Signed Tree Models

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joint works with Julien Duron, John Sylvester, Viktor Zamaraev,
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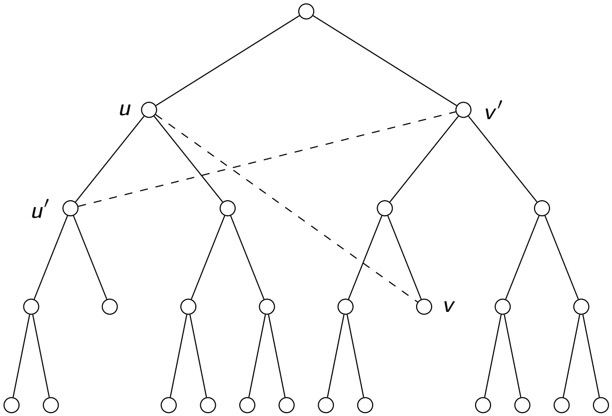
ENS Lyon, LIP

Warsaw Seminar, March 27, 2026

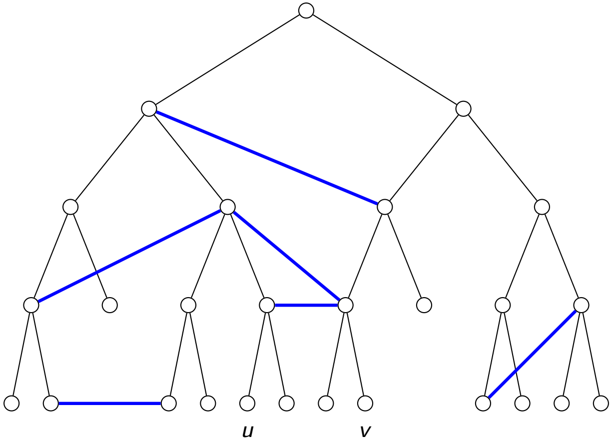
Crossing pairs in a tree

$uv, u'v'$ such that $u \prec u'$ and $v' \prec v$



Unique cover edge of a pair

$u'v' \in E$ covers uv if $u' \preceq u$, $v' \preceq v$,
and $\neg\exists u''v'' \in E \setminus \{u'v'\}$ with $u' \preceq u'' \preceq u$ and $v' \preceq v'' \preceq v$



Tree + set E of non-crossing transversal edges

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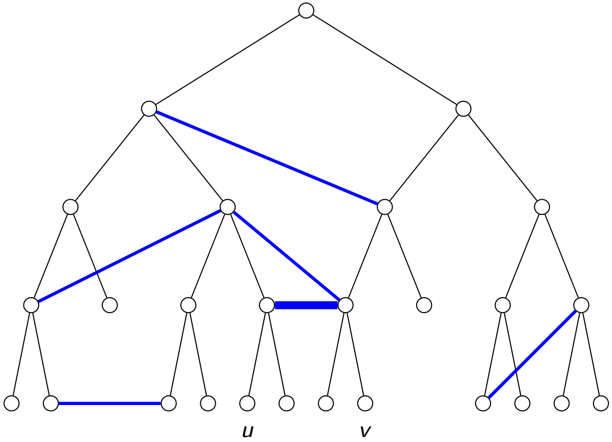

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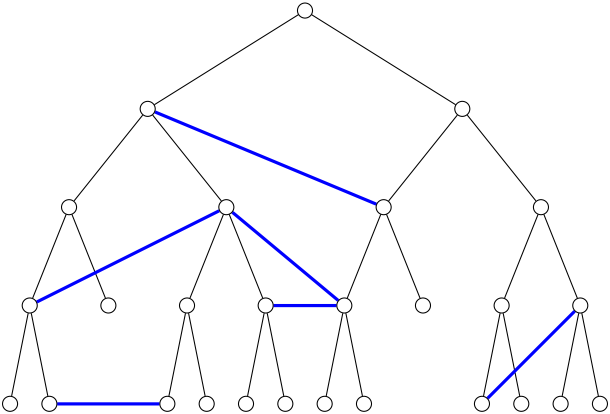
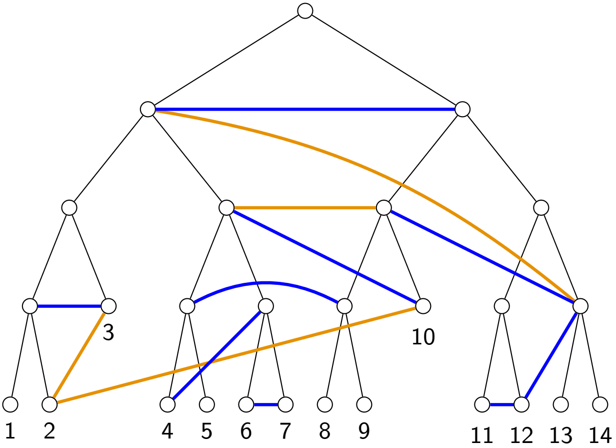


Tree + set E of non-crossing transversal edges

Signed tree model

Tree T + transversal anti-edges A + transversal edges B
No crossing pairs in $A \cup B$



graph G at the leaves with $uv \in E(G) \Leftrightarrow uv$ covered by $u'v' \in B$

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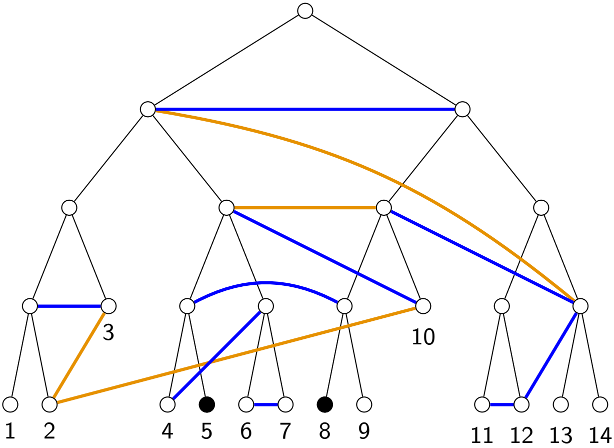

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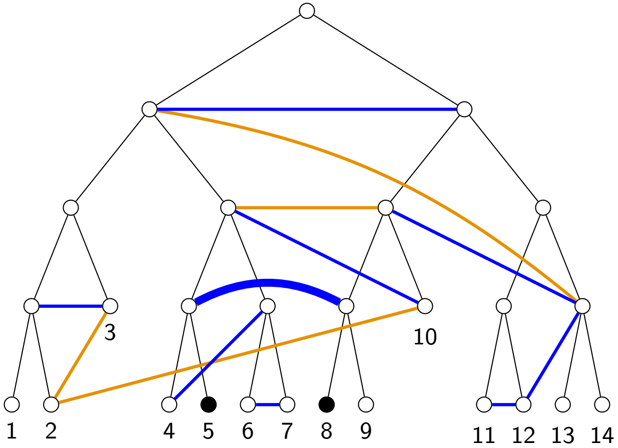

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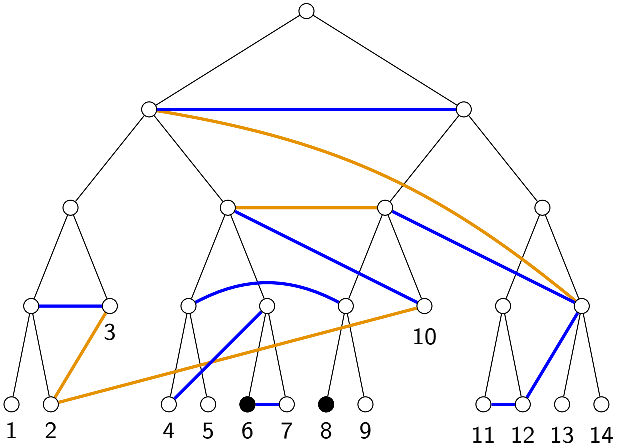

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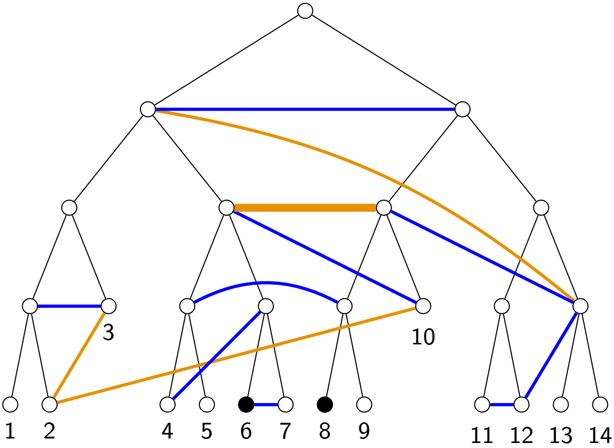

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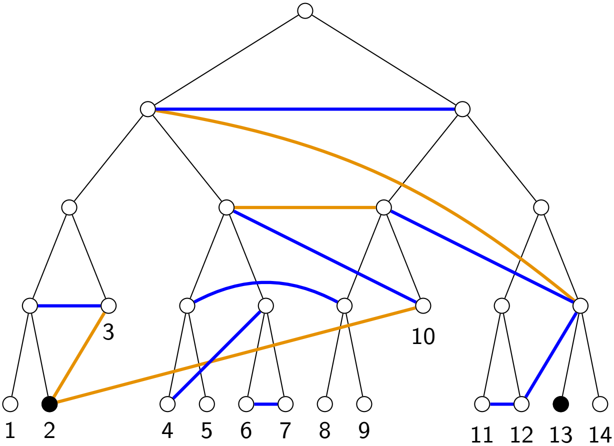

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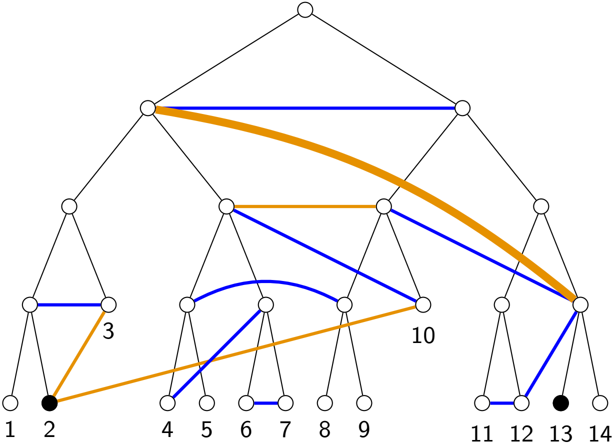

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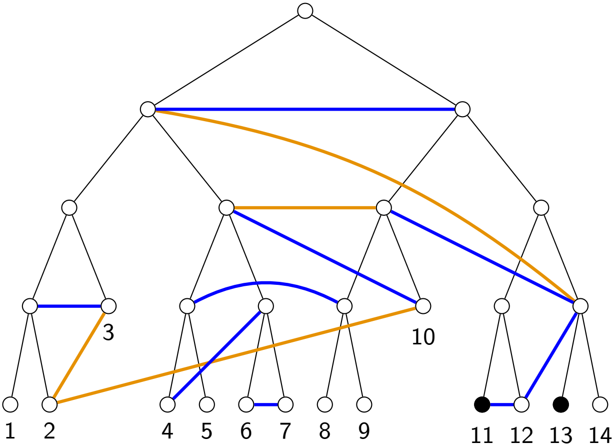

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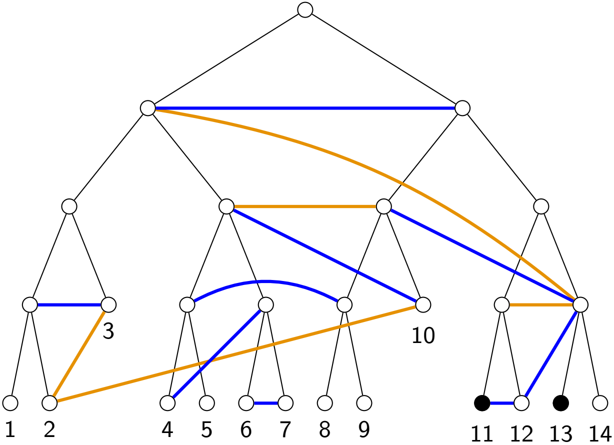

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Sparse signed tree models

Every graph G has a tree model with $|E(G)|$ transversal edges

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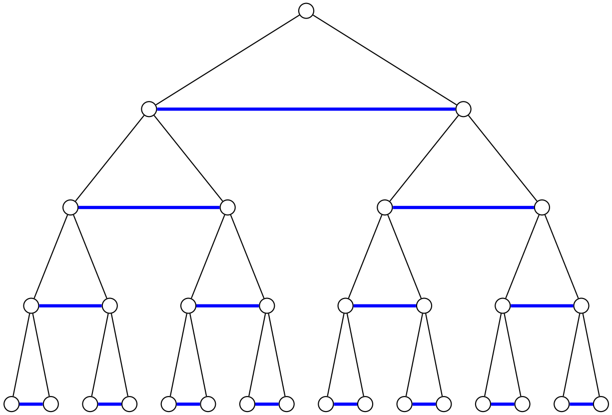
Sparse signed tree model: $\boldsymbol{O(n)}$ transversal pairs
Almost sparse signed tree model $\boldsymbol{n^{1+o(1)}}$ transversal pairs

Examples of sparse signed tree models



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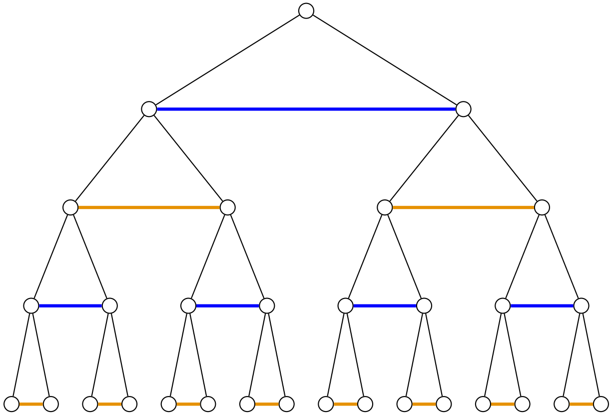
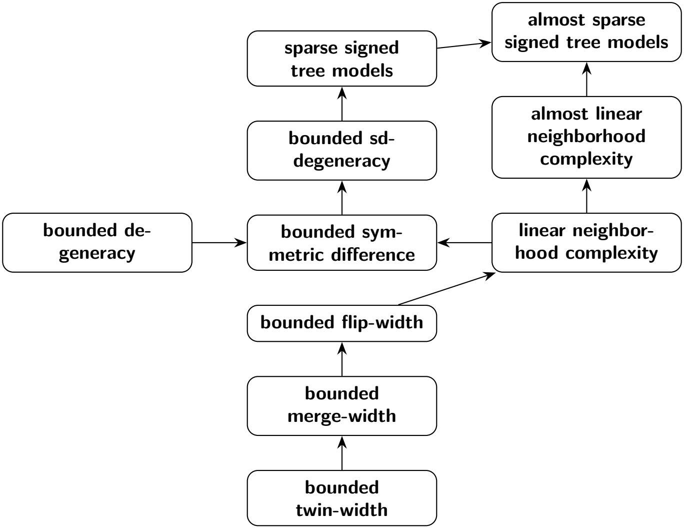
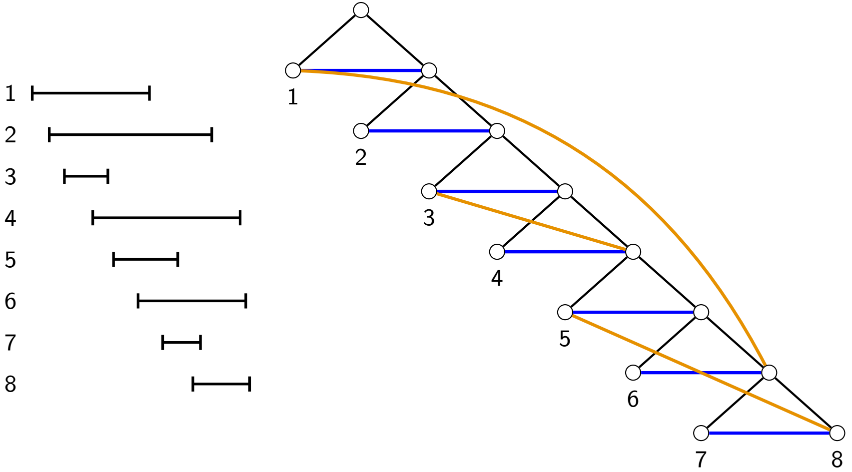


Hasse diagram of our families of interest



Interval graphs admit sparse signed tree models



sort by left endpoints; each interval represents its *right* neighborhood by stopping adjacencices with an negative edge

Symmetric difference and sd-degeneracy

Dense counterparts of degeneracy based on

$$\mathsf{sd}_G(u, v) := |N_G(u) \setminus N_G[v]| + |N_G(v) \setminus N_G[u]|$$

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Symmetric difference:

$$\mathrm{sd}(G) := \max_{H \text{ induced subgraph of } G} \ \min_{u \neq v \in V(H)} \mathrm{sd}_H(u,v)$$

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Sd-degeneracy:

$$\mathrm{sdd}(G) := \min_{v_1,\dots,v_n \text{ ordering of } V(G)} \; \max_{i<n} \; \min_{j>i} \; \mathrm{sd}_{G-\{v_1,\dots,v_{i-1}\}}(v_i, v_j)$$

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Sd-degeneracy sequence of width d: $(u_1, v_1), \dots, (u_{n-1}, v_{n-1})$ s.t.

$$\mathrm{sd}_{G-\{u_1,u_2,\dots,u_{i-1}\}}(u_i, v_i) \leqslant d,$$

and $V(G) = \{u_1, u_2, \dots, u_{n-1}, v_{n-1}\}$

Computing symmetric difference and sd-degeneracy

DEGENERACY is easy whereas:

- SYMMETRIC DIFFERENCE is co-NP-complete
- SD-DEGENERACY is NP-complete

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But computing an sd-degeneracy sequence of width $\mathrm{sd}(G)$ is easy

Linear neighborhood complexity to sd-degeneracy sequence

Theorem

Let $\mathcal{C}$ be a class of linear neighborhood complexity. One can compute an sd-degeneracy sequence of width $O(\log n)$ of an n-vertex m-edge graph G of $\mathcal{C}$ in time $O(m \log n)$.

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- Sample a linear-sized set X
- Partition by neighborhood classes w.r.t. X
- Find many pairs with $\text{sd} = O(\log n)$ within a same part

Sd-degeneracy sequence to signed tree model

Lemma

There is an $O(dn + m)$-time algorithm that, given an sd-degeneracy sequence of width d of an n-vertex m-edge graph G, outputs a signed tree model of G with at most $O(dn)$ transversal pairs.

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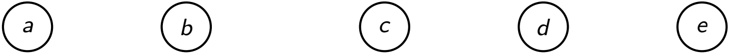

sd-degeneracy sequence: (a,b), (c,d), (d,e), (b,e)

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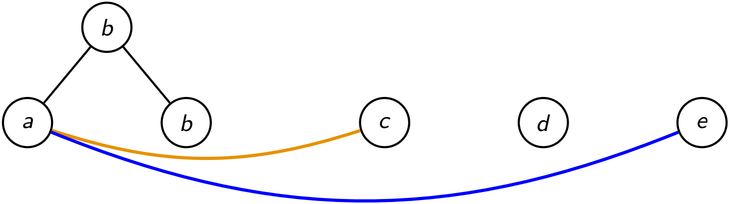

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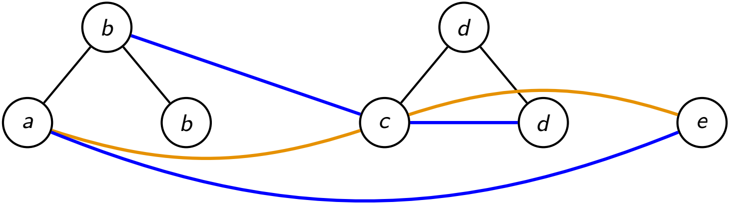

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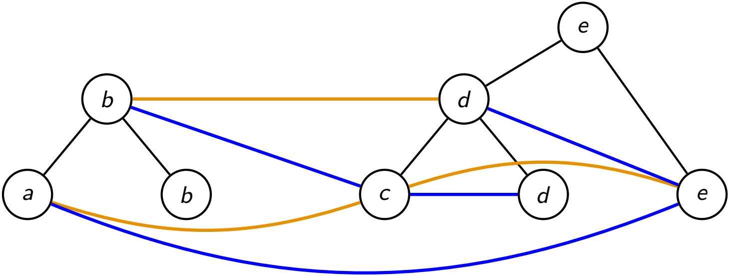

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Shortest Path for graphs with a signed tree model

Our goal now:

Theorem

There is an $O(p \log n)$-time algorithm that, given a signed tree model with p transversal pairs of an n-vertex graph G and $v \in V(G)$, outputs a shortest-path tree of G rooted at v.

Digraph compression

of a graph G: edge-weighted digraph D such that

- $V(G) \subseteq V(D)$, and
- for every $u, v \in V(G)$, $\text{dist}_D(u, v) = \text{dist}_G(u, v)$.

Size: $|E(D)|$

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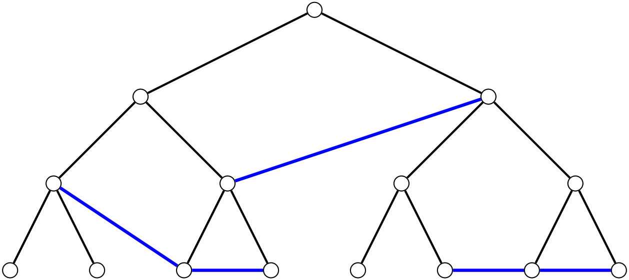
Observation

Given a digraph compression of G of size m, one can solve Single-Source Shortest Path *in G in time $O(m)$.*

Digraph compression of a graph with a positive tree model

Bannach, Marwitz, Tantau '24



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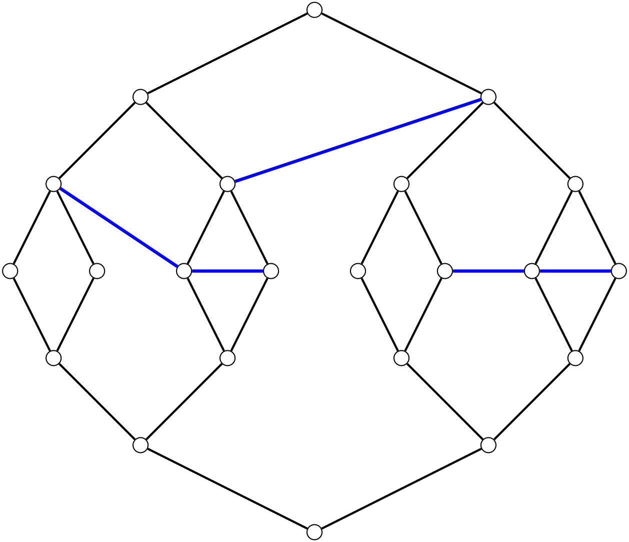

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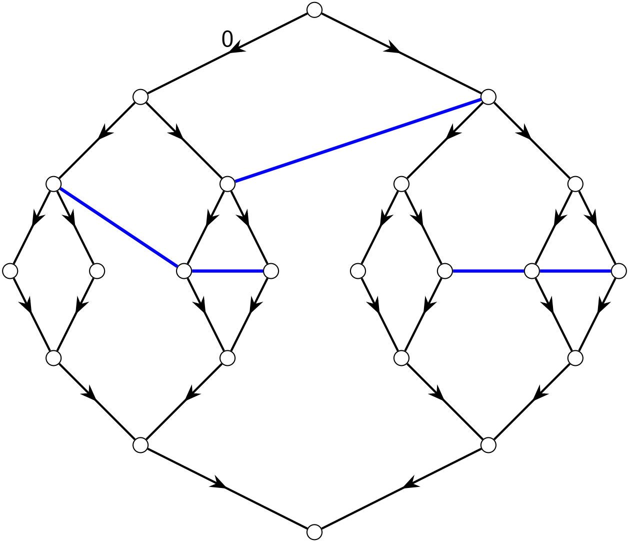

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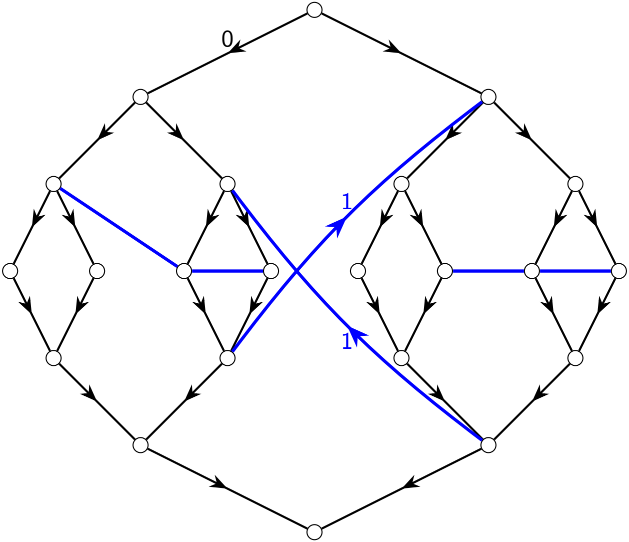

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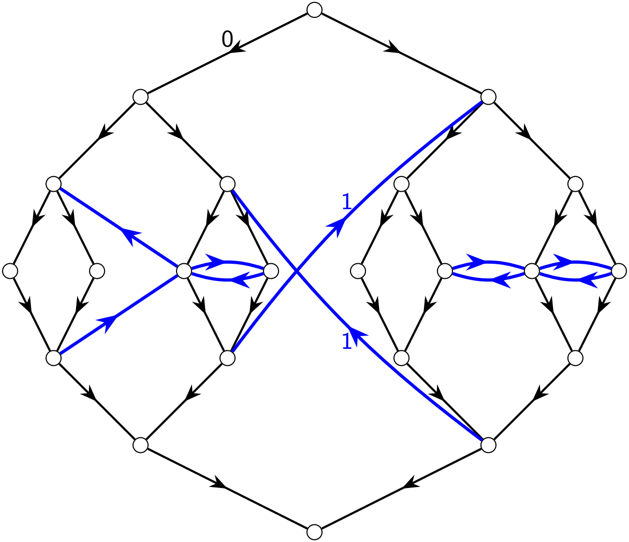

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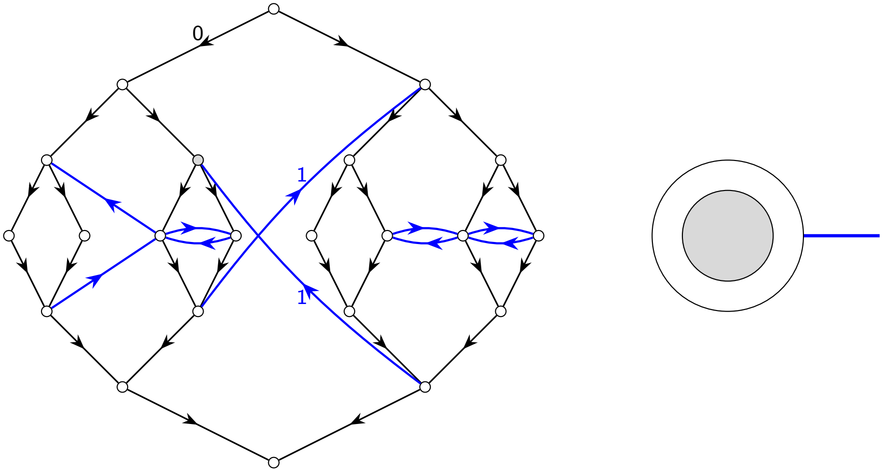

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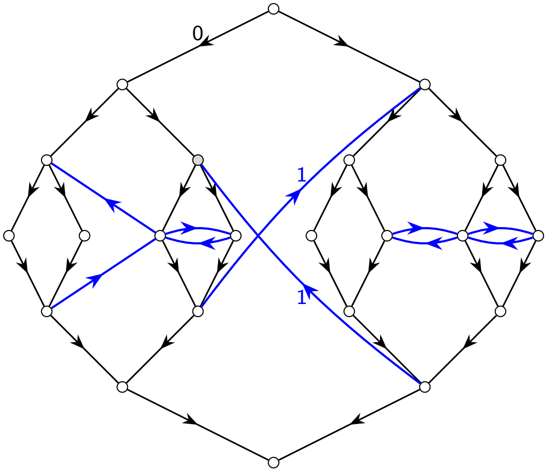

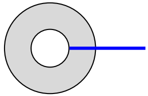

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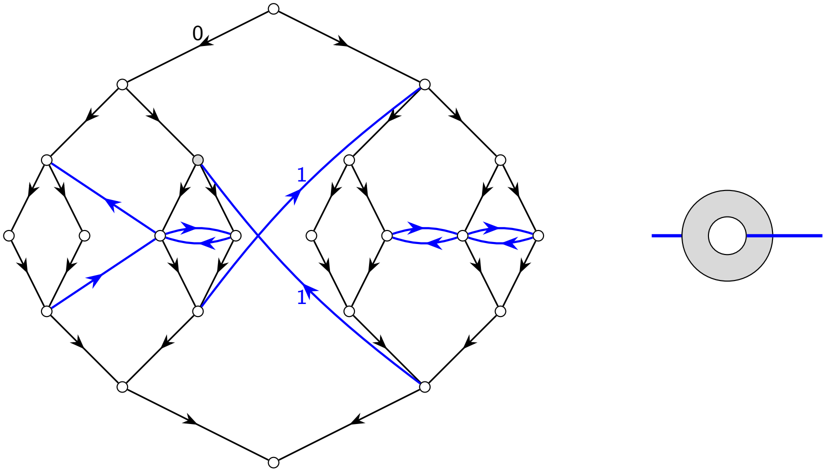

Interval Biclique Partition (IBP)

of a graph G: pair $(<, \mathcal{B})$ where

- $<$ is a total order of $V(G)$, and
- $\mathcal{B}$ is a set of bicliques every side of which is an $<$-interval.

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Theorem (B., Geniet, Kim, Thomassé, Watrigant '21)

Given $(<, \mathcal{B})$ an IBP of an n-vertex graph G, one can solve Single-Source Shortest Path *in G in time $O((n + |\mathcal{B}|) \log n)$.*

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Now improved to $O(n + |\mathcal{B}| \log^* n)$ via digraph compression

Positivizing a signed tree model

Theorem

There is an $O(p \log n)$-time algorithm that converts a signed tree model with p transversal pairs of an n-vertex graph G into any of:

1. *an interval biclique partition of G with $O(p)$ bicliques;*
2. *a digraph compression of size $O(p \log n)$ for G;*
3. *with additional $O(p \log^2 n)$ time, a tree model of G with $O(p \log^2 n)$ transversal edges.*

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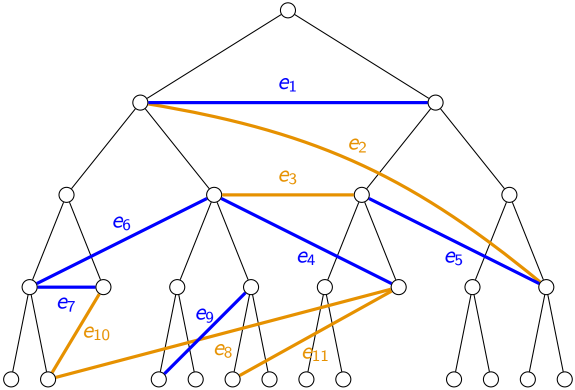
The cover relation on transversal pairs is a forest



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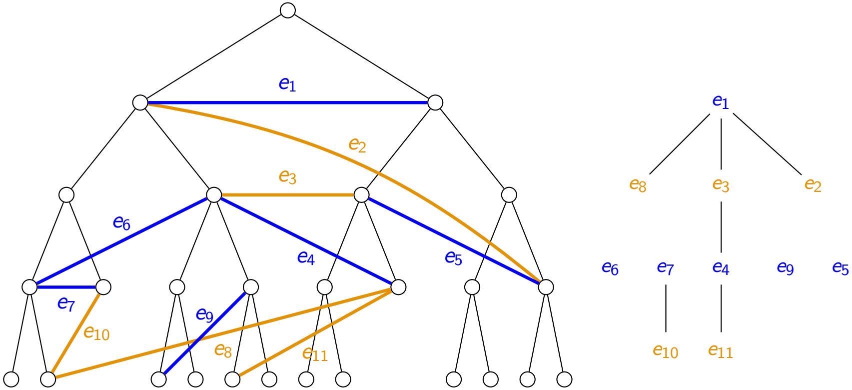

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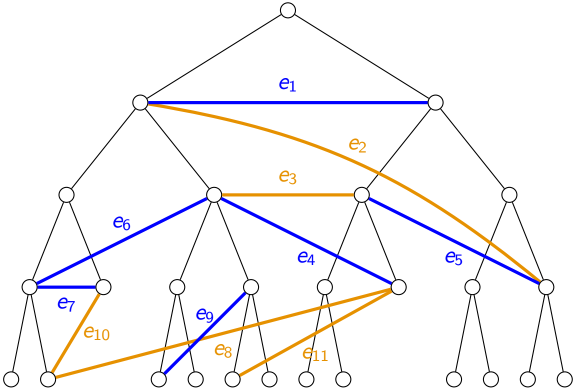
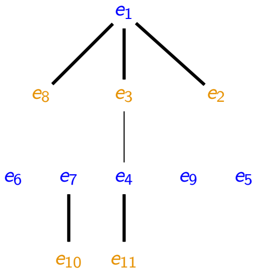
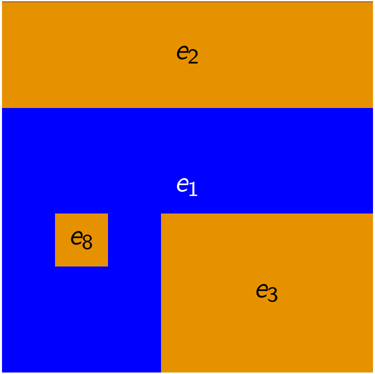




parrallel complementation of bibliques within a biclique

Complementing n disjoint rectangles with $O(n)$ rectangles



Theorem (Rezende, Lee, Wu '89)

One can compute in $O(n \log n)$ time the complement of n axis-parallel ***disjoint*** *rectangles into $O(n)$ rectangles.*

Computing the cover forest

Theorem (Mortensen '06)

Given a point set P, a data structure can be computed in time $O(|P| \log |P|)$ that supports:

1. *adding or deleting a point in P in time $O(\log |P|)$, and*
2. *given a rectangle R, computing the points in $P \cap R$ in time $O(\log |P| + |P \cap R|)$.*

Computing the cover forest

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- Sort the bicliques/rectangles $R_1, \ldots, R_h$ by increasing area
- Pick $p_i \in R_i$, and start with $P = \emptyset$
- For i from 1 to h, R_i is the parent of R_j for every $p_j \in P \cap R_i$
- Remove those contained p_j's from P, and add p_i to P

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3. *with additional $O(p \log^2 n)$ time, a tree model of G with $O(p \log^2 n)$ transversal edges.*

Theorem

All-Pairs Shortest Path *can be solved in $O(n^2 \log^* n)$ time on any n-vertex graph represented by a sparse signed tree model.*

Matrix Multiplication

Theorem

Given $(<, \mathcal{B})$ *an IBP of an n-vertex graph G, one can multiply* $M := Adj_{<}(G)$ *with a vector of size n in time* $O(n + |\mathcal{B}|)$.

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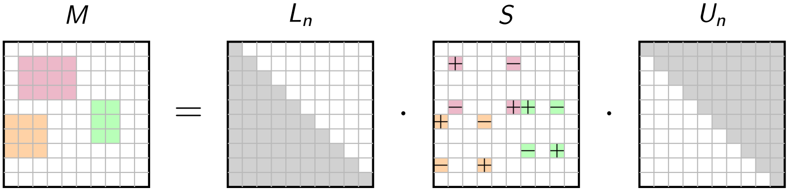

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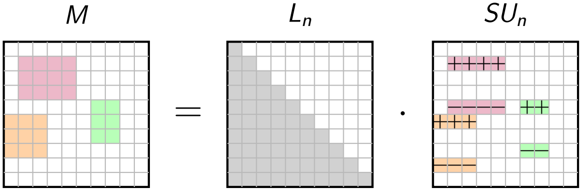

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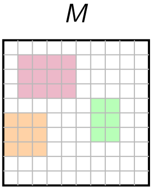

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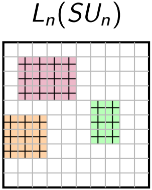

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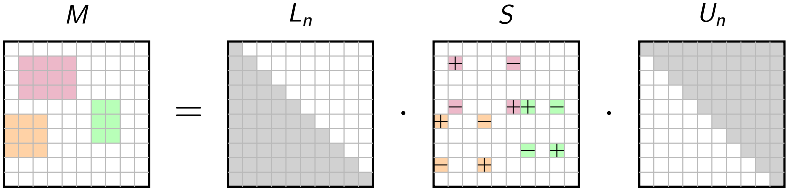
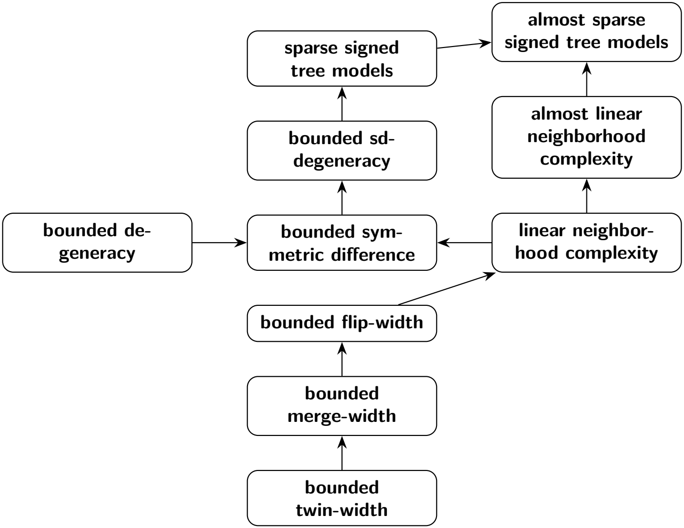


Theorem

MN can be computed in $O(n^2)$ for any adjacency matrix M of an n-vertex graph with a sparse signed tree model and $n \times n$ matrix N.

Hasse diagram of our families of interest



Speed-ups in classes of linear neighborhood complexity

Let $\mathcal{C}$ be any class of linear neighborhood complexity

Theorem

All-Pairs Shortest Path can be solved in time $O(n^2 \log n \log^* n)$ in n-vertex graphs of $\mathcal{C}$.

Theorem

MN can be computed in time $O(n^2 \log n)$ for M the adjacency matrix of an n-vertex graph of $\mathcal{C}$ and N any $n \times n$ matrix.

Speed-ups in classes of linear neighborhood complexity

Let $\mathcal{C}$ be any class of linear neighborhood complexity

Conjecture

All-Pairs Shortest Path *can be solved in time* $O(n^2)$ *in* n*-vertex graphs of* $\mathcal{C}$.

Conjecture

MN *can be computed in time* $O(n^2)$ *for* M *the adjacency matrix of an* n*-vertex graph of* $\mathcal{C}$ *and* N *any* $n \times n$ *matrix.*

Questions

- Is there a factorial class *without* sparse signed tree models?
- Near-quadratic algorithms in graphs of bounded symmetric difference?
- Improving the $O(n^2 \log\log n)$ computation of signed tree models with $O(n \log n)$ transversal pairs in classes of lin. nc?
- Other (algorithmic) uses of signed tree models?
  Dynamic algorithms?
- Comparing bounded merge-width and signed tree models whose tree-minors have bounded degeneracy (expansion)?

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Thank you for your attention!