Using greediness for parameterization

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1. Definitions
   - Local cardinality constraint graph problems
   - Max and min $(k, n-k)$-cut
2. Protective branching
   - An $O^*((\Delta+1)^k)$ algorithm for degrading contribution problems
   - An interesting consequence for max (k,n-k)-cut
3. Non degrading contribution
   - An $O^*((\Delta k)^{2k})$ algorithm
   - Other results

- Local cardinality constraint (lcc) graph problem: find a set V' of k vertices to optimize $f(\delta(V'), |N(V')|, |E(V')|)$ where f is a linear function.

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- W[1]-hard w.r.t k: $O(g(k)n^c)$ algorithms (FPT) are very unlikely.
- What about $O(g(k, \Delta)n^c)$ algorithms?

Yes

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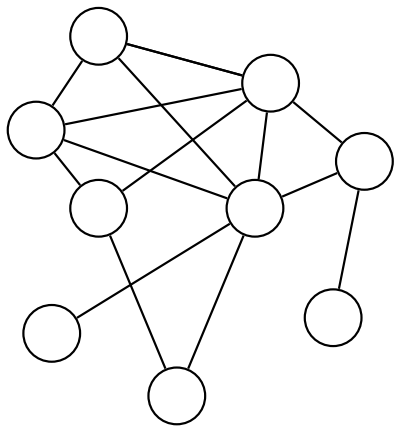

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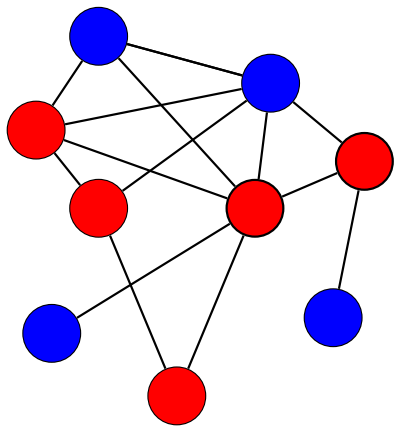

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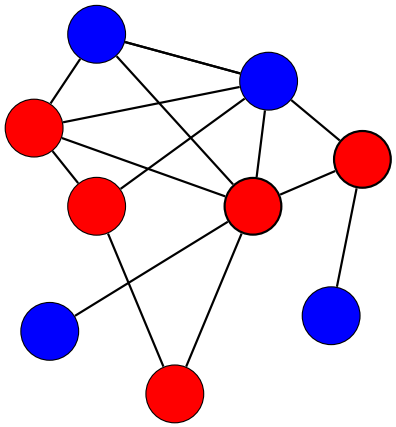

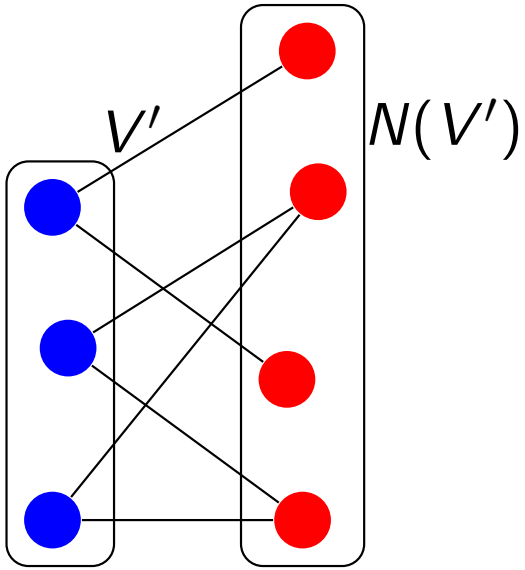

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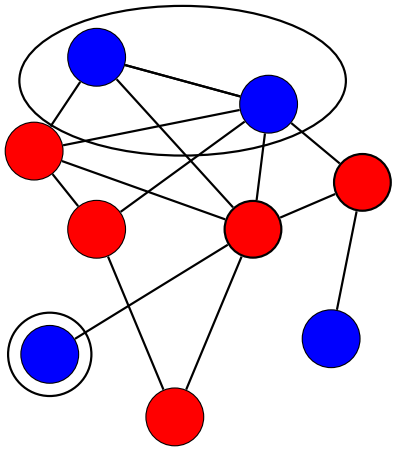

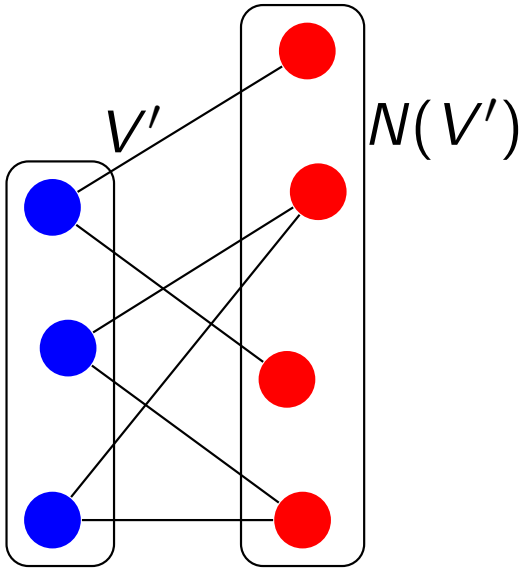

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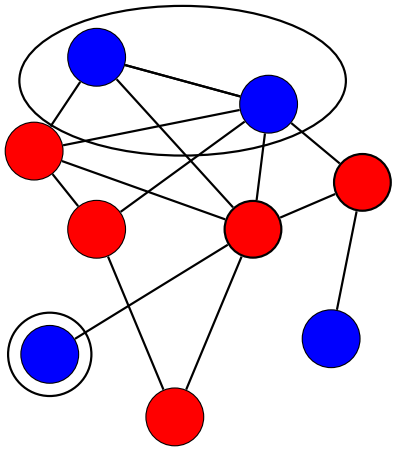

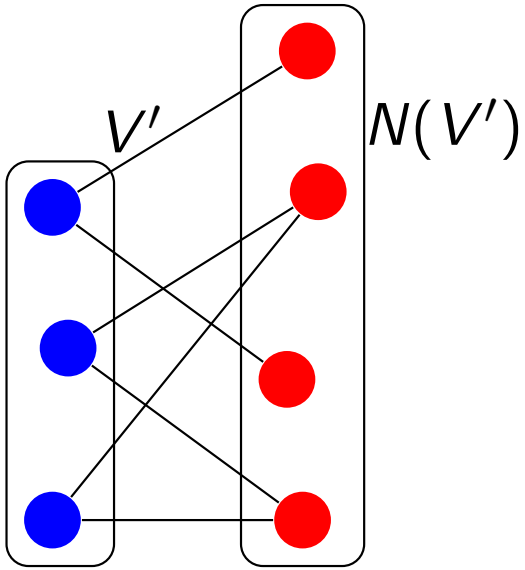

- $g(k, \Delta) \approx 2^{(\Delta+1)k}$

Max $(k, n-k)$-cut

Input: a graph $G = (V, E)$ and two integers p, k
Output: Is there $V' \subseteq V$ such that

- $|V'| = k$
- $val(V') = \delta(V') = |E(V', V \setminus V')| \geqslant p$

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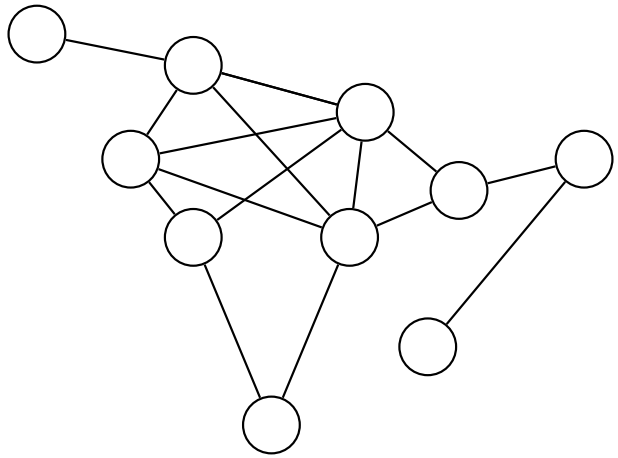
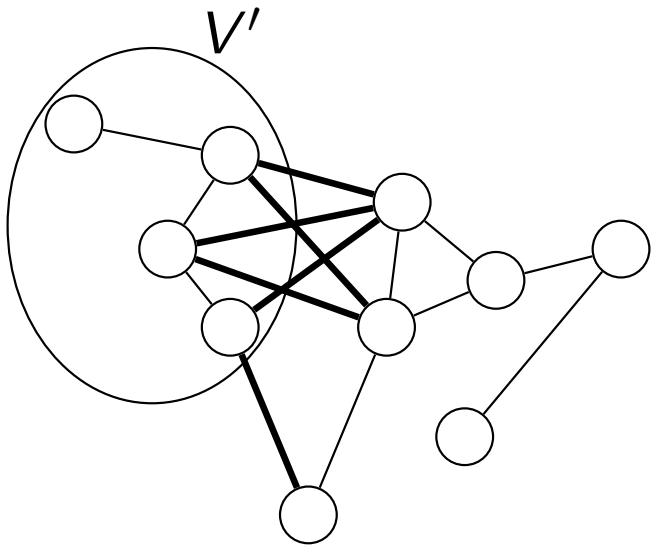
Example



k=4, p=5


V'

Theorem

Max $(k, n-k)$-cut can be solved in $O^*((\Delta+1)^k)$.

Ingredients

- A marking and branching algorithm with a tree of size $f(\Delta, k)$.

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- Degrading contribution: $T \subseteq T' \Rightarrow c_T(v) \leqslant c_{T'}(v)$.

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- $mkc(G, U \setminus \{v\}, T \cup \{v\}, k-1, p)$,
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else output $(T, val(T) \geqslant p)$

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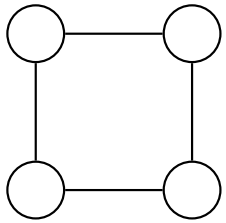

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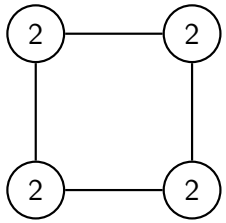

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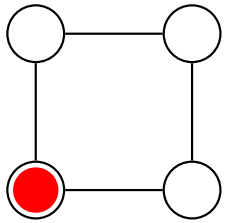

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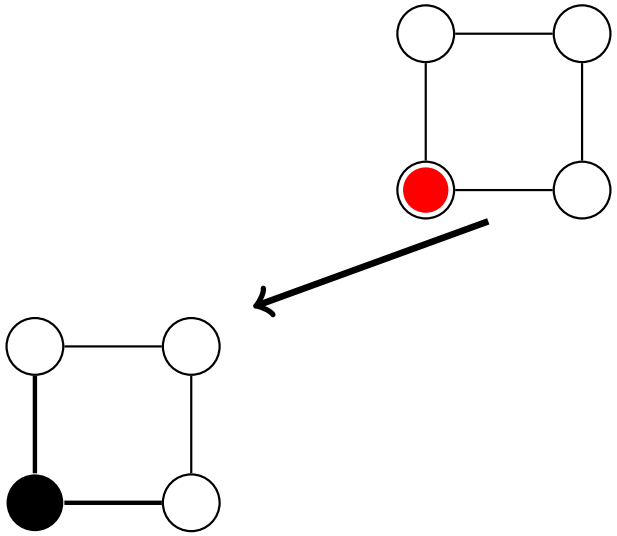

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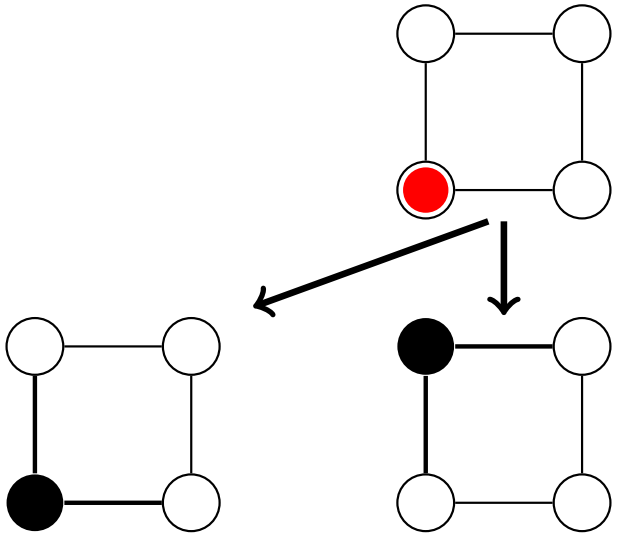

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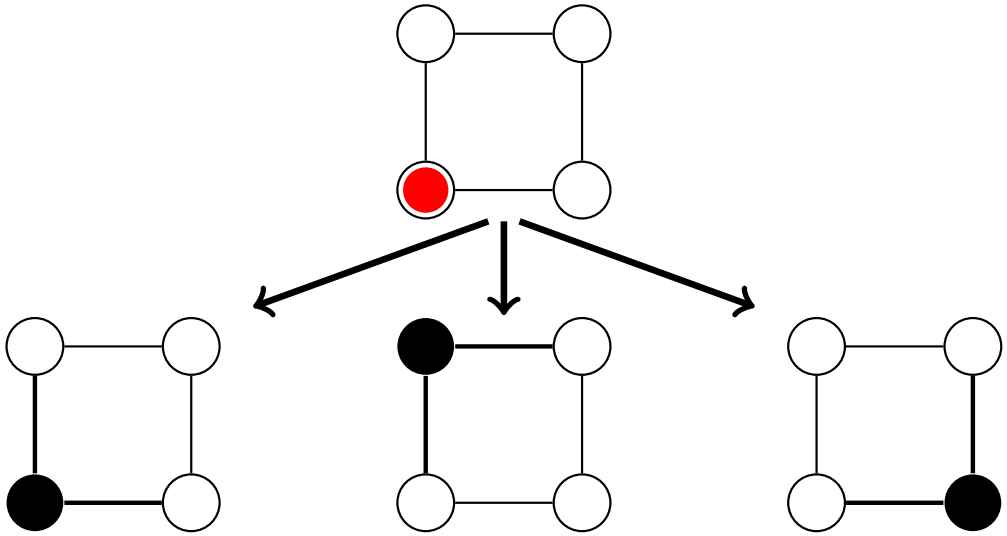

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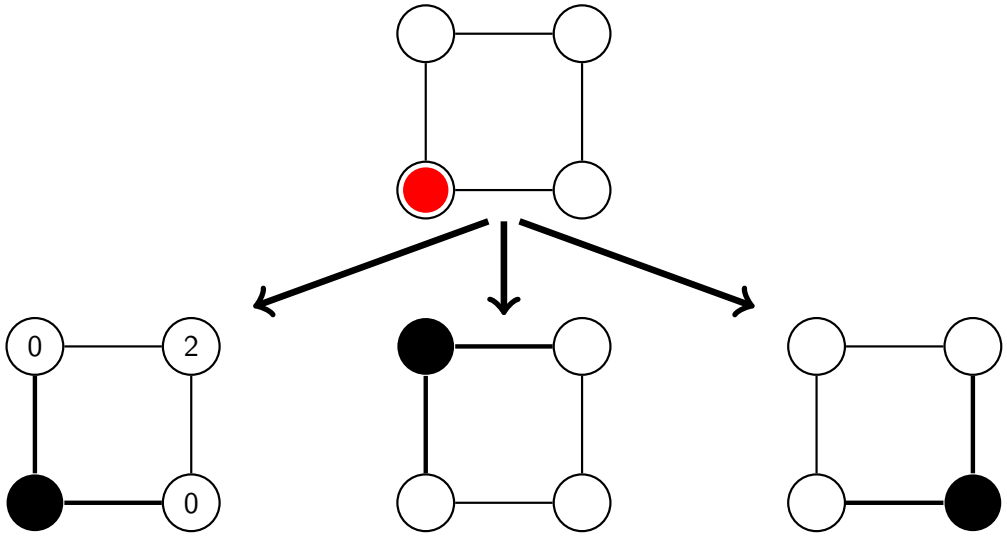

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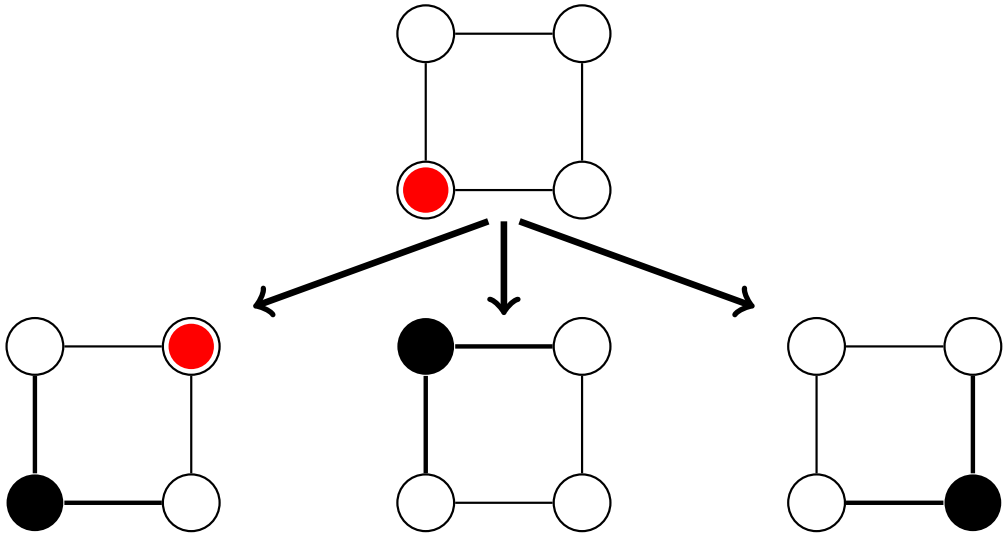

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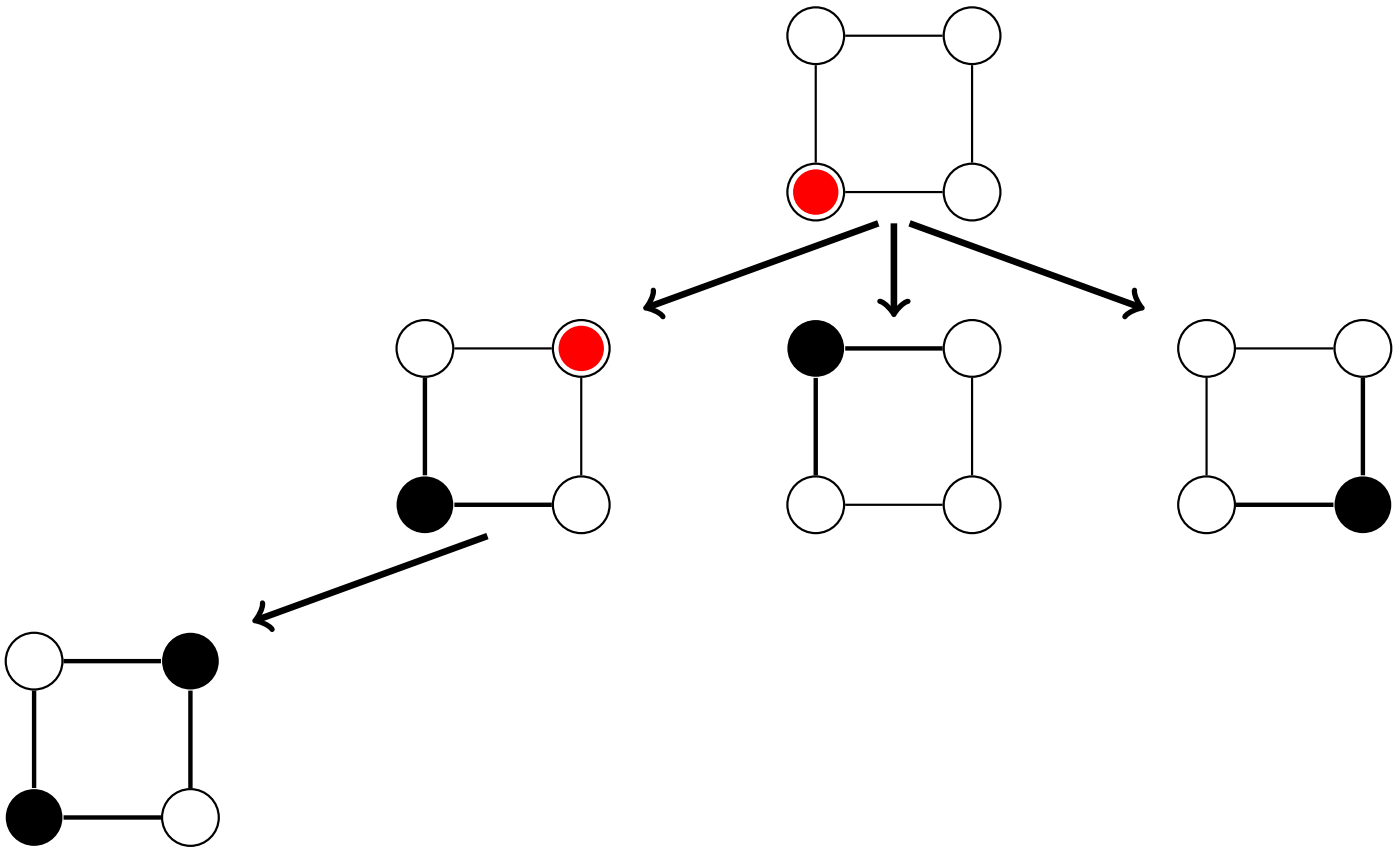

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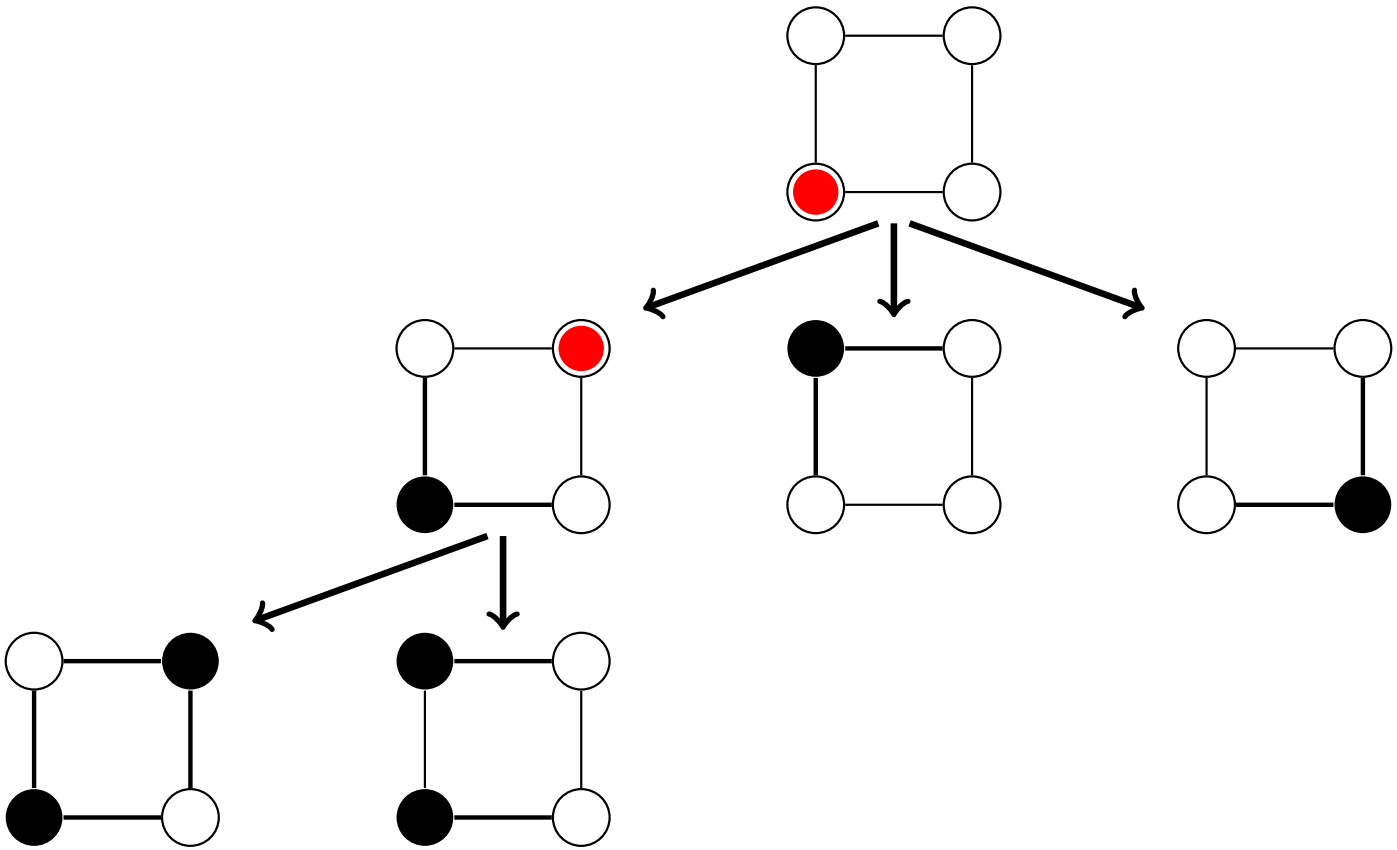

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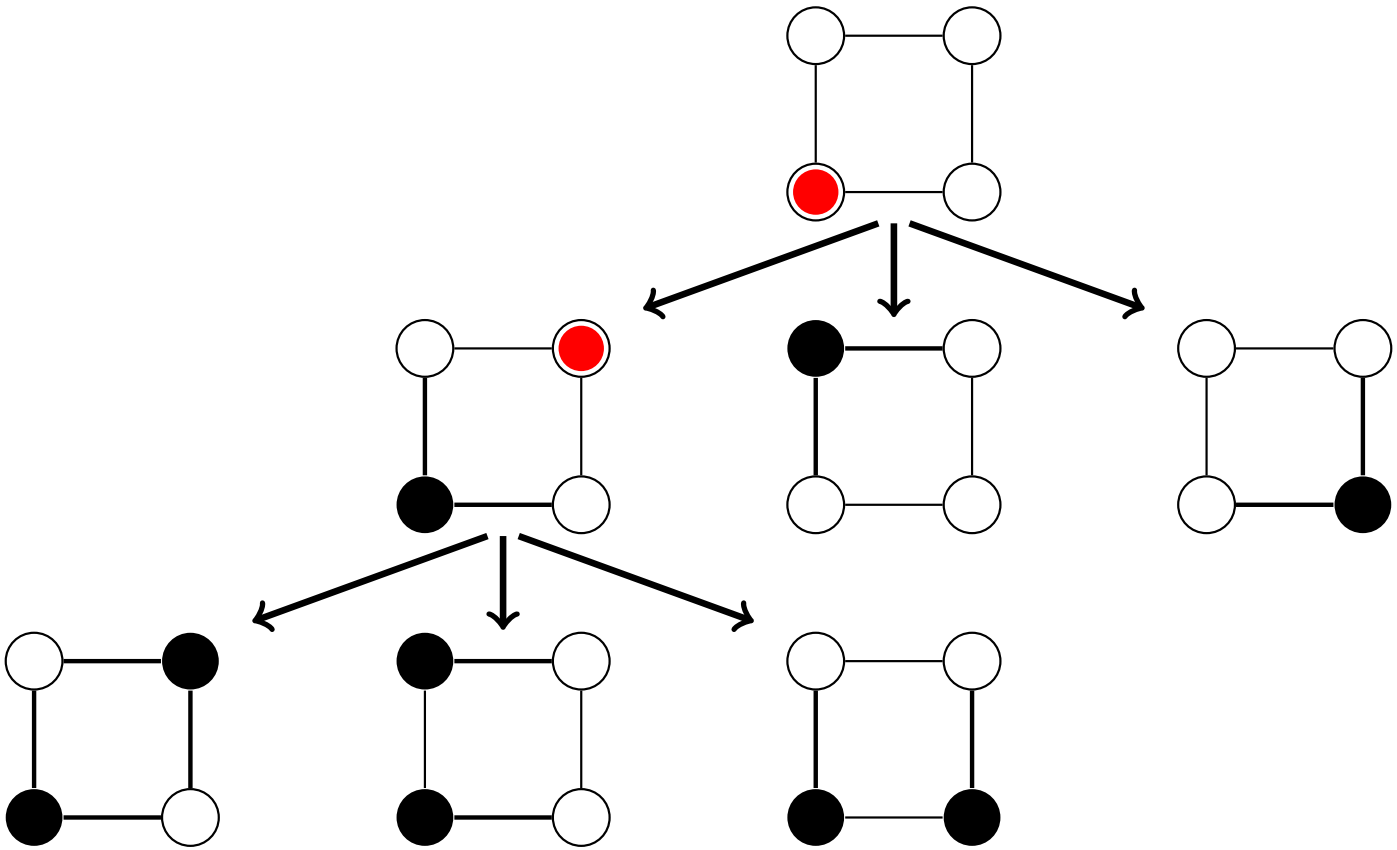

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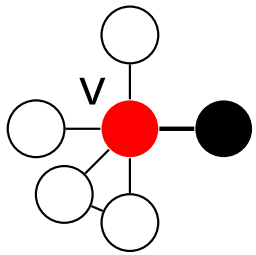

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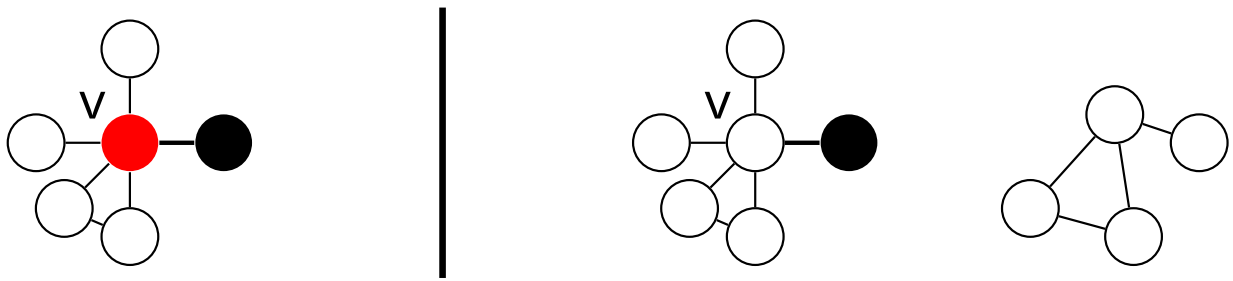

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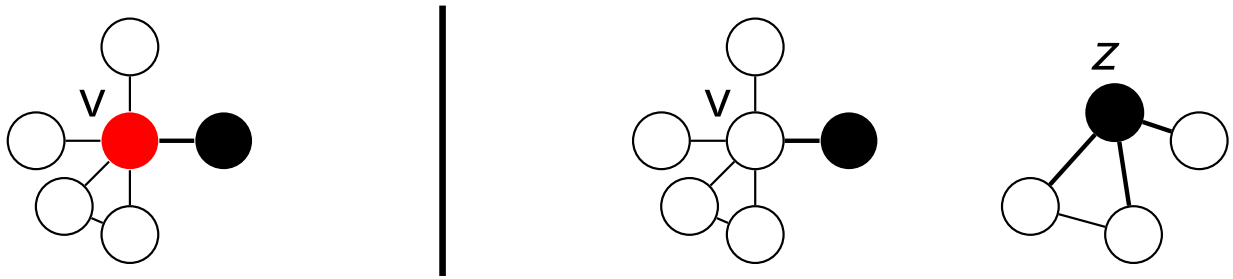

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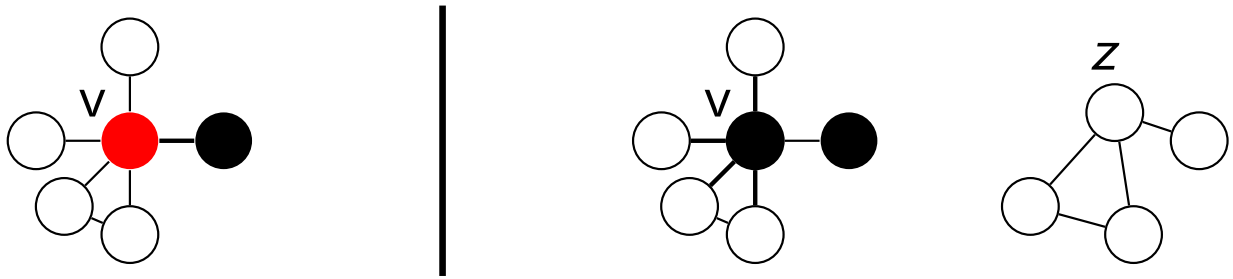

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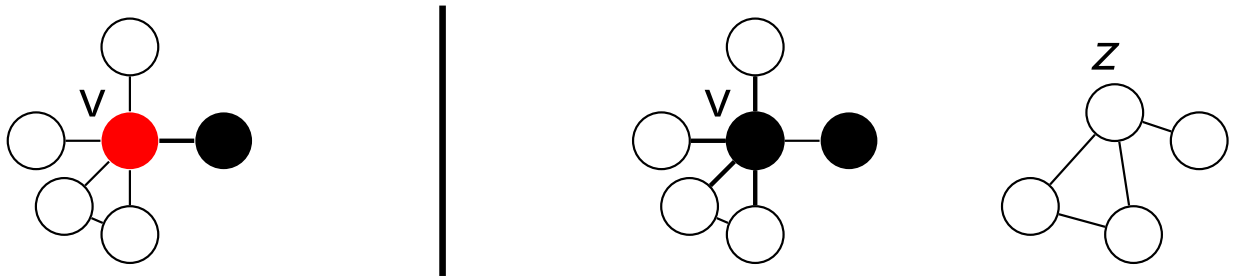

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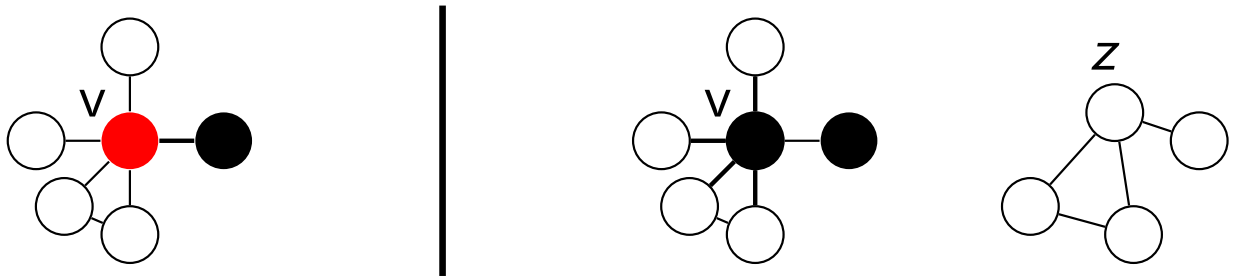

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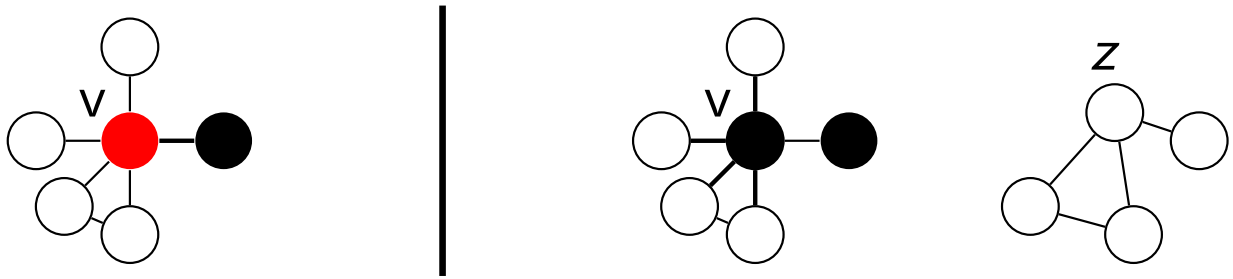
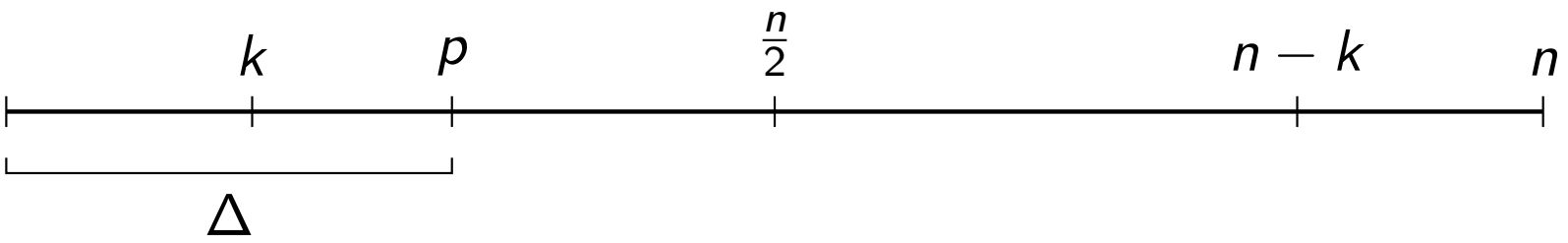


- $val(V_{opt} \setminus \{z\} \cup \{v\}) \geqslant val(V_{opt})$.
- Iterate this principle at most k times.
- Uses degrading contribution.

- Here, branching is not an end in itself but protects greediness.
- Close to Greedy Localization technique.

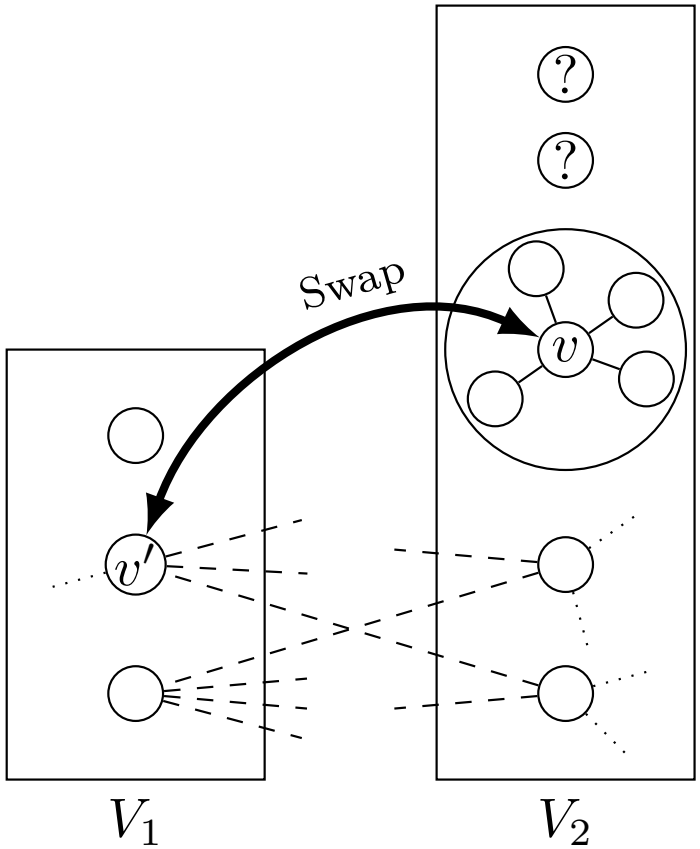
Corollary

Max $(k, n-k)$-cut w.r.t p is FPT

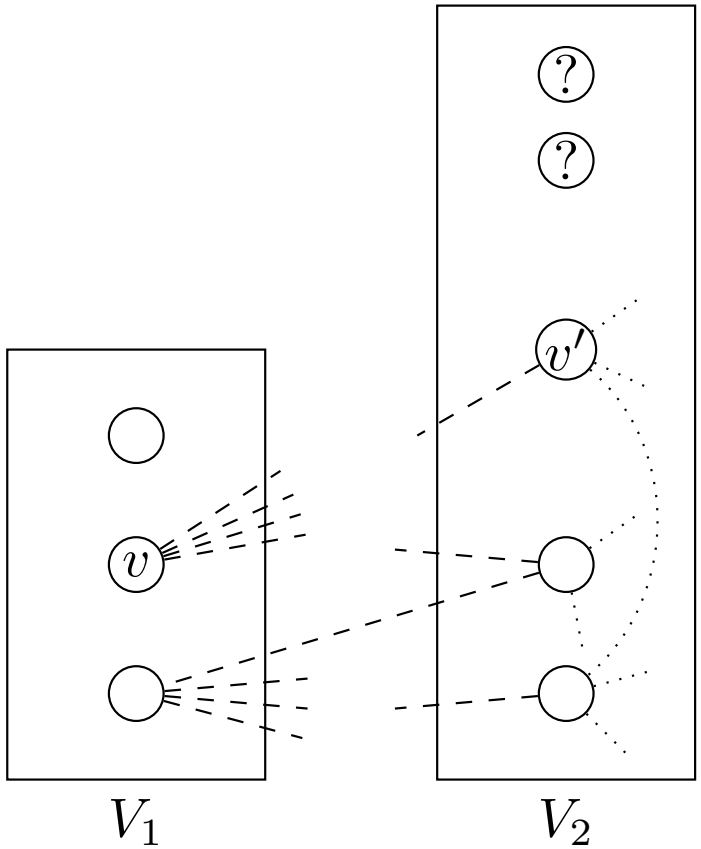


- $p \geqslant \Delta$
- $p \geqslant k$

$$p \geqslant \min(rk, n-k) \geqslant k$$



(a) Vertices $v \in V_2$ and $v' \in V_1$ (that has at least one neighbor in V_1) will be swapped.



(b) With the swapping the cut size increases.

Theorem

Min $(k, n-k)$-cut can be solved in $O^*((\Delta k)^{2k})$.

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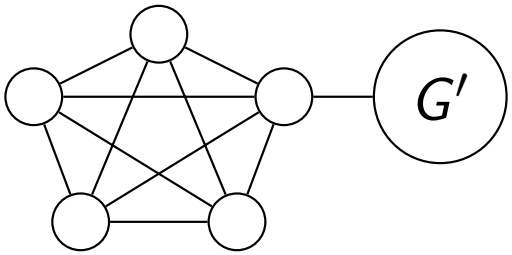

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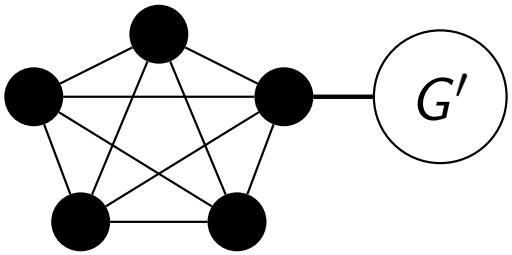

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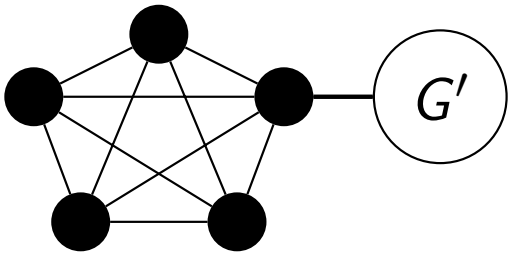

Solution: Consider connected induced subgraph of size up to k.

Good news

Lemma

One can enumerate the connected induced subgraphs of size k in $O^*(\Delta^{2k})$.

Idea: there is an injective function from those connected subgraphs to the binary trees with $k\lceil \log \Delta \rceil$ nodes.

Bad news

Informally: an optimal solution is not necessarily a greedily chosen combination of connected components.

Outline of the algorithm

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- Branch on each vertex of each S_i: the branching tree has size k^{2k}.
- Overall complexity: $O^*((\Delta k)^{2k})$.
- Soundness: For each size of maximal connected component in V_{opt}, one can hybridate with a connected component of the same size.

Theorem

Max (k,n-k)-cut has a fpt approximation schema.

$V' = \{v_1, \ldots, v_k\}$ the k largest-degree vertices $d_1, \ldots, d_k$. Let $B = \Sigma_{i=1\ldots k} d_i$.
SOL $\geqslant B - k^2$, OPT $\leqslant B$.
So, $r \geqslant 1 - \frac{k^2}{B} \geqslant 1 - \frac{k^2}{\Delta}$.

- either $\varepsilon \geqslant \frac{k^2}{\Delta}$, $\rightsquigarrow (1-\varepsilon)$-approximation.
- either $\varepsilon \leqslant \frac{k^2}{\Delta}$, then $\Delta \leqslant \frac{k^2}{\varepsilon}$ $\rightsquigarrow$ fpt algorithm in k.

Theorem

Min (k,n-k)-cut has a randomized fpt approximation schema.

[Feige, Krauthgamer, Nissim '03] If $k < \log n$, there is a randomized polytime $(1+\varepsilon)$-approximation.

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Conclusion and open questions

- Branching to protect local choices.
- Fear the worst to hybridate.
- An $O^*((c_1\Delta)^{c_2 k})$ algorithm for all local cardinality constraint problems?
- ...at least for min (k,n-k)-cut?