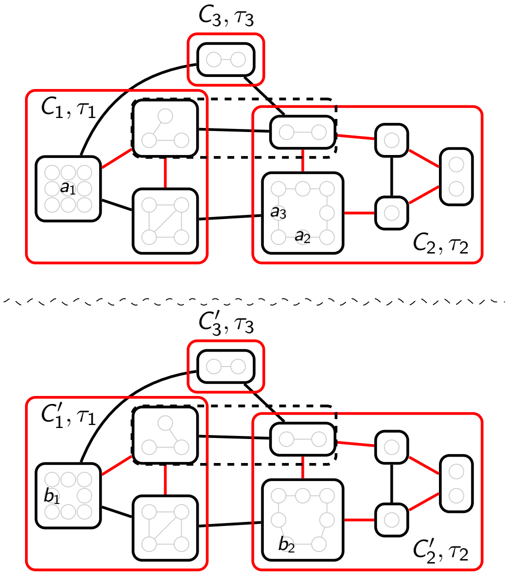

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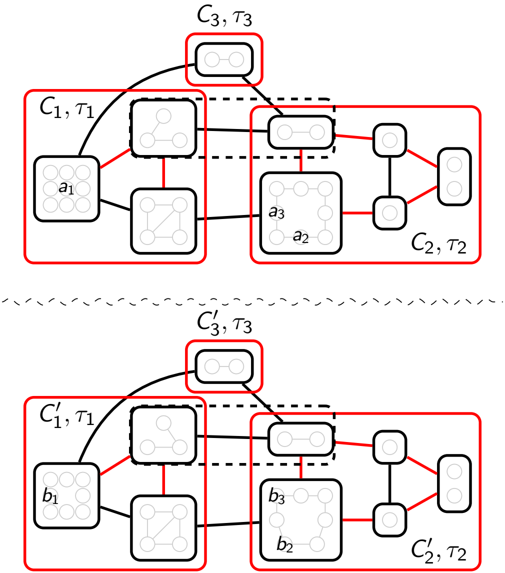

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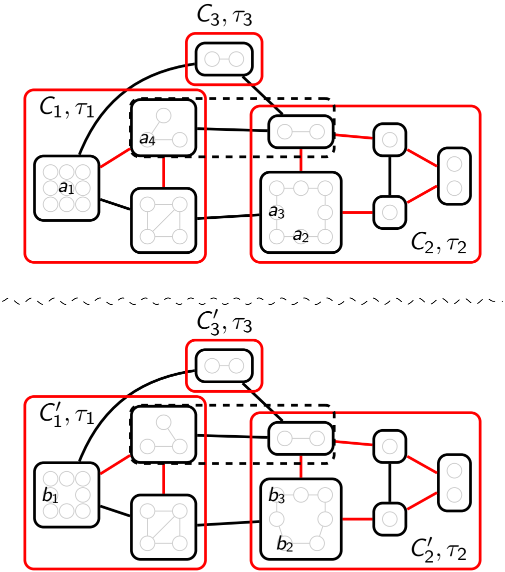
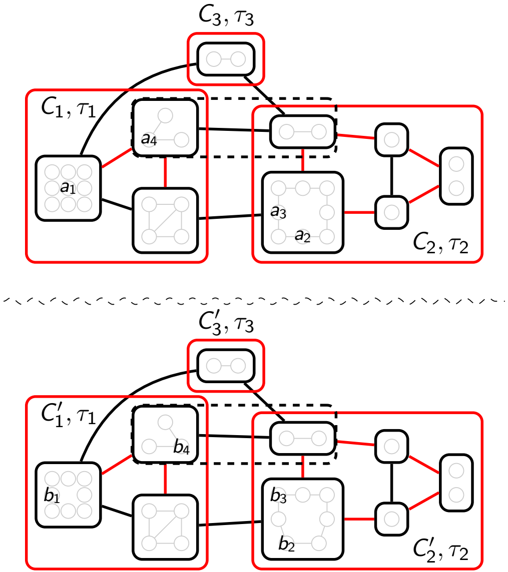


and so on

Showing $\tau = F(\tau_1, \dots, \tau_{d'}, B, X, Y)$ via MSO EF game



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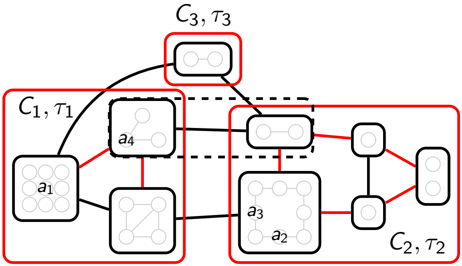

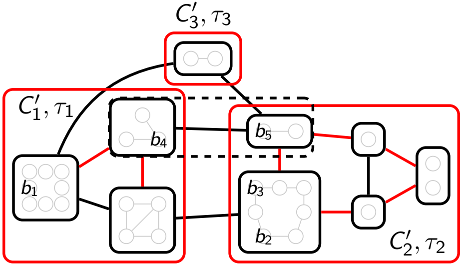

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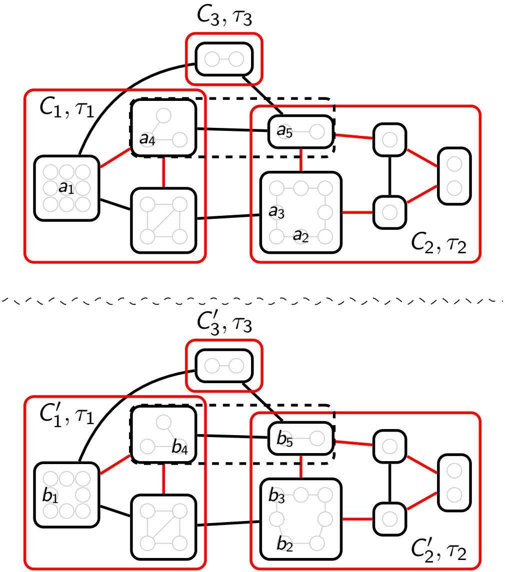

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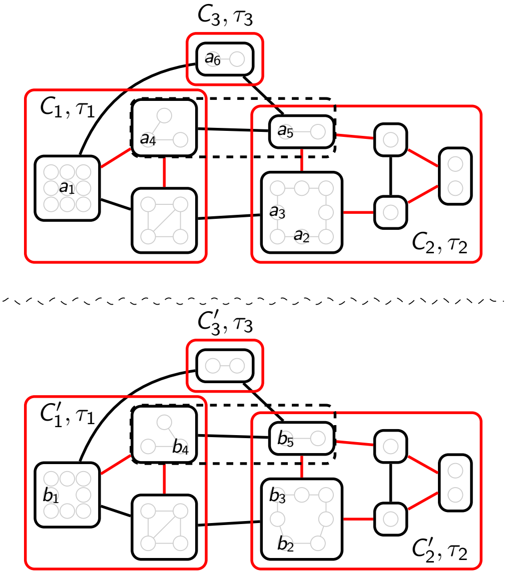

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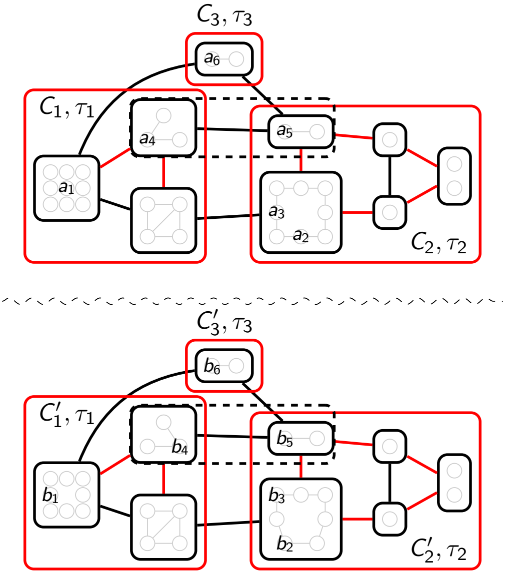

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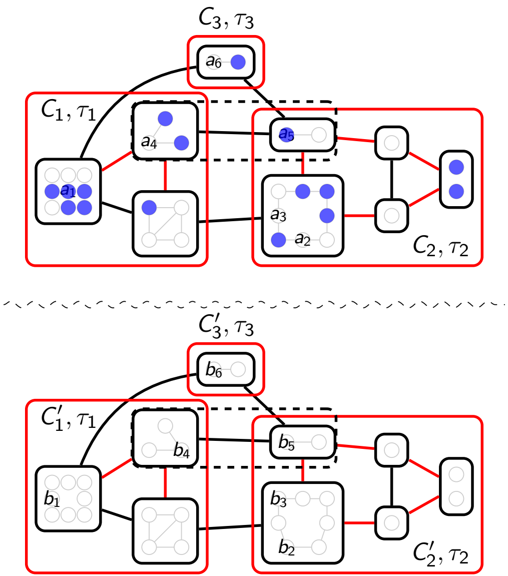
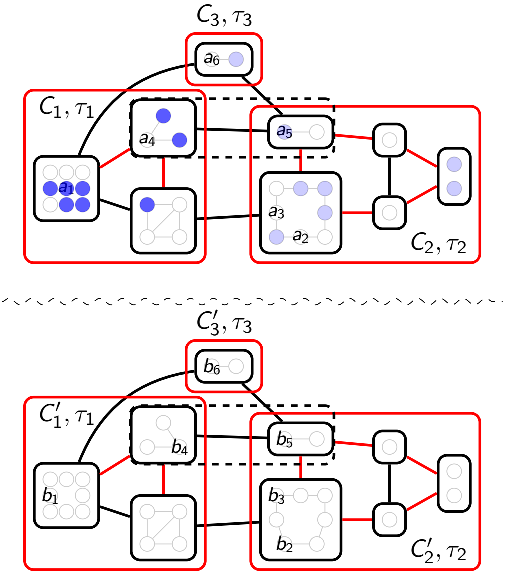


If Spoiler plays a set, Duplicator looks at the intersection with C_1,

Showing $\tau = F(\tau_1, \ldots, \tau_{d'}, B, X, Y)$ via MSO EF game



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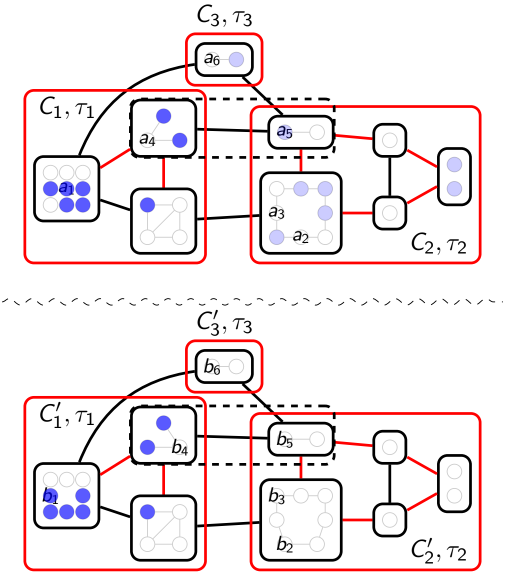

calls her winning strategy in C'_1

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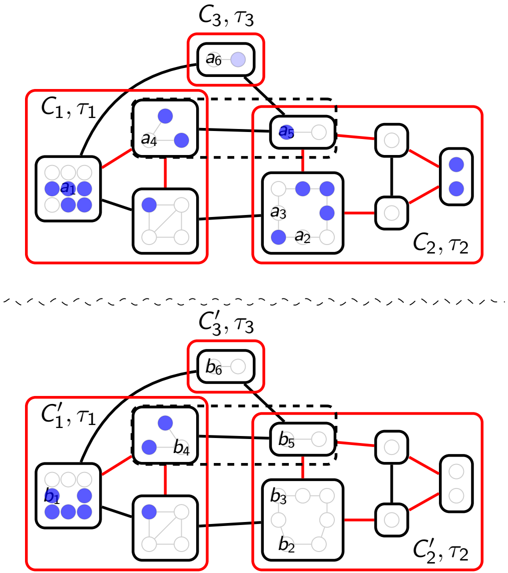
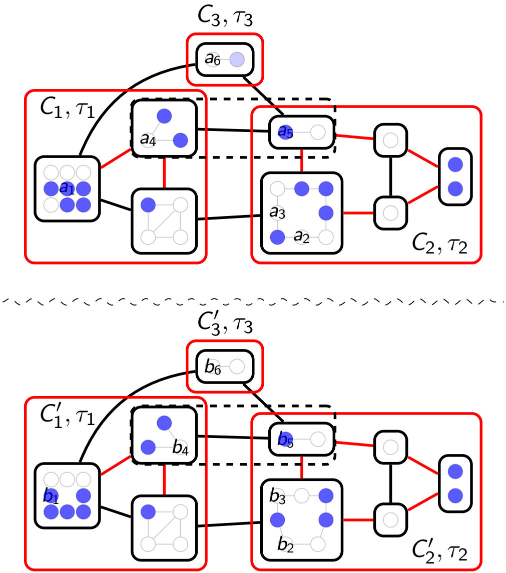


same for the other components

Showing $\tau = F(\tau_1, \ldots, \tau_{d'}, B, X, Y)$ via MSO EF game



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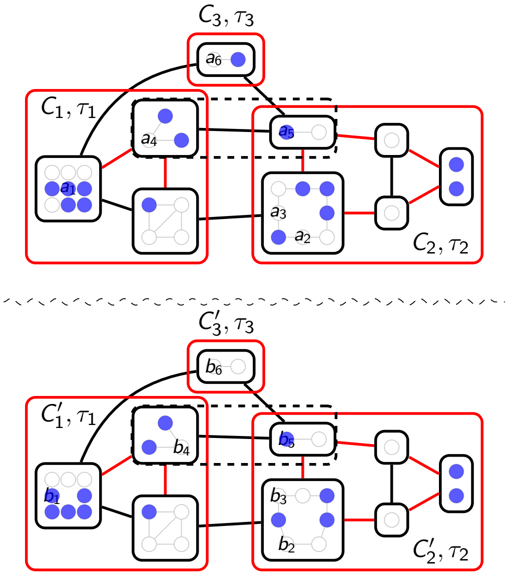

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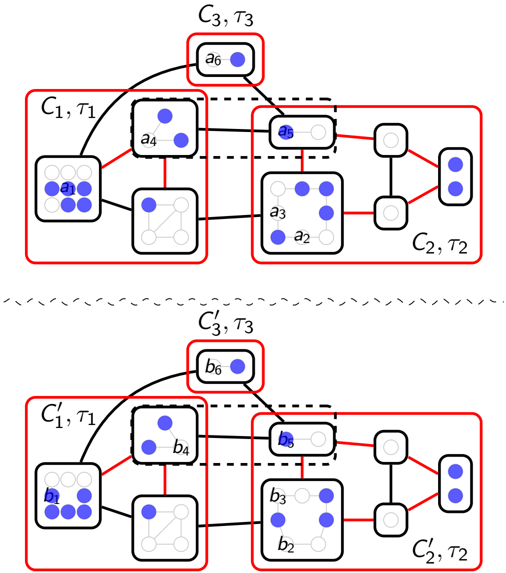

and plays the union

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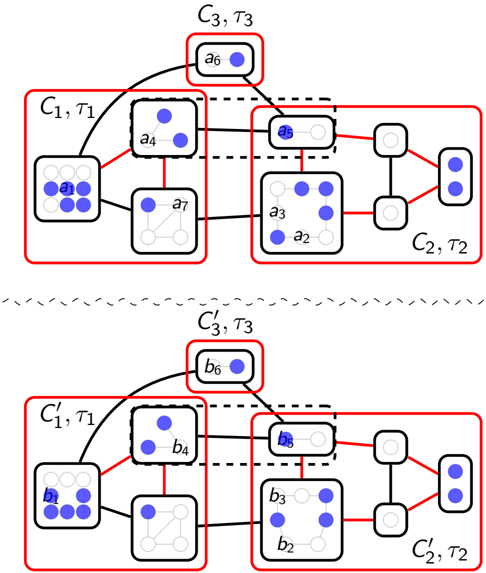
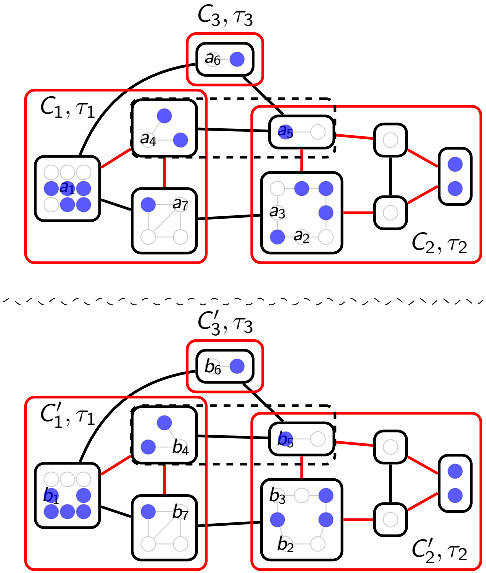


that fully defines the winning strategy of Duplicator

Showing $\tau = F(\tau_1, \ldots, \tau_{d'}, B, X, Y)$ via MSO EF game



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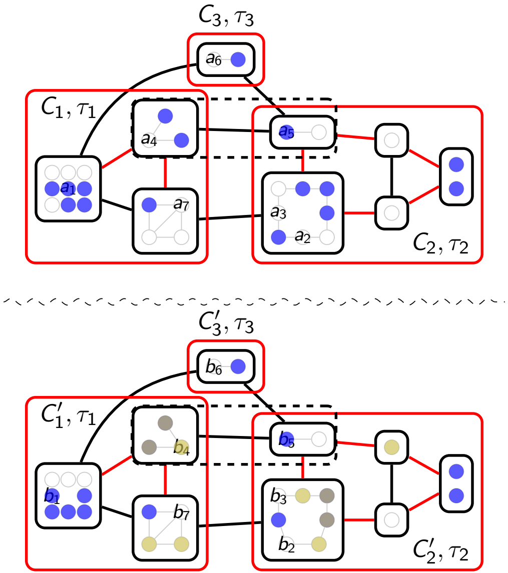

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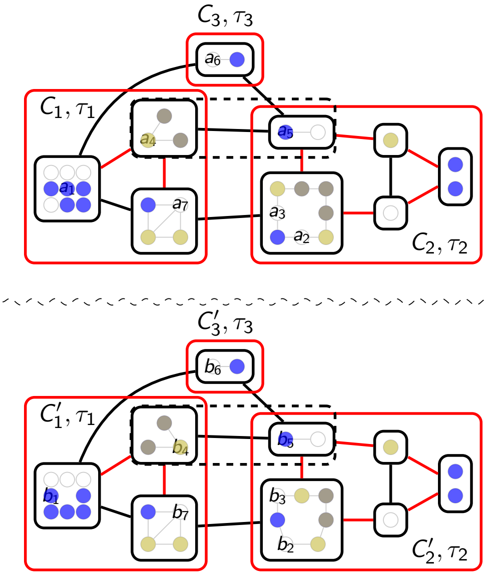

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Turning it into a uniform algorithm

Reminder:

- $\#$non-equivalent partitioned sentences of rank k: $f(d,k)$
- $\#$rank-k partitioned types bounded by $g(d,k) = 2^{f(d,k)}$

For each newly observed type τ,

- keep a representative $(H,\mathcal{P})_\tau$ on at most $(d+1)^{g(d,k)}$ vertices
- determine the $0,1$-vector of satisfied sentences on $(H,\mathcal{P})_\tau$
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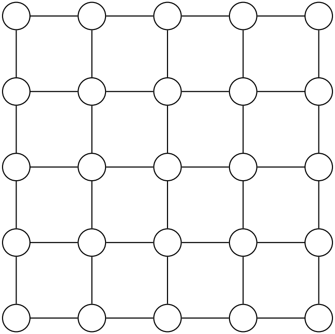
To decide $G \models \varphi$, look at position φ in the $0,1$-vector of $\mathrm{tp}_k^{\mathsf{MSO}}(G)$

Twin-width is more general than the classic widths



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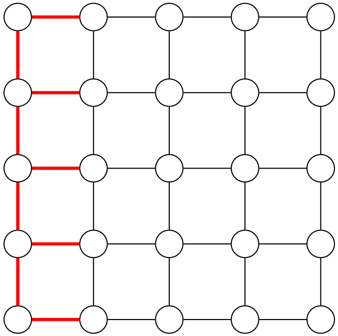

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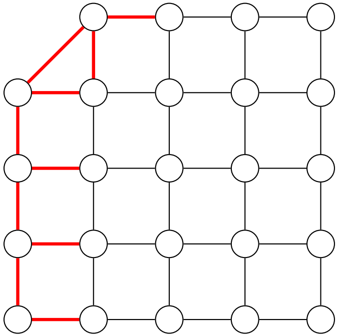

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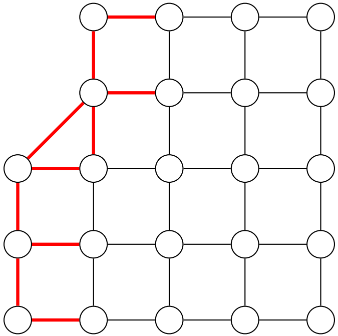

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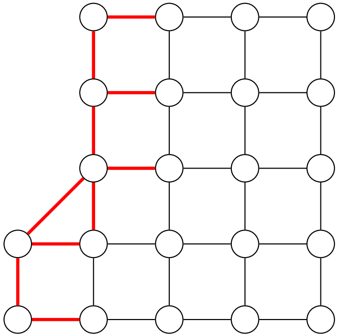

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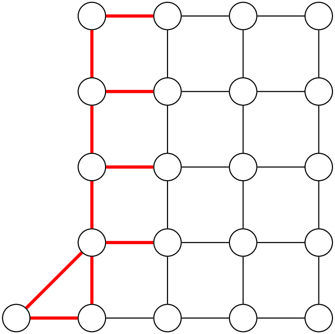

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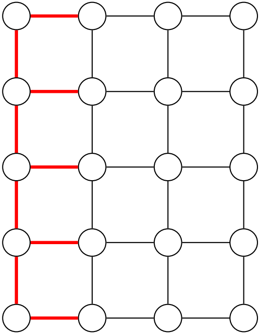

4-sequence for planar grids

Theorem

The following classes have bounded twin-width, and $O(1)$-sequences can be computed in polynomial time.

- *Bounded rank-width, and even, boolean-width graphs,*
- *every hereditary proper subclass of permutation graphs,*
- *posets of bounded antichain size (seen as digraphs),*
- *unit interval graphs,*
- *K_t-minor free graphs,*
- *map graphs,*
- *subgraphs of d-dimensional grids,*
- *K_t-free unit d-dimensional ball graphs,*
- *$\Omega(\log n)$-subdivisions of all the n-vertex graphs,*
- *cubic expanders defined by iterative random 2-lifts from K_4,*
- *strong products of two bounded twin-width classes, one with bounded degree, etc.*

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Can we solve problems faster, given an $O(1)$-sequence?

k-Independent Set given a $d = O(1)$-sequence

d-sequence: $G = G_n, G_{n-1}, \ldots, G_2, G_1 = K_1$

Algorithm: **For every connected subset D of size at most k of the red graph of every G_i, store in $T[D, i]$ one largest independent set in $G\langle D \rangle$ intersecting every vertex of D.**

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Initialization: $T[\{v\}, n] = \{v\}$

End: $T[\{V(G)\}, 1] =$ IS of size at least k or largest IS in G

Running time: $d^{2k}n^2$ red connected subgraphs,
actually only $d^{2k}n = 2^{O_d(k)}n$ updates

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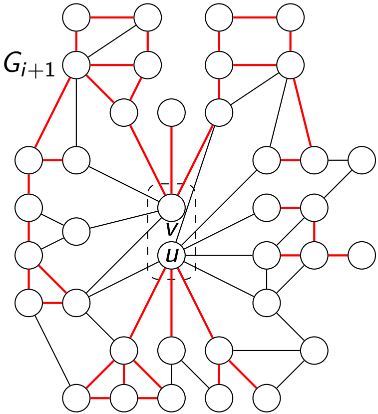
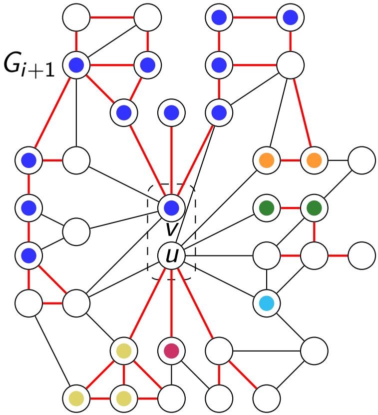
How to compute $T[D, i]$ from all the $T[D', i+1]$?

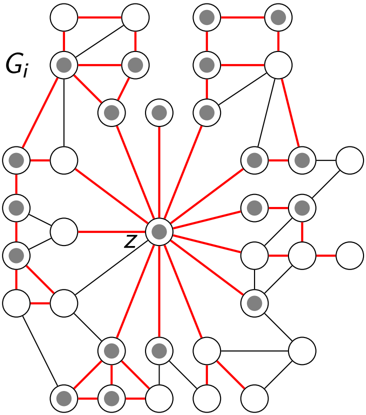
k-Independent Set: Update of partial solutions



Best partial solution inhabiting •?

k-Independent Set: Update of partial solutions





3 unions of $\leqslant d+2$ red connected subgraphs to consider in G_{i+1}
with u, or v, or both

FO model checking on graphs of bounded twin-width

We will now generalize the previous algorithm to:

Theorem (B., Kim, Thomassé, Watrigant '20)

FO model checking can be solved in time $f(|\varphi|, d) \cdot |V(G)|$ on graphs G given with a d-sequence.

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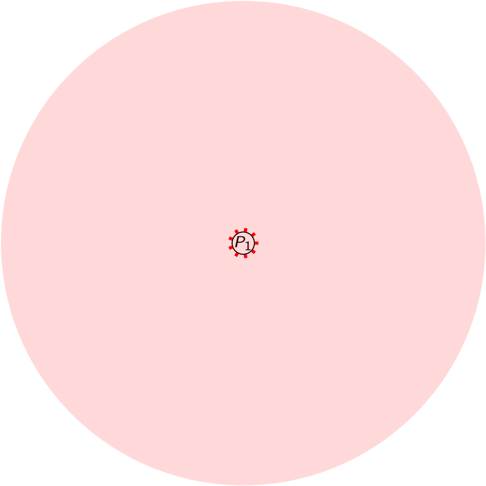
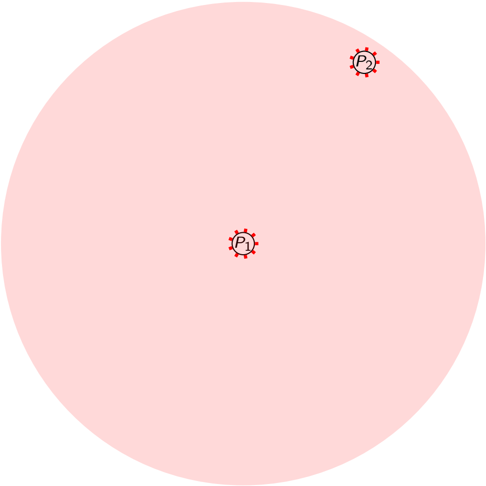
Add **Gaifman's locality** to our MSO model checking algorithm

Local tuple of parts



$(P_1, P_2, \ldots, P_q)$ is *local around* P_1 if...

Local tuple of parts



$(P_1, P_2, \ldots, P_q)$ is *local around* P_1 if...
P_2 is at distance at most 2^{k-2} from $\{P_1\}$ in $(G, \mathcal{P}_i)$

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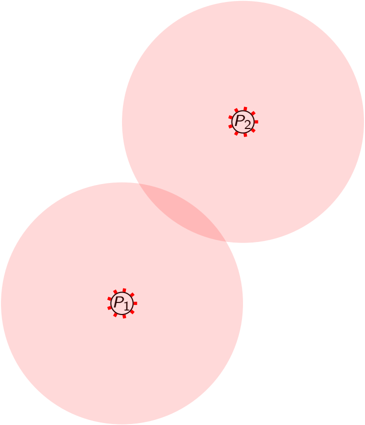

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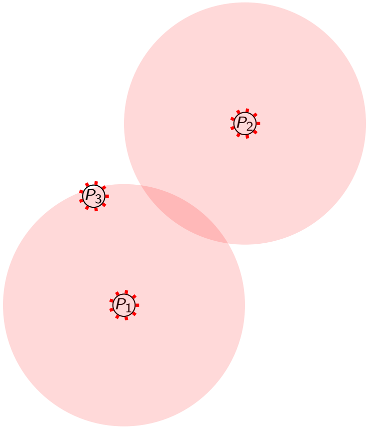

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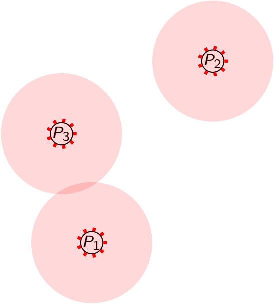

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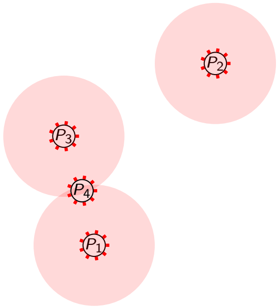

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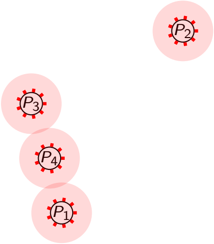
P_4 is at distance at most 2^{k-4} from $\{P_1, P_2, P_3\}$ in $(G, \mathcal{P}_i)$

Local tuple of parts



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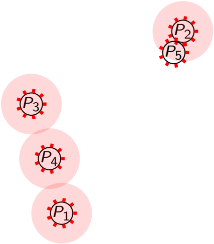

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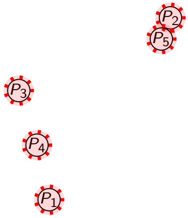

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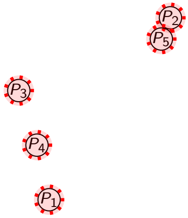

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Partitioned local sentences and types

A prenex sentence is *partitioned local around* X in $(G, \mathcal{P}_i)$ if of the form $Qx_1 \in X \; Qx_2 \in P_2 \; \dots \; Qx_k \in P_k \; \psi(x_1, \dots, x_k)$ with

- ψ is quantifier-free, and
- $(X, P_2, \dots, P_k)$ local around X in $(G, \mathcal{P}_i)$.

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And the corresponding types:

$$\mathrm{ltp}_k^{\mathrm{FO}}(G, \mathcal{P}_i, X) = \{\varphi : \mathrm{qr}(\varphi) \leqslant k,$$

$$\varphi \text{ is partitioned local around } X \text{ in } (G, \mathcal{P}_i),$$

$$(G, \mathcal{P}_i) \models \varphi\}.$$

Partitioned local sentences/types in $(G, \mathcal{P}_n)$ and $(G, \mathcal{P}_1)$

Initialization of the dynamic programming

In $(G, \mathcal{P}_n = \{\{v\} : v \in V(G)\})$: for every $v \in V(G)$,
$Qx_1 \in \{v\}\ Qx_2 \in \{v\}\ \ldots\ Qx_k \in \{v\}\ \psi \equiv \psi(v, v, \ldots, v)$

Partitioned local types are easy to compute in $(G, \mathcal{P}_n)$

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Output of the dynamic programming

In $(G, \mathcal{P}_1 = \{V(G)\})$:
$Qx_1 \in V(G)\ Qx_2 \in V(G)\ \ldots\ Qx_k \in V(G)\ \psi \equiv$ classic sentences

The partitioned local type in $(G, \mathcal{P}_1)$ coincides with the type of G

Partitioned local types give the partitioned types

Isom. $f : \mathcal{P}_i \to \mathcal{P}'_i$ with $\mathsf{ltp}^{\mathsf{FO}}_k(G, \mathcal{P}_i, X) = \mathsf{ltp}^{\mathsf{FO}}_k(G', \mathcal{P}'_i, f(X))$

$(G, \mathcal{P}_i)$

$(G', \mathcal{P}'_i)$

Local strategies win the global game

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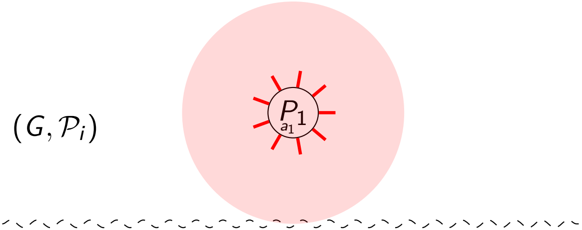

$(G', \mathcal{P}'_i)$

Say, Spoiler plays in P_1

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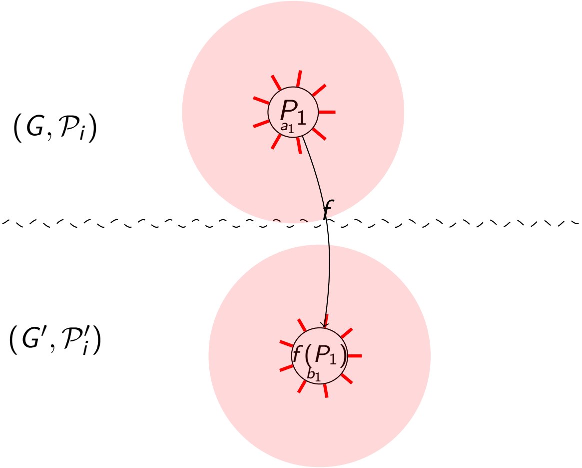

Duplicator answers in $f(P_1)$ following the local game around P_1

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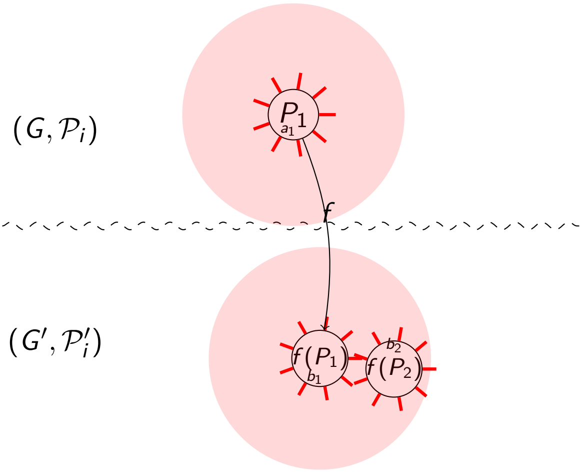

Now when Spoiler plays close to P_1 or $f(P_1)$

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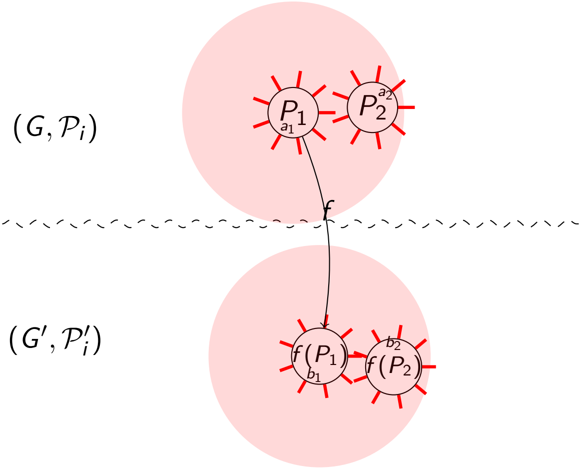

Duplicator follows the winning local strategy

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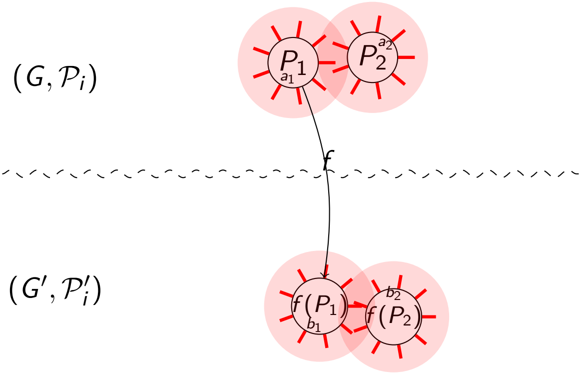

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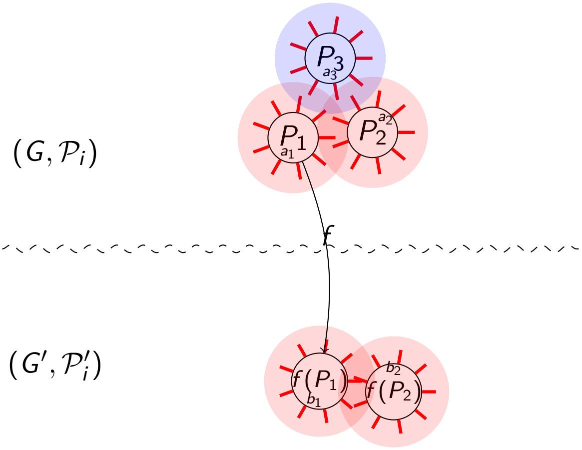

If Spoiler plays too far

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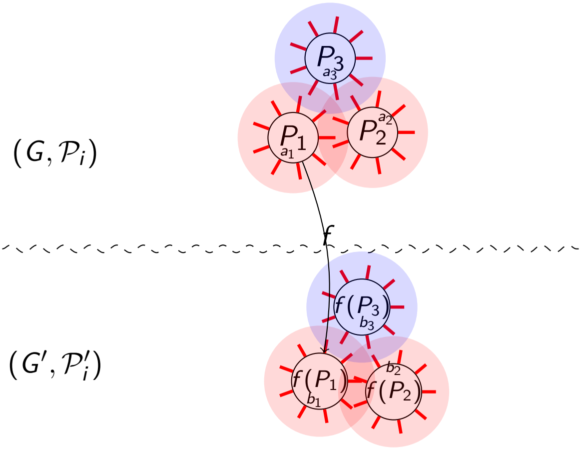

Duplicator starts a new local game around that new part

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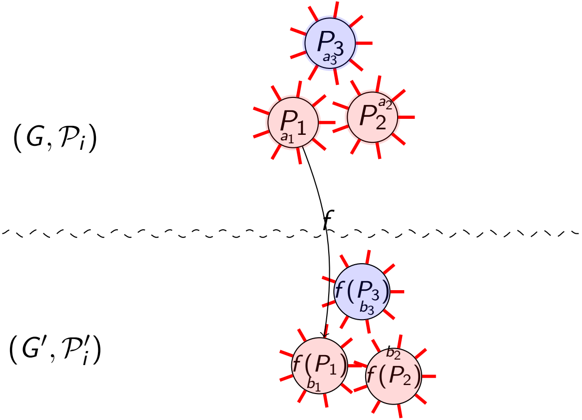

Duplicator starts a new local game around that new part

Concluding as in the MSO model checking algorithm

$(G, \mathcal{P}_{i+1}) \rightsquigarrow (G, \mathcal{P}_i)$: X and Y are merged in Z

Partitioned local types around P

- only needs an update if P is at distance at most 2^{k-1} from Z

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Each contraction: $O_{d,k}(1) = O(d^{2^k})$ updates in $O_{d,k}(1) = f(d,k)$
Total time: $O_{d,k}(n)$

Conclusion

Contraction sequences offer an interesting unifying and generalizing perspective

Class of bounded	MSO tr. of	FO tr. of	seq. constraint	eff. MC
linear rank-width	paths	linear order	bd #edges	MSO
rank-width	trees	tree order	bd component	MSO
twin-width	not closed	perm. subclass	bd degree	FO

Bounded degree, bounded expansion, nowhere denseness?

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Thank you for your attention!