EPTAS for Maximum Clique on Disks and Unit Balls

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and
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LIP, ENS Lyon

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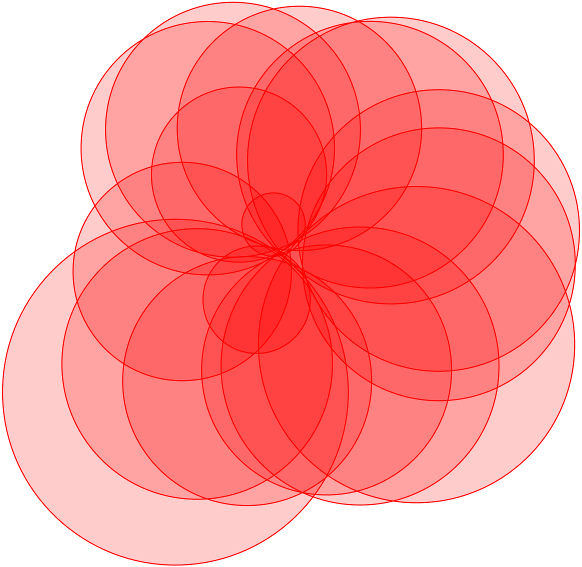
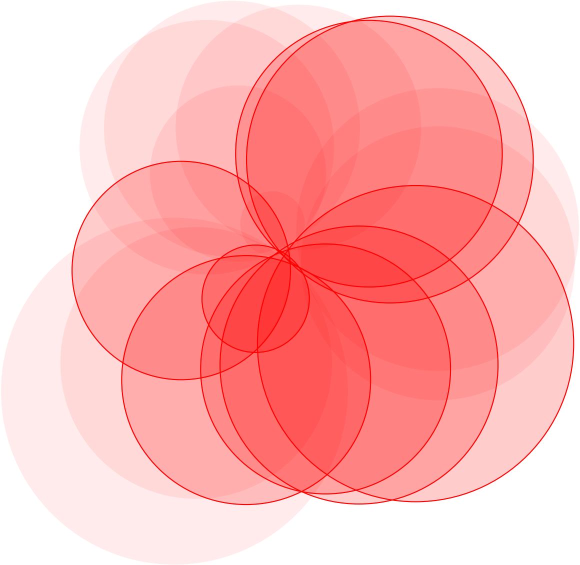
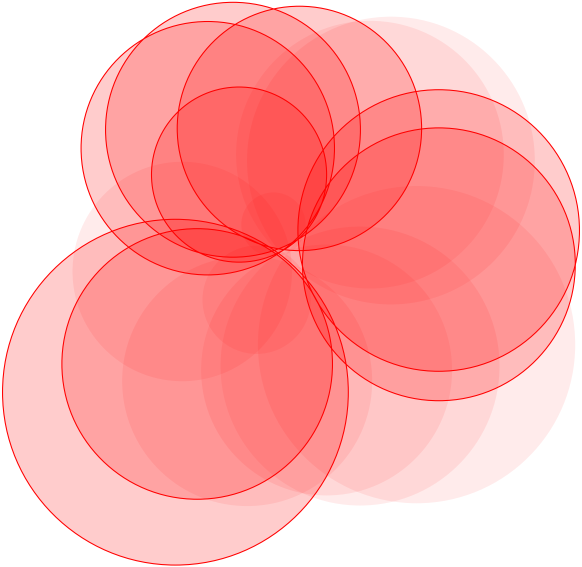


Find a largest collection of disks that pairwise intersect

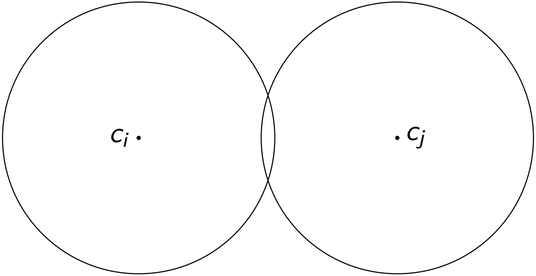


Like this



or that

Polynomial algorithm on unit disks by Clark, Colbourn, and Johnson, 1990.



Guess two farthest disks in an optimum solution S.

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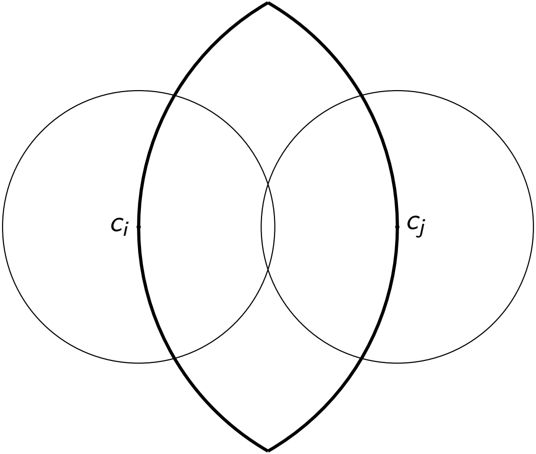

Hence, all the centers of S lie inside the bold digon.

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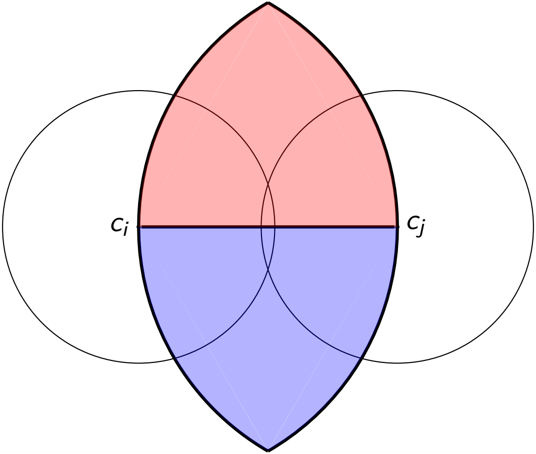
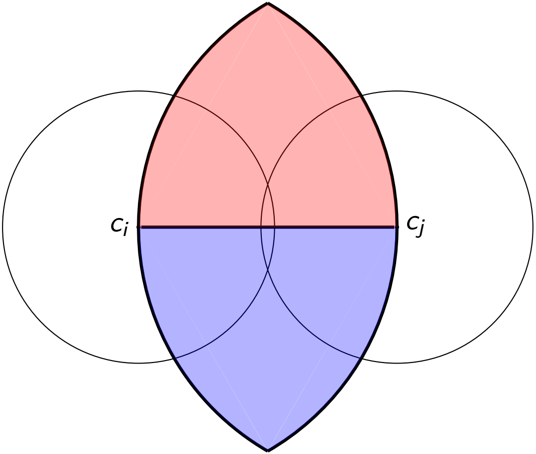


Two disks centered in the same-color region intersect.

Polynomial algorithm on unit disks by Clark, Colbourn, and Johnson, 1990.



We solve $\textsc{Max Clique}$ in a co-bipartite graph.

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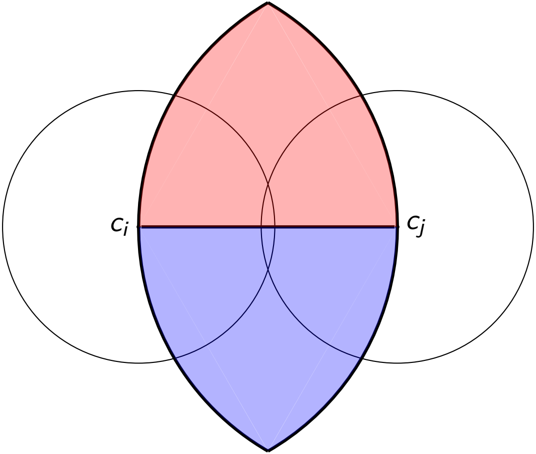
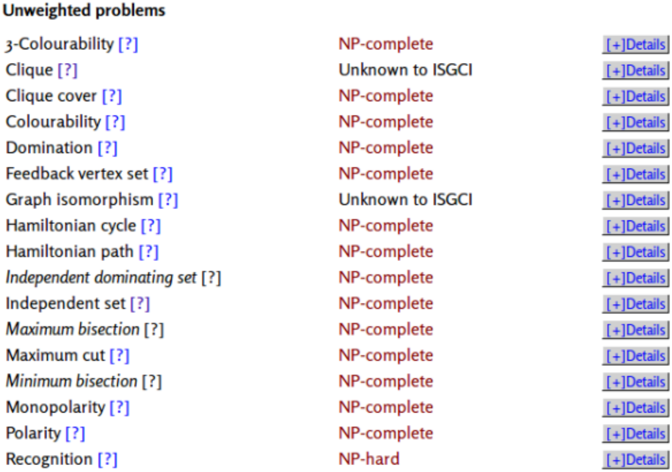


We solve $\textsc{Max Independent Set}$ in a bipartite graph.

Disk graphs



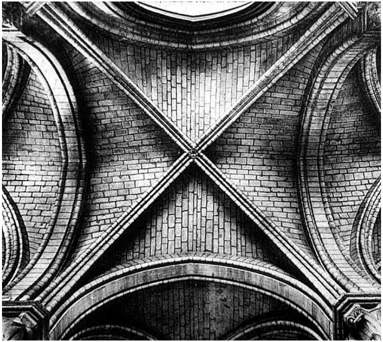
Unweighted problems

3-Colourability [?]	NP-complete	[+]Details
Clique [?]	Unknown to ISGCI	[+]Details
Clique cover [?]	NP-complete	[+]Details
Colourability [?]	NP-complete	[+]Details
Domination [?]	NP-complete	[+]Details
Feedback vertex set [?]	NP-complete	[+]Details
Graph isomorphism [?]	Unknown to ISGCI	[+]Details
Hamiltonian cycle [?]	NP-complete	[+]Details
Hamiltonian path [?]	NP-complete	[+]Details
Independent dominating set [?]	NP-complete	[+]Details
Independent set [?]	NP-complete	[+]Details
Maximum bisection [?]	NP-complete	[+]Details
Maximum cut [?]	NP-complete	[+]Details
Minimum bisection [?]	NP-complete	[+]Details
Monopolarity [?]	NP-complete	[+]Details
Polarity [?]	NP-complete	[+]Details
Recognition [?]	NP-hard	[+]Details

Inherits the NP-hardness of planar graphs.

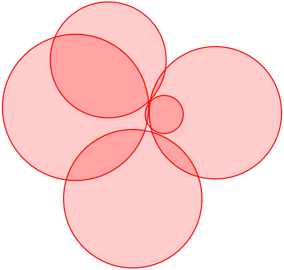
So what is known for MAX CLIQUE on disk graphs?

- Polynomial-time 2-approximation
  - For any clique there are 4 points hitting all the disks.
  - Guess those points and remove the non-hit disks.
  - The resulting graph is partitioned into 2 co-bipartite graphs.
  - Solve exactly on both co-bipartite graphs.
  - Output the best solution.
- No non-trivial exact algorithm known



And what is known about disk graphs?

- Every planar graph is a disk graph.
- Every triangle-free disk graph is planar (centers $\rightarrow$ vertices).
- So a triangle-free non-planar graph like $K_{3,3}$ is not disk.
- A subdivision of a non-planar graph is not a disk graph (more generally not a string graph).
- ...



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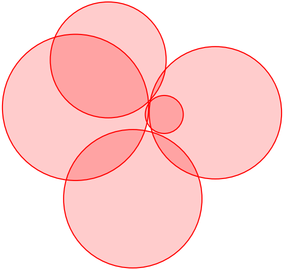
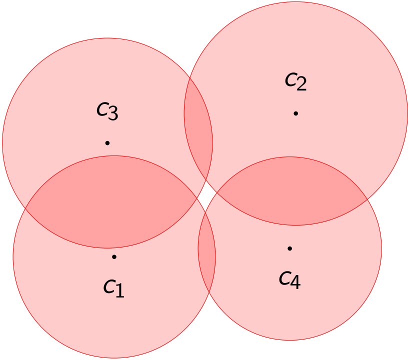


Other ways of showing that a graph is not disk?

Say the 4 centers encoding a $K_{2,2} = \overline{2K_2}$ are in convex position.



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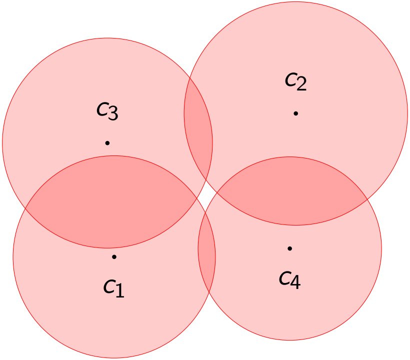

Then the two non-edges should be diagonal.

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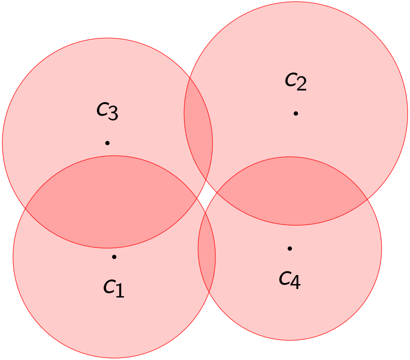
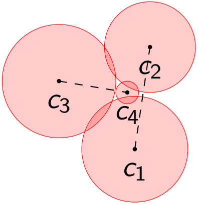


Then the two non-edges should be diagonal.

Suppose $d(c_1, c_3) > r_1 + r_3$ and $d(c_2, c_4) > r_2 + r_4$.
But $d(c_1, c_3) + d(c_2, c_4) \leqslant d(c_1, c_2) + d(c_3, c_4) \leqslant r_1 + r_2 + r_3 + r_4$, a contradiction.

Conclusion: the 4 centers of an induced $\overline{2K_2}$ are either

- not in convex position or
- in convex position with the non-edges being *diagonal*.



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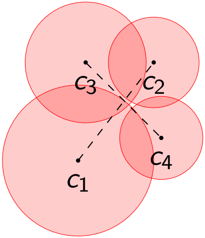

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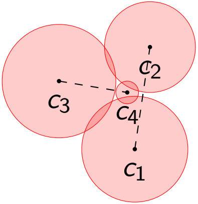

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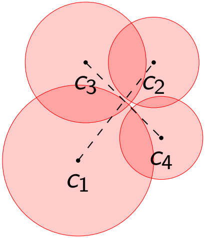
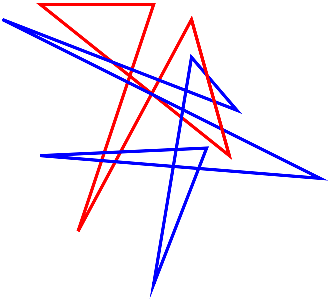


Reformulation: either

- the line $\ell(c_1, c_2)$ crosses the segment c_3c_4, or
- the line $\ell(c_3, c_4)$ crosses the segment c_1c_2, or
- both; equivalently, the segments c_1c_2 and c_3c_4 cross.

Assume $\overline{C_s + C_t}$ is a disk graph.

Link consecutive centers of the two disjoint cycles (non-edges).



For each red segment s_i, we denote by:

- a_i the number of blue segments crossed by $\ell(s_i)$.
- b_i the number of blue segments whose extension cross s_i.
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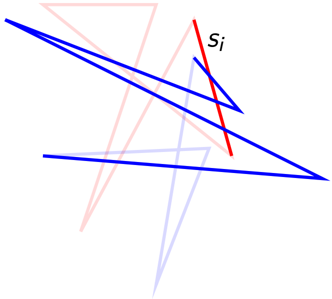

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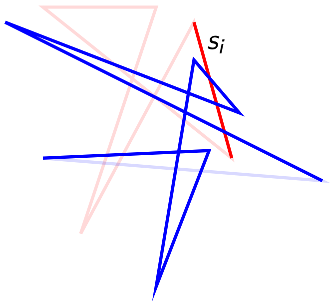

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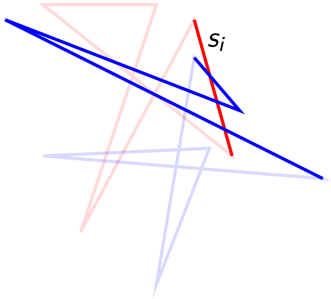

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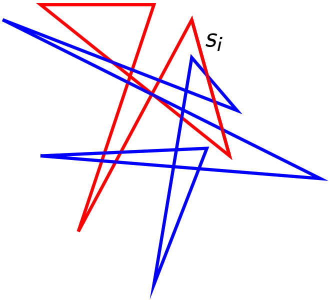

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- a_i the number of blue segments crossed by $\ell(s_i)$.
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It should be that $a_i + b_i - c_i = t$.

$$\sum_{1 \leqslant i \leqslant s} a_i + b_i - c_i = st$$

1) a_i is even:

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2) $\sum_{1\leqslant i\leqslant s} b_i = \sum_{1\leqslant i\leqslant t} a'_i$ is therefore even. (a'_j, b'_j, c'_j same for blue segments)

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$\sum_{1\leqslant i\leqslant s} a_i + b_i - c_i = \sum_{1\leqslant i\leqslant s} a_i + \sum_{1\leqslant i\leqslant t} a'_i - \sum_{1\leqslant i\leqslant s} c_i$ is even.

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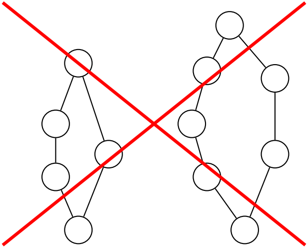
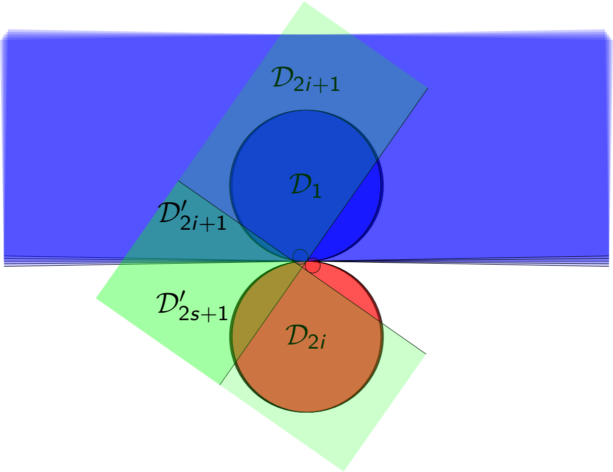
Hence s and t cannot be both odd.

The complement of two odd cycles is not a disk graph.



Are there other graphs of co-degree 2 which are not disk?

Complement of many even cycles and one odd cycle



Can we solve Max Independent Set more efficiently if there are no two vertex-disjoint odd cycles as an induced subgraph?

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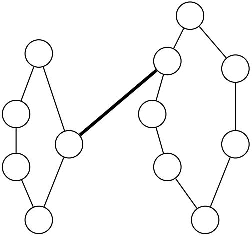
Another way to see it:
at least one edge between two vertex-disjoint odd cycles



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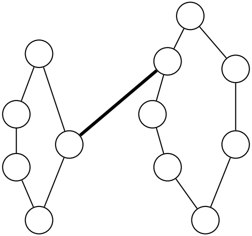

We can get a QPTAS and an exact subexponential algorithm in $2^{\tilde{O}(n^{2/3})}$ with win-wins and known results.

EPTAS

VC dim of $\mathcal{S}$ = maximum size of a set with all intersections with $\mathcal{S}$.
VCdim(G) = VC dimension of the neighborhood set-system.
$\alpha(G)$ = size of a maximum independent set in G.
iocp(G) = same as ocp but induced.

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Theorem

Max Independent Set *can be* $1+\varepsilon$*-approximated in time* $2^{\tilde{O}(1/\varepsilon^3)} n^{O(1)}$ *on graphs G with*

- *VCdim*$(G) = O(1)$,
- $\alpha(G) = \Omega(|V(G)|)$, *and*
- *iocp*$(G) = 1$.

Classic result of Haussler and Welzl in VC dimension theory

Theorem (ε-nets)

A set-system $(\mathcal{S}, U)$ *with VC dimension* d *and only sets of size at least* $\varepsilon|U|$ *has a hitting set of size* $O(\frac{d}{\varepsilon}\log\frac{1}{\varepsilon})$.

Furthermore, any sample of size $\frac{10d}{\varepsilon}\log\frac{1}{\varepsilon}$ is a hitting set w.h.p.

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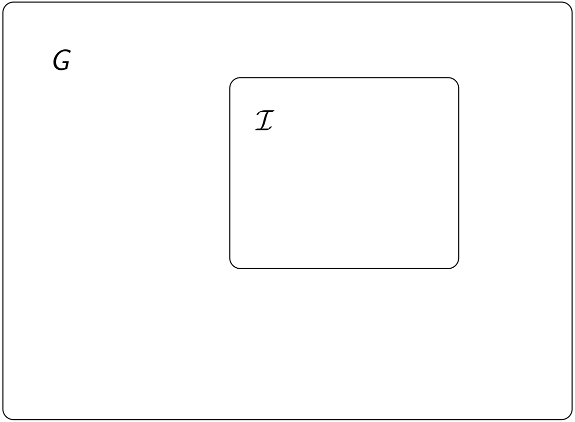
We will apply that result to the set-system
$(\{N(u) \cap \mathcal{I} \mid u \in V(G), |N(u) \cap \mathcal{I}| \geqslant \varepsilon^3 |\mathcal{I}|\}, \mathcal{I})$.
In words, the large neighborhoods over I.

First step: sampling

$\mathcal{I}$ is a fixed maximum independent set.
We can assume that $|\mathcal{I}| = \Theta(n)$.



We pick randomly S of $\tilde{O}(1/\varepsilon^3)$ vertices.

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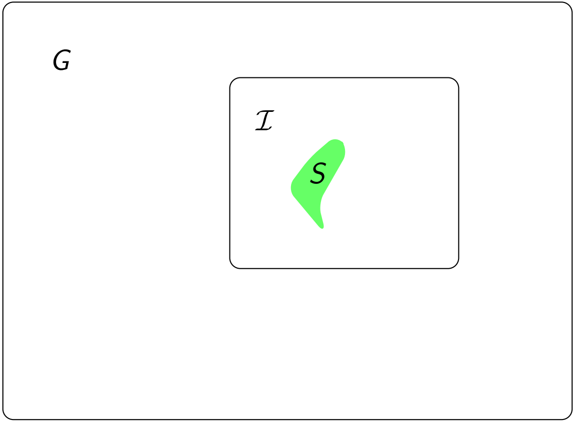

With probability $f(\varepsilon) > 0$, $S \subseteq \mathcal{I}$.

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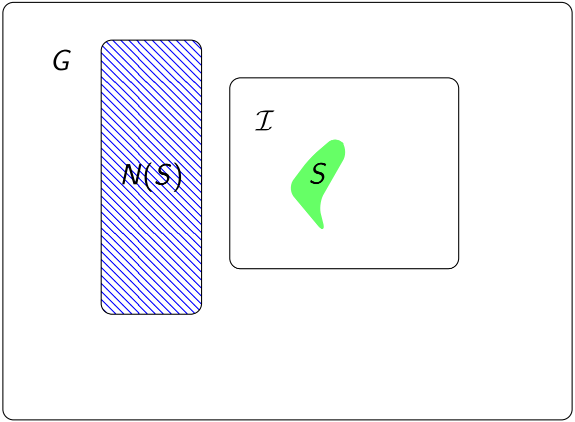

We delete the neighborhood of S.

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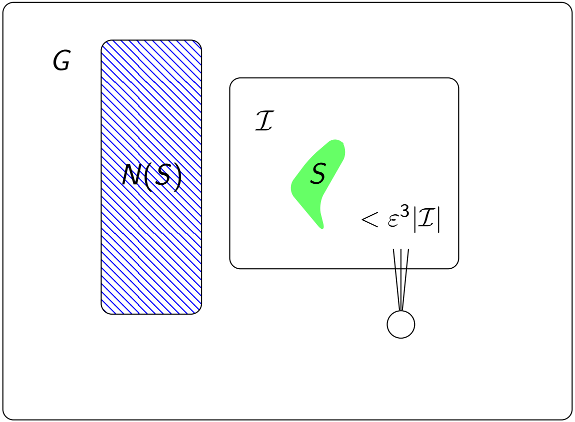

The remaining vertices have few vertices in $\mathcal{I}$.

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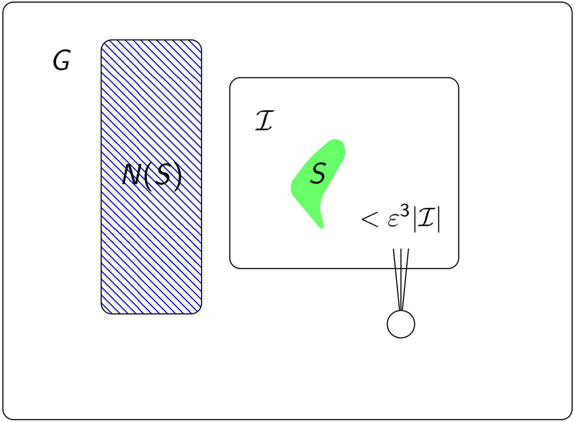

This is due to the theorem of ε-nets.

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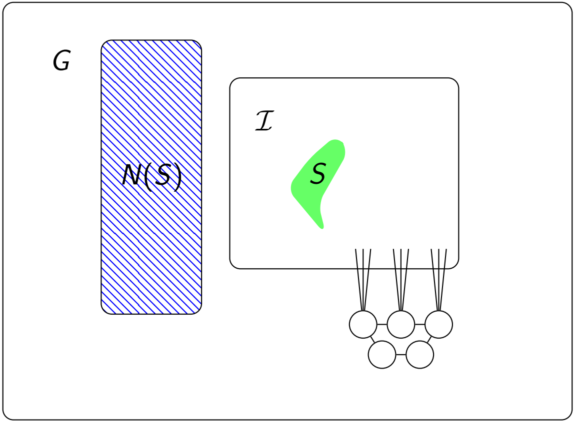

We compute a shortest odd cycle C.

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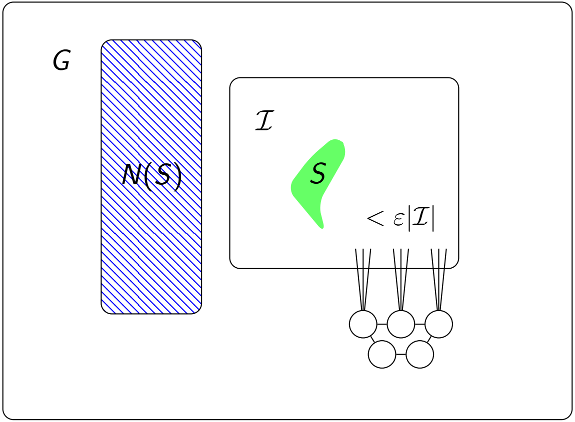

If $|C| \leqslant 1/\varepsilon^2$, we delete its neighborhood.

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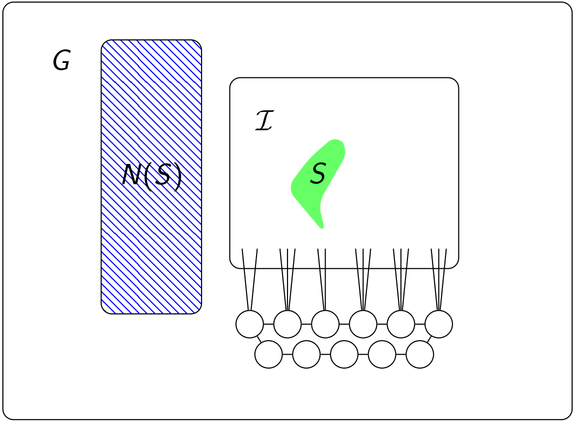
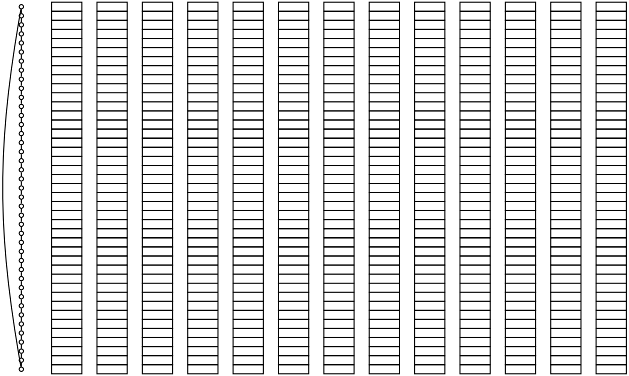


So, we might assume that $|C| > 1/\varepsilon^2$.

Second step: find a small odd cycle transversal



In column, the successive neighborhood of C, *layers*.
Rows indicate the closest neighbor on C, *strata*.

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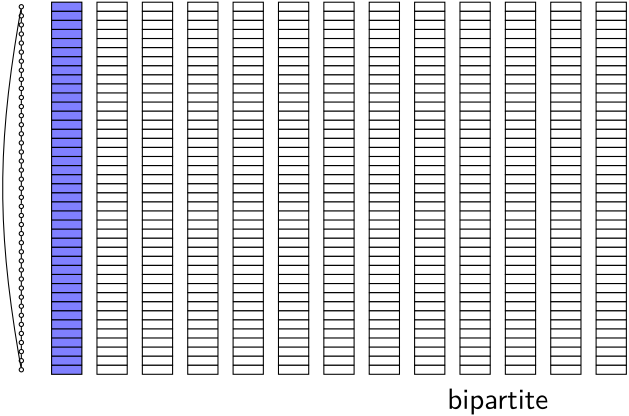
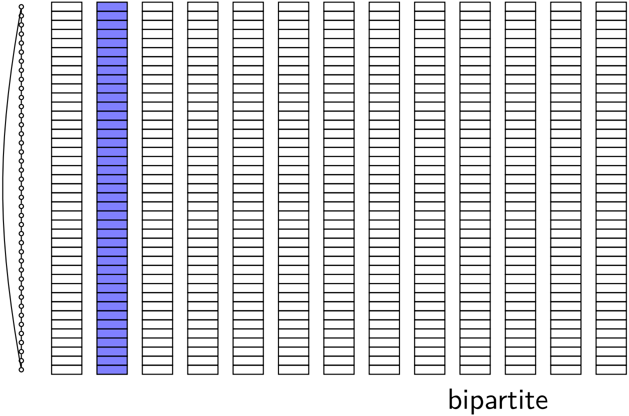


Deleting a *j*-th neighborhood of *C*, leaves a bipartite to the right.

Second step: find a small odd cycle transversal



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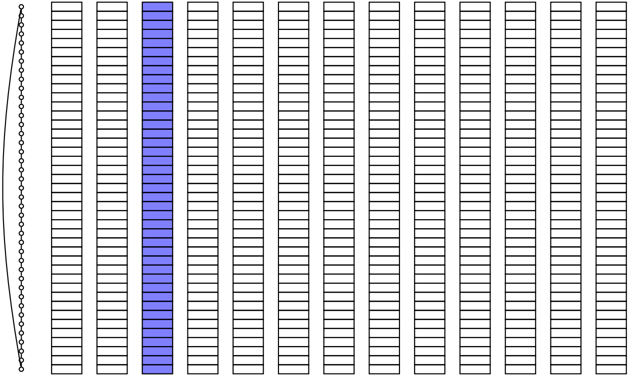

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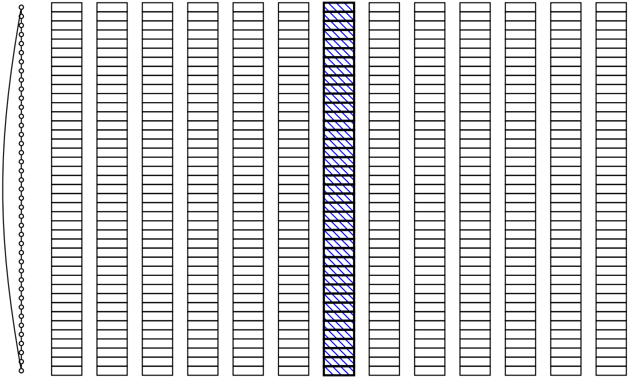
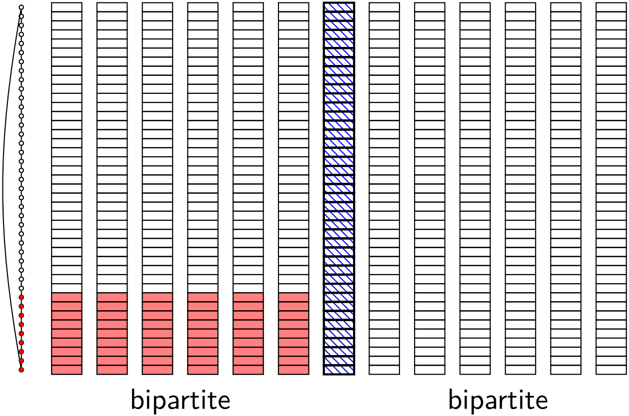


bipartite

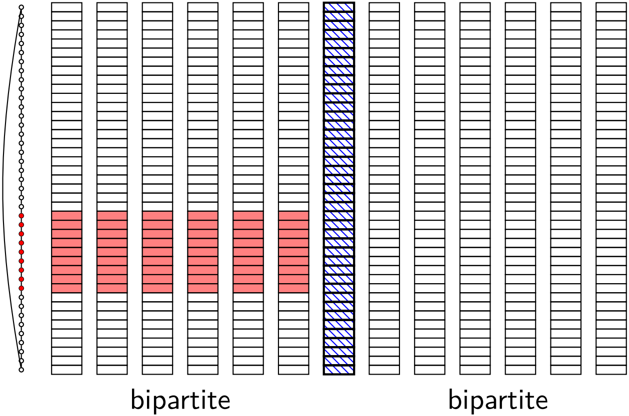
We delete the lightest of the $\approx 1/\varepsilon$ first layers.

Second step: find a small odd cycle transversal



The $\approx 1/\varepsilon$ consecutive layers form an odd cycle transversal.

Second step: find a small odd cycle transversal



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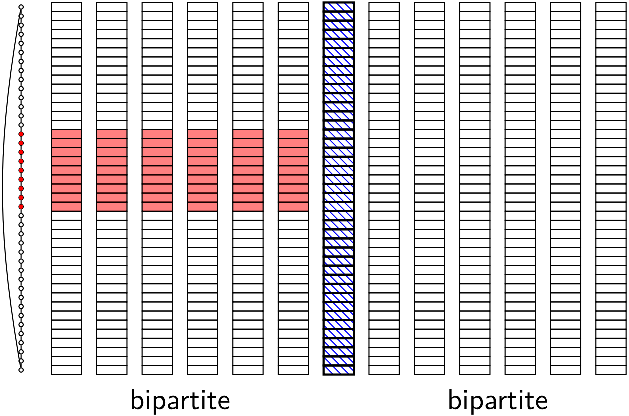

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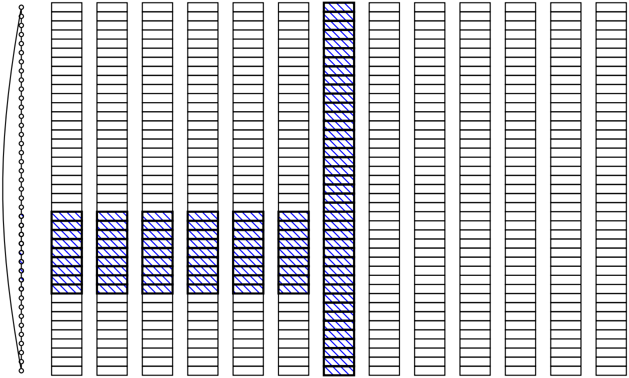

We remove the lightest block of strata.

Filled ellipses and triangles

2-subdivisions: graphs where each edge is subdivided exactly twice
co-2-subdivisions: complements of 2-subdivisions

Lemma

For some $\alpha > 1$, Max Independent Set *on 2-subdivisions is not* α*-approximable algorithm in* $2^{n^{0.99}}$*, unless the ETH fails.*

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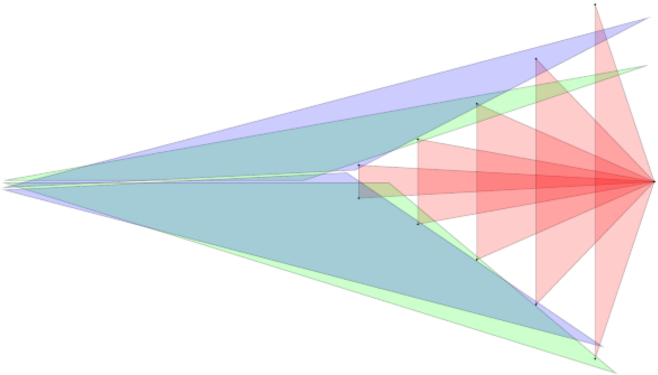
Lemma

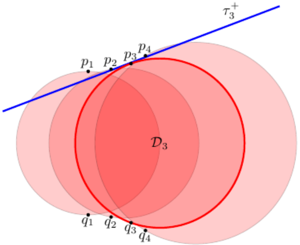
For some $\alpha > 1$, $\textsc{Max Independent Set}$ *on 2-subdivisions is not* α*-approximable algorithm in* $2^{n^{0.99}}$*, unless the ETH fails.*

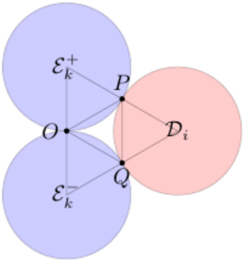
Graphs of filled ellipses or filled triangles contain all the co-2-subdivisions.

Filled triangles



Filled ellipses

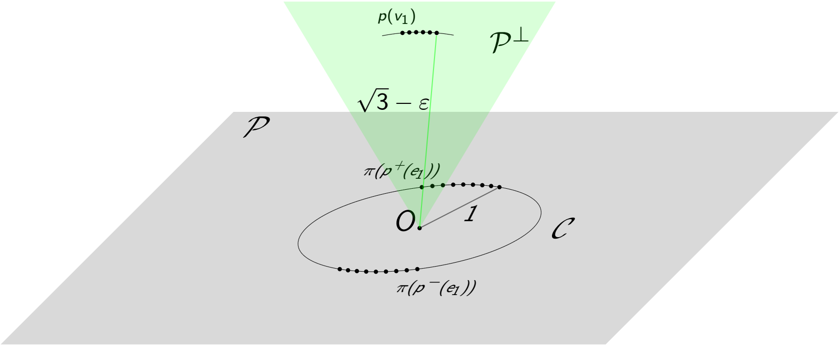




Higher dimensions:

- unit 4D-disk graphs
- ball (3D-disk) graphs with radii arbitrary close to 1

contain all the co-2-subdivisions
(so, no approximation scheme and no subexponential algorithm)



What about unit ball graph?

Let $x_1, \ldots, x_s$ be the consecutive centers in $\mathbb{R}^3$ of a co-odd-cycle. Consider the trace on the 2-sphere of the following vector walk.

- Start at vector $\overrightarrow{ab}$ with $a = x_1$ and $b = x_2$.
- move continuously a from x_1 to x_3 following the segment x_1x_3.
- move continuously b from x_2 to x_4 following the segment x_2x_4.
- and so on, until back to $\overrightarrow{x_1x_2}$.

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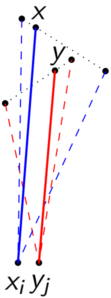
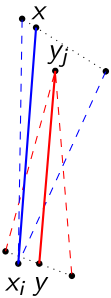
As s is odd, half-way through we reach $\overrightarrow{x_2x_1}$.
Hence the curve drawn on the 2-sphere is antipodal.

As two antipodal curves intersect, we have one of the following configurations:





Open questions

- Is MAX CLIQUE NP-hard on disk and unit ball graphs?
- A first step might be to show NP-hardness for MAX INDEPENDENT SET with iocp 1.
- Actually what about ocp 1?
- What is the complexity of MAX INDEPENDENT SET on the Moebius grid? on quadrangulations of the projective plane?

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Thank you for your attention!