Maximum Independent Set in H-Free Graphs

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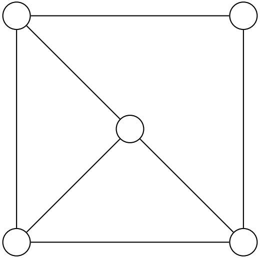
Séminaire GALaC, LRI, Paris-Sud, October 5th, 2018

Independent Set

Problem: Given a graph



and an integer k: Is there an independent set of size at least k?

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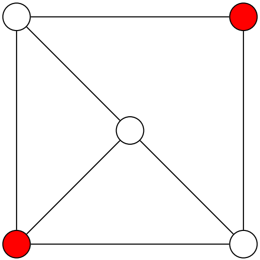

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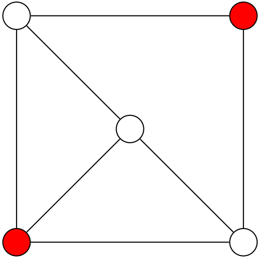

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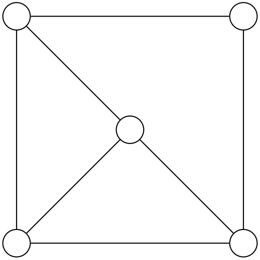
NP-complete even in graphs with maximum degree 3.

What about on graphs excluding an induced subgraph H? (called H-free graphs)

NP-hardness cases [Alekseev '82]



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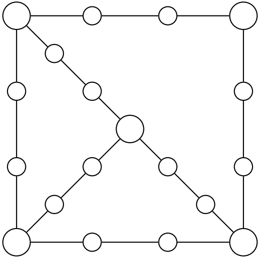

Subdivide every edge twice

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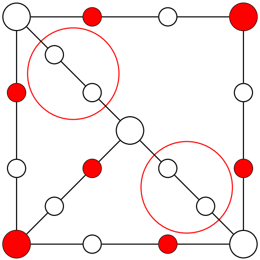

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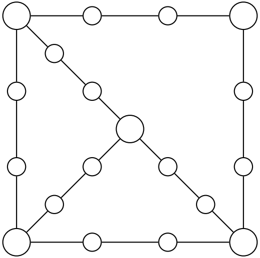

Subdivide every edge any fixed even number of times

This reduction + NP-hardness on graphs of degree 3 ⇒

NP-hardness for graphs of degree 3, with arbitrarily large girth and distance between two vertices with degree 3 (*branching vertices*).

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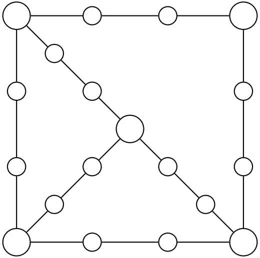

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The constructed graph is H-free except if H is...

P/NP-complete status of MIS on H-free graphs

For H connected:

- NP-complete, if H is not a path or a subdivided claw ($K_{1,3}$)
- in P, if H is a path on up to 6 vertices
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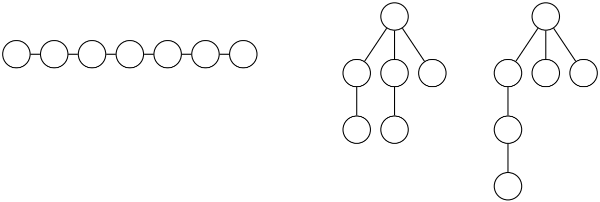
Minimal open cases:



Other dichotomies

The polynomial algorithms for P_5-free and then P_6-free graphs use tools that cannot generalize to P_8-free graphs and beyond.

Understanding P_t-free graphs is a challenge

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Understanding P_t-free graphs is a challenge

there are other goodies/baddies partition:

- PTAS/APX-hard
- SUBEXP/ETH-hard
- **FPT/W[1]-hard**

Parameterized complexity

Fixed-Parameter Tractable (FPT) algorithm:
in time $f(k)n^{O(1)}$ with

- n, the size of the instance,
- k, a parameter such as the solution size, and
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Example:

- Vertex Cover has a simple $2^k n^{O(1)}$-algorithm
- Independent Set is W[1]-hard (hence unlikely FPT)

Convenient definition of W[1]-hard for our purpose:
As hard as Independent Set for FPT reductions

FPT reductions

Reduction from (Π, k) to (Π', k') taking FPT time and such that $\mathbf{k' \leqslant g(k)}$ **for a computable function** $\mathbf{g}$.

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- $(G, k) \mapsto (G, n - k)$, from MIS to VERTEX COVER?

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Why MIS and why forbidden induced subgraphs?

- some hard problems like Dominating Set are almost indifferent to forbidding induced subgraphs
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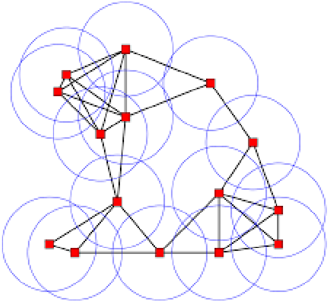

Known results

- FPT for H on at most 4 vertices but C_4 [Dabrowski et al. '12]
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Every K_r-free graphs has either:

- at most Ramsey(k,r) $\approx k^{r-1}$ vertices $\to$ brute-force is FPT
- an independent set of size k $\to$ answer YES

[1] This is why the question is not interesting for subgraphs and minors

Our current goal

Let's try to remove the C_4s with an FPT reduction

First thoughts

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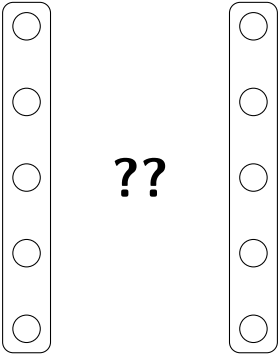
What should we avoid between the cliques?



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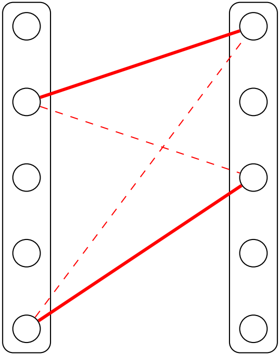

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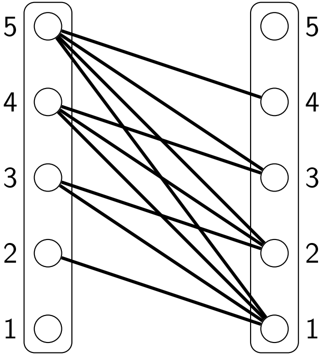
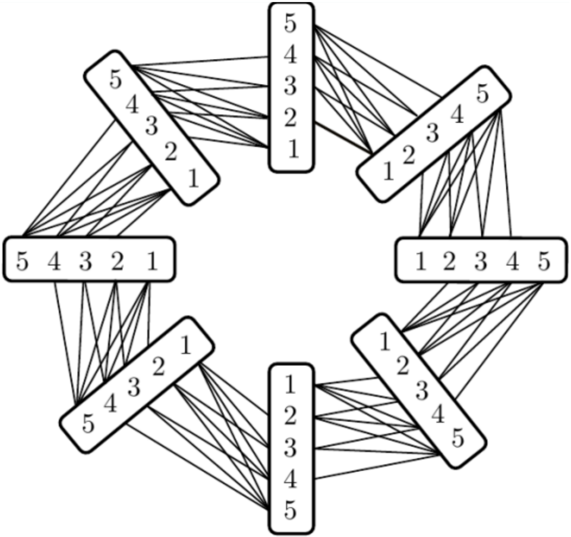


Half-graphs

"Cycle" of half graphs



Two inequalities enforce the equality

Grid Tiling

Input: $k \times k$ grid of cells containing pairs over $[n]^2$

(1,1) (3,1) (2,4)	(5,1) (1,4) (5,3)	(1,1) (2,4) (3,3)
(2,2) (1,4)	(3,1) (1,2)	(2,2) (2,3)
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Example with $k = 3$ cliques/color classes and $n = 5$

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⇒

(1,1) (3,1) **(2,4)**	(5,1) **(1,4)** (5,3)	(1,1) **(2,4)** (3,3)
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Output: select one pair per cell so that

- columns agree on the first coordinate
- rows agree on the second coordinate

Grid Tiling w.r.t the number of cells is W[1]-hard

			$(v_j, \cdot)$	
$(\cdot, v_i)$	(v_i, v_i)	$(\cdot, v_i)$	(v_j, v_i)	$(\cdot, v_i)$
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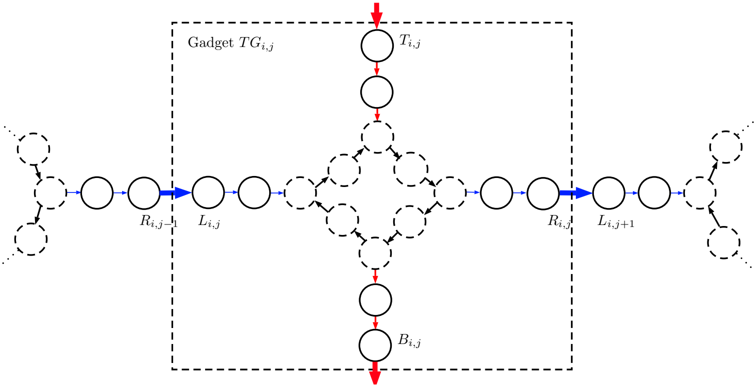
The same with inequalities has the same lower bound

Useful for geometric problems such as Packing Unit Disks

Avoiding C_4 with half graphs everywhere



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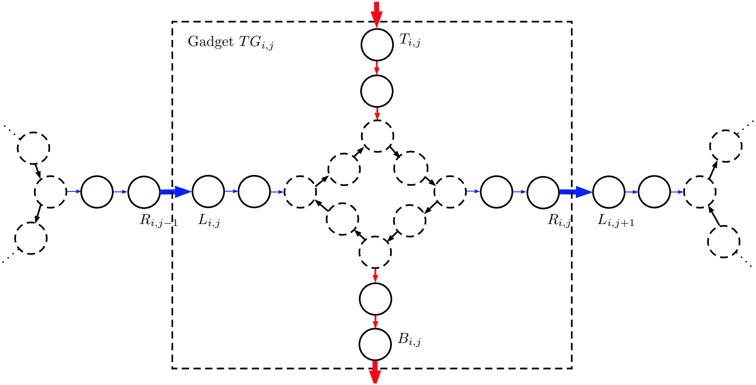
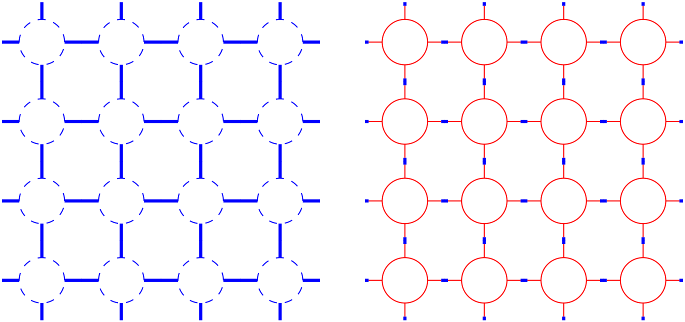


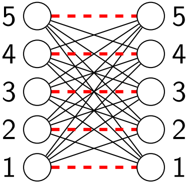
Simultaneously avoiding:

- $C_4, C_5, \ldots, C_s$
- no $K_{1,4}$
- no tree with two branching vertices at distance $\leqslant s/4$

Two variants of the reduction



Variants with half-graphs in blue and antimatchings in red

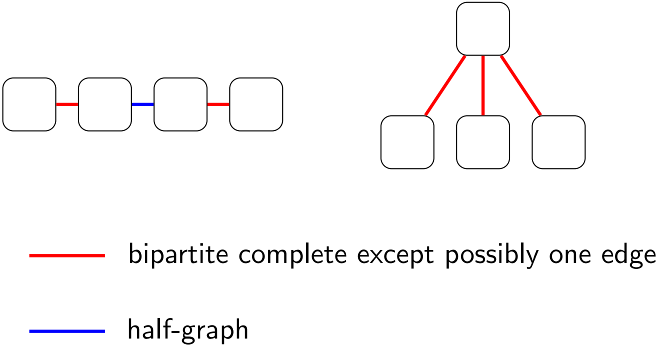


antimatching

FPT candidates

H should be chordal and

- either *a path of cliques* with *simple connections between adjacent cliques*
- or *a subdivided claw of cliques* with *very simple connections between adjacent cliques*



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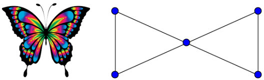
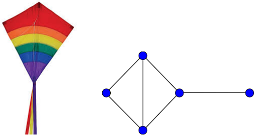
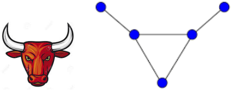
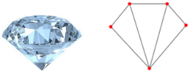
Maybe not so surprising:
notably open for P_t-free graphs
entire papers have been dedicated to -free graphs

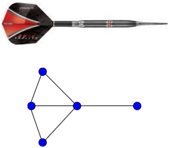
Some candidates on 5 vertices











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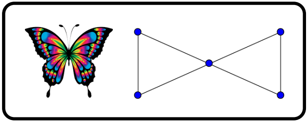

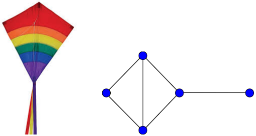

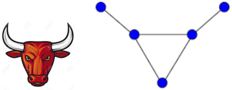

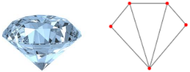

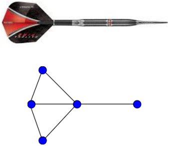

Two tricks to catch the butterfly

Trick 1: we can guess the solution on any subset of $f(k)$ vertices

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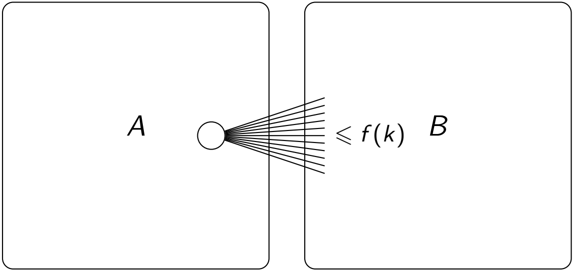

with

- A and B intersecting the solution
- all the vertices in A have at most $f(k)$ neighbors in B

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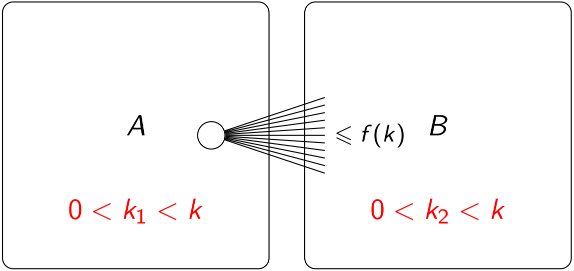

We guess how many vertices a solution contains in A and B

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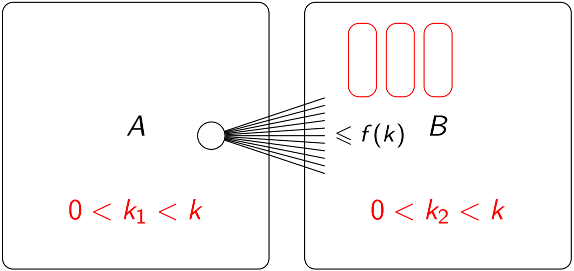

We extract independent sets of size k_2 in $G[B]$

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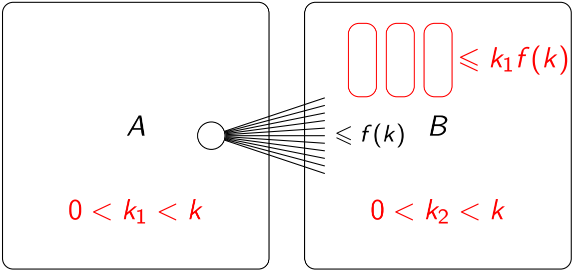

If this stops before $k_1 f(k) + 1$ are extracted, use Trick 1

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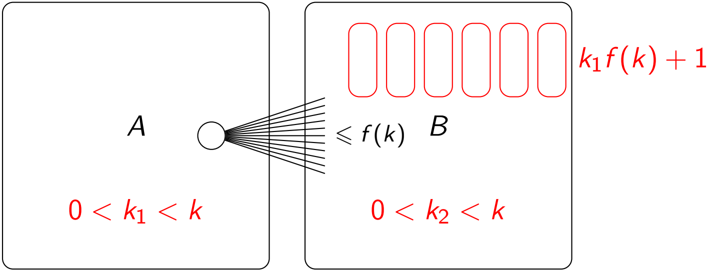

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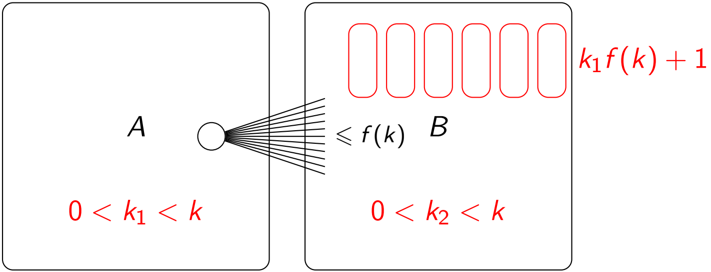

What does this achieve?

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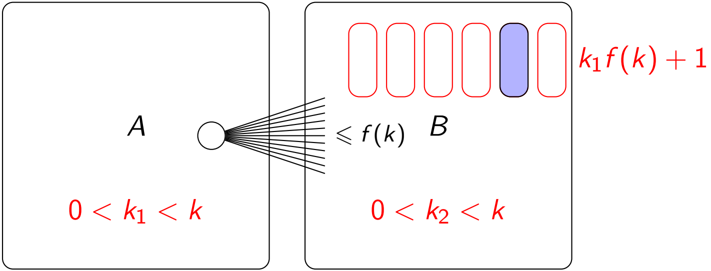
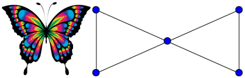
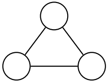


Any independent set of size k_1 in $G[A]$ can be completed

FPT algorithm in butterfly-free graph





Let us consider a triangle and its neighbors

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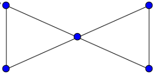
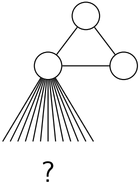
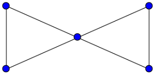


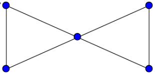


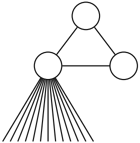
Can there be very many vertices attached to a single vertex?

FPT algorithm in butterfly-free graph



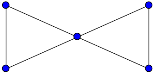
Less than k. Otherwise: easy solution or butterfly

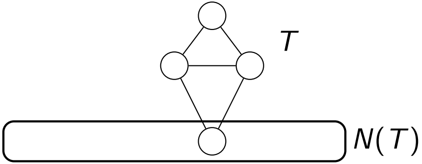
FPT algorithm in butterfly-free graph  



We use Trick 1 to get rid of those particular neighbors

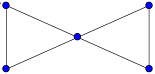
FPT algorithm in butterfly-free graph

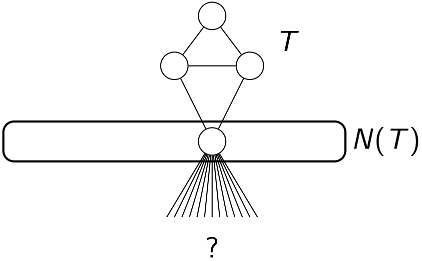




Now, all the vertices in $N(T)$ have at least two neighbors in T

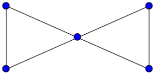
FPT algorithm in butterfly-free graph

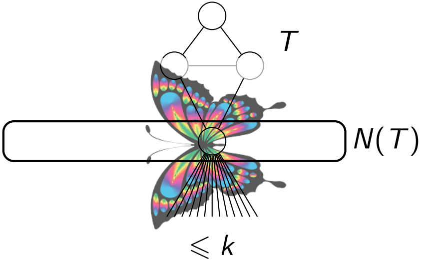




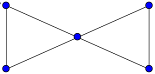
Can they have many neighbors in the rest of the graph?

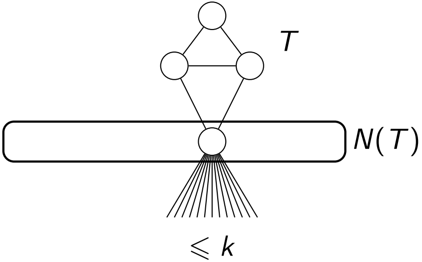
FPT algorithm in butterfly-free graph





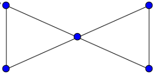
No, less than k; otherwise easy solution or butterfly

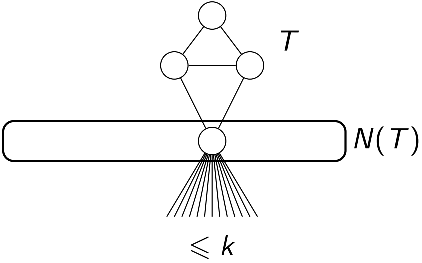
FPT algorithm in butterfly-free graph  



A solution intersects $T \cap N(T)$ (why?)

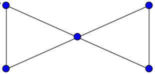
FPT algorithm in butterfly-free graph

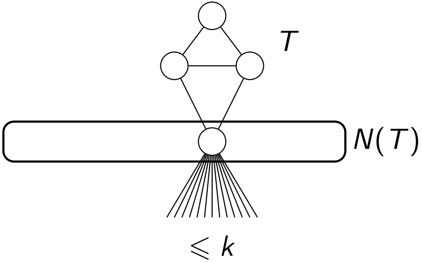




Either it also intersects $\overline{T \cap N(T)}$, and we conclude with Trick 2

FPT algorithm in butterfly-free graph





Or not. And we solve $G[T \cap N(T)]$ since it is $4K_2$-free (Alekseev)

Results and perspectives

FPT algorithms when

- H is a clique minus a bipartite complete graph
  (can be seen as a P_3 of cliques, generalizes the butterfly)
- H is the union of cliques (parameterized version of Alekseev)
- H is a clique minus a triangle ($K_r \setminus K_4$ contrains a $K_{1,4}$)
- H, candidate on 5 vertices: crown, gem, kite, $\overline{P}$, dart, cricket

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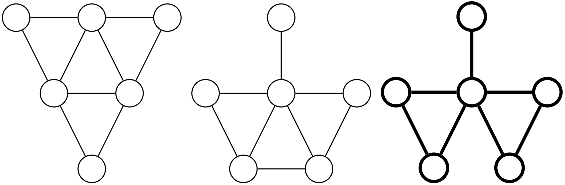
W[1]-hardness cases with a third reduction:



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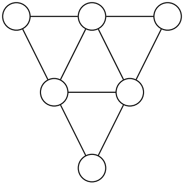
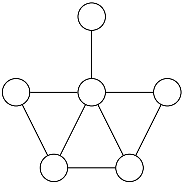
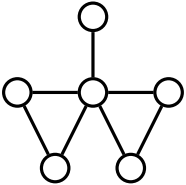

Mainly left with "path of cliques"