When Maximum Stable Set can be solved in FPT time

Édouard Bonnet, Nicolas Bousquet, Stéphan Thomassé, and Rémi Watrigant

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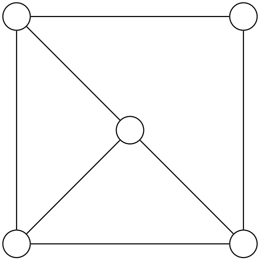
Independent Set

Problem: Given a graph



and an integer k: Is there an independent set of size at least k?

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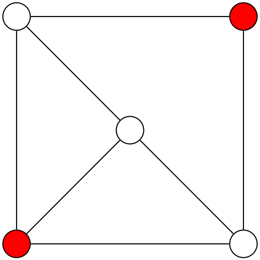

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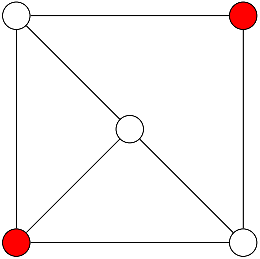

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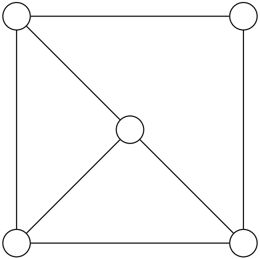
NP-complete even in subcubic graphs

What about on graphs excluding an induced subgraph H? (called H-free graphs)

NP-hard cases [Alekseev '82, Poljak '73]



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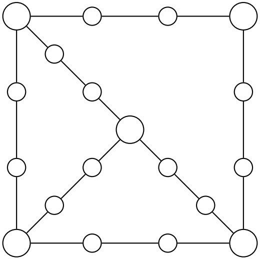

Subdivide every edge twice

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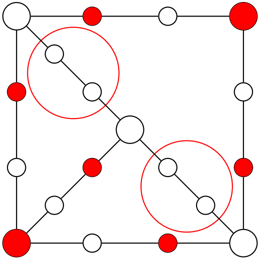

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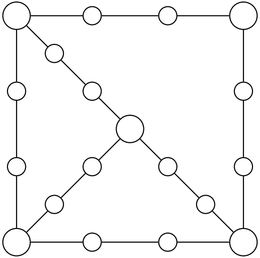

Subdivide every edge any even number of times

This reduction + NP-hardness on subcubic graphs ⇒

NP-hardness for subcubic graphs, with arbitrarily large

- girth, and
- distance between two vertices with degree at least 3.

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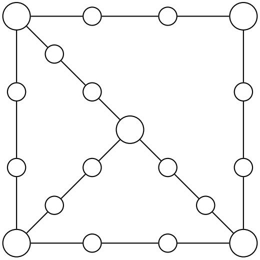

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The constructed graph is H-free except if H is...

P/NP-hard dichotomy

For H connected:

- NP-hard, if H is not a path or a subdivided claw ($K_{1,3}$)
- in P, if H is a path on up to 6 vertices
- in P, if H is a claw with one edge subdivided once
- For other H, the problem is open

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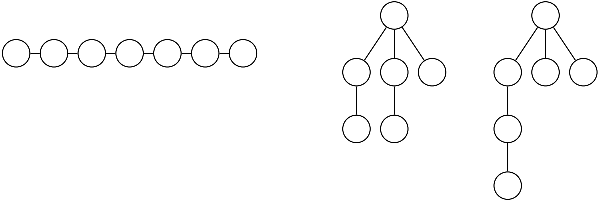
Minimal open cases:



Other dichotomies

There are other goodies/baddies partition:

- PTAS/APX-hard
- SUBEXP/ETH-hard
- **FPT/W[1]-hard**

Parameterized complexity

Fixed-Parameter Tractable (FPT) algorithm:
in time $f(k)n^{O(1)}$ with

- n, the size of the instance,
- k, a parameter such as the solution size, and
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Example:

- Vertex Cover has a simple $2^k n^{O(1)}$-algorithm
- Independent Set is W[1]-hard (hence unlikely FPT)

Convenient definition of W[1]-hard for our purpose:
As hard as Independent Set for FPT reductions

Ultimate goal: Dichotomy classification

For every H,

- if *easy(H)* then Independent Set is FPT on H-free graphs,
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For the P/NP-hard dichotomy, we have at least a natural candidate for the criterion *easy(H)*...

Known results before 2018

Why is Independent Set FPT in K_r-free graphs?[1]

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- FPT for H on at most 4 vertices but C_4 [Dabrowski et al. '12]
- FPT for bull-free graphs [Thomassé et al. '14]
- W[1]-hard in $K_{1,4}$-free graphs [Hermelin et al. '14]

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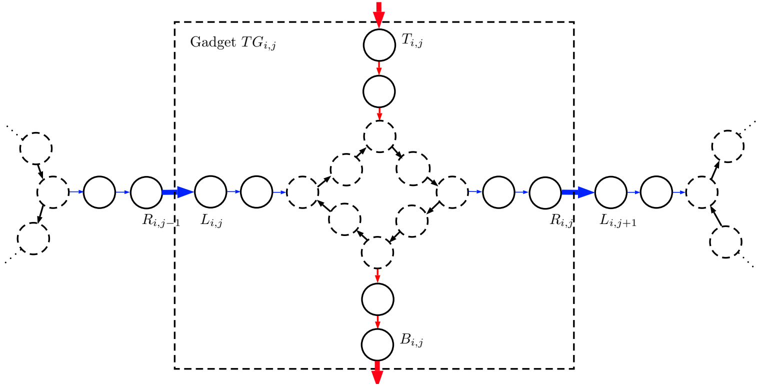
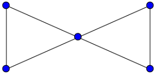
BBCTW '18: W[1]-hardness reduction

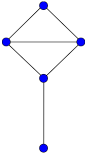


Simultaneously avoiding as induced subgraph:

- $C_4, C_5, \ldots, C_s$
- $K_{1,4}$
- any tree with two degree-3+ vertices at distance at most s

Candidates on 5 vertices

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 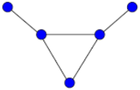

 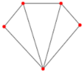

 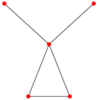

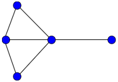

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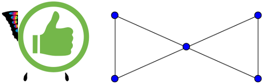

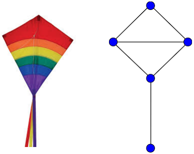

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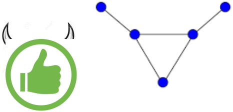
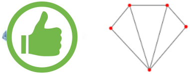
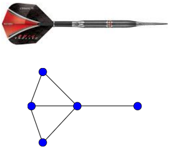
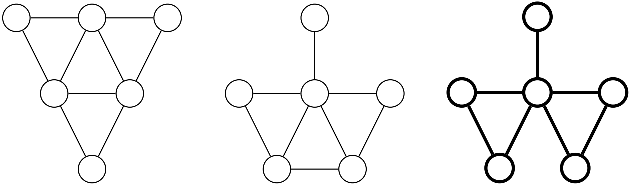






Other W[1]-hard cases due to a variant of the reduction



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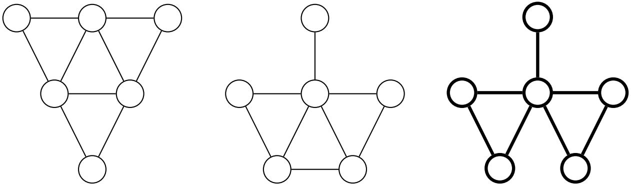

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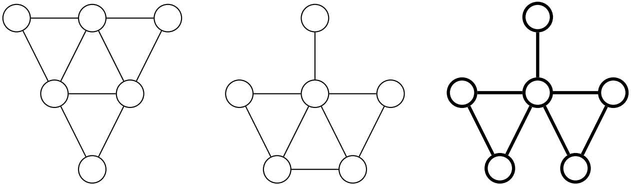

Mainly left with "path of cliques"

$P(a_1, a_2, \ldots, a_s) =$ graph obtained from P_s by replacing the i-th vertex by a clique of size a_i.

Ambitious conjecture

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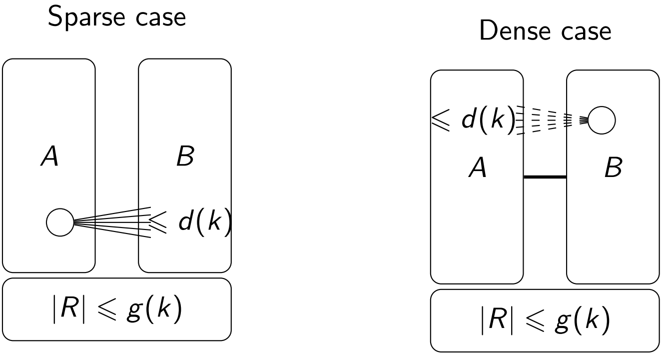
Theorem

INDEPENDENT SET *admits an FPT algorithm in* $P(1, t, t, t)$*-free.*

Main ingredient: introducing *co-graphs with parameterized noise*, and associated FPT subroutines

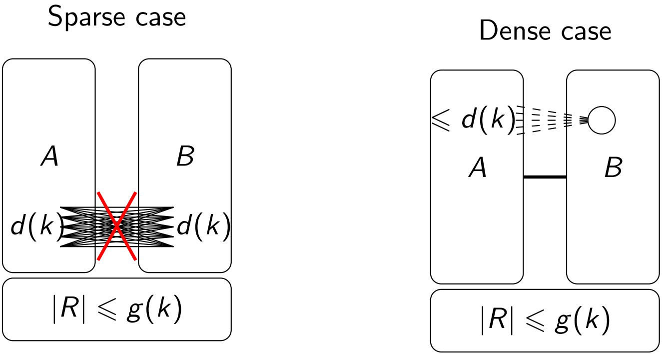
Co-graphs with parameterized noise



Tripartition (A, B, R) of the graph, where R is small, and:

- Sparse case: the degree to the other side is small
- Dense case: the co-degree to the other side is small

Co-graphs with parameterized noise



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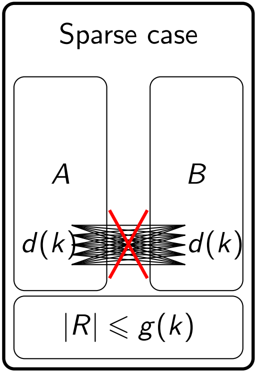

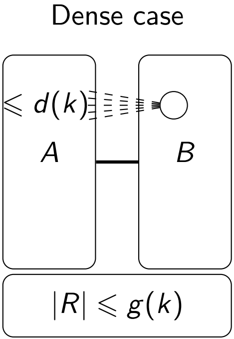

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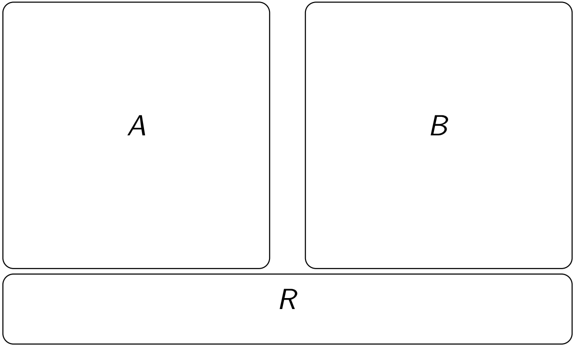
Trick 1: we can guess the solution on any subset of $f(k)$ vertices



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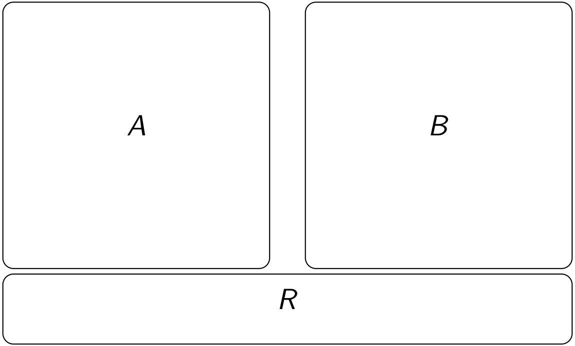

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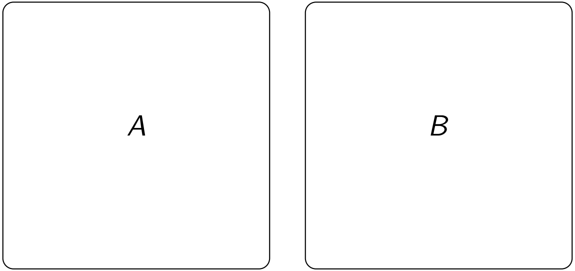

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A	B
$0 < k_1 < k$	$0 < k_2 < k$

We guess how many vertices a solution contains in A and B

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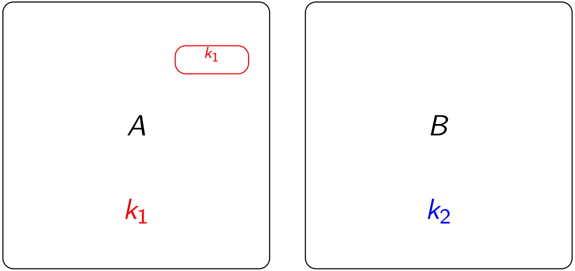

We extract independent sets of size k_1 in $G[A]$

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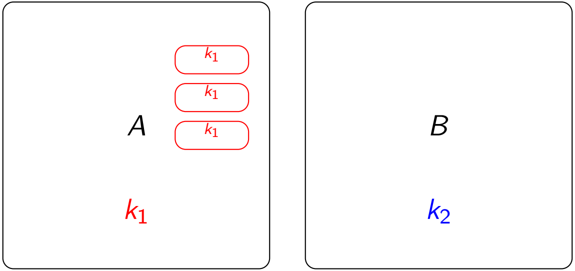

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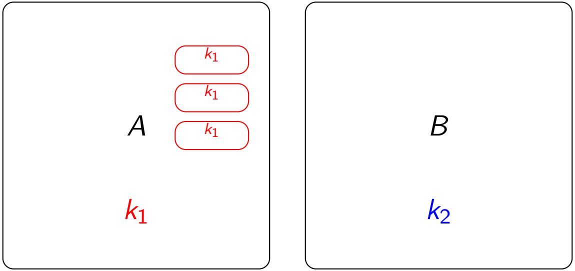

If this process stops quickly, use Trick 1

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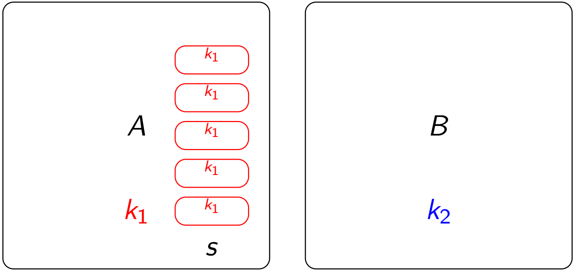

If it goes on, we stop after $s \gg k, d$ steps

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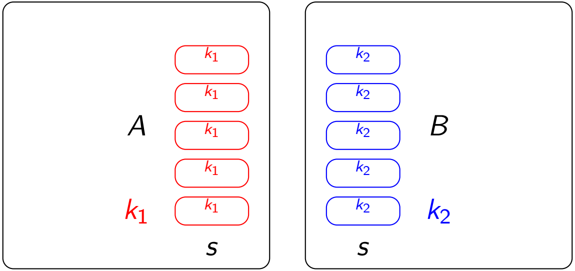

We do the same with independent sets of size k_2 in $G[B]$

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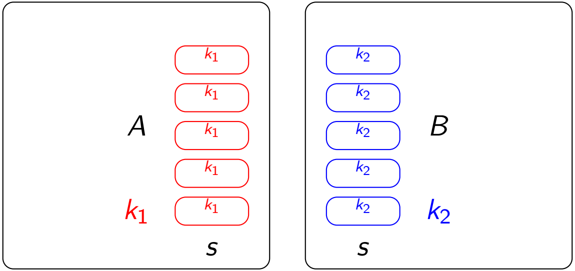

Solution! Except if there is at least one edge between each pair

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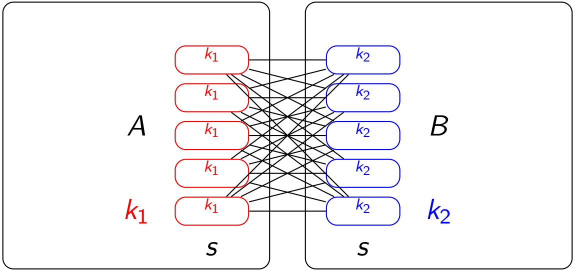

That would be s^2 edges on sk vertices

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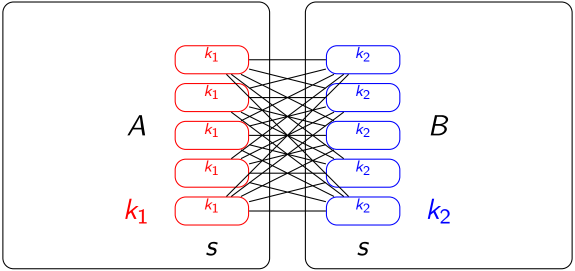

By Kővari-Sós-Turán: less than $d(sk)^{2-1/d} < s^2$ edges

General roadmap for $P(1, t, t, t)$-free graphs

- Build $\mathcal{C}$: a *maximal* collection of independent cliques
- Partition the graph in classes with the same neighborhood in $\mathcal{C}$
- Show: large classes are attached to the cliques *laminarly*

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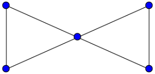
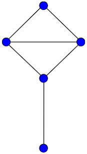
This, the ubiquity of cliques, the $P(1, t, t, t)$-freeness imply

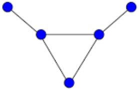
- a sparse tripartition: conclude with previous slide, or
- a dense tripartition: another lemma

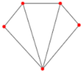
Remaining candidates on 5 vertices

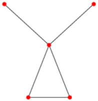



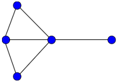


$\overline{P}$





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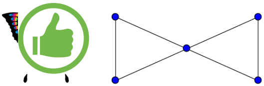
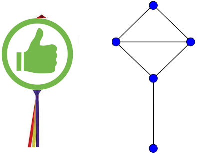
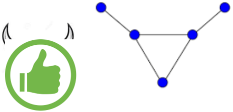
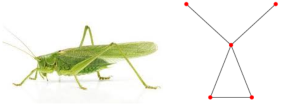
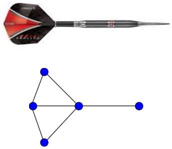


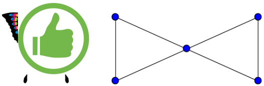


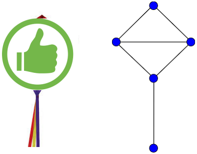


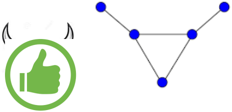


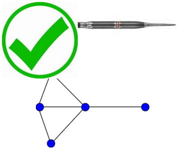


Remaining candidates on 5 vertices









Open questions

- FPT algorithm for $P(t, t, t, t)$-free graphs.
- "easy" FPT algorithm for P_5-free graphs.
- FPT algorithm for P_7-free graphs.
- derandomized algorithms for the cricket and the dart.

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Thank you for your attention!