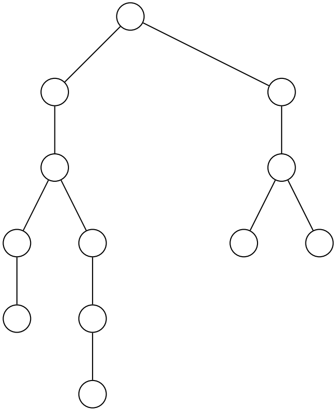
Graph decompositions and their algorithms

Édouard Bonnet

ENS Lyon, LIP

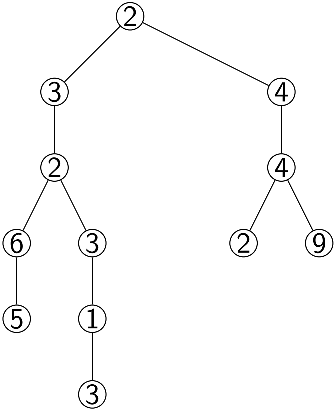
May 5th, 2022, Demi-Journée du Pôle Calcul

Trees



Trees make most NP-hard problems easy

Example of MIN WEIGHTED DOMINATING SET



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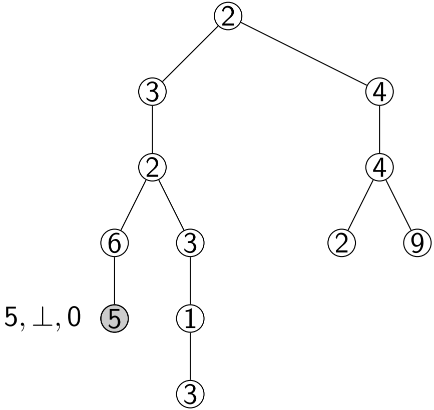

Idea: keep 3 lightest dominating sets of each subtree (rooted at u) one containing u, one not containing u, and one disregarding u

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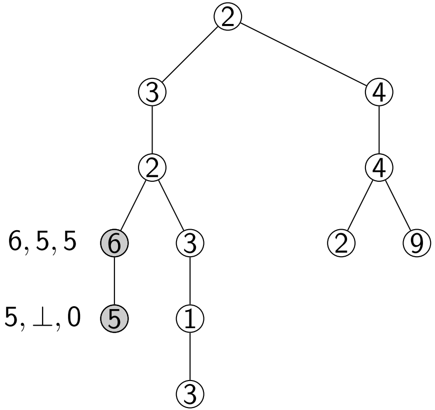

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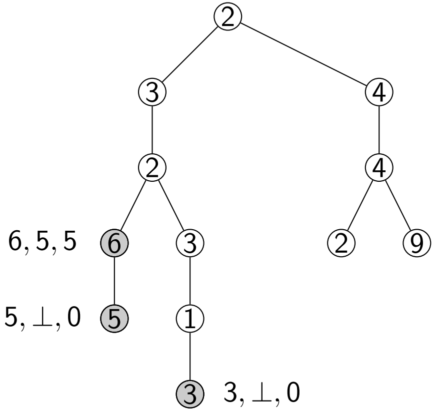

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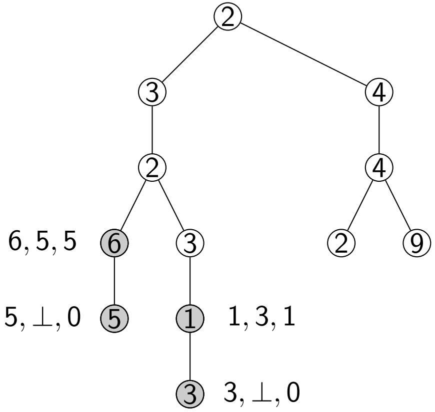

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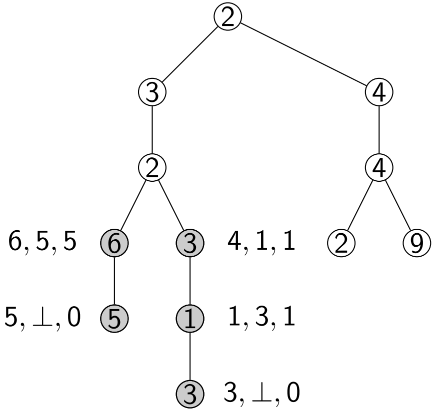

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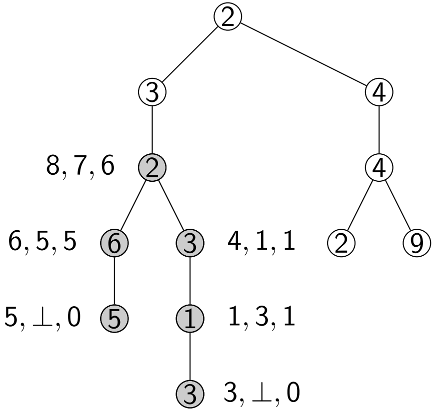

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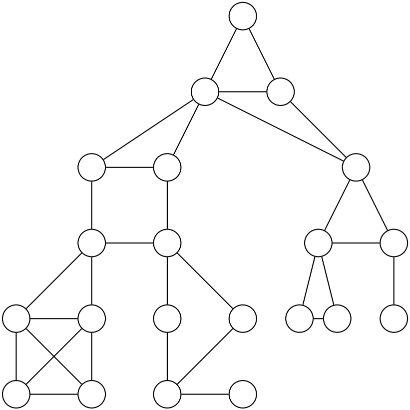
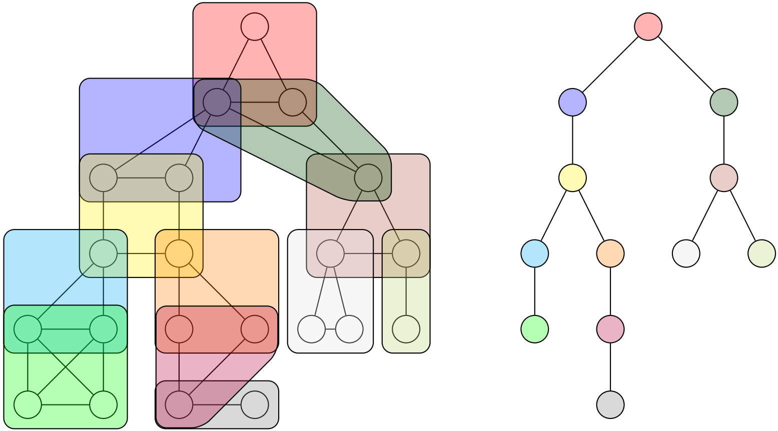
Tree decomposition



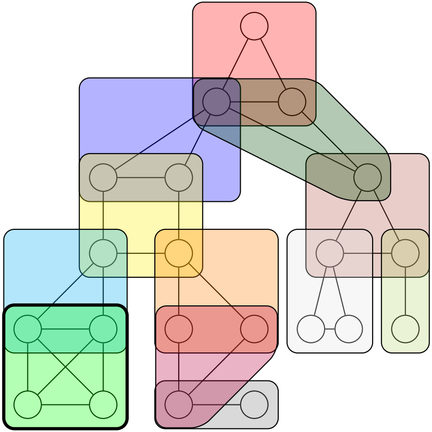
Tree decomposition

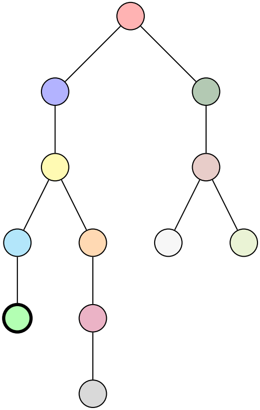
Cover by bags mapping to a tree s.t. each vertex lies in a subtree

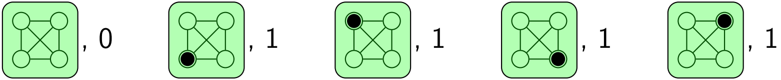


Tree decomposition: solving MAX INDEPENDENT SET

For each trace in each bag, keep a best solution in what is below







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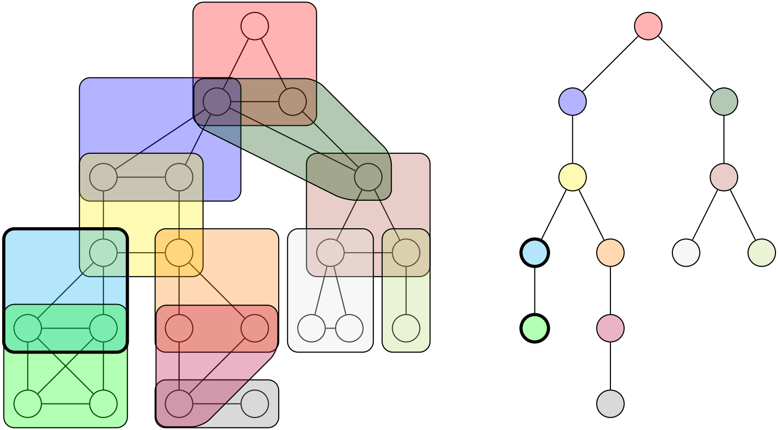

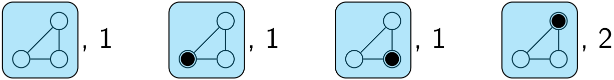

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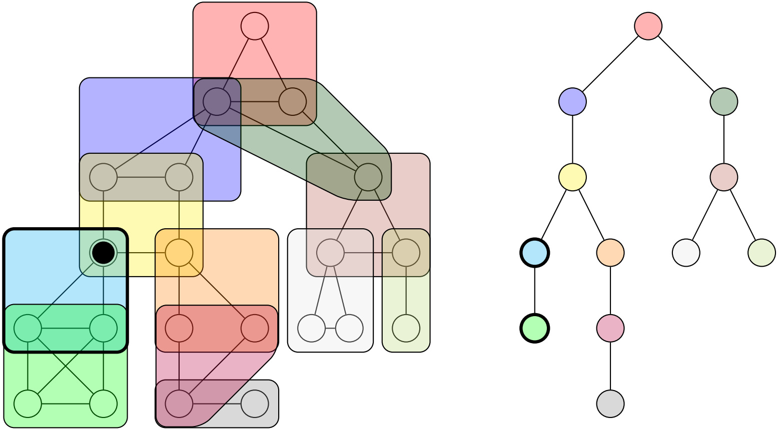

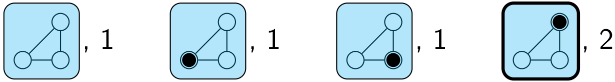

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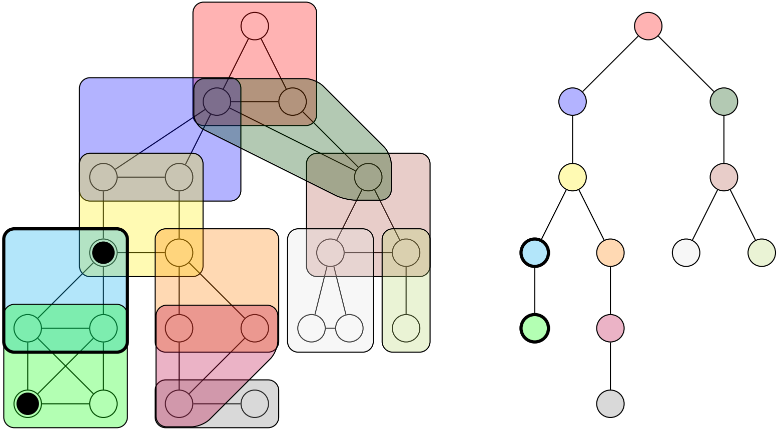

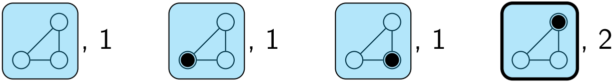

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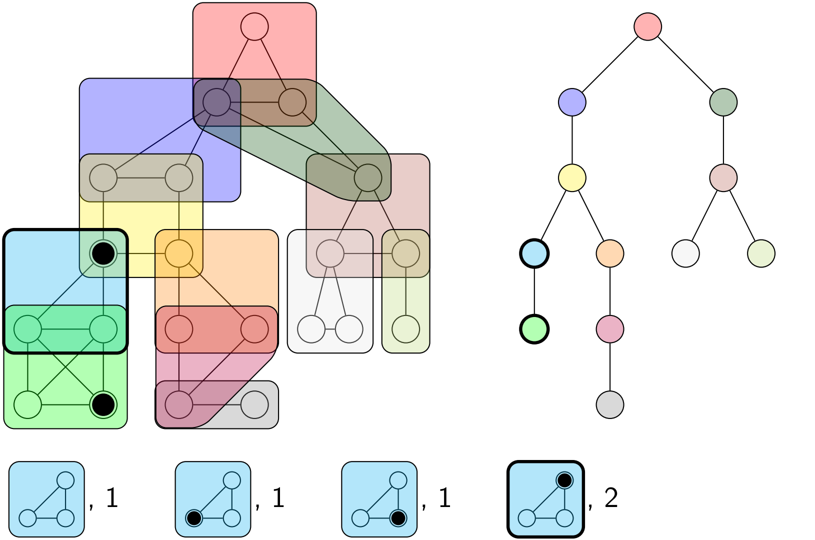

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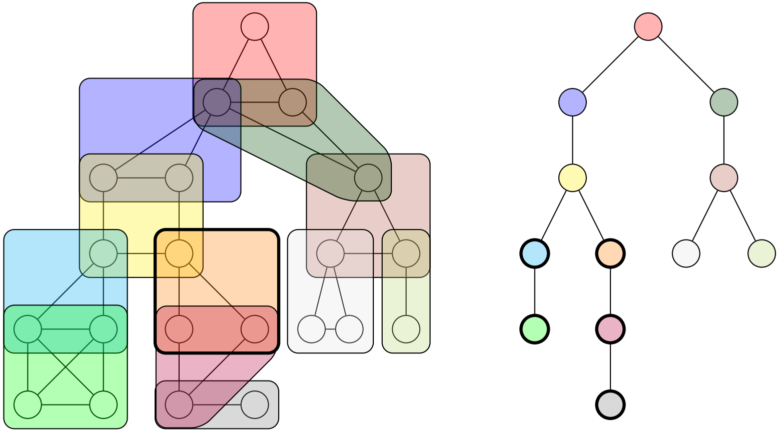

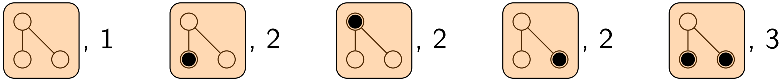

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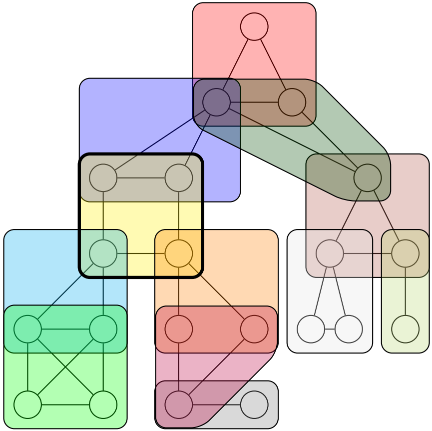

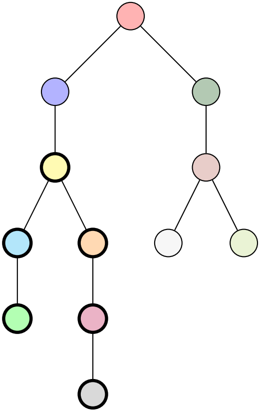

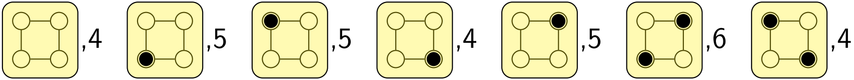

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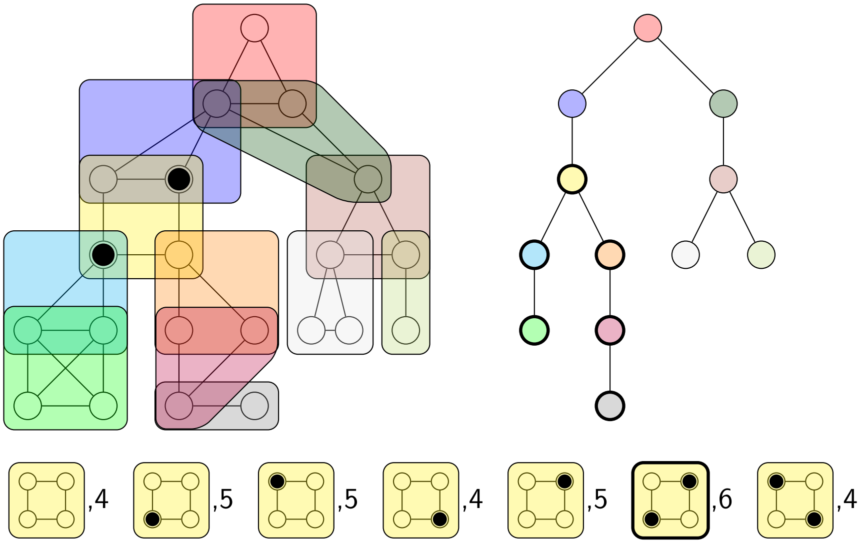

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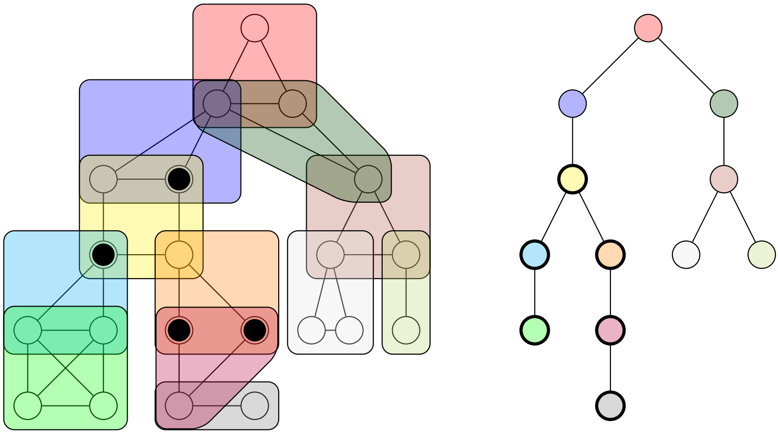

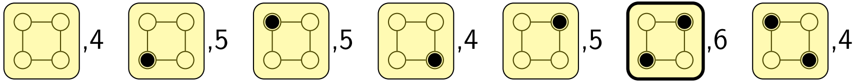

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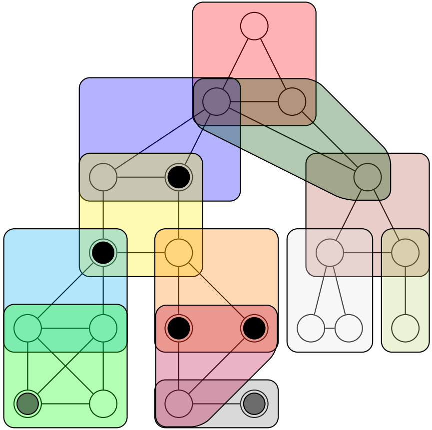

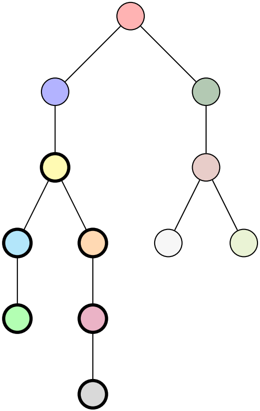

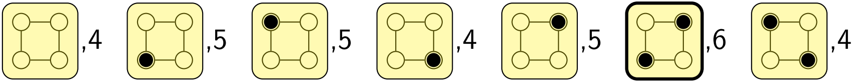

Treewidth

Minimum largest bag size over all tree decompositions minus 1

- rediscovered several times in the 70's and 80's...
- made central by *Graph Minors* and algorithmic graph theory
- previous slide: $2^{O(\mathrm{tw})}n$ time with n bags

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Computing a tree decomposition?

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Computing a tree decomposition? NP-hard but various algorithms

width $2\text{tw}+1$ in $2^{O(\text{tw})}n$

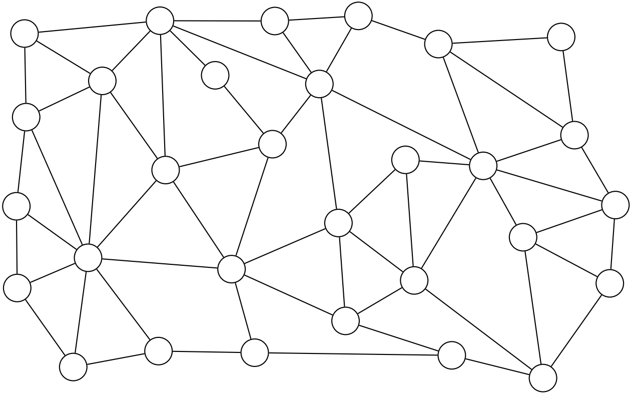
width $5\text{tw}+4$ in $2^{6.76\text{tw}}n\log n$

width tw in $2^{O(\text{tw}^3)}n$

width $\text{tw}\sqrt{\log \text{tw}}$ in $n^{O(1)}$

width tw in 1.74^n

$2^{O(\sqrt{n})}$ time algorithms on planar graphs via Lipton-Tarjan



Planar graphs have treewidth $O(\sqrt{n})$

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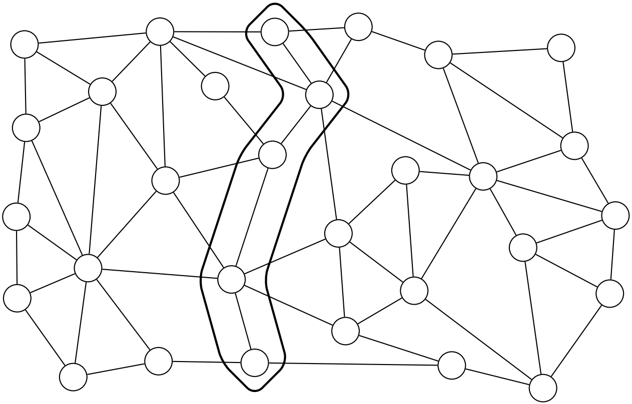

Equivalently $O(\sqrt{n})$ balanced separators, i.e., sides of size $\leqslant 2n/3$

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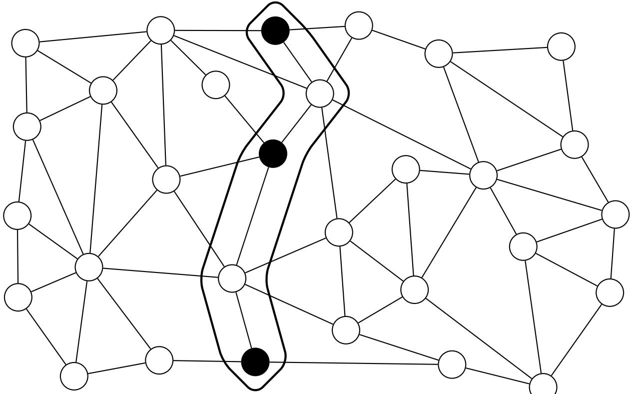

Max Independent Set, 3-Coloring, Hamiltonian Path...

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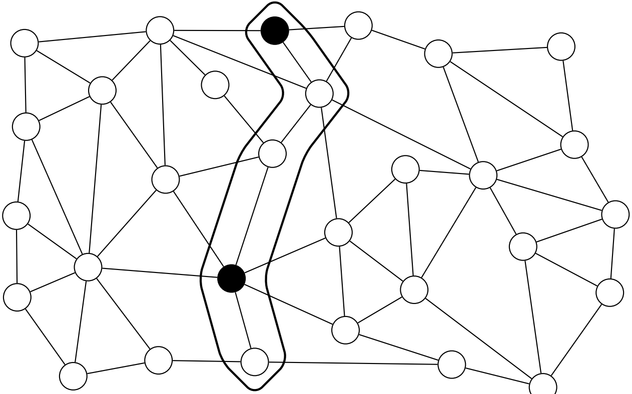

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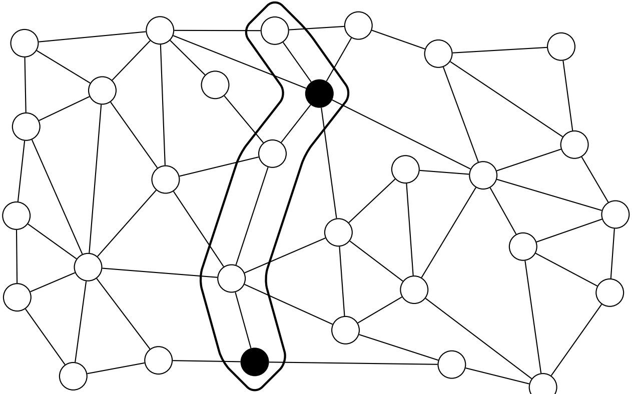

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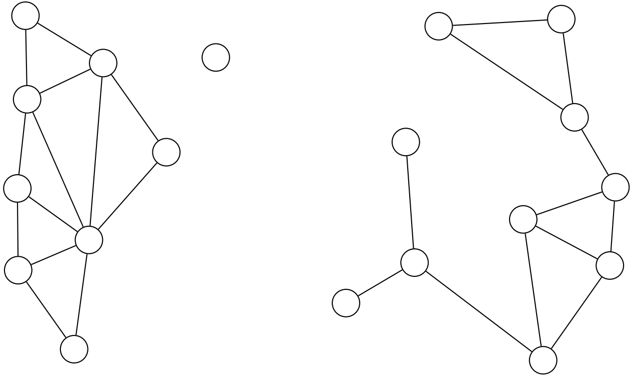
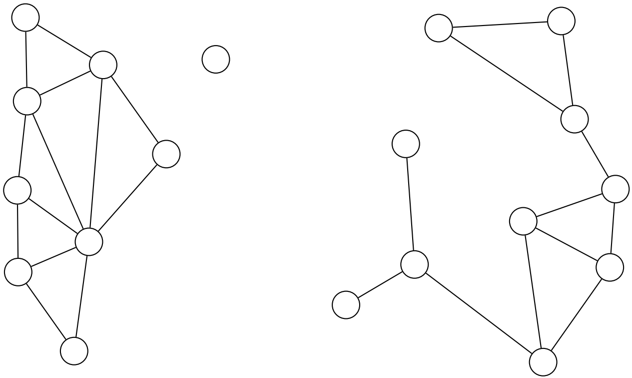


<u>MAX INDEPENDENT SET</u>, 3-COLORING, HAMILTONIAN PATH...

$$T(n) \leqslant 2^{O(\sqrt{n})} T(2n/3) \leqslant \ldots \leqslant 2^{O(\sqrt{n}) \sum_i \sqrt{2/3}^i} = 2^{O(\sqrt{n})}$$

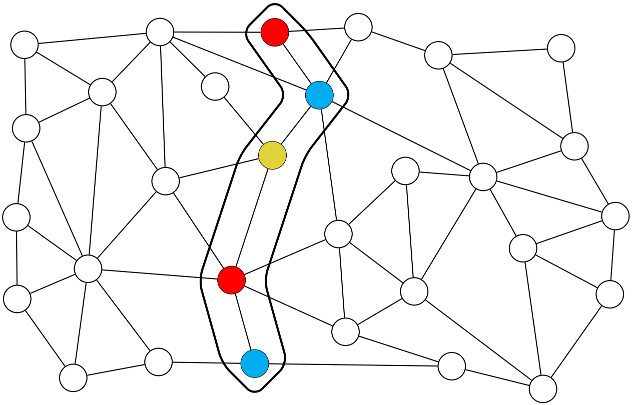
$2^{O(\sqrt{n})}$ time algorithms on planar graphs via Lipton-Tarjan



<u>Max Independent Set</u>, 3-Coloring, Hamiltonian Path...

Even polyspace!

$2^{O(\sqrt{n})}$ time algorithms on planar graphs via Lipton-Tarjan



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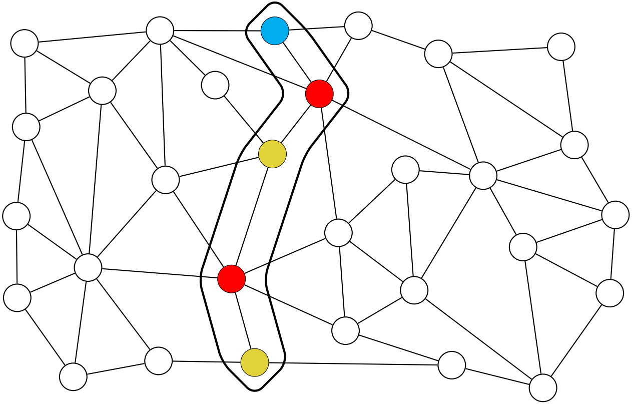

Max Independent Set, 3-Coloring, Hamiltonian Path...

solve the extension List 3-Coloring

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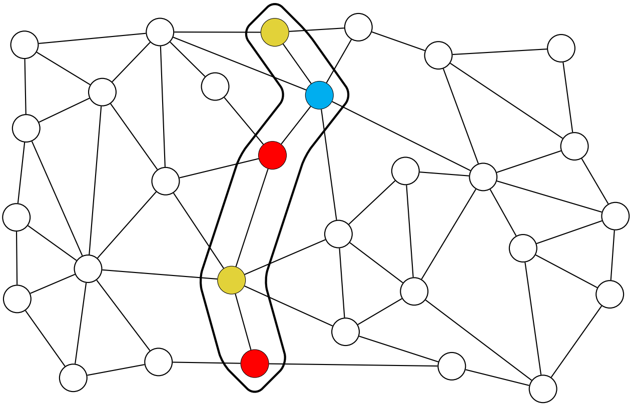


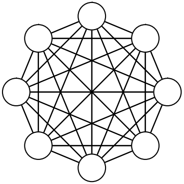
MAX INDEPENDENT SET, <u>3-COLORING</u>, HAMILTONIAN PATH...

solve the extension LIST 3-COLORING

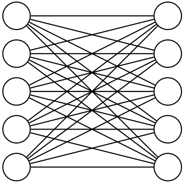
Decomposition of dense graphs?

Graphs with small treewidth have linearly many edges

What about simple dense graphs?



clique



biclique

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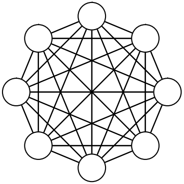
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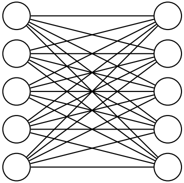
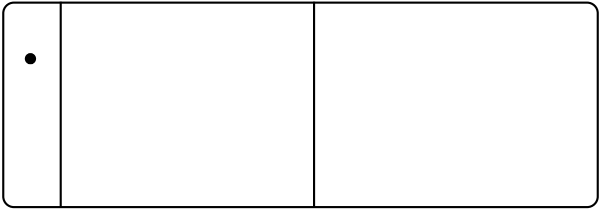

biclique

- cliquewidth defined in the 90's
- allows faster algorithms but hard to compute itself
- rankwidth [Oum, Seymour '05] "equivalent" and approximable

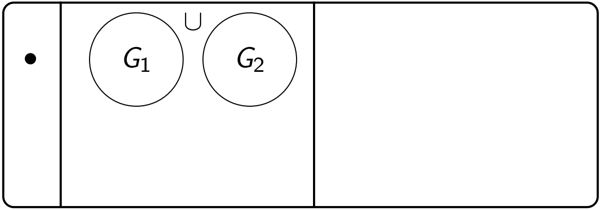
We will see another equivalent definition via *contraction sequences*

Cographs



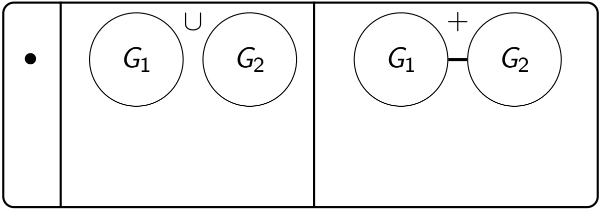
A single vertex is a cograph,

Cographs



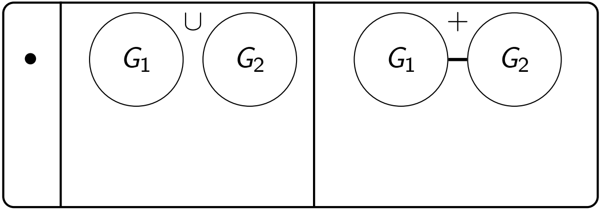
as well as the union of two cographs,

Cographs

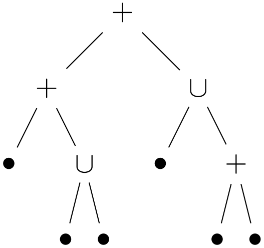


and the complete join of two cographs.

Cographs



Many NP-hard problems are polytime solvable on cographs



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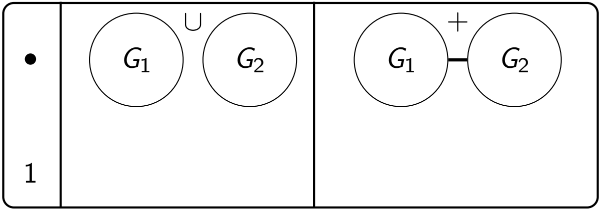
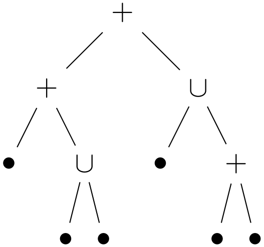


For instance the independence number $\alpha(G)$ is polytime



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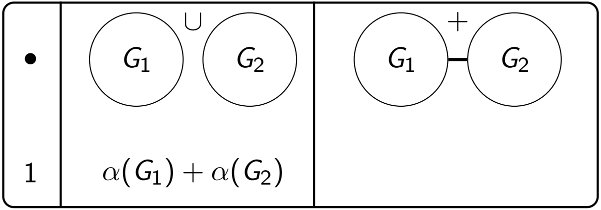
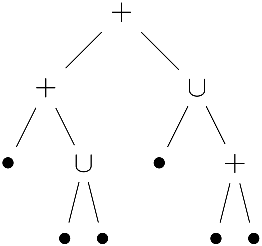


In case of a disjoint union: combine the solutions



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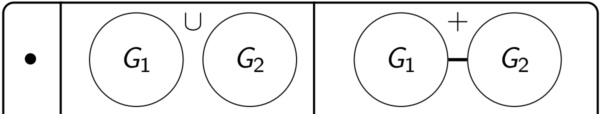
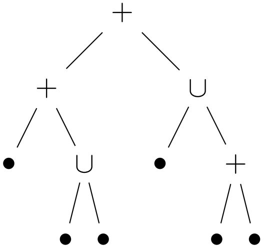


1	$\alpha(G_1) + \alpha(G_2)$	$\max\{\alpha(G_1), \alpha(G_2)\}$

In case of a complete join: pick the larger one



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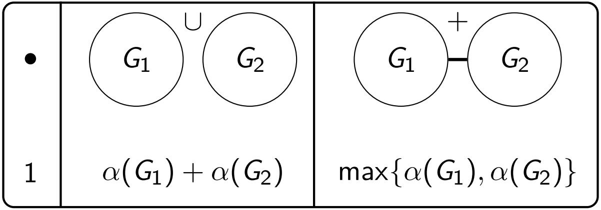
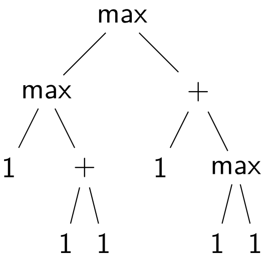

∪
G_1 G_2
+
G_1 G_2
1
$\alpha(G_1) + \alpha(G_2)$
$\max\{\alpha(G_1), \alpha(G_2)\}$


max
max
+
1
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1
max
1 1
1 1

Another cograph definition

Every induced subgraph has two twins

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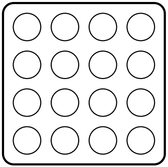
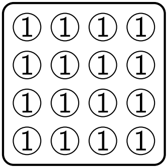


Is there another algorithmic scheme based on this definition?

Another cograph definition

Every induced subgraph has two twins



We store in each vertex its inner max independent set

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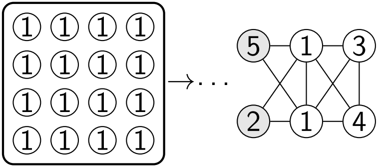

We can find a pair of false/true twins

Another cograph definition

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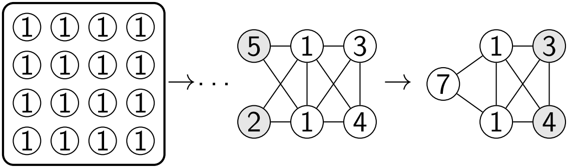

Sum them if they are false twins

Another cograph definition

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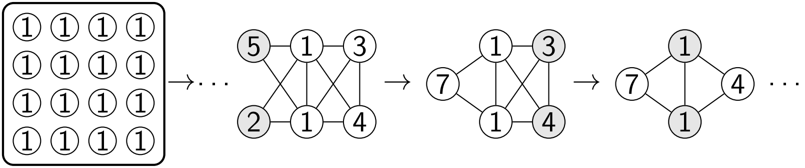
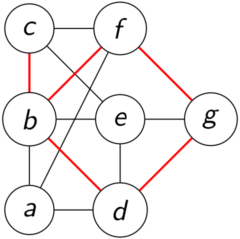


Max them if they are true twins

Trigraphs



Three outcomes between a pair of vertices:
edge, or non-edge, or red edge

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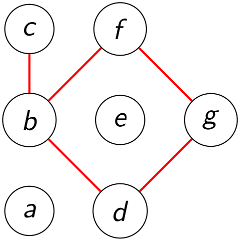

Three outcomes between a pair of vertices:
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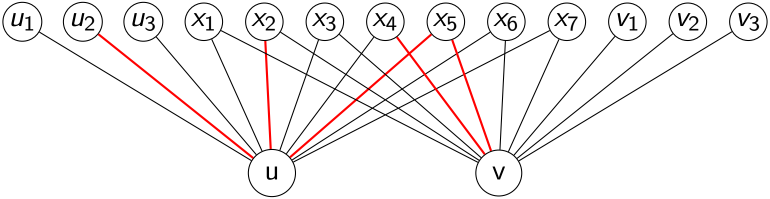
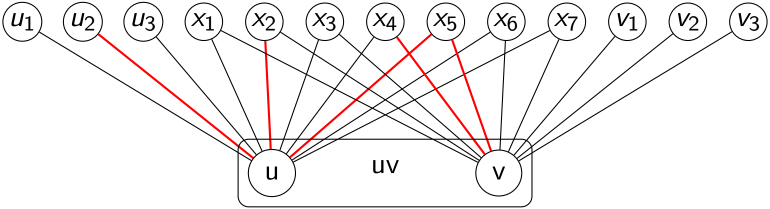
Red graph: trigraph minus its black edges

Contractions in trigraphs



Identification of two non-necessarily adjacent vertices

Contractions in trigraphs



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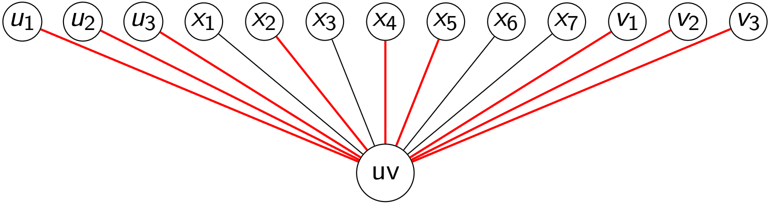
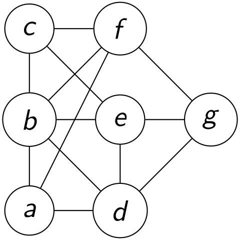


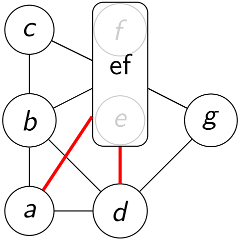
edges to $N(u) \triangle N(v)$ turn red, for $N(u) \cap N(v)$ red is absorbing

Contraction sequence



A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that
G_i is obtained by performing one contraction in G_{i+1}.

Contraction sequence



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$G\langle S \rangle = G[\bigcup \text{vertices of } G \text{ contracted into a vertex of } S]$

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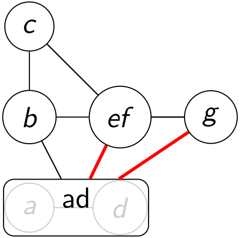

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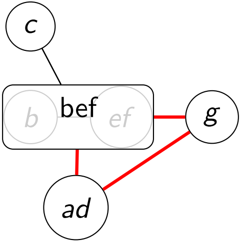

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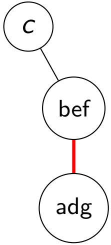

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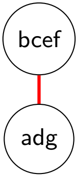

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Reduced parameters

A graph class has bounded reduced X if all its members admit a contraction sequence whose red graphs have bounded X

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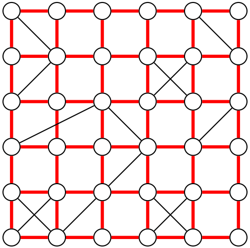
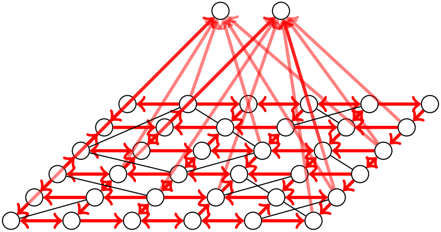
red graphs have bounded ...	characterize bounded ...
degree	**twin-width**
component size	**cliquewidth** (sparse: treewidth)
number of edges*	linear cliquewidth (sparse: pathwidth)
outdegree	(oriented) twin-width
degree + treewidth	?
cutwidth	?
bandwidth	?

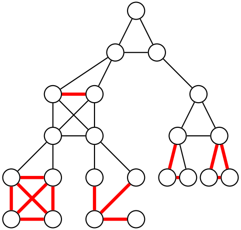
Different conditions imposed in the sequence of red graphs



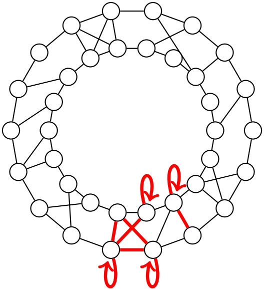
bd degree: defines bd twin-width



bd outdegree: defines bd oriented twin-width

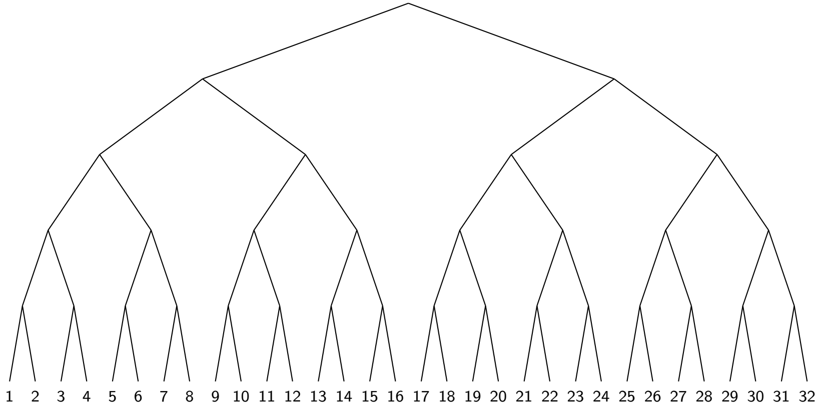


bd component: redefines bd cliquewidth



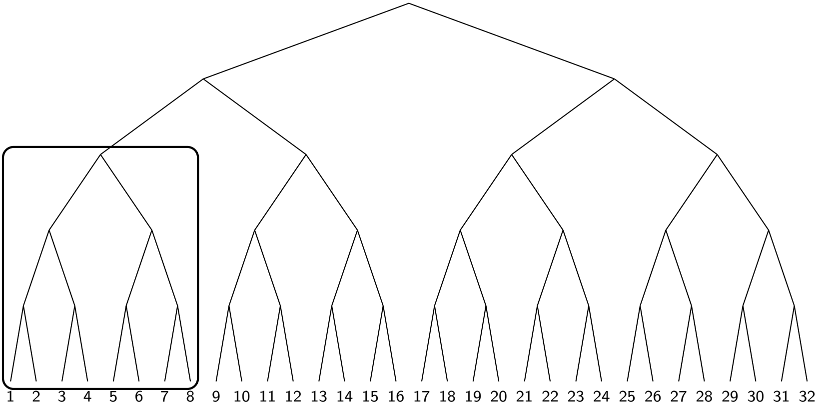
bd #edges: redefines bd linear cliquewidth

Bd boolean-width ⇒ bd component twin-width



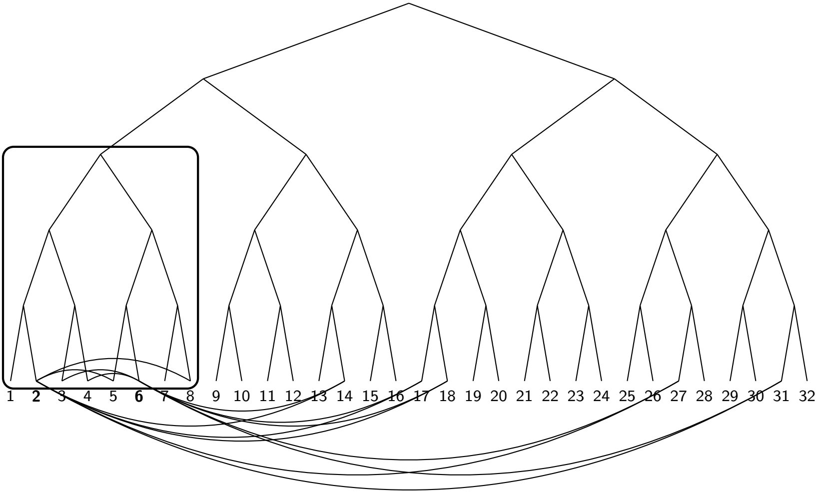
Bd boolean-width: binary tree layout s.t. every edge cut in the tree induces a bipartition with bd # distinct neighborhoods

Bd boolean-width ⇒ bd component twin-width



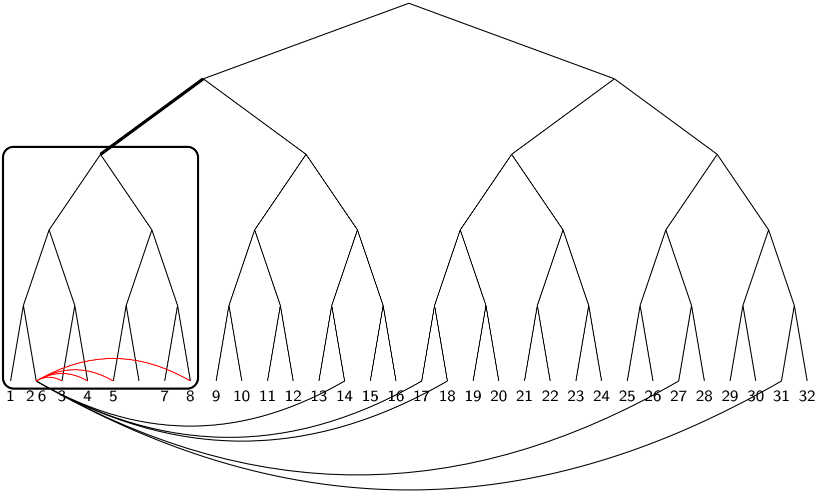
There is a subtree on $\ell \in [d+1, 2d]$ leaves, where d bounds the number of single-vertex neighborhoods in a bipartition

Bd boolean-width ⇒ bd component twin-width



Two vertices have the same neighborhood outside of this subtree

Bd boolean-width ⇒ bd component twin-width



Contracting them preserves the upper bound at $2d$
on the size of red connected components

Component twin-width and boolean-width are tied

Theorem (B., Kim, Reinald, Thomassé '22)

A class has bounded component twin-width iff it has bounded boolean-width/cliquewidth/rank-width.

Proof.

We just saw one direction.

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When red components merge, their subtree gets a same parent. □

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Theorem (B., Kim, Reinald, Thomassé '22)

A class has bounded total twin-width iff it has bounded linear boolean-width/cliquewidth/rank-width.

Is it easier to design algorithms via this characterization?

Solve 3-Coloring on a graph G with a contraction sequence s.t. all red graphs have components of size at most d

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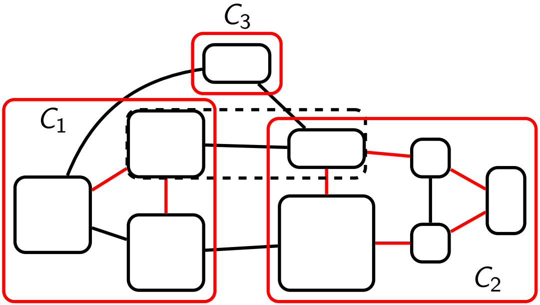

For every red component C keep every profile $V(C) \to 2^{\{1,2,3\}} \setminus \{\emptyset\}$ realizable by a proper 3-coloring of $G\langle C \rangle$

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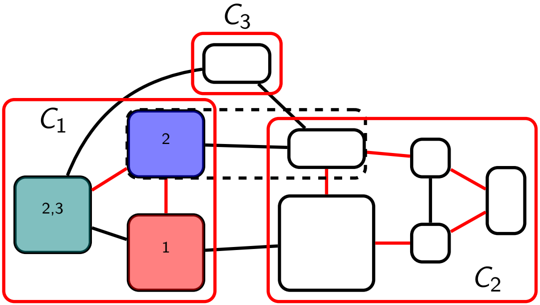

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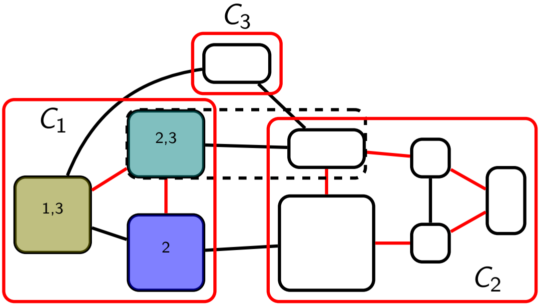

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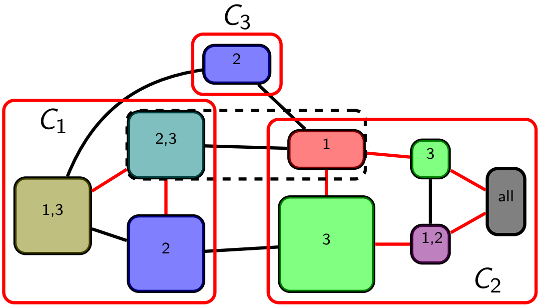

Some tuples of the at most $d+1$ profiles corresponding to merging red components are compatible

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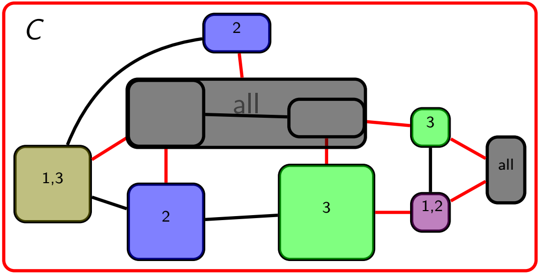

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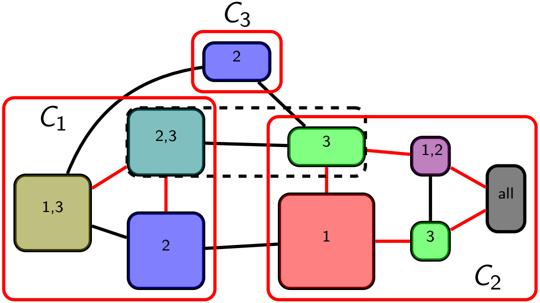

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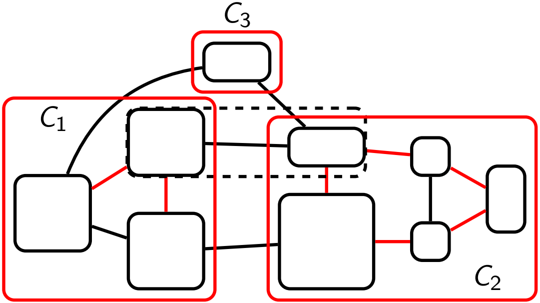

Initialization: time $3n$
Update: time $7^d d^2$
Total: time $7^d d^2 n$
End: still a profile on the single vertex *containing* the whole graph?

Formulas, sentences, and model checking

GRAPH FO/MSO MODEL CHECKING	**Parameter:** $|\varphi|$

Input: A graph G and a first-order/monadic second-order sentence $\varphi \in FO/MSO(\{E\})$
Question: $G \models \varphi$?

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Formulas, sentences, and model checking

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$G \models \varphi$? $\Leftrightarrow$ 3-Coloring

Courcelle's theorems

We will reprove with contraction sequences:

Theorem (Courcelle, Makowsky, Rotics '00)

MSO model checking can be solved in time $f(|\varphi|, d) \cdot |V(G)|$ *given a witness that the clique-width/component twin-width of the input* G *is at most* d.

generalizes

Theorem (Courcelle '90)

MSO model checking can be solved in time $f(|\varphi|, t) \cdot |V(G)|$ *on graphs* G *of treewidth at most* t.

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MSO model checking can be solved in time $f(|\varphi|, t) \cdot |V(G)|$ on graphs G of treewidth at most t.

Instead of maintaining all the possible profiles of 3-colorings, mantain all the sentences of quantifier depth $\leqslant q$ satisfied by a red component!

Rank-k m-types

Sets of non-equivalent formulas/sentences of quantifier rank at most k satisfied by a fixed structure:

$$\mathrm{tp}_k^{\mathcal{L}}(\mathscr{A}, \vec{a} \in A^m) = \{\varphi(\vec{x}) \in \mathcal{L}[k] : \mathscr{A} \models \varphi(\vec{a})\},$$

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Theorem (folklore)

For $\mathcal{L} \in \{FO, MSO\}$, the number of rank-$k$ m-types is bounded by a function of k and m only.

Proof.

"$\mathcal{L}[k+1]$ are Boolean combinations of $\exists x \mathcal{L}[k]$." □

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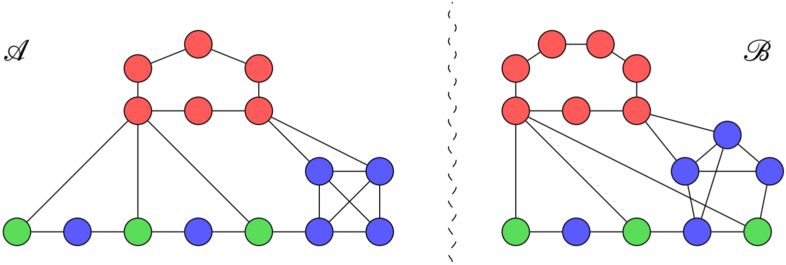
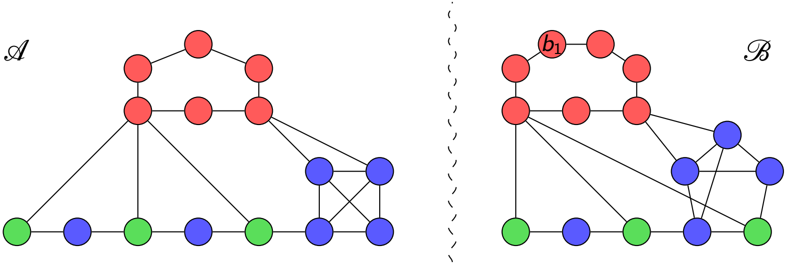
Rank-k types partition the graphs into $g(k)$ classes.
Efficient Model Checking = quickly finding the class of the input.

FO Ehrenfeucht-Fraissé game



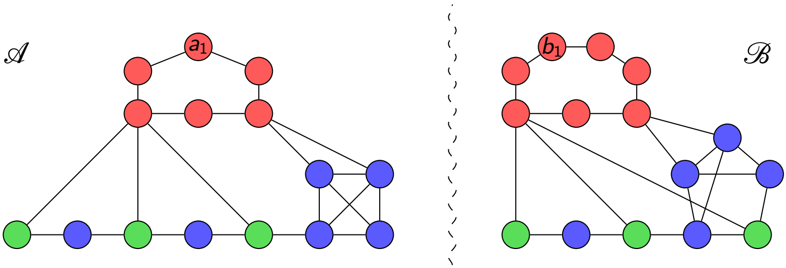
2-player game on two σ-structures $\mathscr{A}, \mathscr{B}$ (for us, colored graphs)

FO Ehrenfeucht-Fraissé game



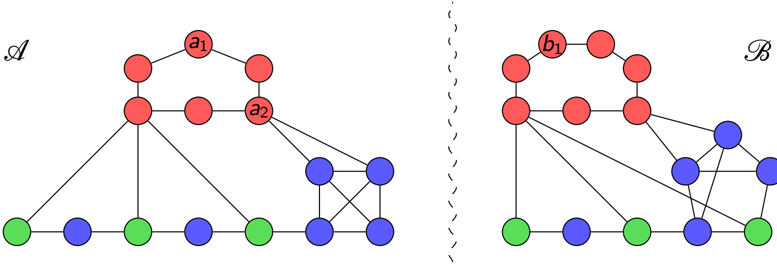
At each round, Spoiler picks a structure ($\mathscr{B}$) and a vertex therein

FO Ehrenfeucht-Fraissé game



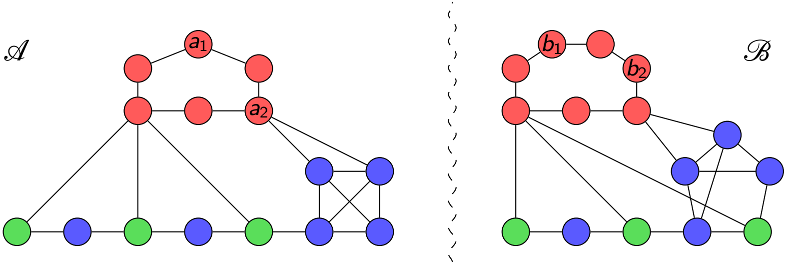
Duplicator answers with a vertex in the other structure

FO Ehrenfeucht-Fraissé game



After q rounds, Duplicator wishes that $a_i \mapsto b_i$ is an isomorphism between $\mathscr{A}[a_1, \ldots, a_k]$ and $\mathscr{B}[b_1, \ldots, b_k]$

FO Ehrenfeucht-Fraissé game



After q rounds, Duplicator wishes that $a_i \mapsto b_i$ is an isomorphism between $\mathscr{A}[a_1, \ldots, a_k]$ and $\mathscr{B}[b_1, \ldots, b_k]$

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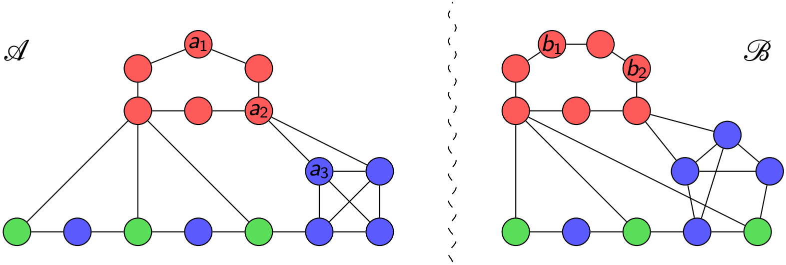

When no longer possible, Spoiler wins

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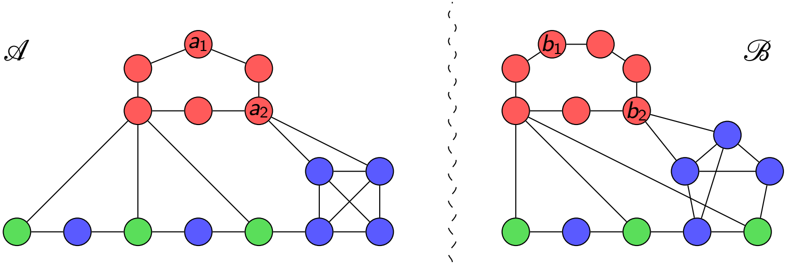

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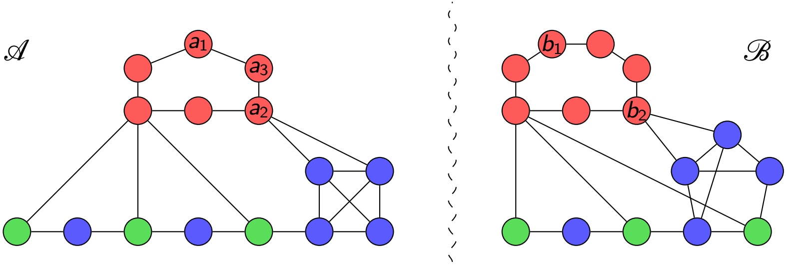
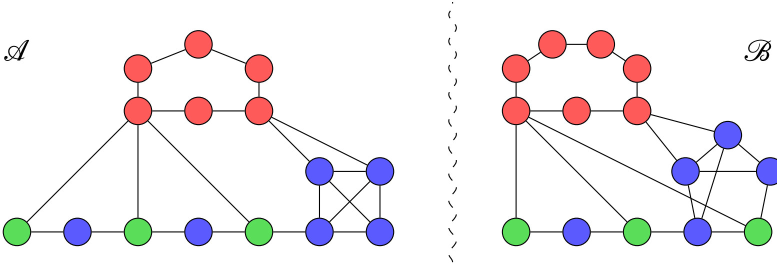


If Duplicator can survive k rounds, we write $\mathscr{A} \equiv_k^{\mathsf{FO}} \mathscr{B}$

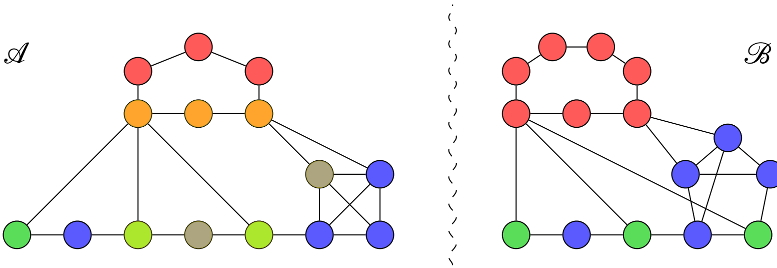
Here $\mathscr{A} \equiv_2^{\mathsf{FO}} \mathscr{B}$ and $\mathscr{A} \not\equiv_3^{\mathsf{FO}} \mathscr{B}$

MSO Ehrenfeucht-Fraissé game



Same game but Spoiler can now play set moves

MSO Ehrenfeucht-Fraissé game



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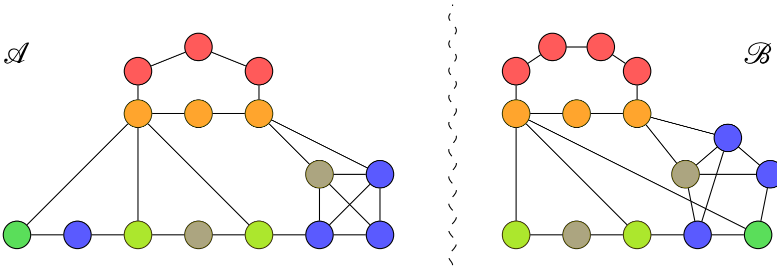
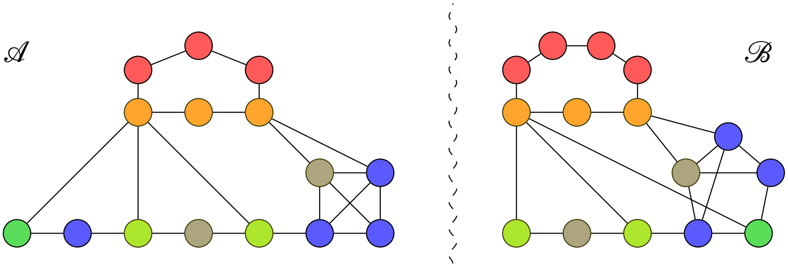


To which Duplicator answers a set in the other structure

MSO Ehrenfeucht-Fraissé game



Again we write $\mathscr{A} \equiv_k^{\mathsf{MSO}} \mathscr{B}$ if Duplicator can survive k rounds

k-round EF games capture rank-k types

Theorem (Ehrenfeucht-Fraissé)

For every σ-structures $\mathscr{A}, \mathscr{B}$ and logic $\mathcal{L} \in \{FO, MSO\}$,

$$\mathscr{A} \equiv_k^{\mathcal{L}} \mathscr{B} \text{ if and only if } tp_k^{\mathcal{L}}(\mathscr{A}) = tp_k^{\mathcal{L}}(\mathscr{B}).$$

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Proof.

Induction on k.

($\Rightarrow$) $\mathcal{L}[k+1]$ formulas are Boolean combinations of $\exists x \varphi$ or $\exists X \varphi$ where $\varphi \in \mathcal{L}[k]$. Use the answer of Duplicator to $x = a$ or $X = A$.

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($\Leftarrow$) If $\text{tp}_{k+1}^{\mathcal{L}}(\mathcal{A}) = \text{tp}_{k+1}^{\mathcal{L}}(\mathcal{B})$, then the type $\text{tp}_k^{\mathcal{L}}(\mathcal{A}, a)$ is equal to some $\text{tp}_k^{\mathcal{L}}(\mathcal{B}, b)$. Move a can be answered by playing b. □

MSO model checking for component twin-width d

Partitioned sentences: sentences on $(E, U_1, \ldots, U_d)$-structures, interpreted as a graph vertex partitioned in d parts

Maintain for every red component C of every trigraph G_i

$$\text{tp}_k^{\text{MSO}}(G, \mathcal{P}_i, C) = \{\varphi \in \text{MSO}_{E,U_1,\ldots,U_d}(k) : (G\langle C\rangle, \mathcal{P}_i\langle C\rangle) \models \varphi\}.$$

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For each $v \in V(G)$, $\text{tp}_k(G, \mathcal{P}_n, \{v\}) =$ type of K_1
$\text{tp}_k(G, \mathcal{P}_1, \{V(G)\}) =$ type of G

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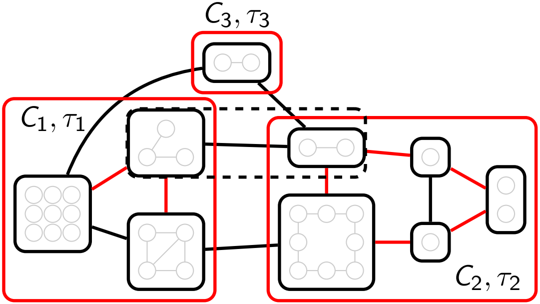

$\tau = \text{tp}_k^{\text{MSO}}(G, \mathcal{P}_i, C)$ based on the $\tau_j = \text{tp}_k^{\text{MSO}}(G, \mathcal{P}_{i+1}, C_j)$?

MSO model checking for component twin-width d

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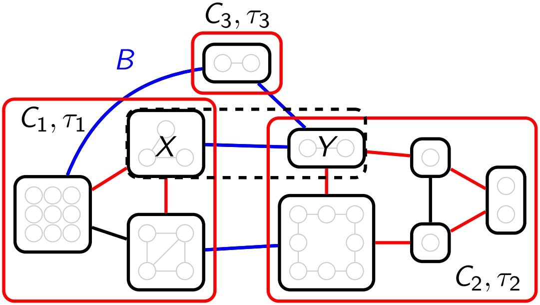
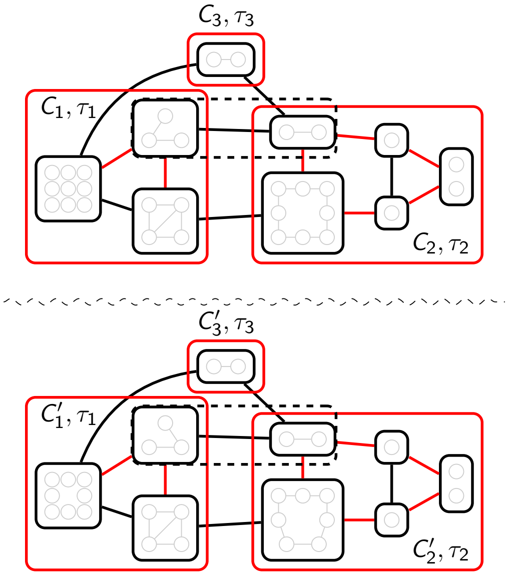


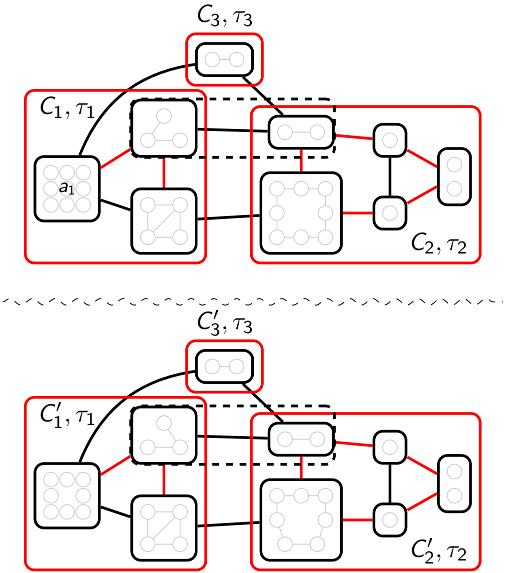
C arises from $C_1, \ldots, C_{d'}$: $\tau = F(\tau_1, \ldots, \tau_{d'}, B, X, Y)$

Showing $\tau = F(\tau_1, \ldots, \tau_{d'}, B, X, Y)$ via MSO EF game



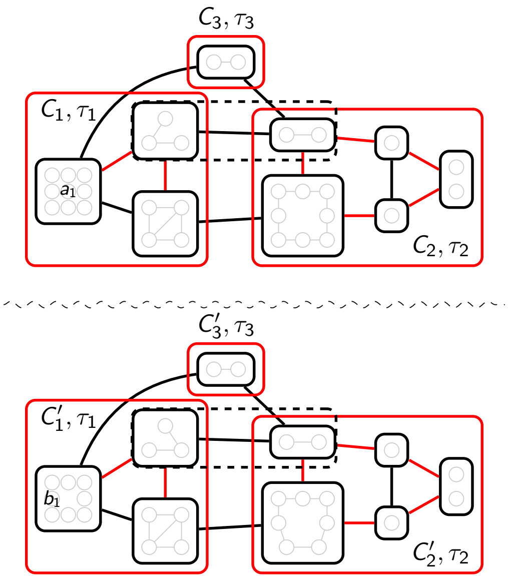
Duplicator combines her strategies in the red components

Showing $\tau = F(\tau_1, \ldots, \tau_{d'}, B, X, Y)$ via MSO EF game



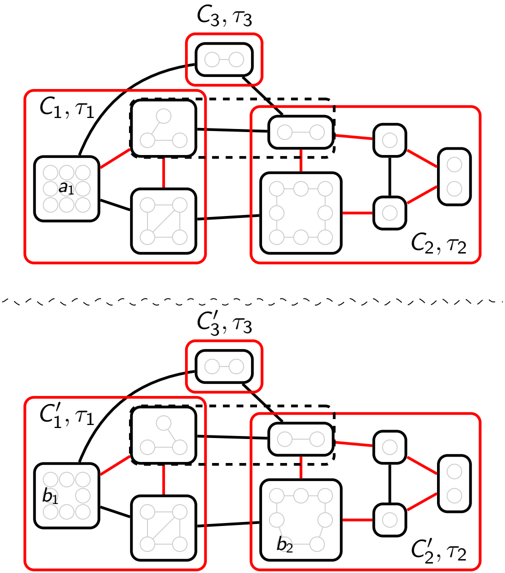
If Spoiler plays a vertex in the component of type τ_1,

Showing $\tau = F(\tau_1, \ldots, \tau_{d'}, B, X, Y)$ via MSO EF game



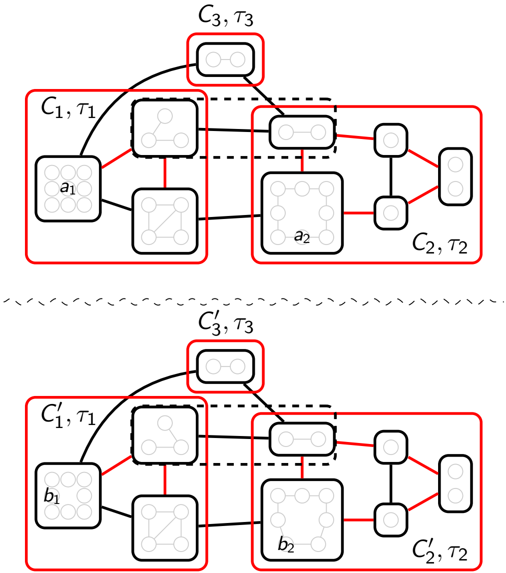
Duplicator answers the corresponding winning move

Showing $\tau = F(\tau_1, \ldots, \tau_{d'}, B, X, Y)$ via MSO EF game



Same in the component of type τ_2

Showing $\tau = F(\tau_1, \ldots, \tau_{d'}, B, X, Y)$ via MSO EF game



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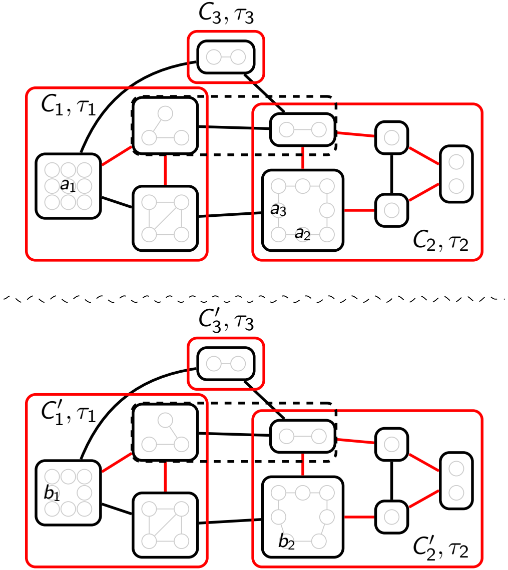

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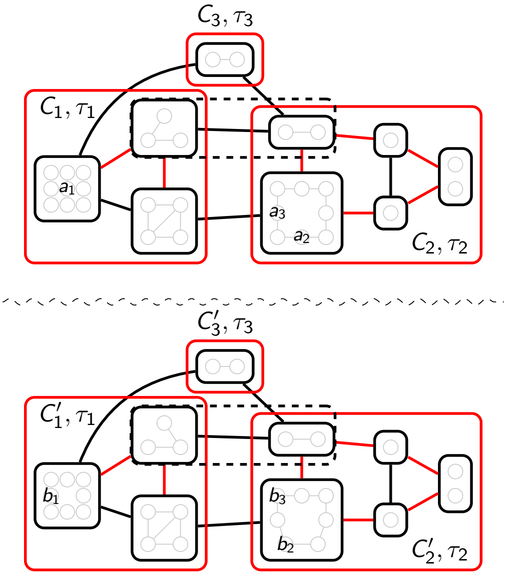

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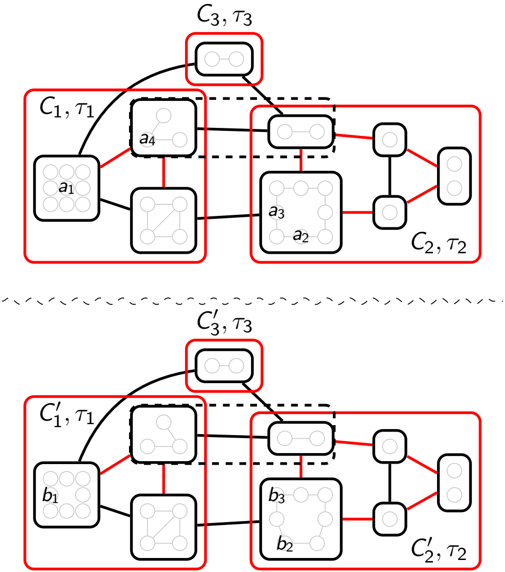

and so on

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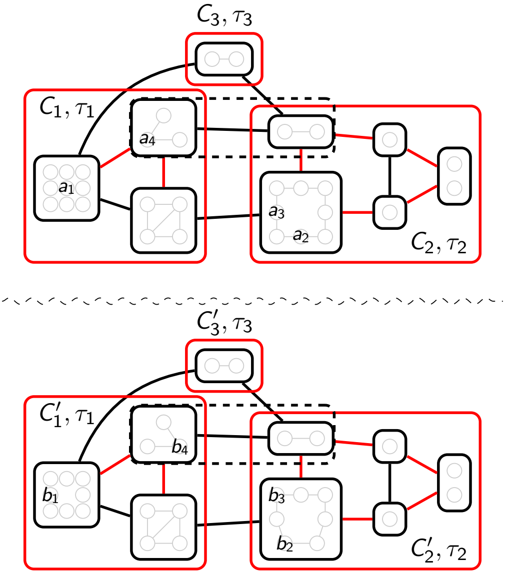

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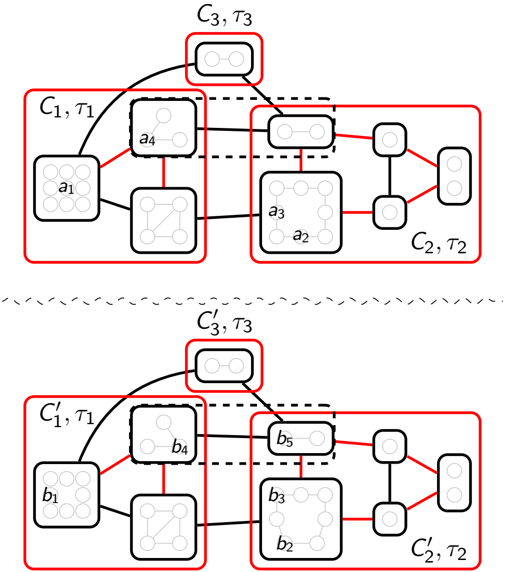

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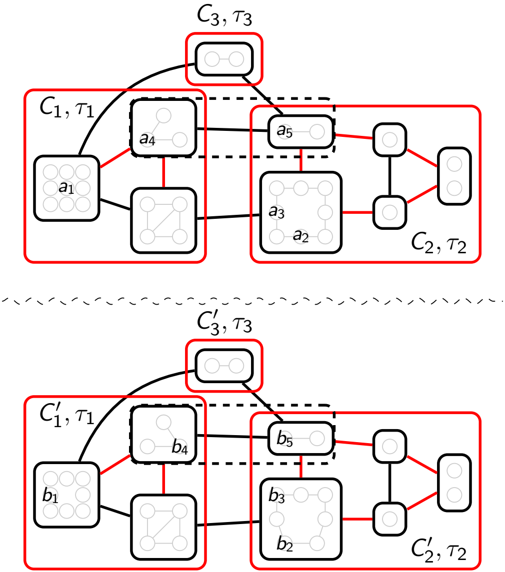

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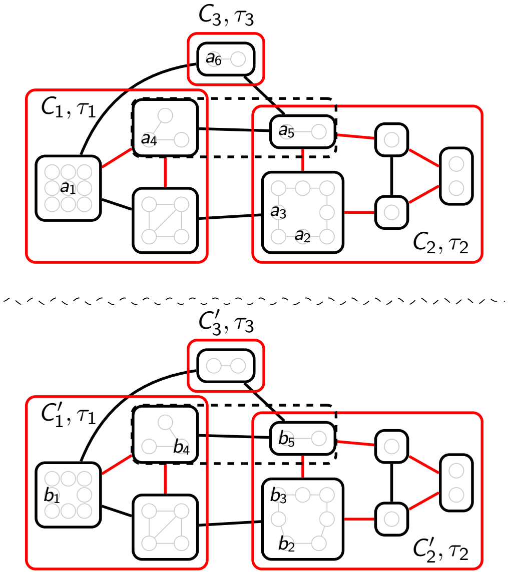

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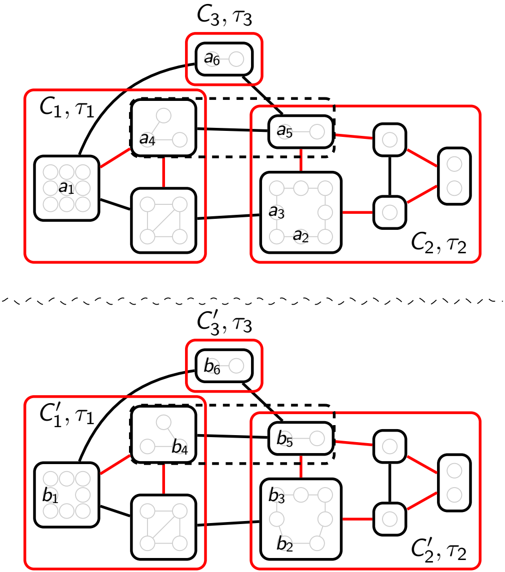

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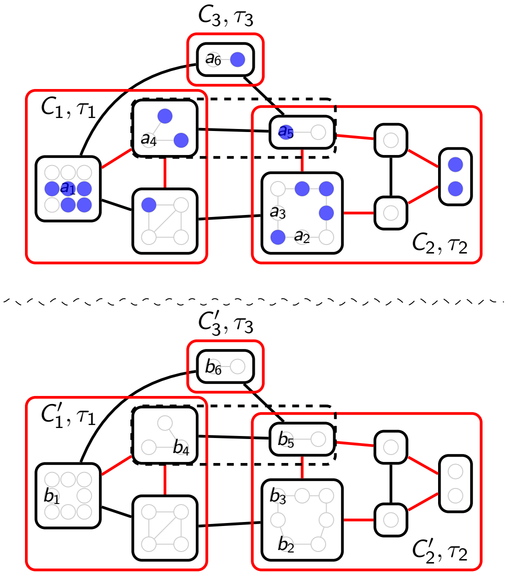
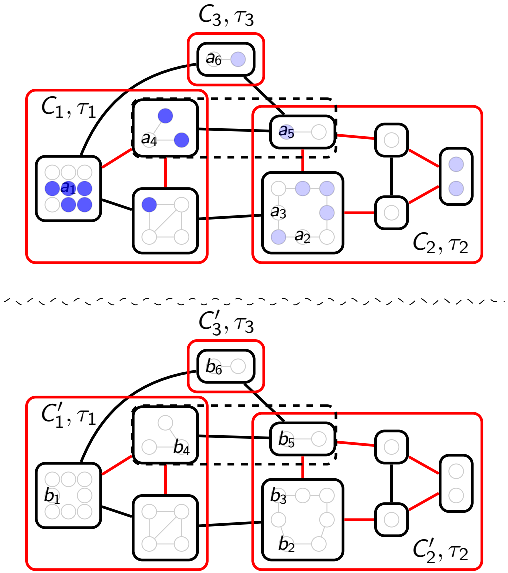


If Spoiler plays a set, Duplicator looks at the intersection with C_1,

Showing $\tau = F(\tau_1, \ldots, \tau_{d'}, B, X, Y)$ via MSO EF game



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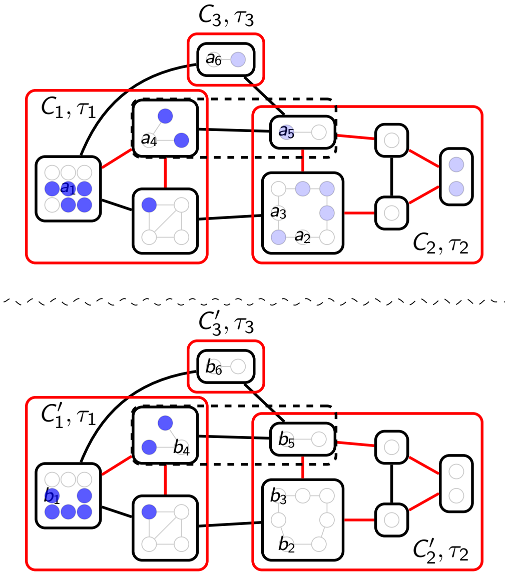

calls her winning strategy in C'_1

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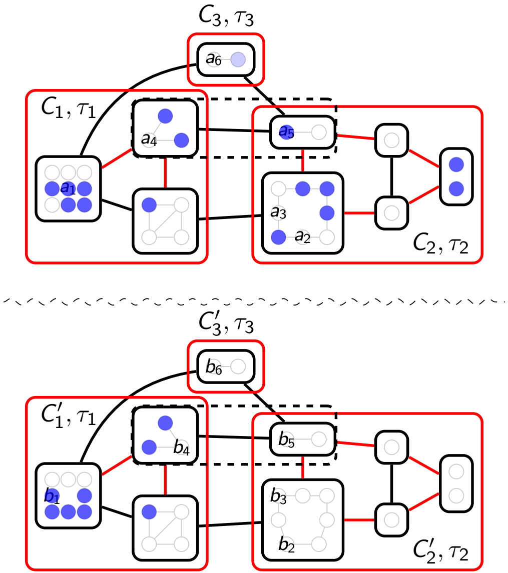

same for the other components

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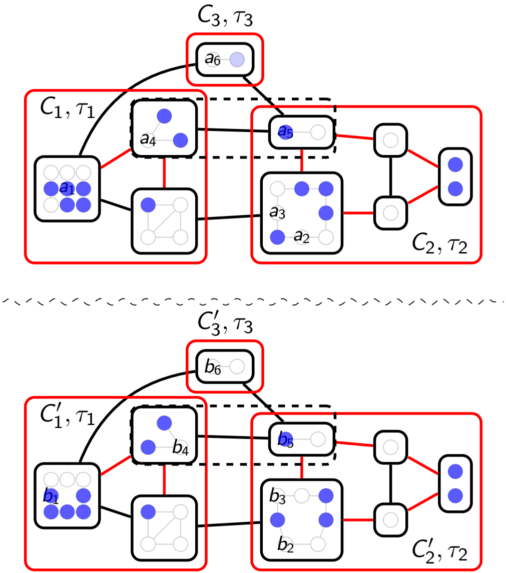

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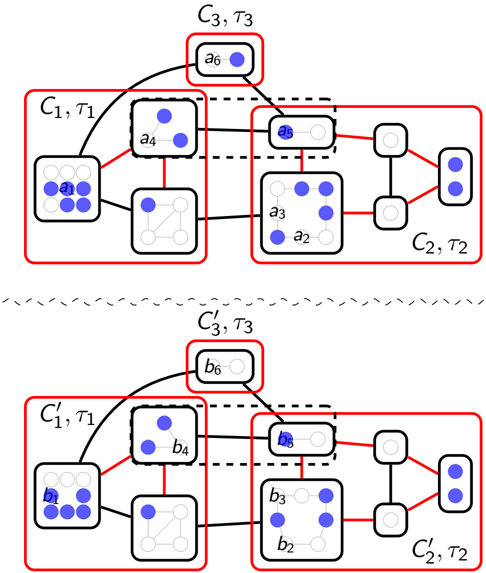

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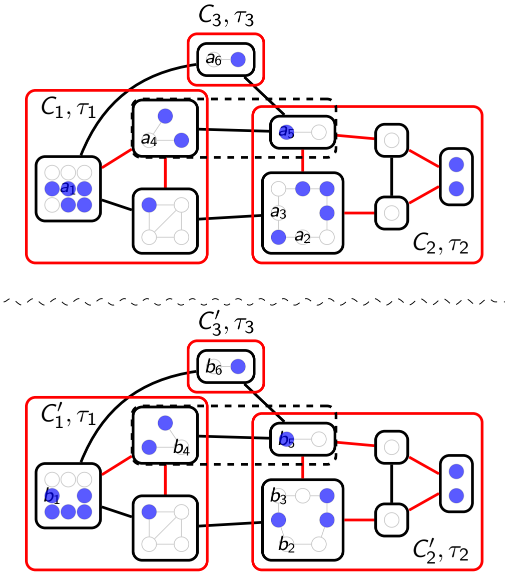

and plays the union

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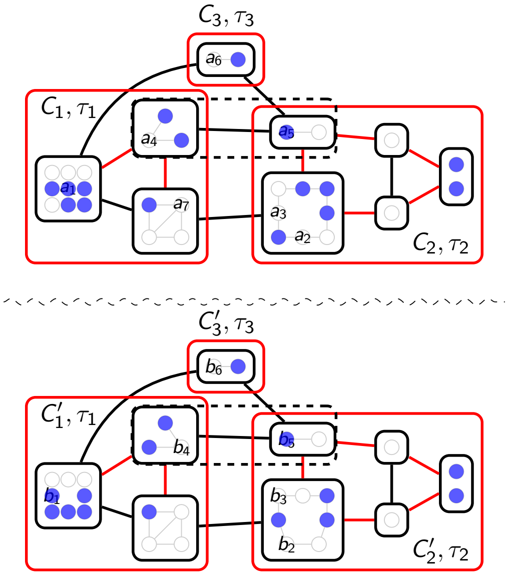

that fully defines the winning strategy of Duplicator

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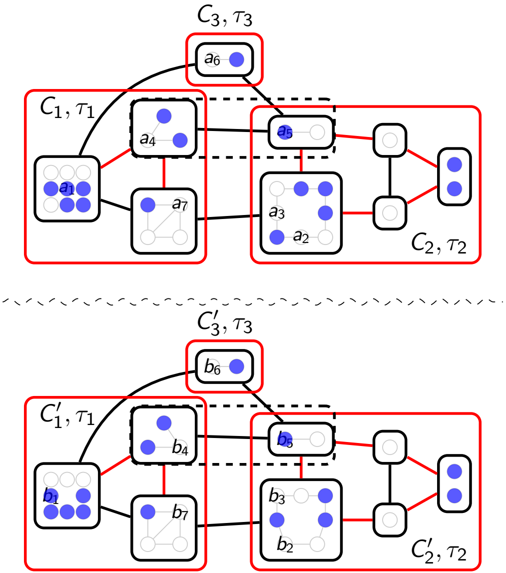

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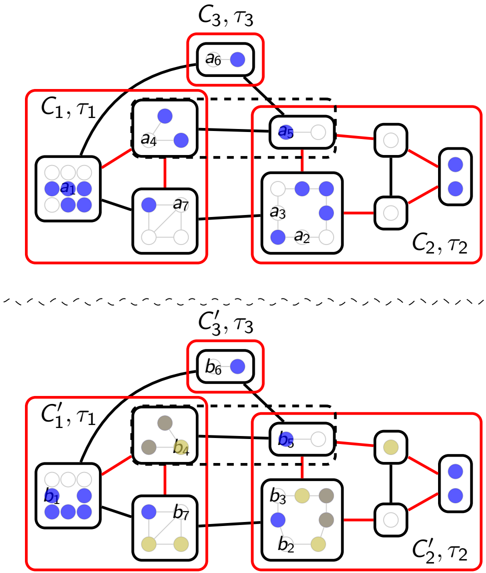

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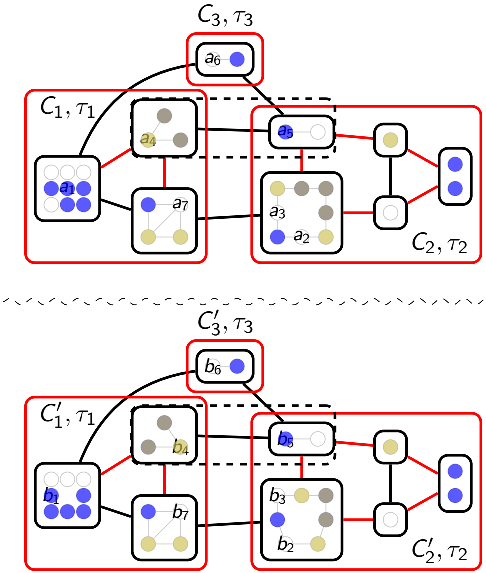

that fully defines the winning strategy of Duplicator

Turning it into a uniform algorithm

Reminder:

- #non-equivalent partitioned sentences of rank k: $f(d,k)$
- #rank-k partitioned types bounded by $g(d,k) = 2^{f(d,k)}$

For each newly observed type τ,

- keep a representative $(H,\mathcal{P})_\tau$ on at most $(d+1)^{g(d,k)}$ vertices
- determine the $0,1$-vector of satisfied sentences on $(H,\mathcal{P})_\tau$
- record the value of $F(\tau_1,\ldots,\tau_{d'},B,X,Y)$ for future uses

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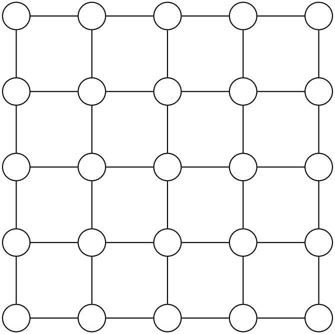
To decide $G \models \varphi$, look at position φ in the $0,1$-vector of $\mathrm{tp}_k^{\mathsf{MSO}}(G)$

Twin-width is more general than the classic widths



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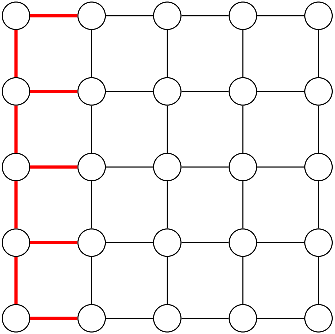

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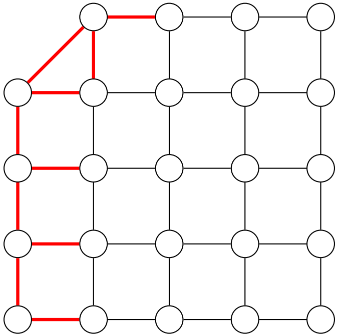

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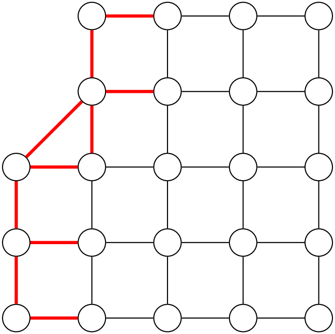

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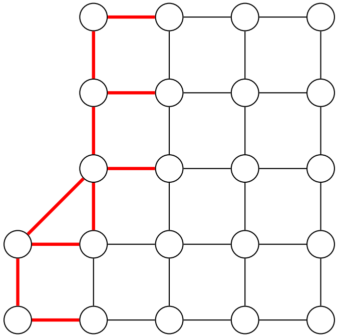

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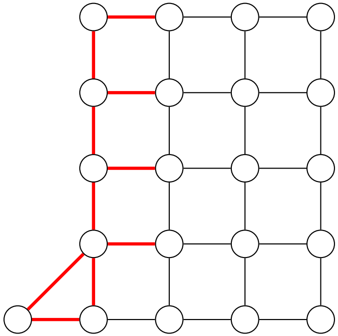

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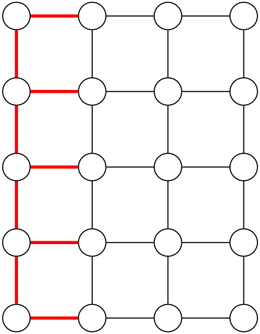
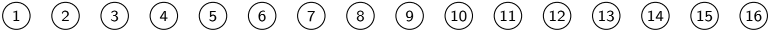
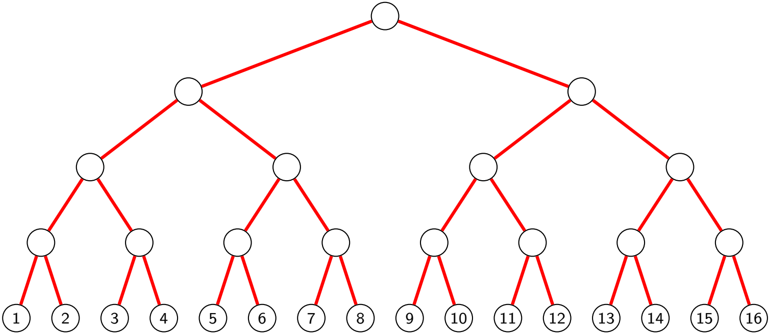


4-sequence for planar grids, but unbounded cliquewidth

($\geqslant 2\log n$)-subdivisions have twin-width at most 4

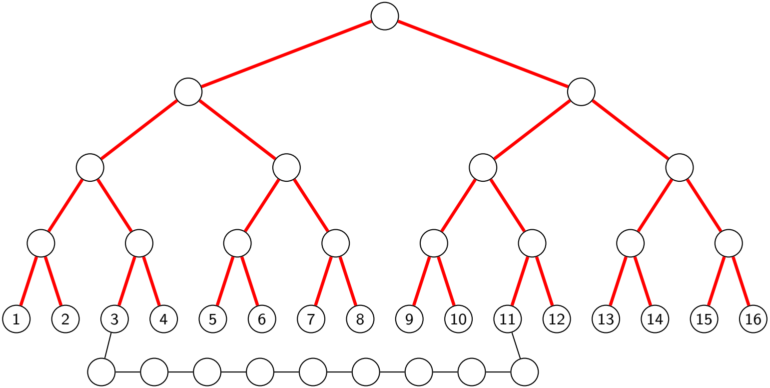


$(\geqslant 2 \log n)$-subdivisions have twin-width at most 4



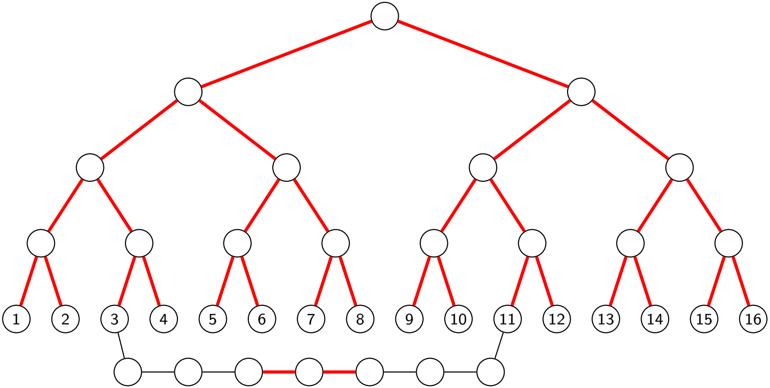
Add a red full binary tree whose leaves are the vertex set

($\geqslant 2\log n$)-subdivisions have twin-width at most 4



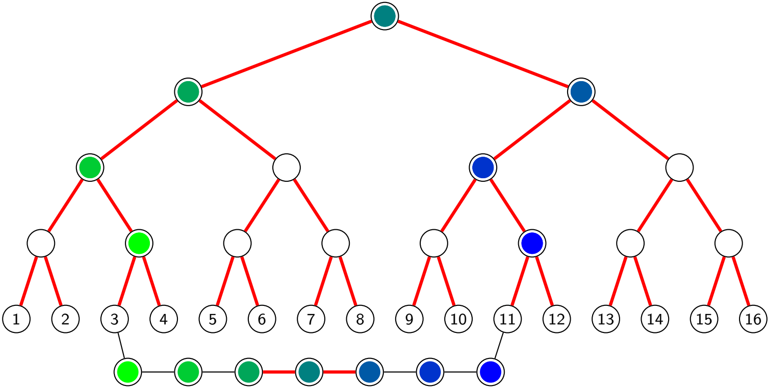
Take any subdivided edge

$(\geqslant 2 \log n)$-subdivisions have twin-width at most 4



Shorten it to the length of the path in the red tree

$(\geqslant 2 \log n)$-subdivisions have twin-width at most 4



Zip the subdivided edge in the tree

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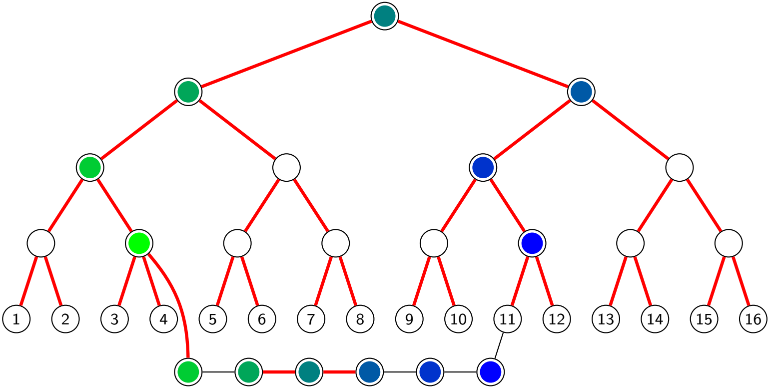
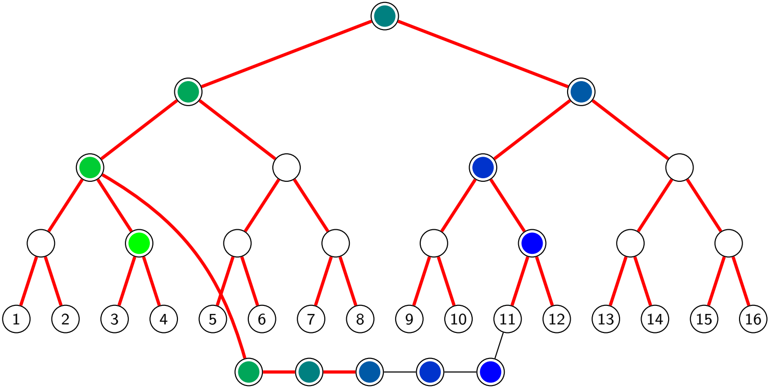


Zip the subdivided edge in the tree

($\geqslant 2 \log n$)-subdivisions have twin-width at most 4



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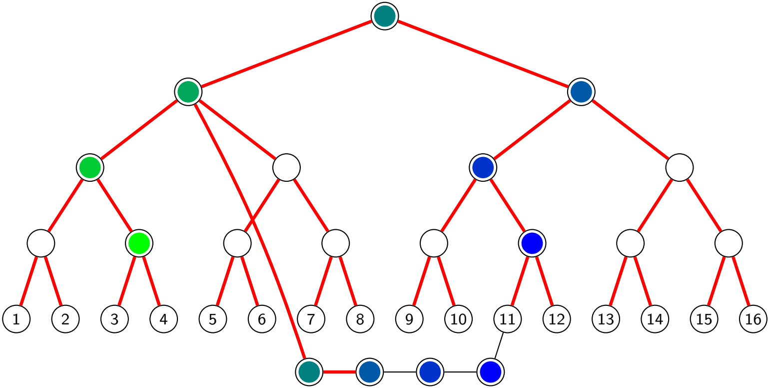

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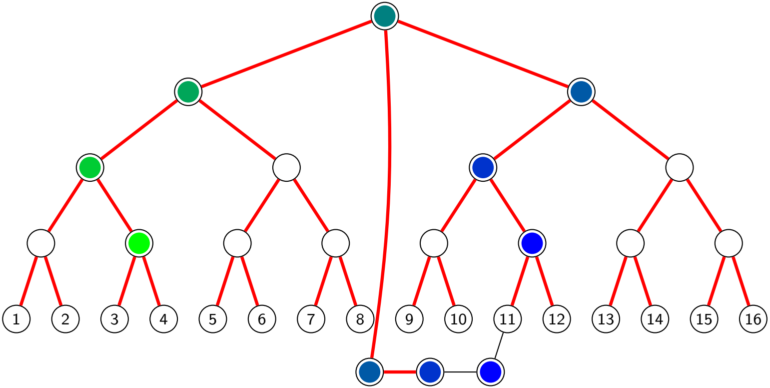

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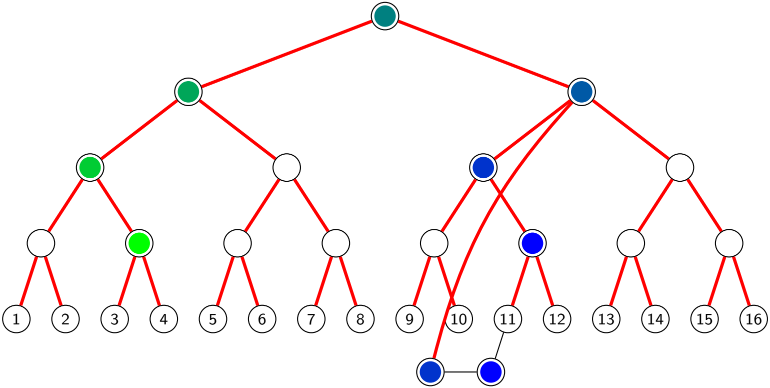

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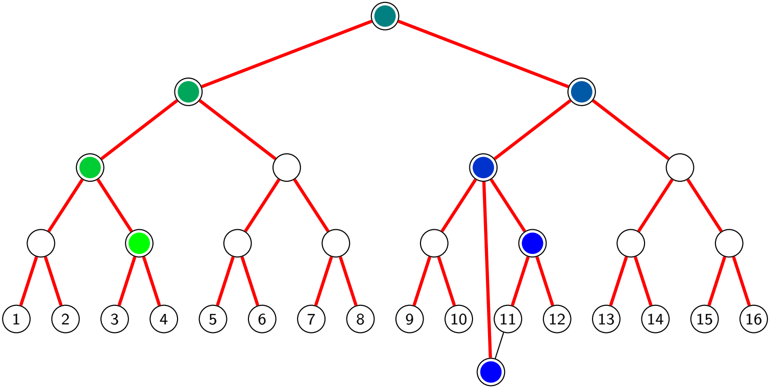

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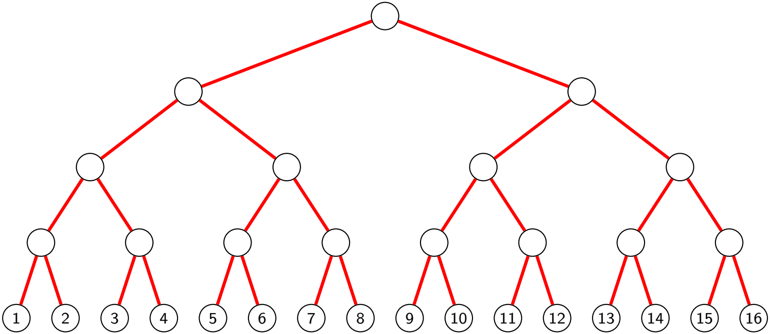

Move to the next subdivided edge also of unbounded cliquewidth

Theorem

The following classes have bounded twin-width, and $O(1)$-sequences can be computed in polynomial time.

- *Bounded rank-width, and even, boolean-width graphs,*
- *every hereditary proper subclass of permutation graphs,*
- *posets of bounded antichain size (seen as digraphs),*
- *unit interval graphs,*
- *K_t-minor free graphs,*
- *map graphs,*
- *subgraphs of d-dimensional grids,*
- *K_t-free unit d-dimensional ball graphs,*
- *$\Omega(\log n)$-subdivisions of all the n-vertex graphs,*
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Can we solve problems faster, given an $O(1)$-sequence?

k-Independent Set given a $d = O(1)$-sequence

d-sequence: $G = G_n, G_{n-1}, \ldots, G_2, G_1 = K_1$

Algorithm: **For every connected subset D of size at most k of the red graph of every G_i, store in $T[D, i]$ one largest independent set in $G\langle D \rangle$ intersecting every vertex of D.**

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Initialization: $T[\{v\}, n] = \{v\}$

End: $T[\{V(G)\}, 1] =$ IS of size at least k or largest IS in G

Running time: $d^{2k}n^2$ red connected subgraphs,
actually only $d^{2k}n = 2^{O_d(k)}n$ updates

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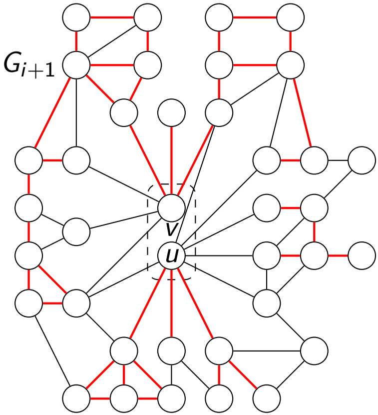
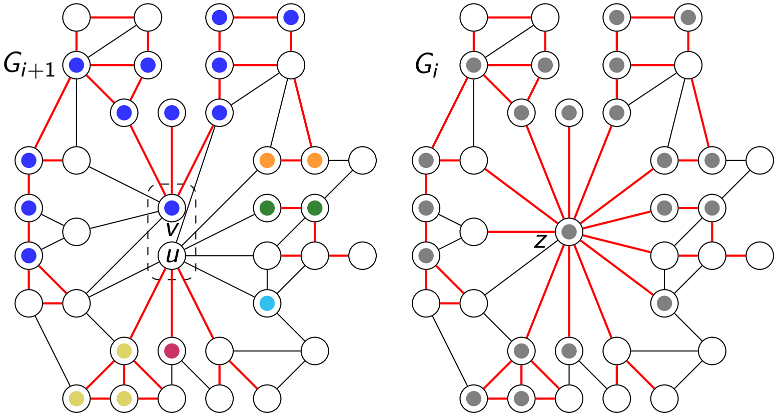
How to compute $T[D, i]$ from all the $T[D', i+1]$?

k-Independent Set: Update of partial solutions



Best partial solution inhabiting •?

k-Independent Set: Update of partial solutions



3 unions of $\leqslant d+2$ red connected subgraphs to consider in G_{i+1} with u, or v, or both

FO model checking on graphs of bounded twin-width

The previous algorithm generalizes to:

Theorem (B., Kim, Thomassé, Watrigant '20)

FO model checking can be solved in time $f(|\varphi|, d) \cdot |V(G)|$ on graphs G given with a d-sequence.

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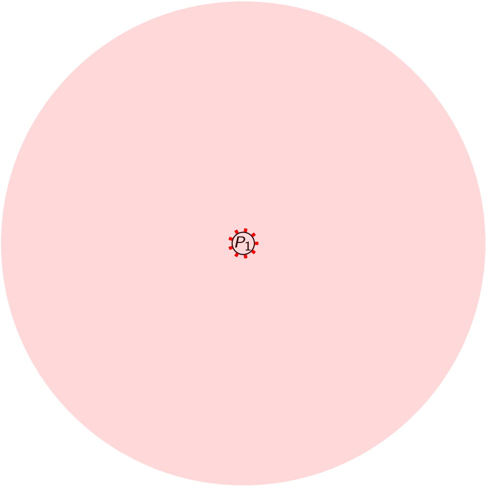
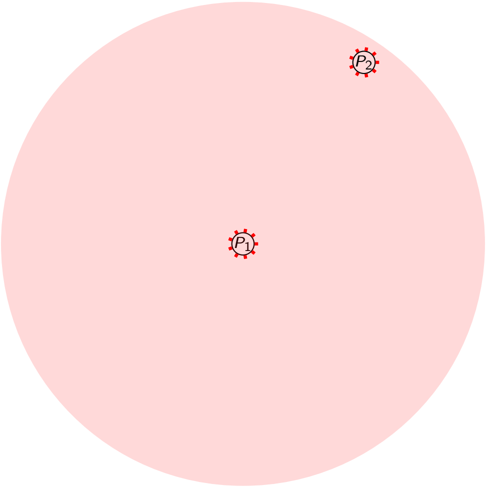
Thank you for your attention!

Local tuple of parts



$(P_1, P_2, \ldots, P_q)$ is *local around* P_1 if...

Local tuple of parts



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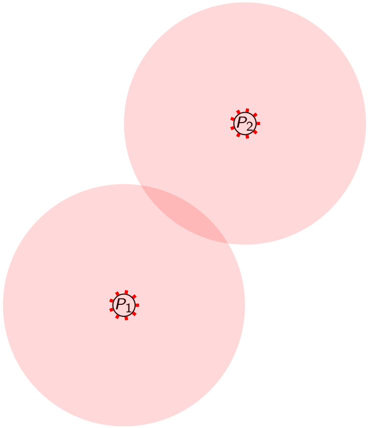

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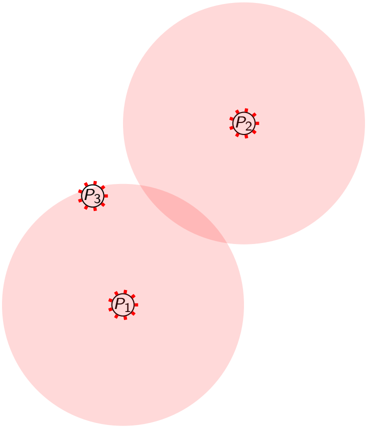

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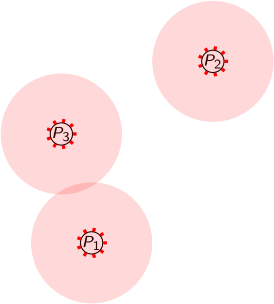

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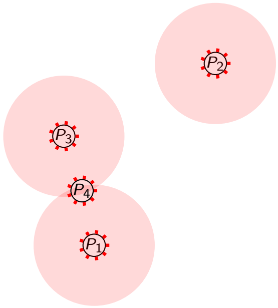

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P_4 is at distance at most 2^{k-4} from $\{P_1, P_2, P_3\}$ in $(G, \mathcal{P}_i)$

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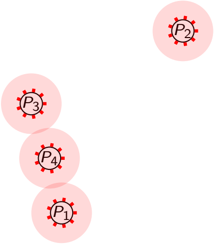

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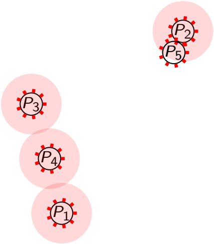

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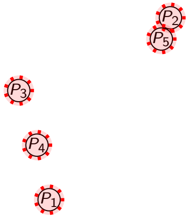

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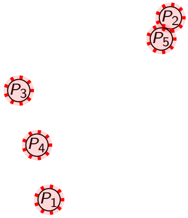

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Partitioned local sentences and types

A prenex sentence is *partitioned local around* X in $(G, \mathcal{P}_i)$ if of the form $Qx_1 \in X \; Qx_2 \in P_2 \; \ldots \; Qx_k \in P_k \; \psi(x_1, \ldots, x_k)$ with

- ψ is quantifier-free, and
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And the corresponding types:

$$\mathsf{ltp}_k^{\mathsf{FO}}(G, \mathcal{P}_i, X) = \{\varphi : \mathsf{qr}(\varphi) \leqslant k,$$

$$\varphi \text{ is partitioned local around } X \text{ in } (G, \mathcal{P}_i),$$

$$(G, \mathcal{P}_i) \models \varphi\}.$$

Partitioned local sentences/types in $(G, \mathcal{P}_n)$ and $(G, \mathcal{P}_1)$

Initialization of the dynamic programming

In $(G, \mathcal{P}_n = \{\{v\} : v \in V(G)\})$: for every $v \in V(G)$,
$Qx_1 \in \{v\}\ Qx_2 \in \{v\}\ \ldots\ Qx_k \in \{v\}\ \psi \equiv \psi(v, v, \ldots, v)$

Partitioned local types are easy to compute in $(G, \mathcal{P}_n)$

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Partitioned local types are easy to compute in $(G, \mathcal{P}_n)$

Output of the dynamic programming

In $(G, \mathcal{P}_1 = \{V(G)\})$:
$Qx_1 \in V(G)\ Qx_2 \in V(G)\ \ldots\ Qx_k \in V(G)\ \psi \equiv$ classic sentences

The partitioned local type in $(G, \mathcal{P}_1)$ coincides with the type of G

Partitioned local types give the partitioned types

Isom. $f : \mathcal{P}_i \to \mathcal{P}'_i$ with $\mathsf{ltp}_k^{\mathsf{FO}}(G, \mathcal{P}_i, X) = \mathsf{ltp}_k^{\mathsf{FO}}(G', \mathcal{P}'_i, f(X))$

$(G, \mathcal{P}_i)$

$(G', \mathcal{P}'_i)$

Local strategies win the global game

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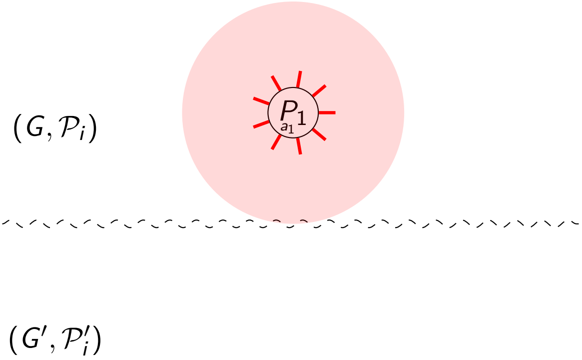

Say, Spoiler plays in P_1

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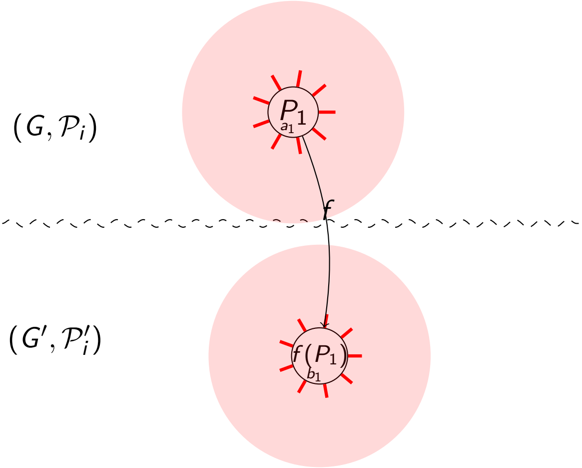

Duplicator answers in $f(P_1)$ following the local game around P_1

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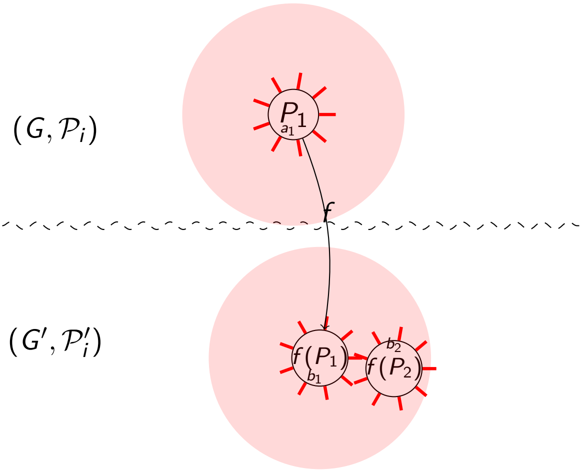

Now when Spoiler plays close to P_1 or $f(P_1)$

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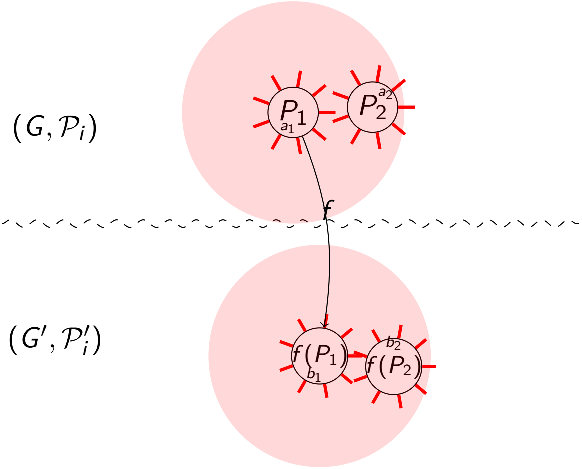

Duplicator follows the winning local strategy

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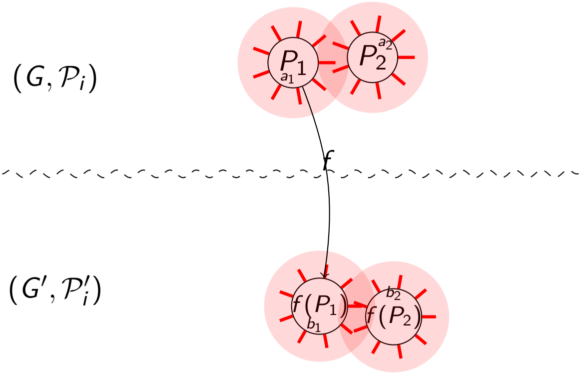

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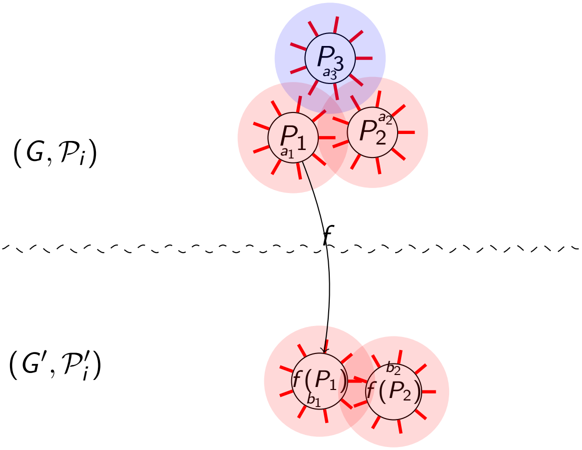

If Spoiler plays too far

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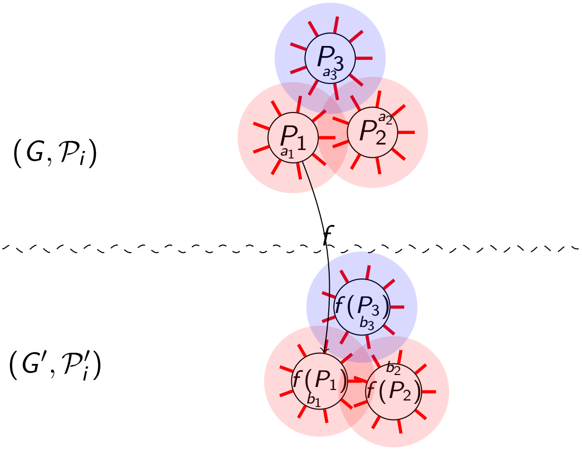

Duplicator starts a new local game around that new part

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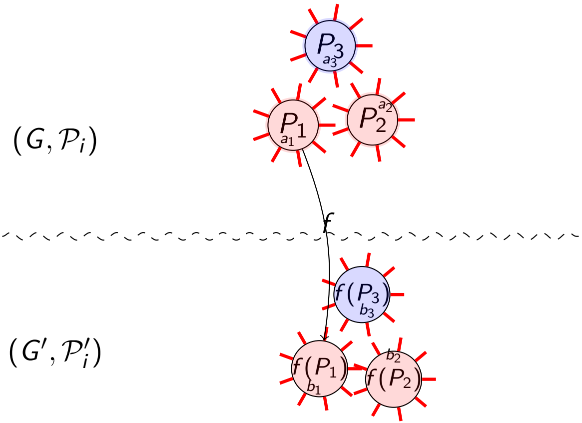

Duplicator starts a new local game around that new part

Concluding as in the MSO model checking algorithm

$(G, \mathcal{P}_{i+1}) \rightsquigarrow (G, \mathcal{P}_i)$: X and Y are merged in Z

Partitioned local types around P

- only needs an update if P is at distance at most 2^{k-1} from Z

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Each contraction: $O_{d,k}(1) = O(d^{2^k})$ updates in $O_{d,k}(1) = f(d, k)$
Total time: $O_{d,k}(n)$