Developments in Geometric Intersection Graphs

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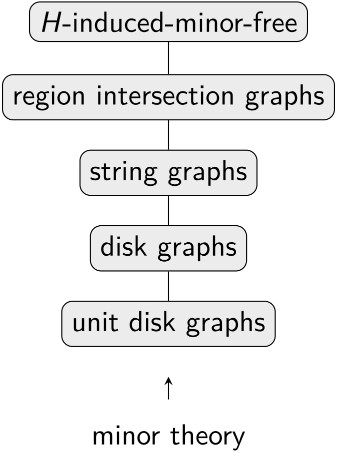
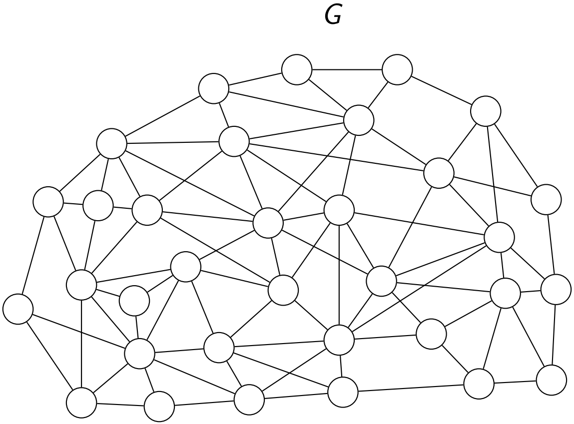
Outline

Survey a few developments in geometric intersection graphs:
between minor and induced minor theory



Minor and Induced Minor



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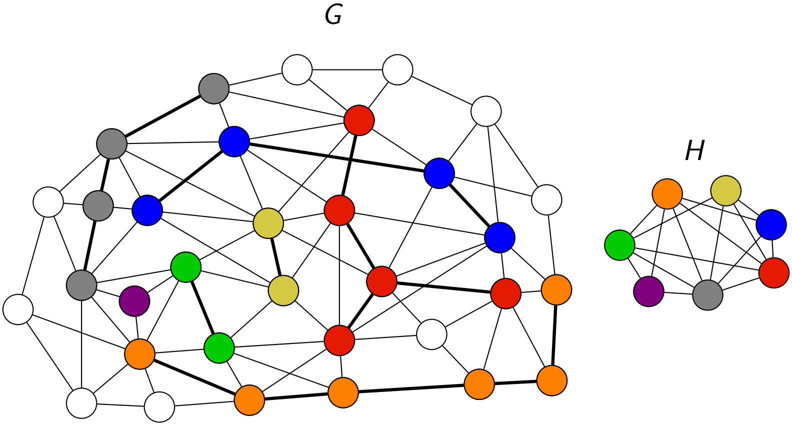
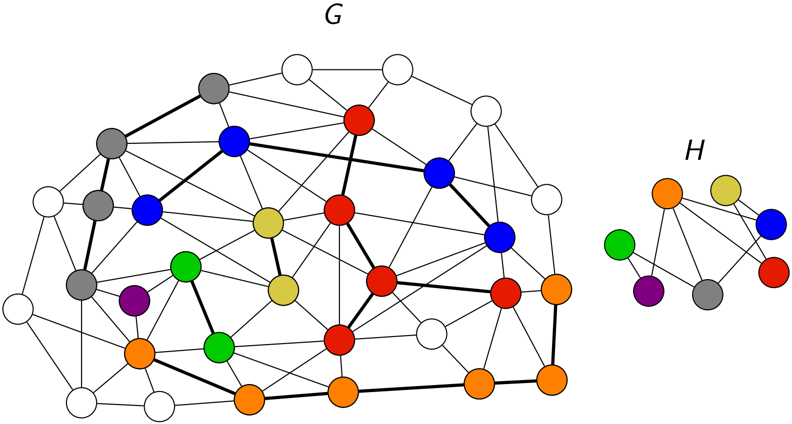


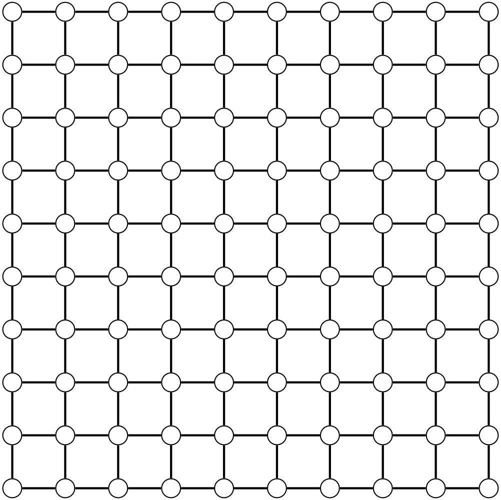
H is an induced minor of *G*

Minor and Induced Minor



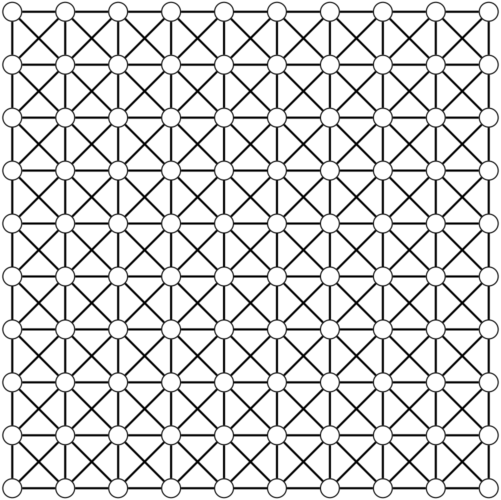
H is a minor of G

Excluding minors vs. excluding induced minors



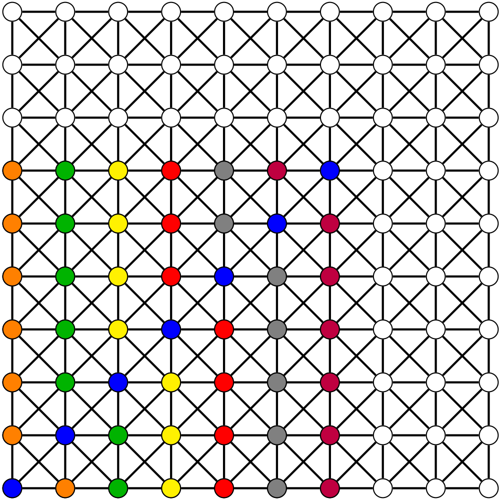
Excludes K_5 as a minor

Excluding minors vs. excluding induced minors



Does not exclude *any* minor

Excluding minors vs. excluding induced minors



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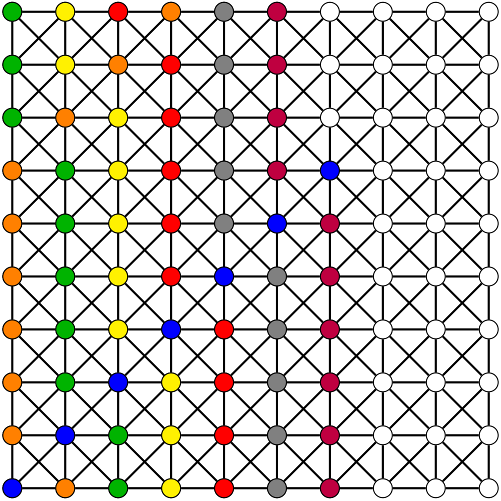
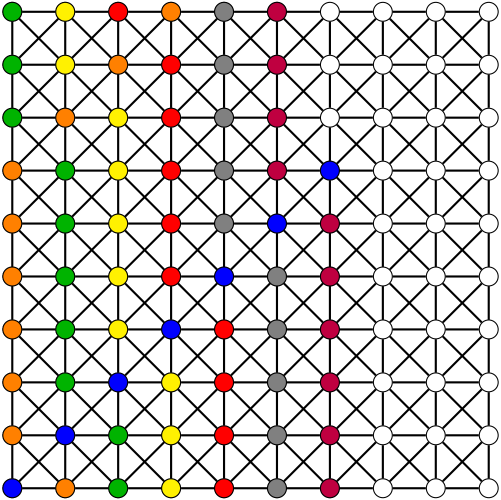


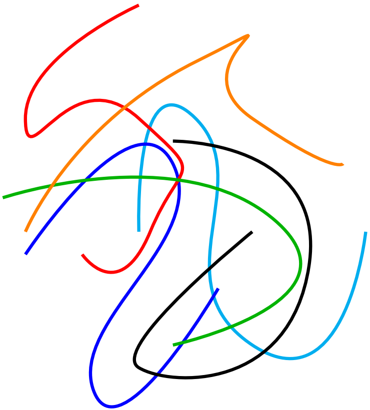
Yet is a bounded-degree unit disk graph

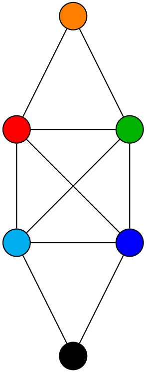
Excluding minors vs. excluding induced minors



Excludes the 1-subdivision of K_5 as an induced minor

String graphs





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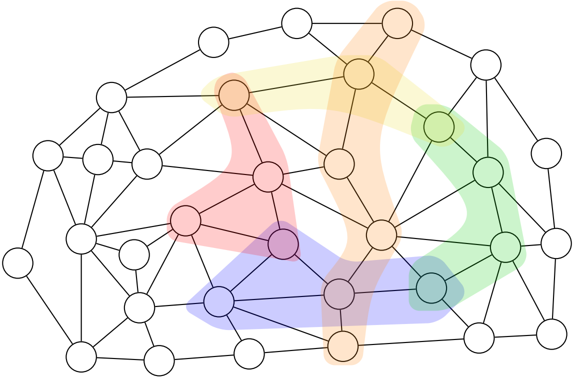
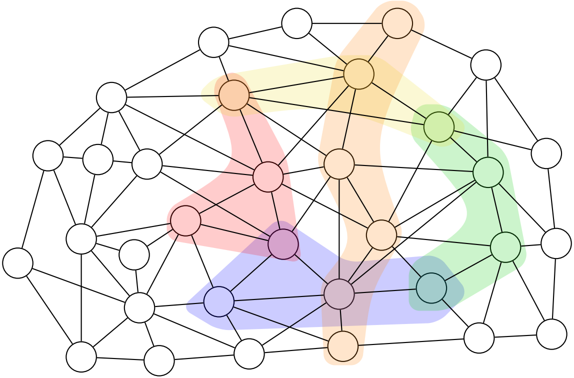
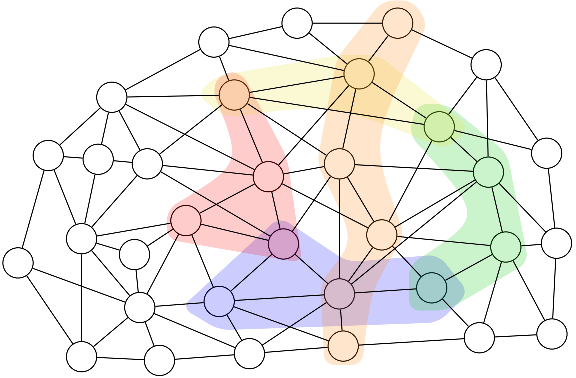


String graphs and region intersection graphs



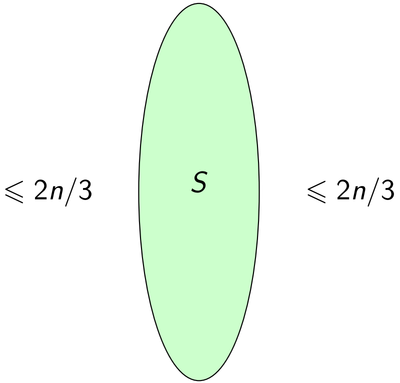
Region Intersection Graph (RIG) over a graph or graph class

String graphs and region intersection graphs



RIG($\{K_t$-minor-free$\}$) excludes $K_t^{(1)}$ as induced minor

Unbounded balanced separators



Clique-based separators

Balanced separator S partitioned into "few" cliques $C_1, C_2, \ldots$

Weight of S: $w(S) := \sum_i \log(|V(C_i)| + 1)$

Theorem (de Berg, Bodlaender, Kisfaludi-Bak, Marx, van der Zanden '20)

Intersection graphs of fat objects in $\mathbb{R}^d$ admit clique-based separator of weight $O(n^{1-\frac{1}{d}})$.

Clique-based separators – subexponential algorithms

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At most $\prod_i (|V(C_i)| + 1) = 2^{w(S)}$ independent sets within S

$T(n) \leqslant 2^{w(S)} T(2n/3) \to 2^{O(w(S))}$-time algorithm for MIS

Clique-based separators – subexponential algorithms

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At most $\prod_i (|V(C_i)|^2 + 1) \leqslant 2^{2w(S)}$ induced forests within S
$\rightarrow 2^{O(w(S))}$-time algorithm for Feedback Vertex Set

Clique-based separators – subexponential algorithms

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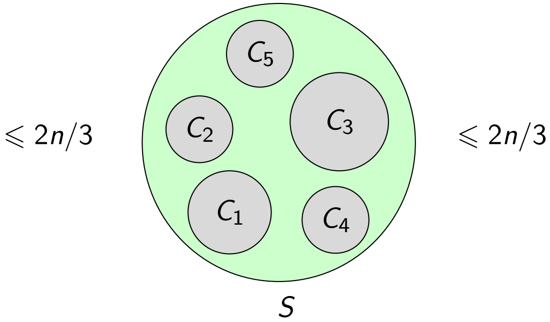
Clique-based separators of sublinear weight on pseudodisk graphs, map graphs, geodesic disks in subsets of $\mathbb{R}^2$, etc.

Clique-based separators – off-by-1 distance oracle (DO)

Space/query time trade-off between distance matrix and Dijkstra

Following [Aronov, de Berg, Theocharous '24]



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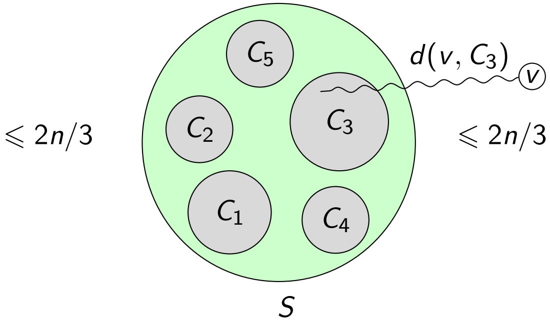

Store distance of every $v \in V(G)$ to every C_i of S: $n \cdot n^{\beta} = n^{1+\beta}$

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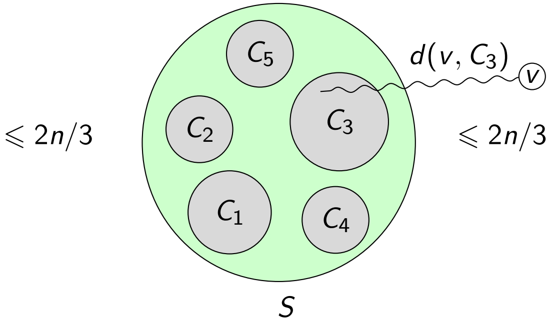

Recurse on the components of $G - S \to O(n^{1+\beta})$ stored values

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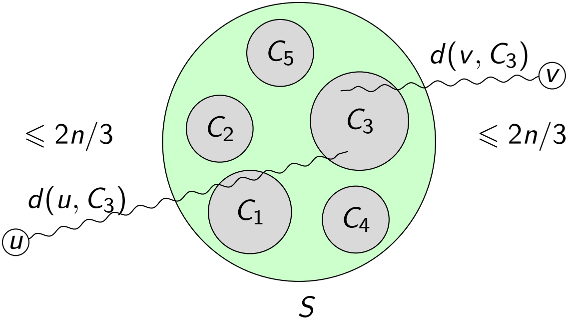

$d(u, v)$? If u and v are on $\neq$ comp. of $G - S$:

$$\min_i d(u, C_i) + d(v, C_i) + 1$$

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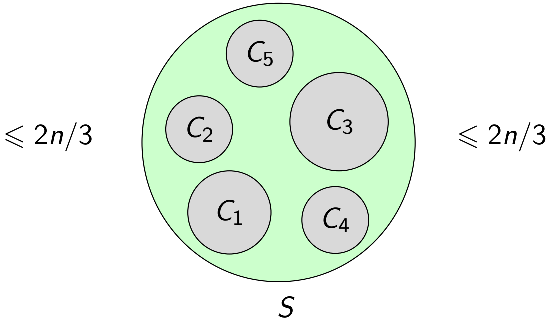

$d(u, v)$? Otherwise, recurse and take the min with

$$\min_i d(u, C_i) + d(v, C_i) + 1$$

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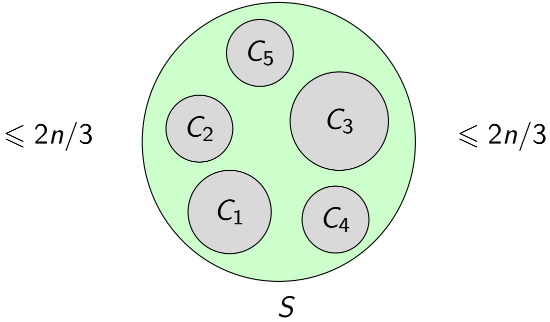

Subquadratic space $O(n^{1+\beta})$ and sublinear query time $O(n^{\beta})$

Clique-based separators – off-by-1 distance oracle (DO)

Space/query time trade-off between distance matrix and Dijkstra

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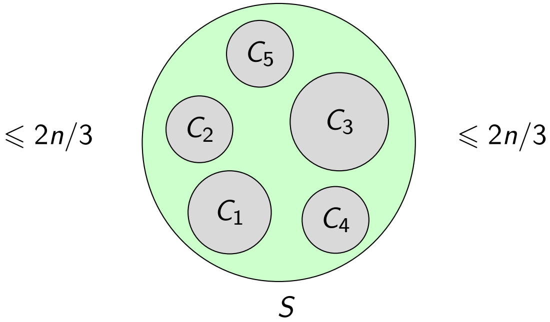

Each C_i has weak diameter at most d $\rightarrow$ off-by-d DO

Tree-independence number

Treewidth where bag "size" is max independent set within the bag

Theorem (Dallard, Fomin, Golovach, Korhonen, Milanič '24)

Tree-independence number k is 8-approximable in $2^{O(k^2)} n^{O(k)}$.

$n^{\Omega(k)}$ is likely needed (GAP-ETH) but not the $2^{O(k^2)}$ factor

Given such a decomposition, MIS can be solved in $n^{O(k)}$

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"Treewidth vs. clique number" and *"Tree-independence number"*

Balanced separators dominated by few vertices

Theorem (Robertson, Seymour '86)

Graphs excluding a grid as minor have balanced separators of constant size.

Conjecture (Gartland–Lokshtanov)

Graphs excluding a grid as induced minor have balanced separators dominated by a constant number of vertices.

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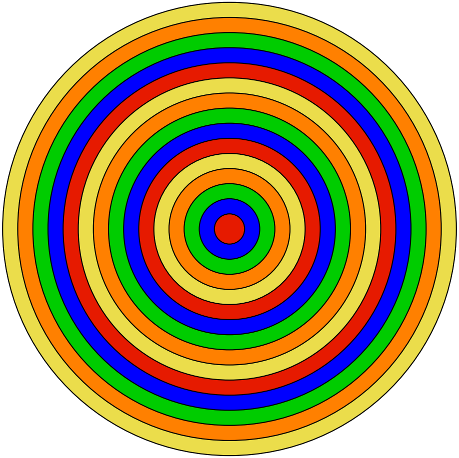
Known in some classes: P_t-free graphs, even-hole-free graphs, etc.

Geometric intersection classes?

Contraction Decomposition – parameterized algorithms

Baker's approach by contracting instead of deleting



Edge Contraction Decomposition:
Partition $E_1, \ldots, E_p$ of $E(G)$ s.t. $\mathrm{tw}(G/E_i) = O(p), \quad \forall i \in [p]$.

Contraction Decomposition – parameterized algorithms

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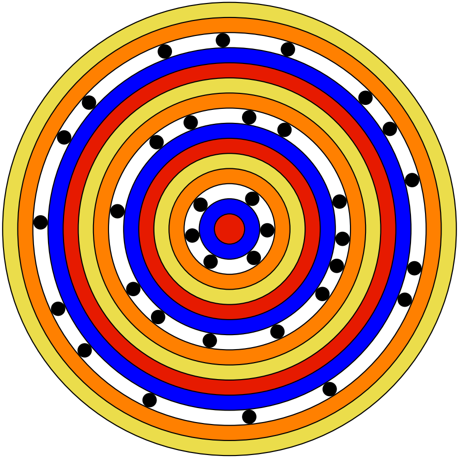

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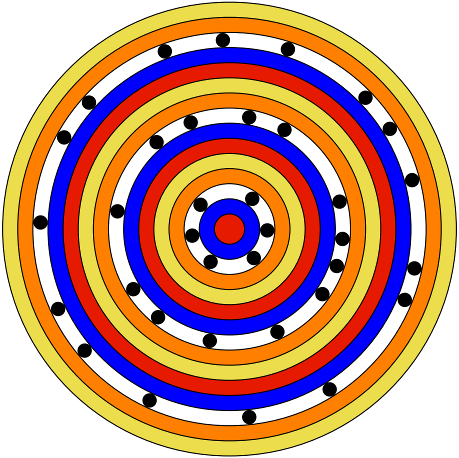

Π: remove k edges such that... (think EDGE MULTIWAY CUT)

Set $p = \sqrt{k}$, guess i and $S \cap E_i$ for a smallest $S \cap E_i$ in $\sqrt{k} \cdot n^{2\sqrt{k}}$

Contraction Decomposition – parameterized algorithms

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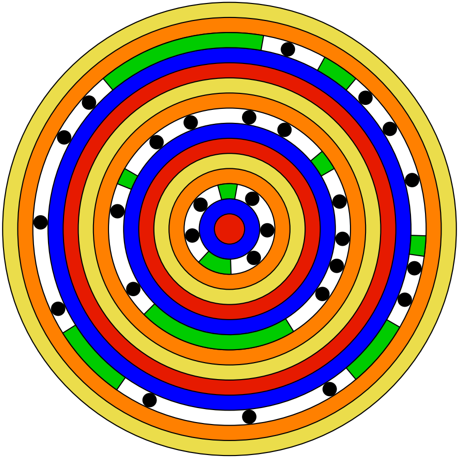

Π: remove k edges such that... (think EDGE MULTIWAY CUT)

Solve $G/(E_i \setminus S)$ as its treewidth is at most $O(p) + \sqrt{k} = O(p)$

Contraction Decomposition – parameterized algorithms

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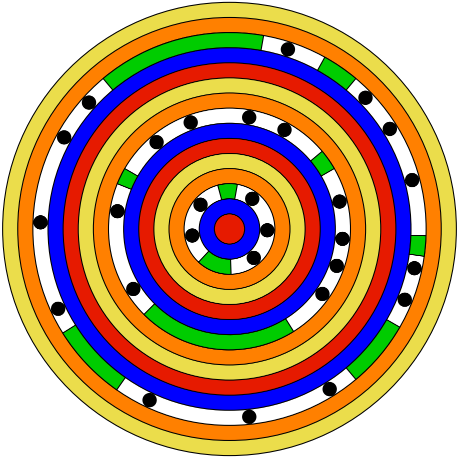
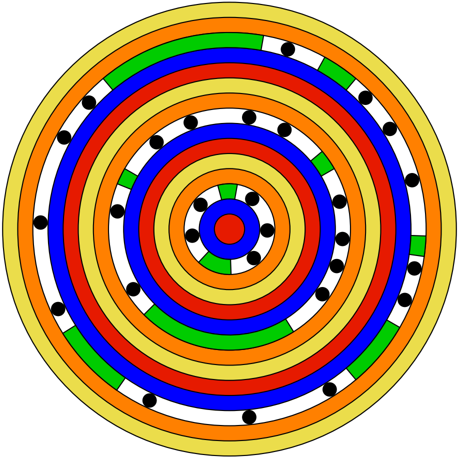


Π: remove k edges such that...

$n^{O(\sqrt{k})}$ if $n^{O(\mathrm{tw})}$ algorithm, $2^{O(\sqrt{k})} n^{O(1)}$ if polynomial kernel, too

Contraction Decomposition – parameterized algorithms

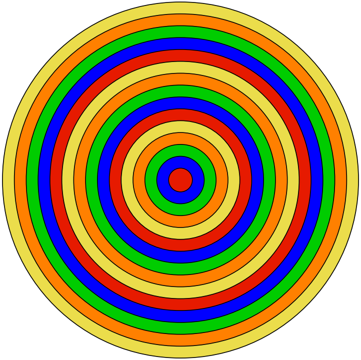
Baker's approach by contracting instead of deleting



Π: remove k edges such that...

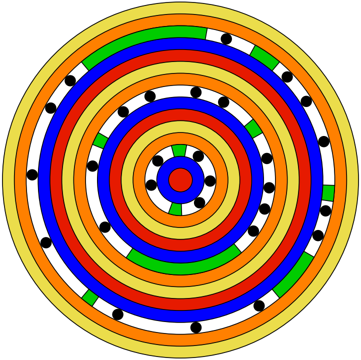
It breaks for vertex variants

Robust Vertex Contraction Decomposition



Vertex Contraction Decomposition: Partition $V_1, \ldots, V_p$ of $V(G)$
s.t. $\mathrm{tw}(G/E_{V_i \setminus S}) = O(p + |S|), \ \forall i \in [p]$ and $\forall S \subseteq V_i$.

Robust Vertex Contraction Decomposition



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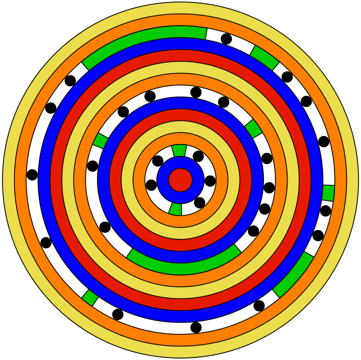

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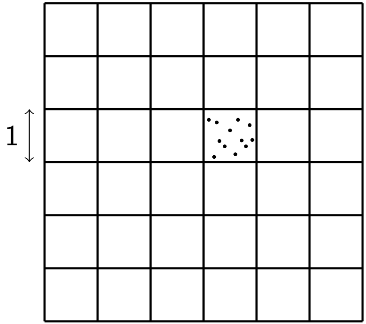
Theorem (Bandyapadhyay, Lochet, Lokshtanov, Marx, Misra, Neuen, Saurabh, Tale, Xue '25)

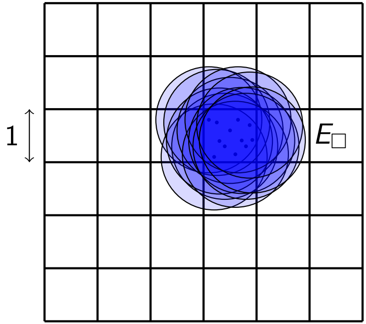
H-minor-free graphs admit a Vertex Contraction Decomposition.

Robust Vertex Contraction Decomposition – UDGs



UDG Vertex Contraction Decomposition: □-preserving partition $V_1, \ldots, V_p$ s.t. $\mathrm{tw}(G/E_{V_i \setminus S} \cup E_{\square}) = O(p + |S|), \forall i \in [p], S \subseteq V_i$.

Robust Vertex Contraction Decomposition – UDGs



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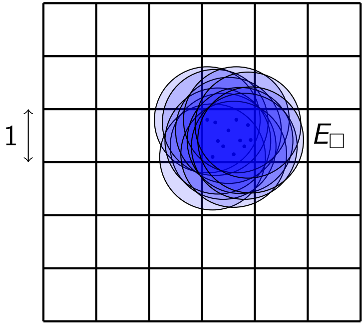

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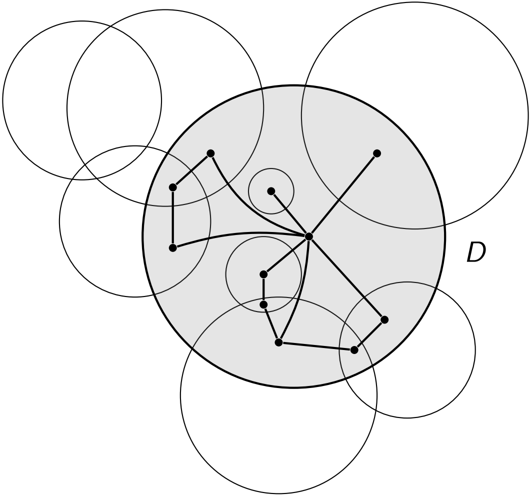
Theorem (Bandyapadhyay, Lochet, Lokshtanov, Saurabh, Xue '24)

Unit disk graphs admit a UDG Vertex Contraction Decomposition.

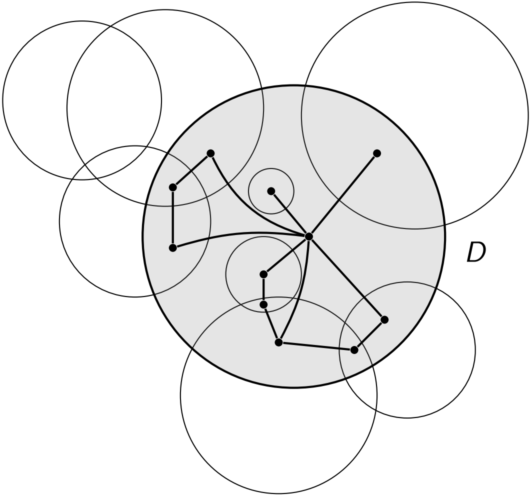
Local radius of disk graphs – EPTASes



Local radius of D = radius of dual graph of arrangement $\cap D$

Local radius of $G = \min_{\mathcal{R} \text{ of } G} \max_{D \in \mathcal{R}}$ local radius of D

Local radius of disk graphs – EPTASes

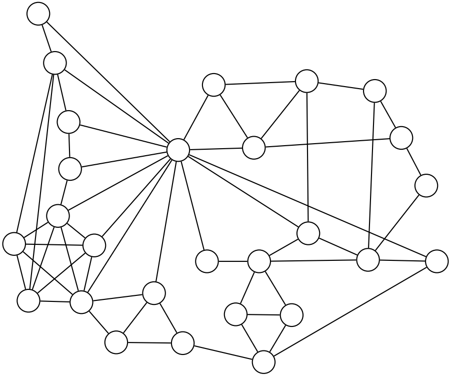


Theorem (Lokshtanov, Panolan, Saurabh, Xue, Zehavi '23)

- *low local radius → linearly bounded local treewidth (Baker)*
- *EPTAS-preserving reduction to low local radius*

Bounded weak-diameter colorings

Weak-diameter-d k-coloring of G: (improper) k-coloring of G such that every pair of vertices in a same monochromatic component is at distance at most d in G.



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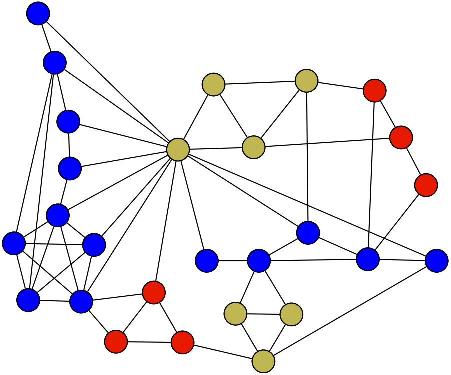
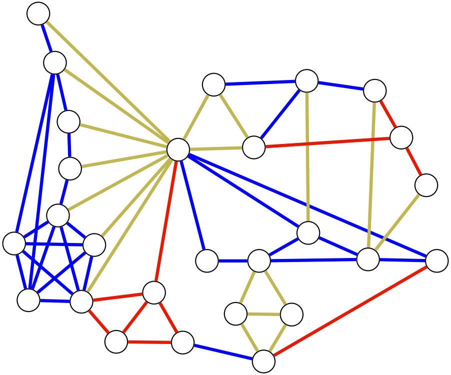


Weak-diameter-2 3-coloring

Bounded weak-diameter colorings

Weak-diameter-d k-edge-coloring of G: (improper) k-edge-coloring of G such that every pair of vertices in a same monochromatic component is at distance at most d in G.

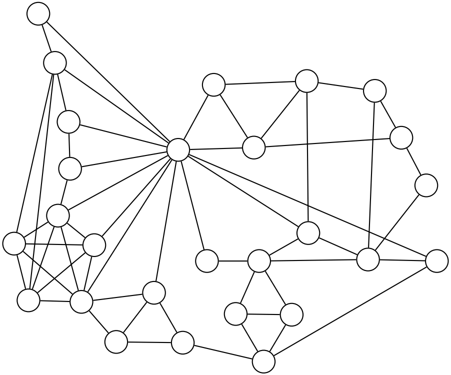


Weak-diameter-2 3-edge-coloring

Applications of bounded weak-diameter colorings

Closely related to padded and low-diameter decompositions

Various applications in approximation algorithms, distributed algorithms, spanners, routing, induced minor theory, etc.



Weak-diameter-d k-coloring $\rightarrow$ $(\Delta + 1)$-coloring in $O(dk)$ rounds

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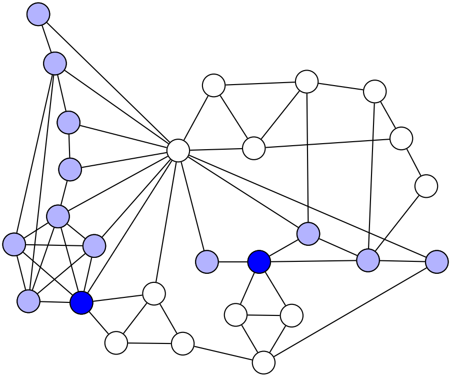

In $O(d)$ rounds, a delegate per component of the first color collects their induced subgraph and broadcasts the $(\Delta + 1)$-coloring

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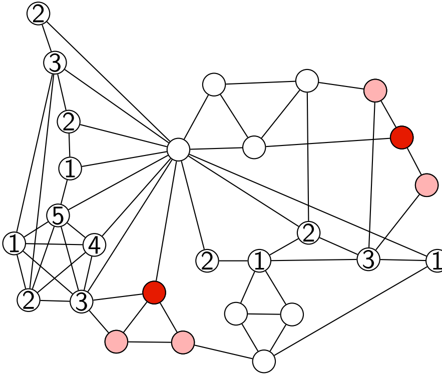

Then we move to the second color, etc.

Bounded weak-diameter colorings for minor-free classes

Theorem (Klein, Plotkin, Rao '93)

Every K_h-minor-free graph admits a weak-diameter-$f(h)$ $2^{O(h)}$(-edge)-coloring with $f(h) = O(h^2)$.

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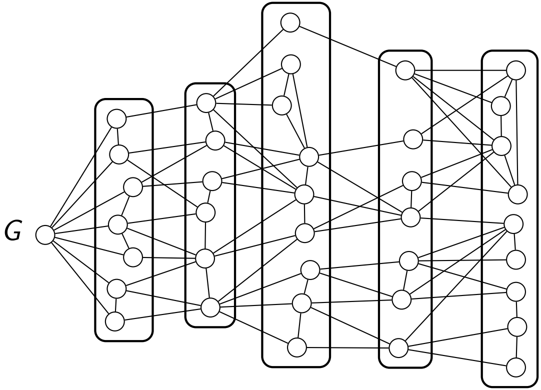

While the *processed* component has weak diameter $> f(h)$,
start a BFS at an arbitrary vertex

Bounded weak-diameter colorings for minor-free classes

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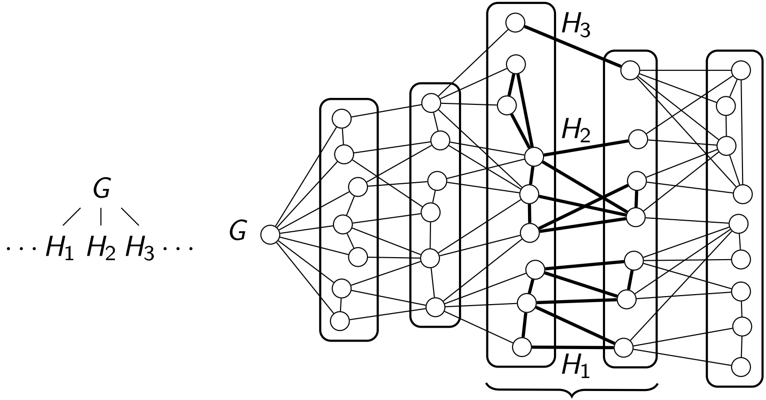

and recurse on every connected component of every subgraph induced by $O(1)$ consecutive layers.

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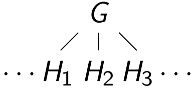
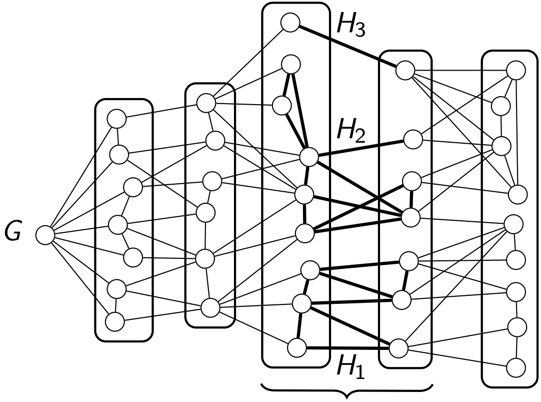
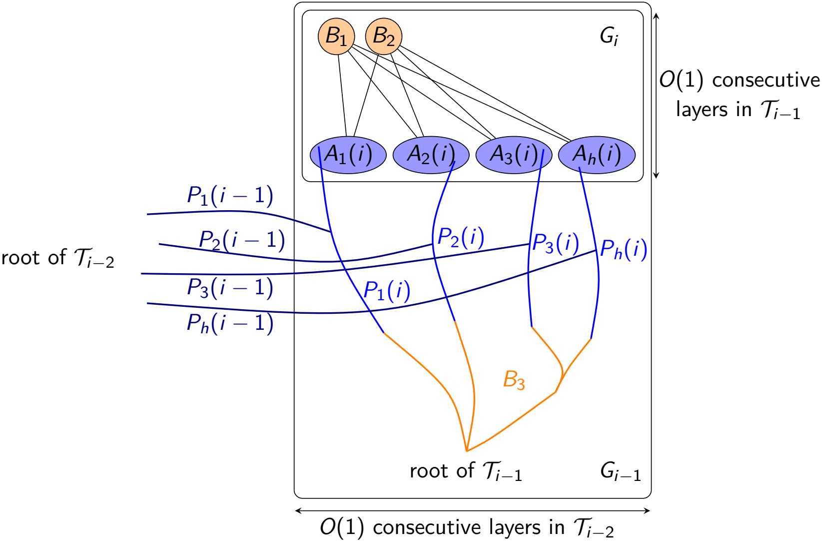
branching tree





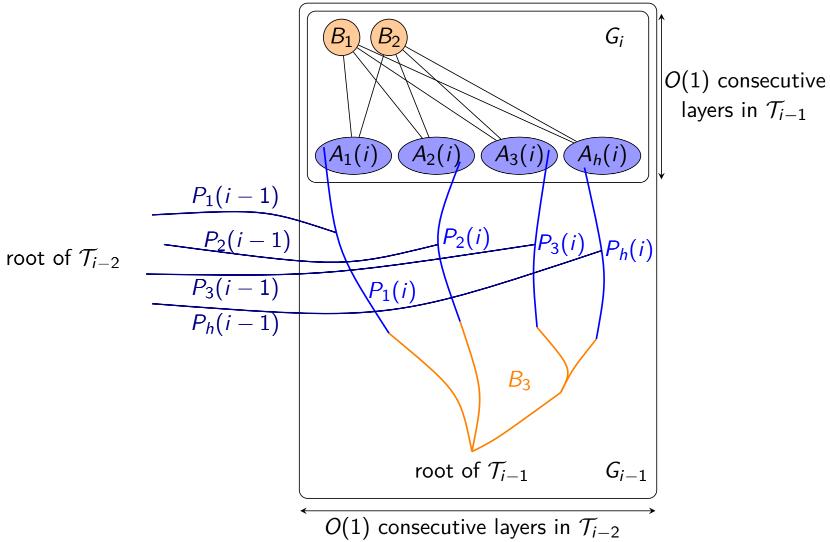
Claim: The branching tree has depth at most $h + 1$.

Bounded weak-diameter colorings for minor-free classes



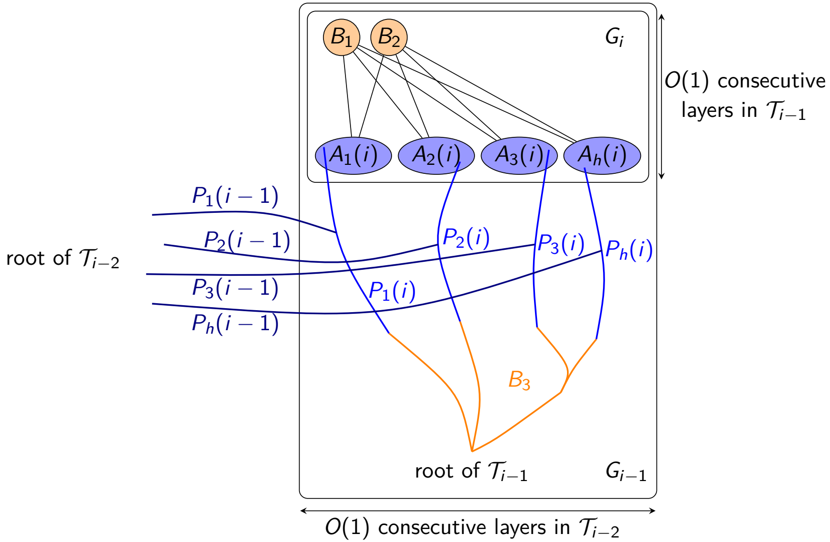
Say there is a branch $G_1, G_2, \ldots, G_{h+2}$ in the branching tree

Bounded weak-diameter colorings for minor-free classes



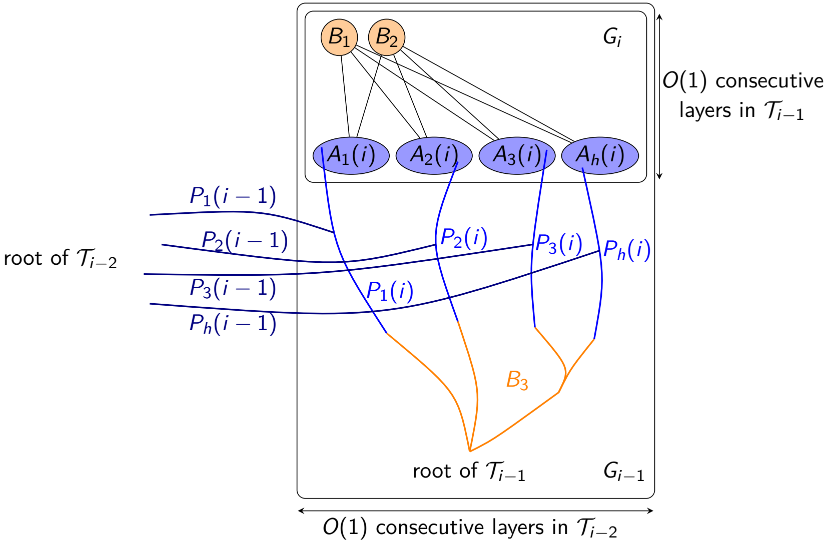
$A_1(h+1), \ldots, A_h(h+1) \subset V(G_{h+1})$ far apart in G is a $K_{h,0}$ minor

Bounded weak-diameter colorings for minor-free classes



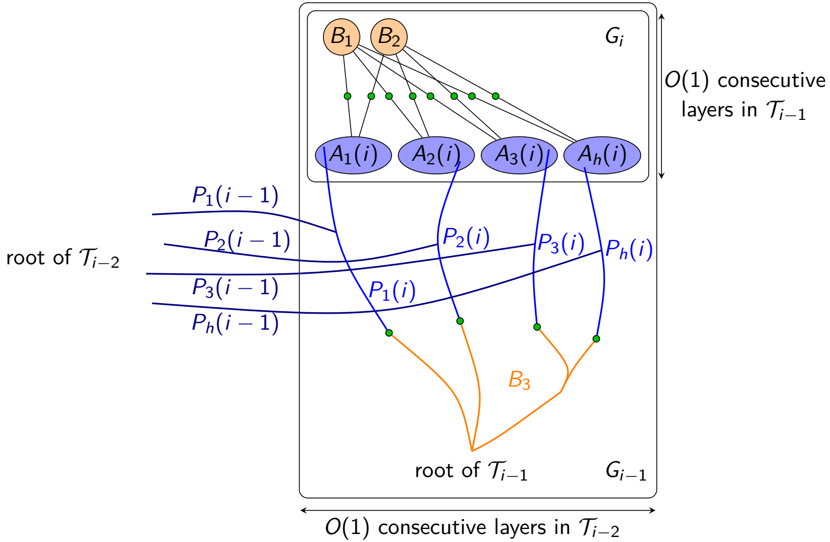
Rewinding back to G_1 builds a $K_{h,h}$ minor

Bounded weak-diameter colorings for minor-free classes



Rewinding back to G_1 builds a $K_{h,h}$ (induced!) minor

Bounded weak-diameter colorings for ind-minor-free classes



Can we build an induced minor model of the 1-subdivision of $K_{h,h}$?

Weak-diameter colorings beyond minor-free classes

Question

Are there f, g such that every $K_h^{(1)}$-induced-minor-free graph has a weak-diameter-$f(h)$ $g(h)$(-edge)-coloring?

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Every region intersection graph G over some K_h-minor free H has a weak-diameter-$O(h^2)$ $2^{O(h)}$(-edge)-coloring.

Twist the metric in G based on its representation in H

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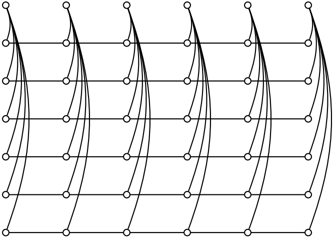
Can it be made robust?

Could it be that every class excluding an induced minor is contained in RIG($\{K_h$-minor-free$\}$) for some fixed h?

Extending the Pohoata-Davies grid

Theorem (B., Hickingbotham '25+)

For every h, there is a $K_6^{(1)}$-induced-minor-free graph that is not a region intersection graph over K_h-minor free graphs.



Excludes three cycles with paths bridging any two cycles and avoiding the neighborhood of the third

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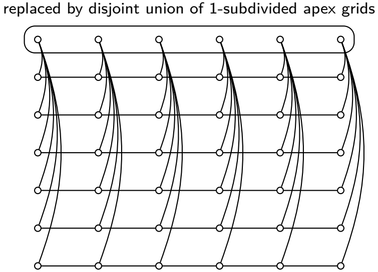

Still excludes an induced minor, but is no longer in RIG($\{K_h$-minor-free$\}$) for any h

Balanced separators of string graphs

Theorem (Matoušek '13)

Every m-edge string graph has treewidth $O(\sqrt{m}\log m)$.

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Fox and Pach first observed that this yields a subexponential algorithm for Max Independent Set

1. While there is a vertex of degree at least $n^{1/3}$, branch on adding it to the solution or removing it from the graph.
2. At the leaves, graphs have treewidth $O(\sqrt{n^{1/3} \cdot n}) = O(n^{2/3})$.

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1. takes time $2^{\widetilde{O}(n^{2/3})}$, and each leaf of 2. takes time $2^{O(n^{2/3})}$

Tight ETH bounds in string graphs

Theorem (Fox, Pach '11; B., Rzążewski '19)

MAX INDEPENDENT SET, FEEDBACK VERTEX SET, 3-COLORING *can be solved in time* $2^{O(n^{2/3})}$, *and requires* $2^{\Omega(n^{1/2})}$ *under the ETH.*

Theorem (Marx, Pilipczuk '15)

MAX INDEPENDENT SET *in n-vertex string graphs given with a representation of size s can be solved in time* $2^{\widetilde{O}(n^{1/2})} s^{O(1)}$.

What is the correct exponent?

Approximating Max Independent Set in string graphs

Theorem (Adamaszek, Har-Peled, Wiese '19)

Max Independent Set *in string graphs given with a polynomial-size representation admits a QPTAS.*

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We currently cannot rule out:

Conjecture (most optimistic)

For every H, Max Independent Set *in H-induced-minor-free graphs admits a PTAS.*

Balanced separators of induced-minor-free graphs

Theorem (Korhonen, Lokshtanov '24)

Every m-edge H-induced-minor-free graph has treewidth $\widetilde{O}_H(\sqrt{m})$.

$\approx$ same algorithmic applications

Question

Can it be improved to $O_H(\sqrt{m})$?

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- the use of Voronoi diagrams of graphs
- the use of (distance) VC dimension for DIAMETER
- new parameters and how they relate to intersection graphs

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Thank you for your attention!