# Twin-width and Logic

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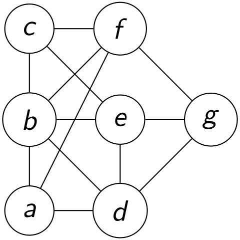
ENS Lyon, LIP

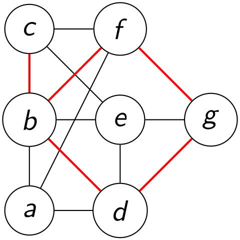
July 24th, HIGHLIGHTS'23, Kassel, Germany

# Graphs



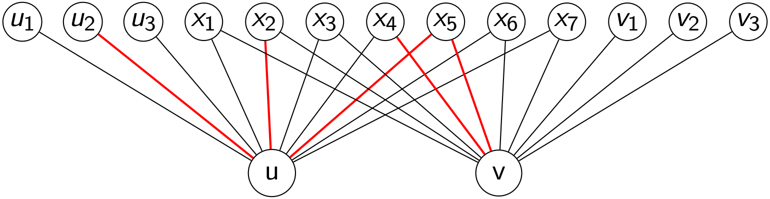

Two outcomes between a pair of vertices:
edge or non-edge

# Trigraphs



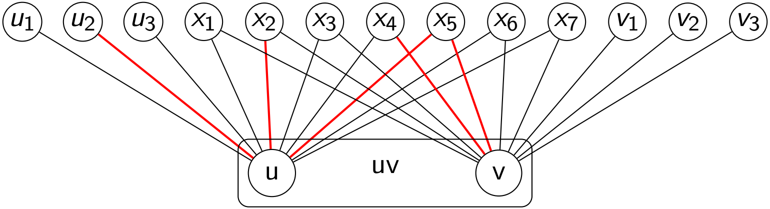

Three outcomes between a pair of vertices:
edge, or non-edge, or red edge (error edge)

# Contractions in trigraphs




Identification of two non-necessarily adjacent vertices

# Contractions in trigraphs

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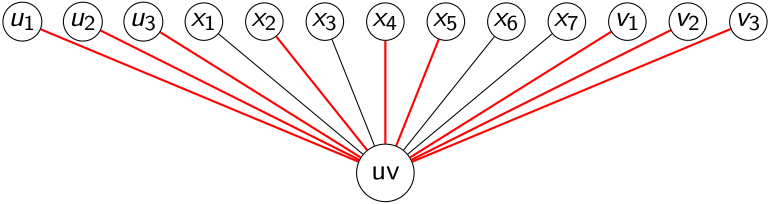
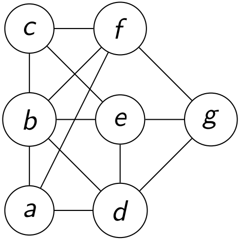
# Contractions in trigraphs




edges to $N(u) \triangle N(v)$ turn red, for $N(u) \cap N(v)$ red is absorbing

# Contraction sequence




A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that
$G_i$ is obtained by performing one contraction in $G_{i+1}$.

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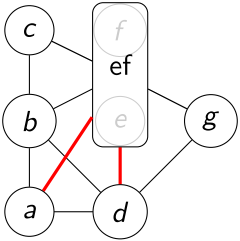


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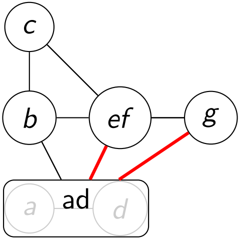


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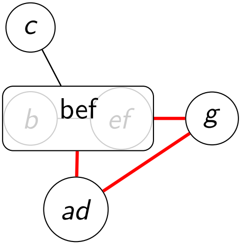


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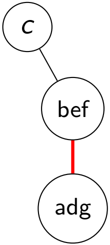


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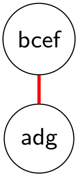


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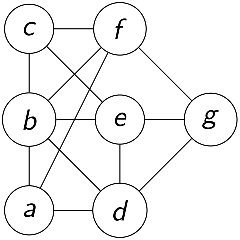
# Twin-width

tww($G$): Least integer $d$ such that $G$ admits a contraction sequence where all trigraphs have *maximum red degree* at most $d$.




Maximum red degree = 0
**overall maximum red degree = 0**

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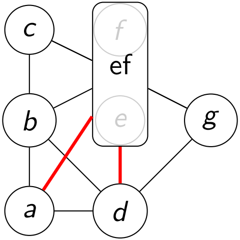


Maximum red degree = 2

**overall maximum red degree = 2**

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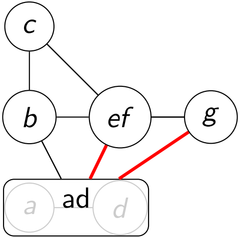


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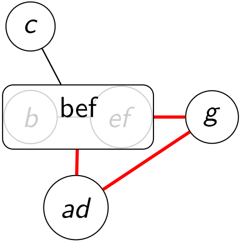


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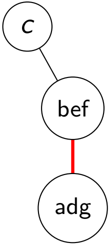


Maximum red degree $= 1$

**overall maximum red degree = 2**

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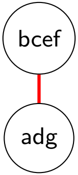


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**overall maximum red degree = 2**

# Twin-width

tww($G$): Least integer $d$ such that $G$ admits a contraction sequence where all trigraphs have *maximum red degree* at most $d$.

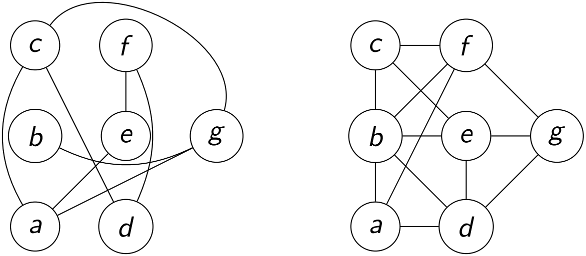

Maximum red degree $= 0$

**overall maximum red degree = 2**

# Simple operations preserving small twin-width

- complementation: remains the same
- taking induced subgraphs: may only decrease
- adding one vertex linked arbitrarily: at most "doubles"
- substitution, lexicographic product: max of the twin-widths

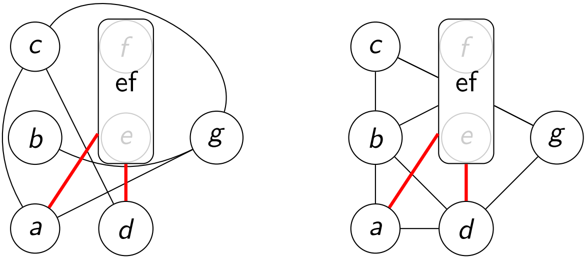
# Complementation




$\overline{G}$ $G$

$$\text{tww}(\overline{G}) = \text{tww}(G)$$

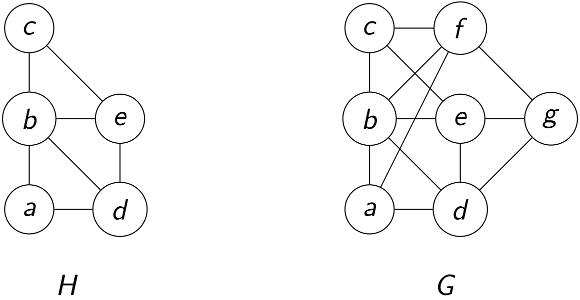
# Complementation




$\overline{G_6}$ $G_6$

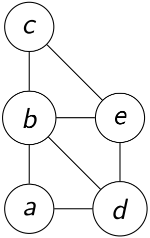
$$\text{tww}(\overline{G}) = \text{tww}(G)$$

# Induced subgraph

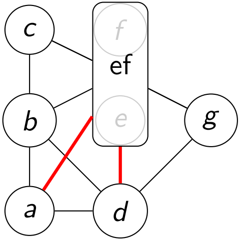



$$\text{tww}(H) \leqslant \text{tww}(G)$$

# Induced subgraph

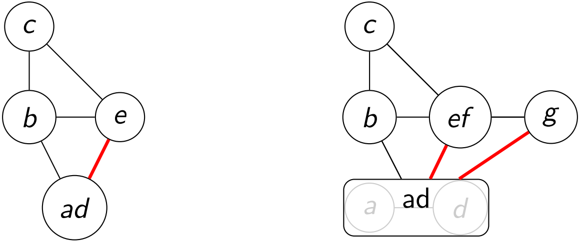



$H$



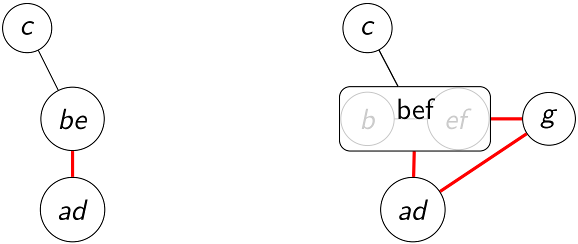

Ignore absent vertices

# Induced subgraph




Mimic the contractions otherwise

# Induced subgraph

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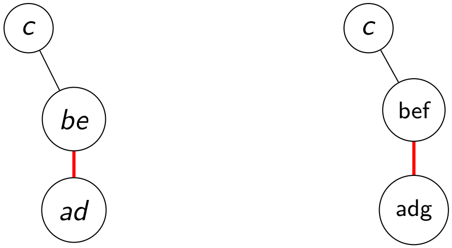


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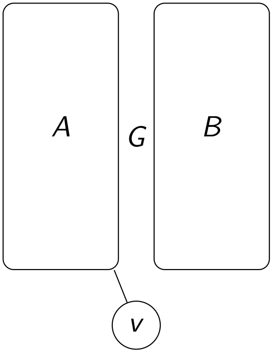
# Adding one vertex $v$ (arbitrarily linked)




Split every part into their part in $A$ and in $B$ until the very end

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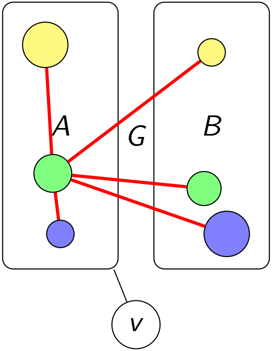
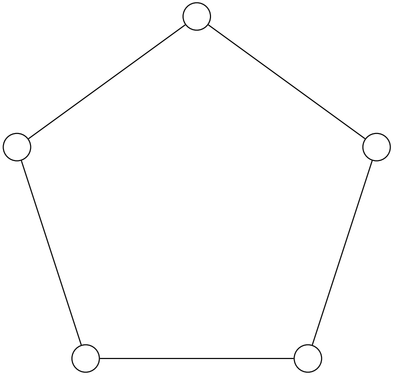



Split every part into their part in $A$ and in $B$ until the very end

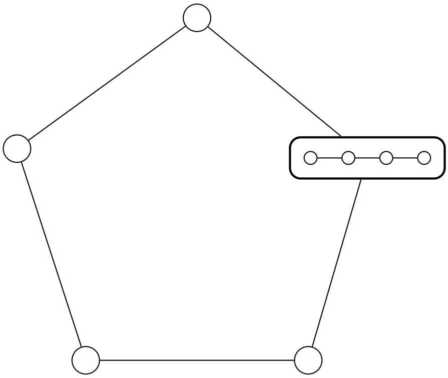
$$\text{tww}(G+v) \leqslant 2 \cdot \text{tww}(G) + 1$$

# Substitution and lexicographic product



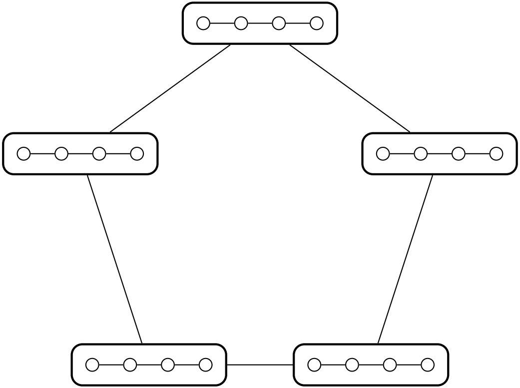
$G = C_5$

# Substitution and lexicographic product



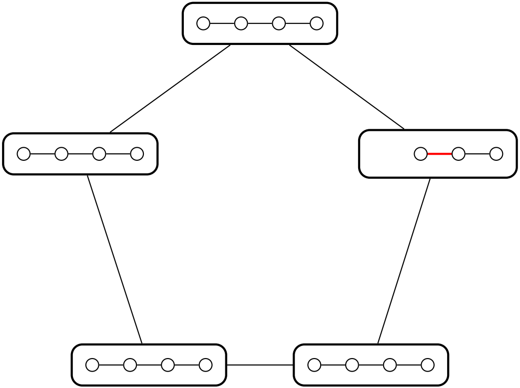
$G = C_5$, $H = P_4$, substitution $G[v \leftarrow H]$

# Substitution and lexicographic product



$G = C_5$, $H = P_4$,   lexicographic product $G[H]$

# Substitution and lexicographic product



More generally any modular decomposition

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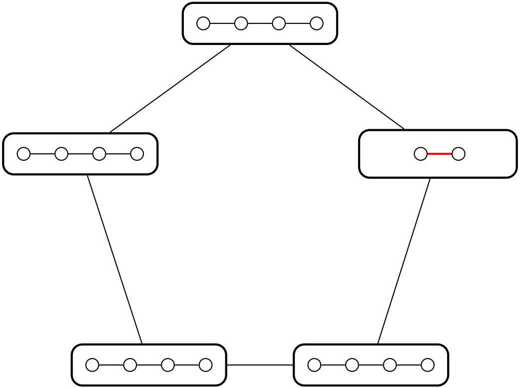

More generally any modular decomposition

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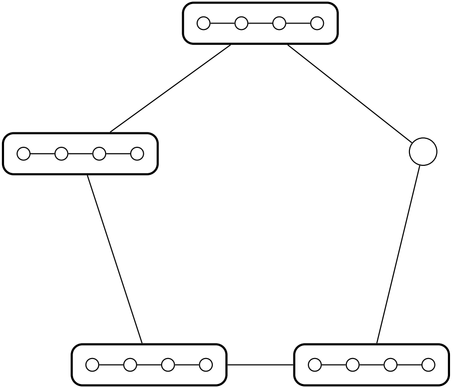
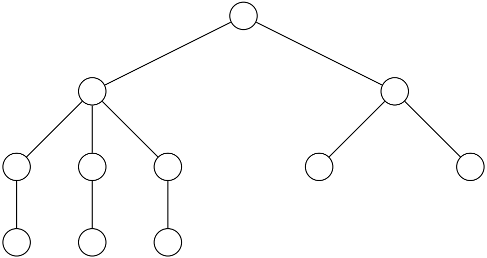


$$\text{tww}(G[H]) = \max(\text{tww}(G), \text{tww}(H))$$

# Classes with bounded twin-width

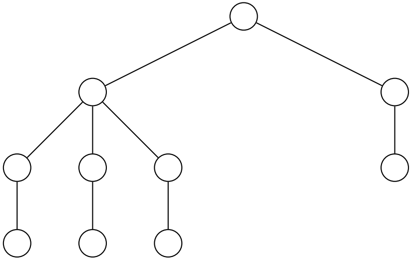
- cographs = twin-width 0
- trees, bounded treewidth, clique-width/rank-width
- grids
- ...

# Trees



If possible, contract two twin leaves

# Trees



If not, contract a deepest leaf with its parent

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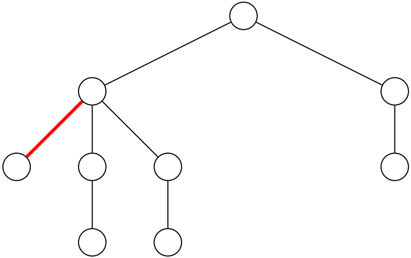

If not, contract a deepest leaf with its parent

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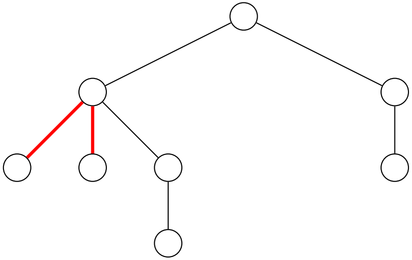
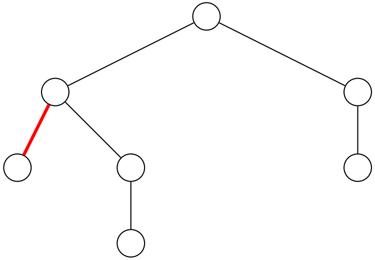


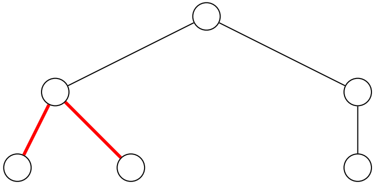
If possible, contract two twin leaves

# Trees



Cannot create a red degree-3 vertex

# Trees



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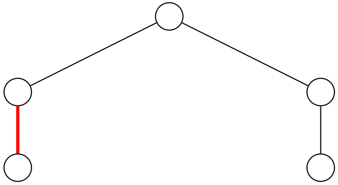

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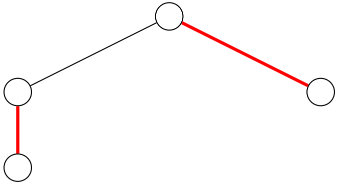

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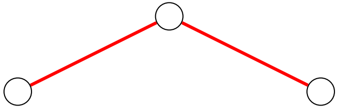

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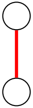

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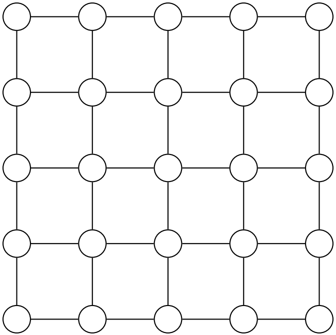
# Trees

Generalization to bounded *treewidth* and even bounded *rank-width*

# Grids



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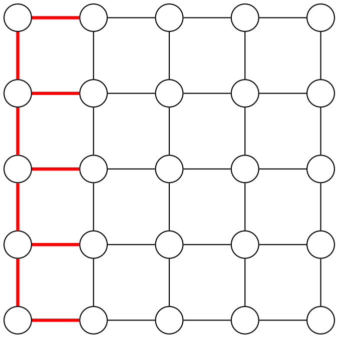

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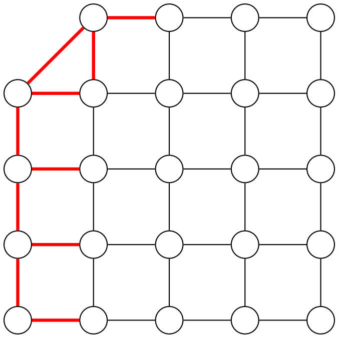

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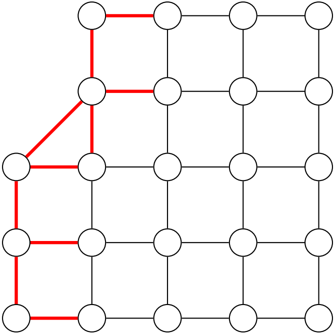

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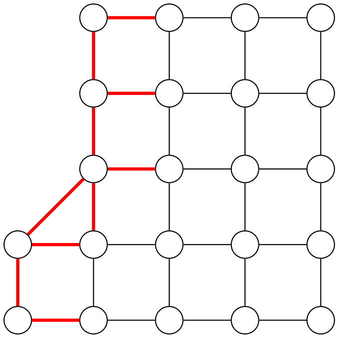

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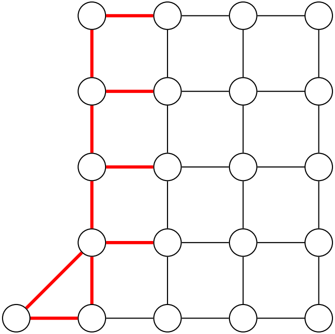

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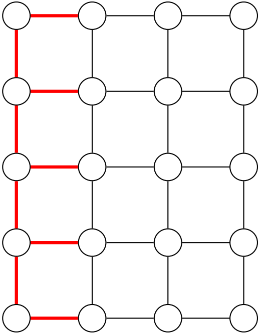
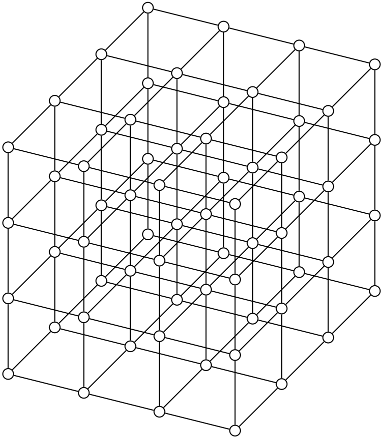


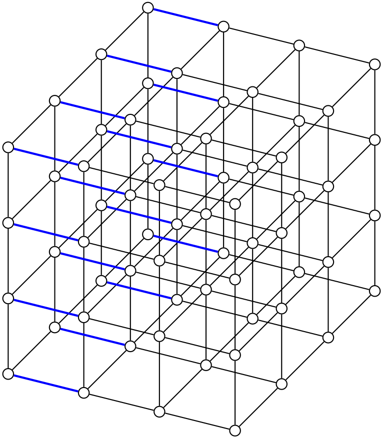
4-sequence for planar grids

# 3-dimensional grids



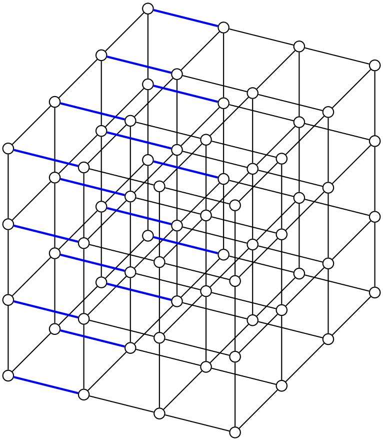
Contains arbitrary large clique minors

# 3-dimensional grids



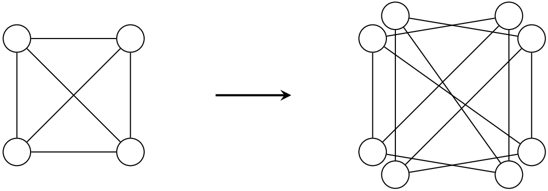
Contract the blue edges in any order → 12-sequence

# 3-dimensional grids



The $d$-dimensional grid has twin-width $\leqslant 4d$ (even $3d$)

# 2-lifts, expanders with bounded twin-width



split each vertex in 2, replace each edge by 1 of the 2 matchings

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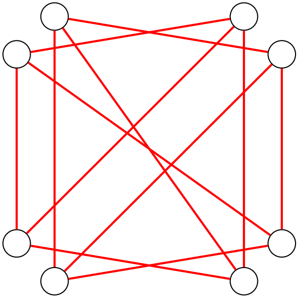

Iterated 2-lifts of $K_4$ have twin-width at most 6

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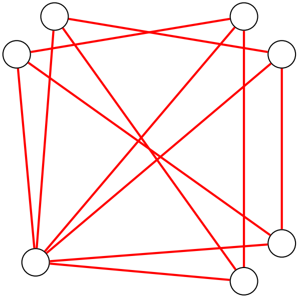

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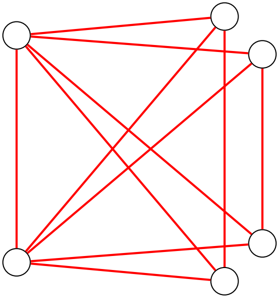

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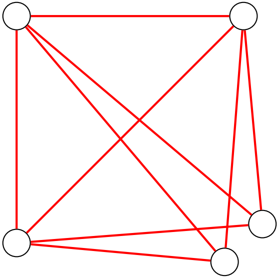
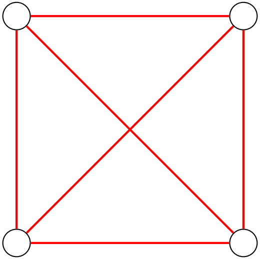


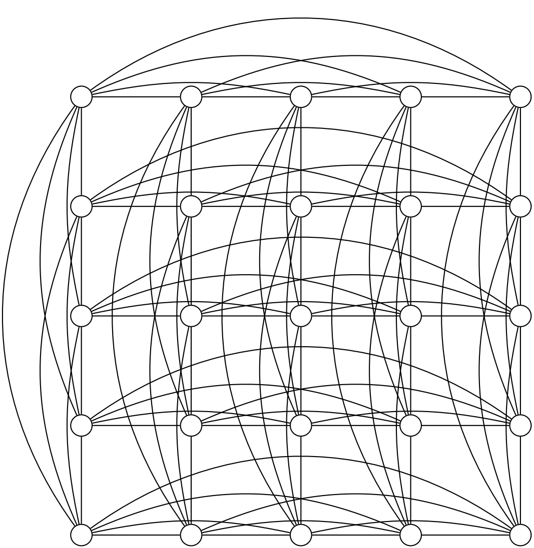
Iterated 2-lifts of $K_4$ have twin-width at most 6

# 2-lifts, expanders with bounded twin-width



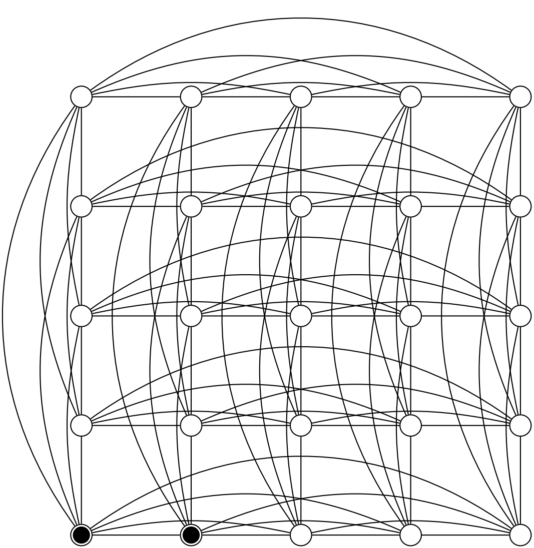
Iterated 2-lifts of $K_4$ have twin-width at most 6
but no balanced separators of size $o(n)$

# First example of unbounded twin-width



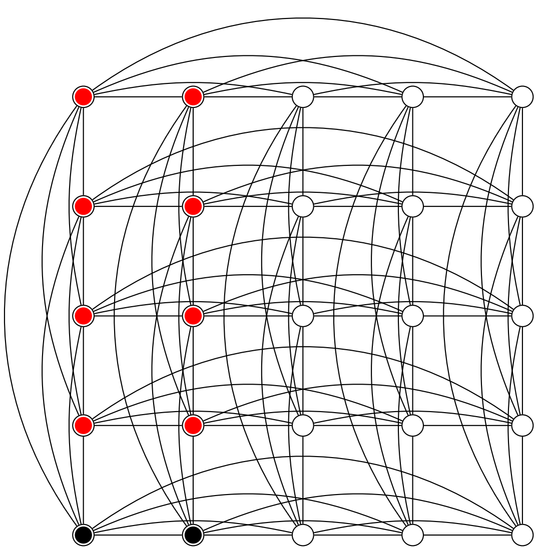
Line graph of a biclique a.k.a. rook graph

# First example of unbounded twin-width



No pair of near twins

# First example of unbounded twin-width



No pair of near twins

# Universal bipartite graph

No $O(1)$-contraction sequence:
**twin-width is *not* an iterated identification of near twins.**

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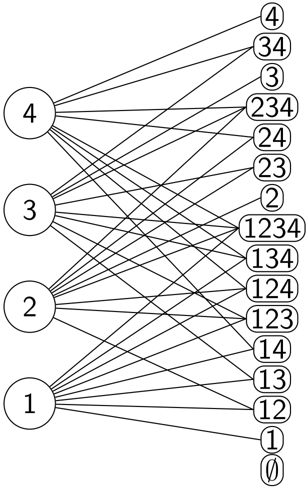

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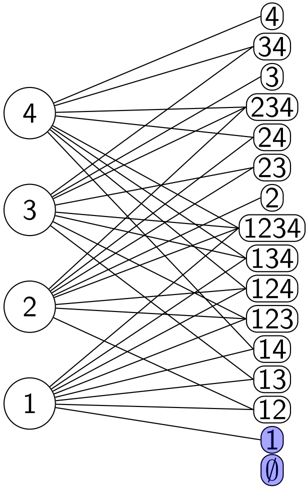

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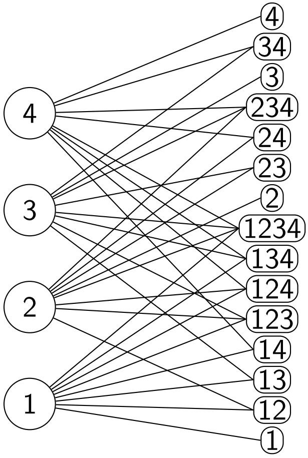

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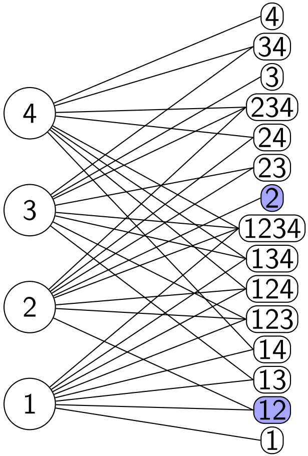

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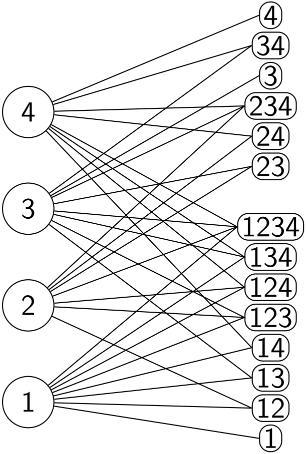

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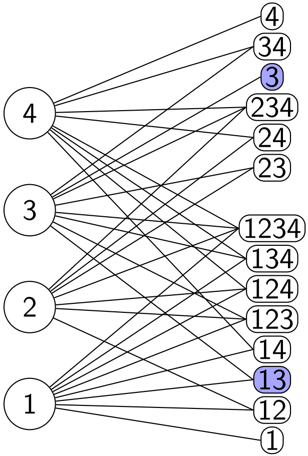

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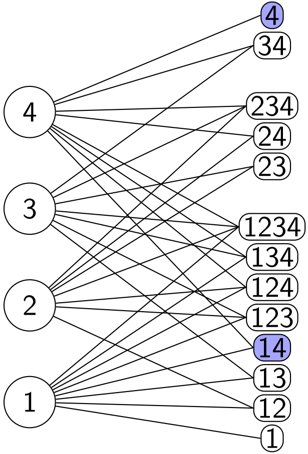

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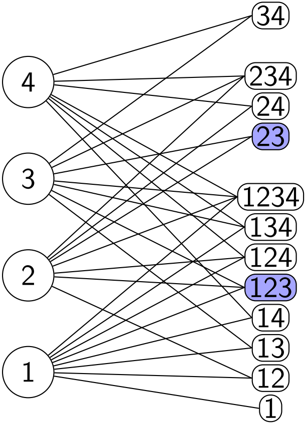

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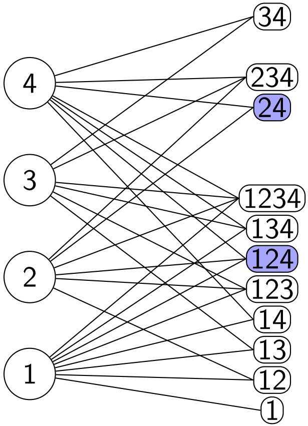

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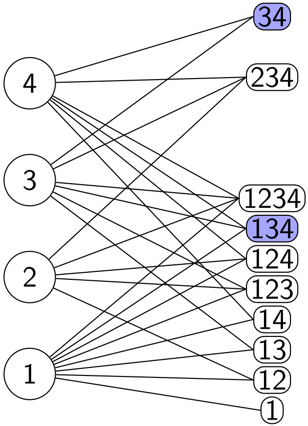

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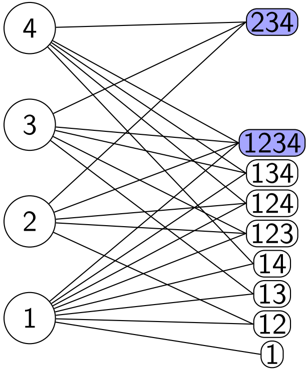

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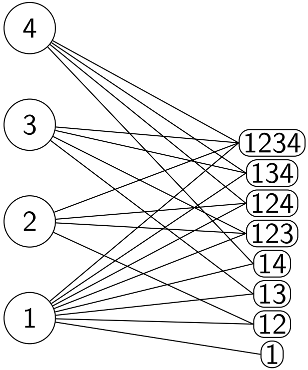

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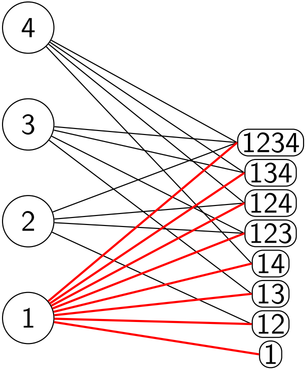

# Characterizing *bounded twin-width* via a vertex ordering

Our next goal:

**Theorem ((informal) B., Kim, Thomassé, Watrigant '20)**

*If each graph of a class $\mathcal{C}$ admits at least one simple (in a sense that we will define) adjacency matrix then $\mathcal{C}$ has bounded twin-width.*

# Twin-width in the language of matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Encode a bipartite graph (or, if symmetric, any graph)

# Twin-width in the language of matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Contraction of two columns (similar with two rows)

# Twin-width in the language of matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & r & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & r & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & r & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

How is the twin-width (re)defined?

# Twin-width in the language of matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & r & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & r & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & r & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

How to tune it for non-bipartite graph?

# Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

$$\left[\begin{array}{c|c|c|c|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

# Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |

Maximum number of non-constant zones per column or row part
= **error value**

# Partition viewpoint

Matrix partition: partitions of the row set and of the column set
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| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |

Maximum number of non-constant zones per column or row part
. . . until there are a single row part and column part

# Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |

**Twin-width as maximum error value of a contraction/division sequence**

# Grid minor

$t$-grid minor: $t \times t$-division where every cell is non-empty

Non-empty cell: contains at least one 1 entry

$$\left[\begin{array}{cc|cc|cc|cc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

4-grid minor

# Grid minor

$t$-grid minor: $t \times t$-division where every cell is non-empty

Non-empty cell: contains at least one 1 entry

$$
\left[\begin{array}{cc|cc|cc|cc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]
$$

4-grid minor

A matrix is said $t$-**grid free** if it does not have a $t$-grid minor

# Mixed minor

Mixed cell: not horizontal nor vertical

$$\left[\begin{array}{cc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

3-mixed minor

# Mixed minor

Mixed cell: not horizontal nor vertical

$$\left[\begin{array}{cc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

3-mixed minor

Every mixed cell is witnessed by a $2 \times 2$ square = **corner**

# Mixed minor

Mixed cell: not horizontal nor vertical

$$\left[\begin{array}{cc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

3-mixed minor

A matrix is said $t$-**mixed free** if it does not have a $t$-mixed minor

# Mixed value

$$
\begin{array}{c}
R_4 \\ R_3 \\ \\ R_2 \\ \\ R_1 \\ \\
\end{array}
\left[\begin{array}{cc|ccc|c|cc}
1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & \color{red}{1} & \color{red}{1} & \color{red}{0} & 1 & 0 & 0 \\
1 & 0 & \color{red}{1} & \color{red}{0} & \color{red}{1} & 0 & 0 & 1
\end{array}\right]
$$

$C_2$

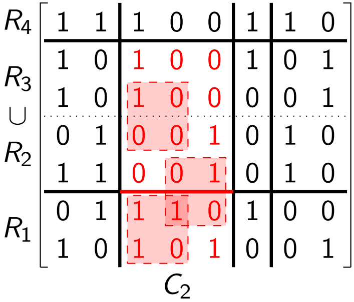
$\approx$ (maximum) number of cells with a corner per row/column part

# Mixed value

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $R_4$ | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 0 |
| $R_3$ | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| $R_2$ | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 |
| | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 |
| $R_1$ | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 |
| | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 |
| | | | | $C_2$ | | | | |

But we add the number of *boundaries* containing a corner

# Mixed value




$\therefore$ merging row parts do not increase mixed value of column part

# Twin-width and mixed freeness

Theorem (B., Kim, Thomassé, Watrigant '20)

*If $G$ admits* ***a*** *$t$-mixed free adjacency matrix, then* $tww(G) = 2^{2^{O(t)}}$.

# Twin-width and mixed freeness

Theorem (B., Kim, Thomassé, Watrigant '20)

*If* $\exists \sigma$ *s.t.* $Adj_\sigma(G)$ *is t-mixed free, then* $tww(G) = 2^{2^{O(t)}}$.

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*If* $\exists\sigma$ *s.t.* $Adj_\sigma(G)$ *is* $t$*-mixed free, then* $tww(G) = 2^{2^{O(t)}}$.

**Step 1: find a division sequence** $(\mathcal{D}_i)_i$ **with mixed value** $f(t)$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |

Merge consecutive parts greedily

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| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |

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| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |

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**Step 1: find a division sequence $(\mathcal{D}_i)_i$ with mixed value $f(t)$**

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Stuck, removing every other separation $\rightarrow \frac{f(t)}{2}$ mixed cells per part

# Stanley-Wilf conjecture / Marcus-Tardos theorem

Auxiliary 0,1-matrix with one entry per cell: a 1 iff the cell is mixed

### Question

*For every $k$, is there a $c_k$ such that every $n \times m$ $0,1$-matrix with at least $c_k$ 1 per row and column admits a $k$-grid minor?*

# Stanley-Wilf conjecture / Marcus-Tardos theorem

Auxiliary 0,1-matrix with one entry per cell: a 1 iff the cell is mixed

### Conjecture (reformulation of Füredi-Hajnal conjecture '92)

*For every $k$, there is a $c_k$ such that every $n \times m$ $0,1$-matrix with at least $c_k \max(n, m)$ 1 entries admits a $k$-grid minor.*

# Stanley-Wilf conjecture / Marcus-Tardos theorem

Auxiliary 0,1-matrix with one entry per cell: a 1 iff the cell is mixed

## Conjecture (reformulation of Füredi-Hajnal conjecture '92)

*For every $k$, there is a $c_k$ such that every $n \times m$ $0,1$-matrix with at least $c_k \max(n, m)$ 1 entries admits a $k$-grid minor.*

## Conjecture (Stanley-Wilf conjecture '80s)

*Any proper permutation class contains only $2^{O(n)}$ $n$-permutations.*

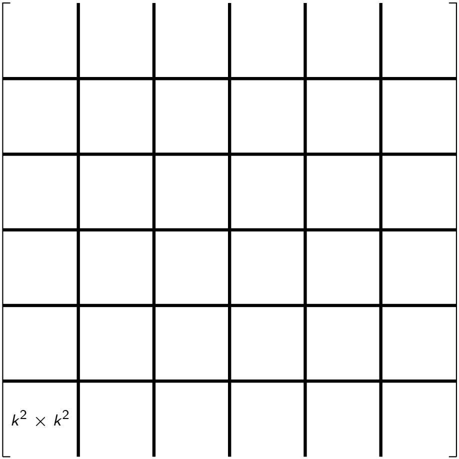
Klazar showed Füredi-Hajnal $\Rightarrow$ Stanley-Wilf in 2000
**Marcus and Tardos showed Füredi-Hajnal in 2004**

# Marcus-Tardos one-page inductive proof

$$M = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Let $M$ be an $n \times n$ 0,1-matrix without $k$-grid minor

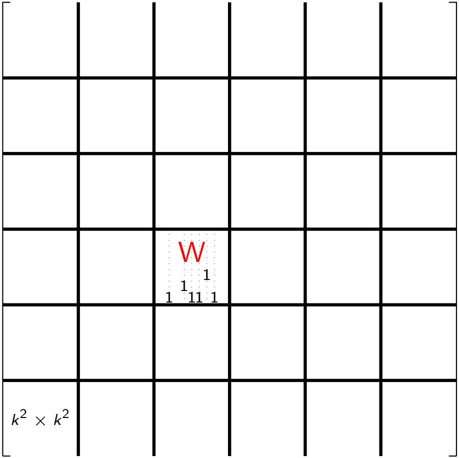
# Marcus-Tardos one-page inductive proof

$M =$ 


Draw a regular $\frac{n}{k^2} \times \frac{n}{k^2}$ division on top of $M$

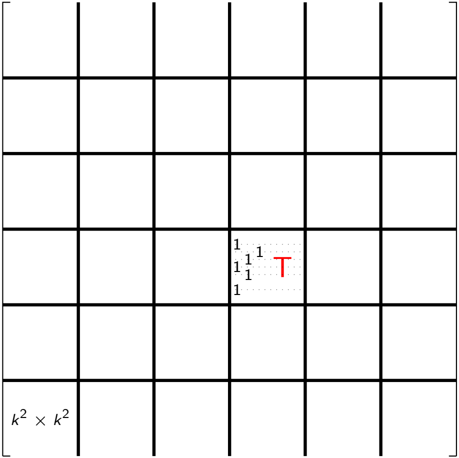
# Marcus-Tardos one-page inductive proof

$M =$



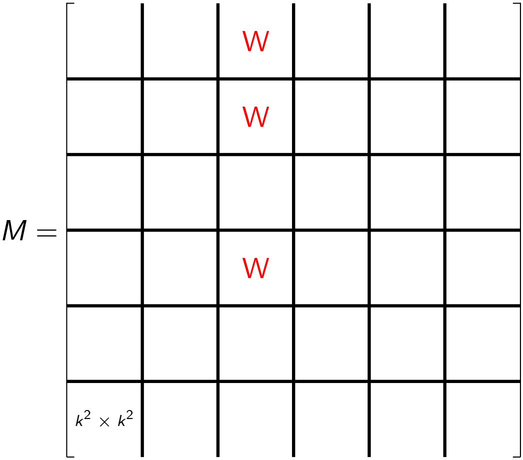

A cell is *wide* if it has at least $k$ columns with a 1

# Marcus-Tardos one-page inductive proof

$M =$ 

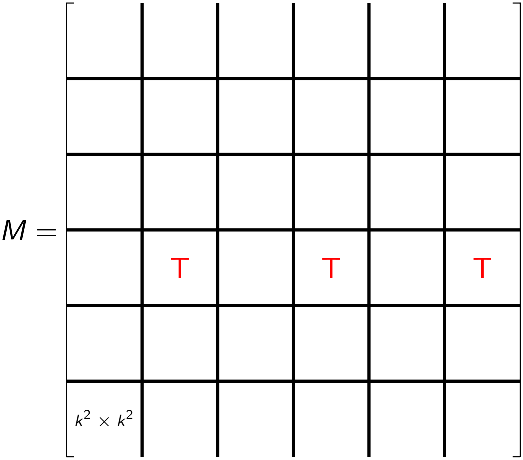

A cell is *tall* if it has at least $k$ rows with a 1

# Marcus-Tardos one-page inductive proof




There are less than $k\binom{k^2}{k}$ wide cells per column part. Why?

# Marcus-Tardos one-page inductive proof



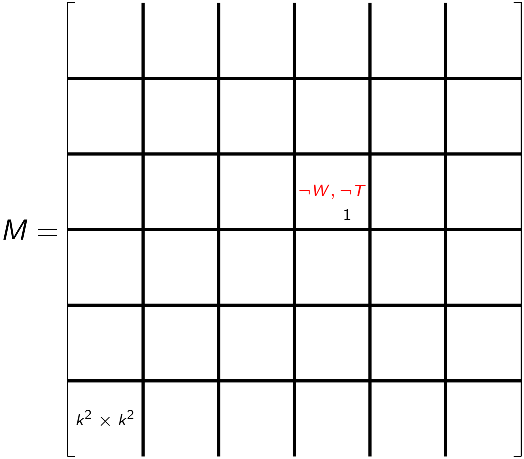

There are less than $k\binom{k^2}{k}$ tall cells per row part

# Marcus-Tardos one-page inductive proof

$$M = \begin{bmatrix} & & W & & & \\ & W & W & & & T \\ & & & & & \\ & T & W & T & & T \\ & & T & & & \\ k^2 \times k^2 & & & & & W \end{bmatrix}$$

In W and T, at most $2 \cdot \frac{n}{k^2} \cdot k\binom{k^2}{k} \cdot k^4 = 2k^3\binom{k^2}{k}n$ entries 1

# Marcus-Tardos one-page inductive proof




There are at most $(k-1)^2 c_k \frac{n}{k^2}$ remaining 1. Why?

# Marcus-Tardos one-page inductive proof

$$M = \left[\begin{array}{c|c|c|c|c|c} & & \mathrm{W} & & & \\ \hline & \mathrm{W} & \mathrm{W} & & & \mathrm{T} \\ \hline & & & \neg W, \neg T \;\; 1 & & \\ \hline & \mathrm{T} & \mathrm{W} & \mathrm{T} & & \mathrm{T} \\ \hline & & \mathrm{T} & & & \\ \hline k^2 \times k^2 & & & & & \mathrm{W} \end{array}\right]$$

Choose $c_k = 2k^4\binom{k^2}{k}$ so that $(k-1)^2 c_k \frac{n}{k^2} + 2k^3\binom{k^2}{k} n \leqslant c_k n$

# Twin-width and mixed freeness

Theorem (B., Kim, Thomassé, Watrigant '20)

*If $\exists \sigma$ s.t. $Adj_\sigma(G)$ is $t$-mixed free, then $tww(G) = 2^{2^{O(t)}}$.*

**Step 1: find a division sequence $(\mathcal{D}_i)_i$ with mixed value $f(t)$**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |

Stuck, removing every other separation $\rightarrow \frac{f(t)}{2}$ mixed cells per part

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**Impossible!**

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**Step 1: find a division sequence $(\mathcal{D}_i)_i$ with mixed value $f(t)$**
**Step 2: find a contraction sequence with error value $g(t)$**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |

Refinement of $\mathcal{D}_i$ where each part coincides on the non-mixed cells

# Twin-width and mixed freeness

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# Twin-width and mixed freeness

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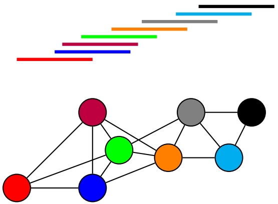
*If $\exists\sigma$ s.t. $Adj_\sigma(G)$ is $t$-mixed free, then $tww(G) = 2^{2^{O(t)}}$.*

Now to bound the twin-width of a class $\mathcal{C}$:

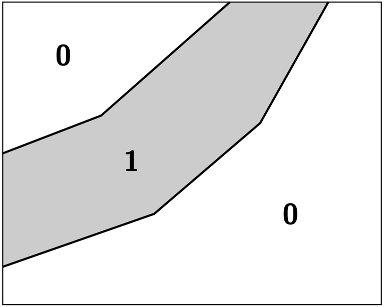
1) Find a *good* vertex-ordering procedure
2) Argue that, in this order, a $t$-mixed minor would conflict with $\mathcal{C}$

# Unit interval graphs

Intersection graph of unit segments on the real line

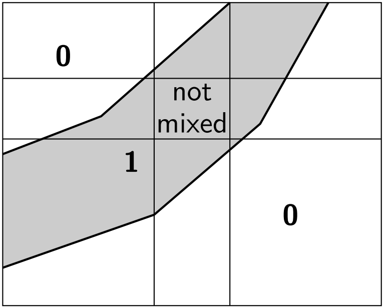


# Unit interval graphs




order by left endpoints

# Unit interval graphs




No 3-by-3 grid has all 9 cells crossed by two non-decreasing curves

# Graph minors

Formed by **vertex deletion**, **edge deletion**, and **edge contraction**

A graph $G$ is *H-minor free* if $H$ is not a minor of $G$

A graph class is *H-minor free* if all its graphs are

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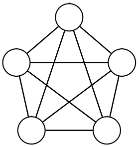
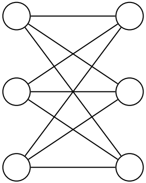
A graph class is *H-minor free* if all its graphs are

Planar graphs are exactly the graphs without $K_5$ or $K_{3,3}$ as a minor



$K_5$

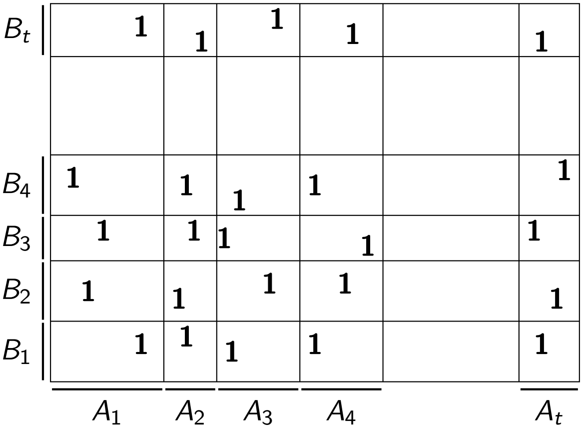


$K_{3,3}$

# Bounded twin-width – $K_t$-minor free graphs

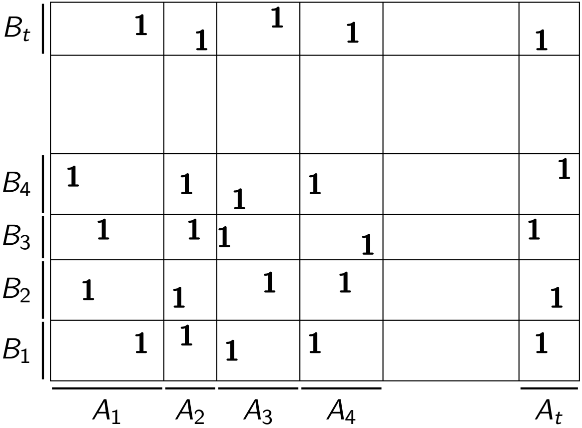
Given a hamiltonian path, we would just use this order

# Bounded twin-width – $K_t$-minor free graphs




Contracting the $2t$ subpaths yields a $K_{t,t}$-minor, hence a $K_t$-minor

# Bounded twin-width – $K_t$-minor free graphs




Instead we use a specially crafted lex-DFS discovery order

Theorem (B., Geniet, Kim, Thomassé, Watrigant '20 & '21)

*The following classes have bounded twin-width, and $O(1)$-sequences can be computed in polynomial time.*

- *Bounded rank-width, and even, boolean-width graphs,*
- *every hereditary proper subclass of permutation graphs,*
- *posets of bounded antichain size (seen as digraphs),*
- *unit interval graphs,*
- *$K_t$-minor free graphs,*
- *map graphs,*
- *subgraphs of $d$-dimensional grids,*
- *$K_t$-free unit $d$-dimensional ball graphs,*
- *$\Omega(\log n)$-subdivisions of all the $n$-vertex graphs,*
- *cubic expanders defined by iterative random 2-lifts from $K_4$,*
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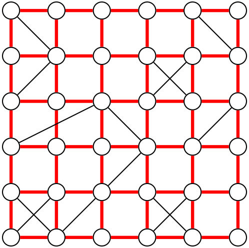
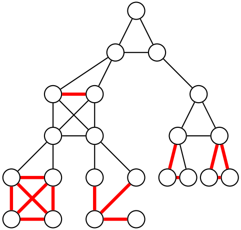
**Can we solve problems faster, given an $O(1)$-sequence?**

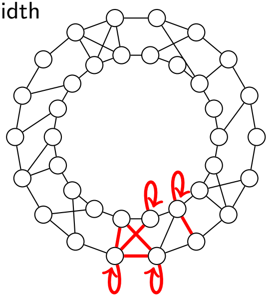
# Different conditions imposed in the sequence of red graphs



bd degree: defines bd twin-width

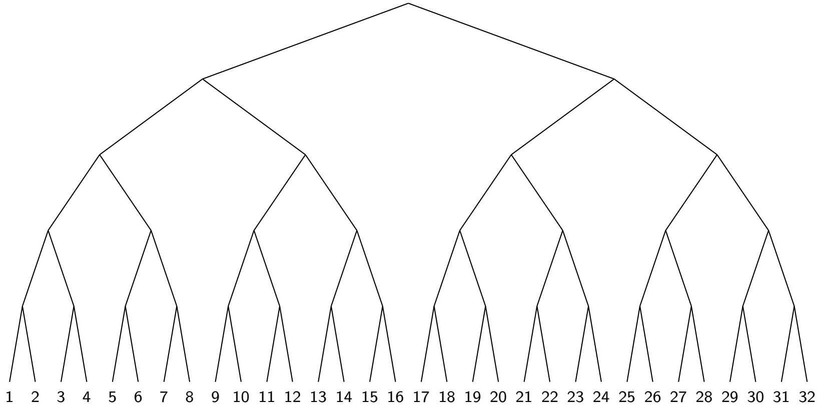


bd component: redefines bd cliquewidth



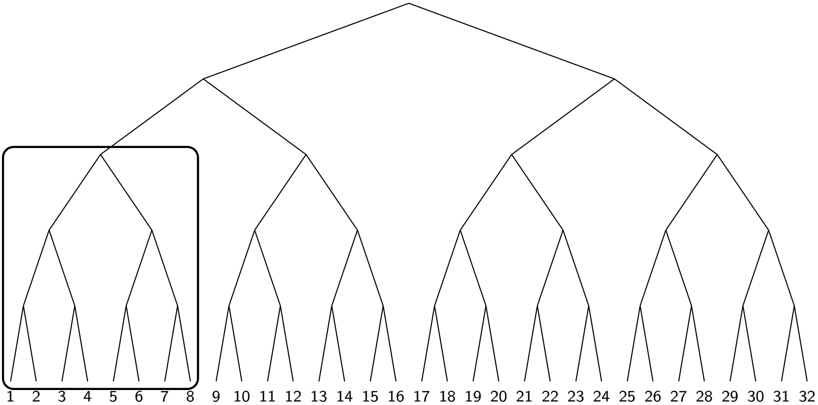
bd #edges: redefines bd linear cliquewidth

# Bd boolean-width ⇒ bd component twin-width



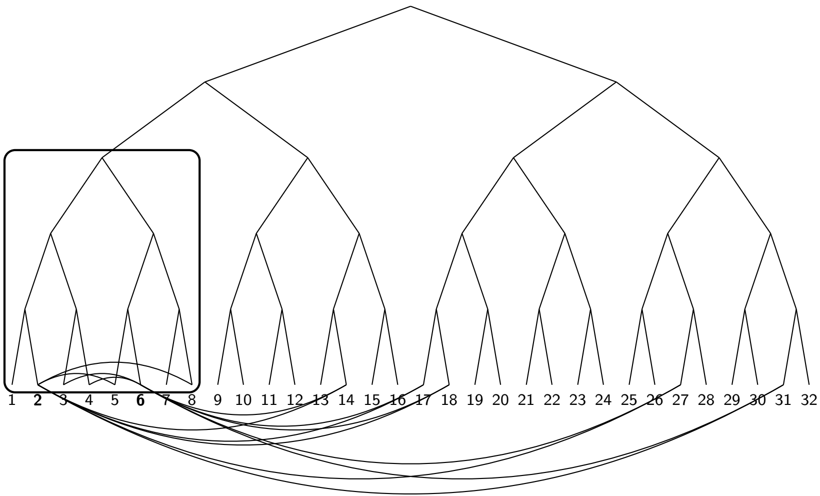

Bd boolean-width: binary tree layout s.t. every edge cut in the tree induces a bipartition with bd # distinct neighborhoods

# Bd boolean-width ⇒ bd component twin-width



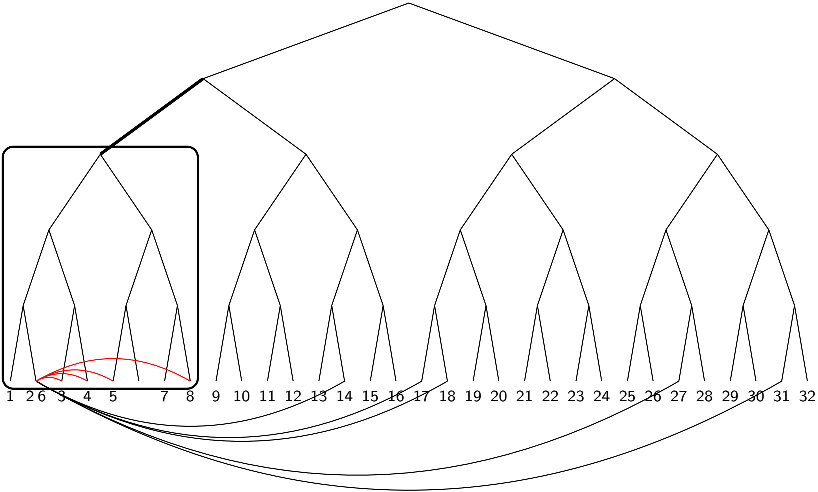

There is a subtree on $\ell \in [d+1, 2d]$ leaves, where $d$ bounds the number of single-vertex neighborhoods in a bipartition

# Bd boolean-width ⇒ bd component twin-width




Two vertices have the same neighborhood outside of this subtree

# Bd boolean-width ⇒ bd component twin-width




Contracting them preserves the upper bound at $2d$
on the size of red connected components

# Component twin-width and boolean-width are tied

### Theorem (B., Kim, Reinald, Thomassé '22)

*A class has bounded component twin-width iff it has bounded boolean-width/cliquewidth/rank-width.*

### Proof.

We just saw one direction.

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When red components merge, their subtree gets a same parent. □

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Conversely, build the binary tree layout based on the contractions.
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**Theorem (Baril, Couceiro, Lagerkvist '22)**

*The component twinwidth plus one is at least the cliquewidth and at most twice the cliquewidth.*

# Efficient parameterized algorithms via this characterization

Solve 3-Coloring on a graph $G$ with a contraction sequence s.t. all red graphs have components of size at most $d$

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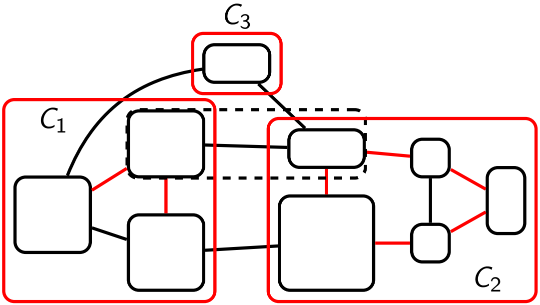


For every red component $C$ keep every profile $V(C) \to 2^{\{1,2,3\}} \setminus \{\emptyset\}$ realizable by a proper 3-coloring of $G\langle C \rangle$

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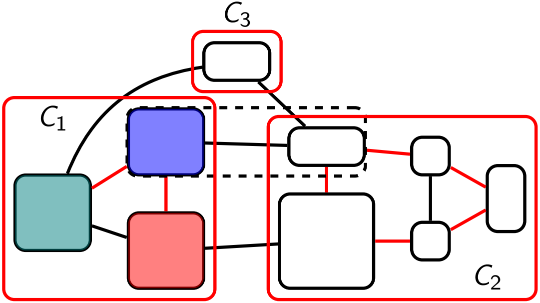


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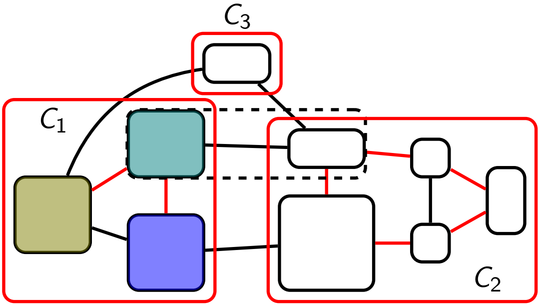


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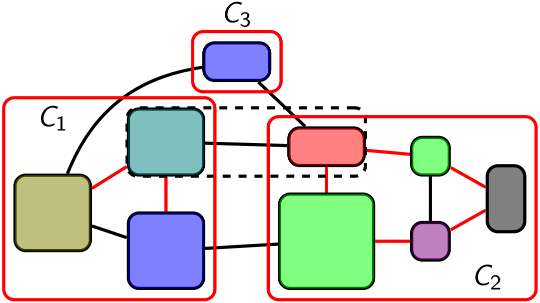


Some tuples of the at most $d+1$ profiles
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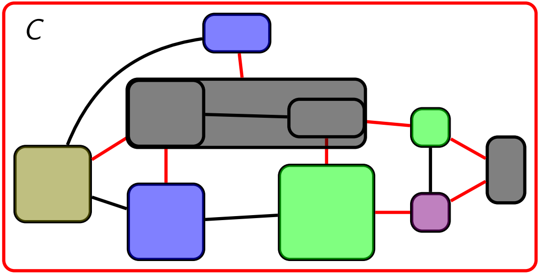


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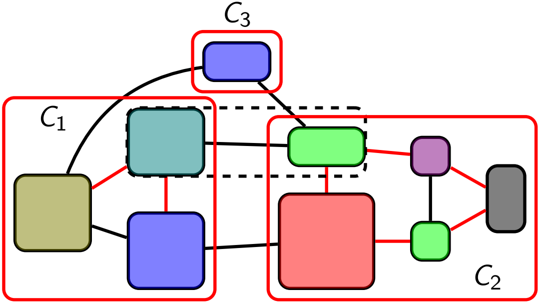


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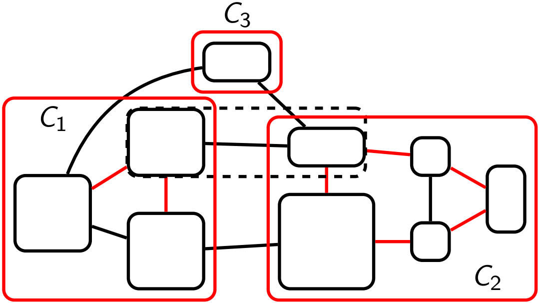


Initialization: time $3n$
Update: time $7^d d^2$
Total: time $7^d d^2 n$
End: still a profile on the single vertex *containing* the whole graph?

# Formulas, sentences, and model checking

GRAPH FO/MSO MODEL CHECKING **Parameter:** $|\varphi|$
**Input:** A graph $G$ and a first-order/monadic second-order sentence $\varphi \in FO/MSO(\{E\})$
**Question:** $G \models \varphi$?

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Example:

$$\varphi = \exists x_1 \exists x_2 \cdots \exists x_k \forall x \bigvee_{1 \leqslant i \leqslant k} x = x_i \vee \bigvee_{1 \leqslant i \leqslant k} E(x, x_i) \vee E(x_i, x)$$

$G \models \varphi$? $\Leftrightarrow$

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$G \models \varphi$? $\Leftrightarrow$ 3-COLORING

# The lens of contraction sequences

| Class of bounded | constraint on red graphs | efficient model-checking |
|---|---|---|
| linear rank-width | bd #edges | MSO |
| rank-width | bd component | **MSO** |
| twin-width | bd degree | **?** |

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We will reprove the result in bold, and fill the **?**

# Courcelle's theorems

We will reprove with contraction sequences:

**Theorem (Courcelle, Makowsky, Rotics '00)**

*MSO model checking can be solved in time* $f(|\varphi|, d) \cdot |V(G)|$ *given a witness that the clique-width/component twin-width of the input* $G$ *is at most* $d$*.*

*generalizes*

**Theorem (Courcelle '90)**

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- as the incidence graph preserves bounded treewidth, possible edge-set quantification
- linear FPT approximation for treewidth
- (polynomial) FPT approximation for clique-width

# Rank-$k$ $m$-types

Sets of non-equivalent formulas/sentences of quantifier rank at most $k$ satisfied by a fixed structure:

$$\mathrm{tp}_k^{\mathcal{L}}(\mathscr{A}, \vec{a} \in A^m) = \{\varphi(\vec{x}) \in \mathcal{L}[k] : \mathscr{A} \models \varphi(\vec{a})\},$$

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## Theorem (folklore)

*For $\mathcal{L} \in \{FO, MSO\}$, the number of rank-$k$ $m$-types is bounded by a function of $k$ and $m$ only.*

## Proof.

"$\mathcal{L}[k+1]$ are Boolean combinations of $\exists x \mathcal{L}[k]$." □

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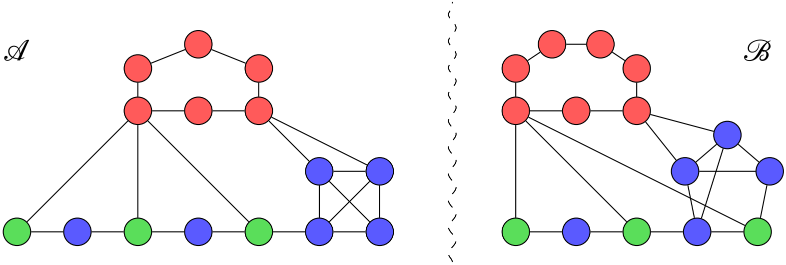
**Rank-$k$ types partition the graphs into $g(k)$ classes.**

Efficient Model Checking = quickly finding the class of the input.

# FO Ehrenfeucht-Fraissé game




2-player game on two $\sigma$-structures $\mathscr{A}, \mathscr{B}$ (for us, colored graphs)

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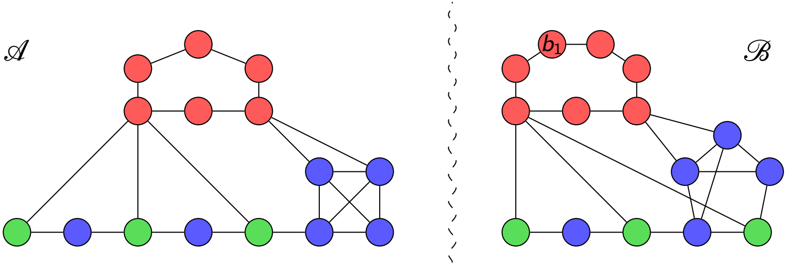


At each round, Spoiler picks a structure ($\mathscr{B}$) and a vertex therein

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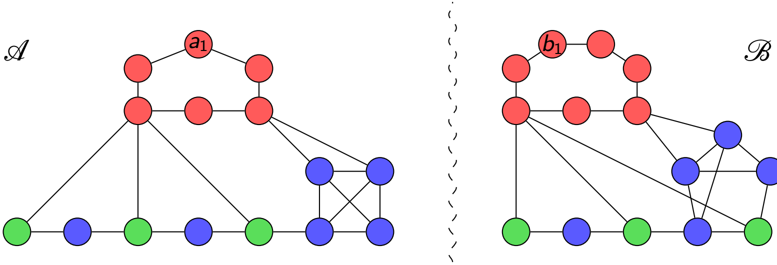


Duplicator answers with a vertex in the other structure

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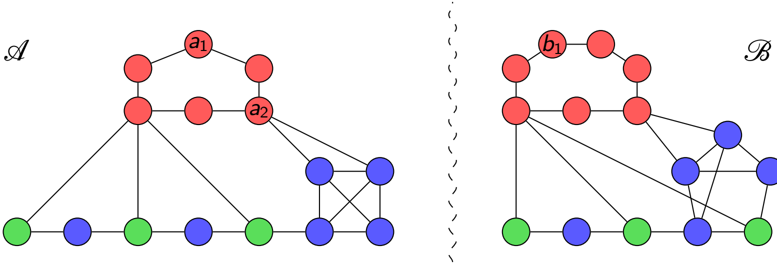


After $q$ rounds, Duplicator wishes that $a_i \mapsto b_i$ is an isomorphism between $\mathscr{A}[a_1, \ldots, a_k]$ and $\mathscr{B}[b_1, \ldots, b_k]$

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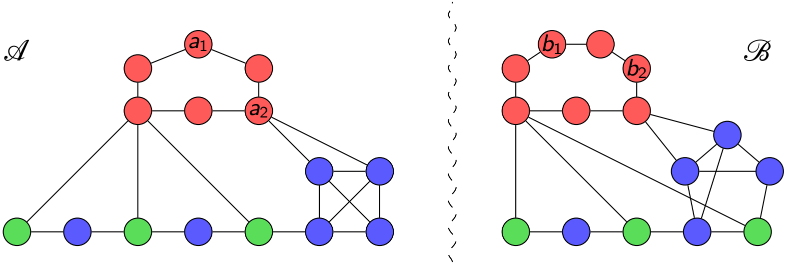


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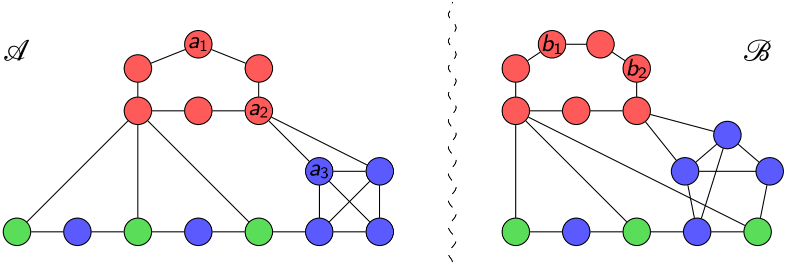


When no longer possible, Spoiler wins

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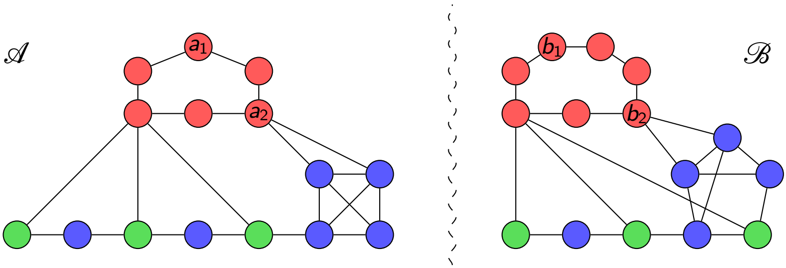


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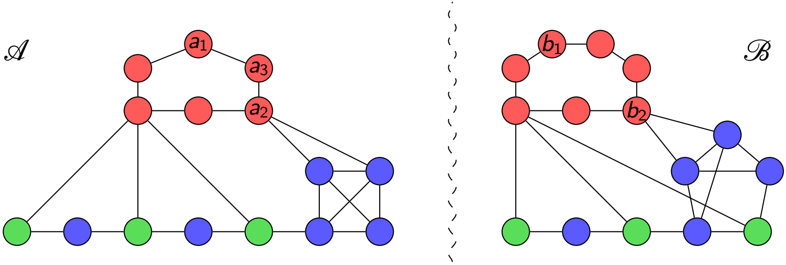
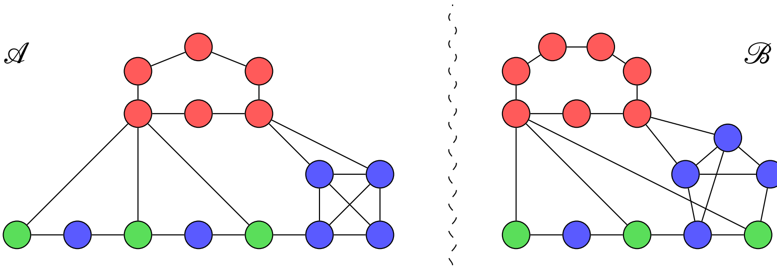


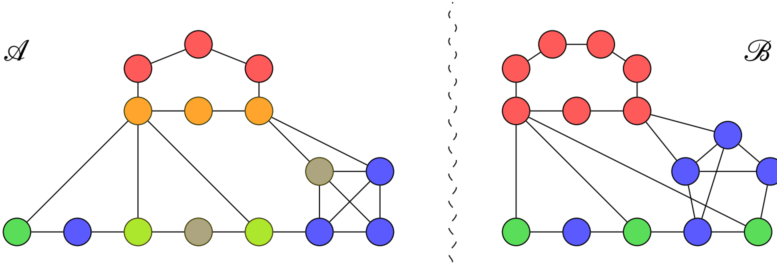

If Duplicator can survive $k$ rounds, we write $\mathscr{A} \equiv_k^{\mathsf{FO}} \mathscr{B}$
Here $\mathscr{A} \equiv_2^{\mathsf{FO}} \mathscr{B}$ and $\mathscr{A} \not\equiv_3^{\mathsf{FO}} \mathscr{B}$

# MSO Ehrenfeucht-Fraissé game




Same game but Spoiler can now play set moves

# MSO Ehrenfeucht-Fraissé game

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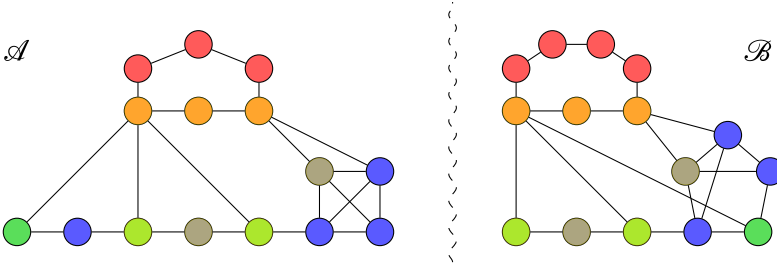
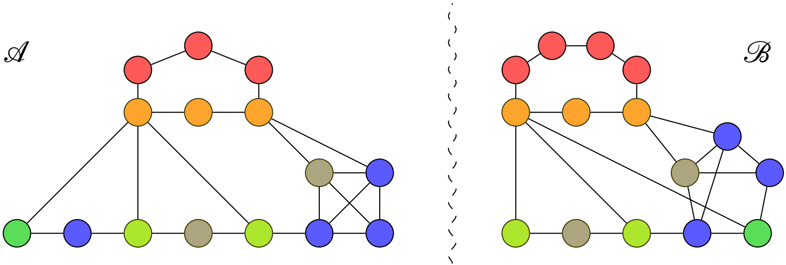



To which Duplicator answers a set in the other structure

# MSO Ehrenfeucht-Fraissé game




Again we write $\mathscr{A} \equiv_k^{\mathsf{MSO}} \mathscr{B}$ if Duplicator can survive $k$ rounds

# $k$-round EF games capture rank-$k$ types

Theorem (Ehrenfeucht-Fraïssé)

*For every $\sigma$-structures $\mathscr{A}, \mathscr{B}$ and logic $\mathcal{L} \in \{FO, MSO\}$,*

$$\mathscr{A} \equiv_k^{\mathcal{L}} \mathscr{B} \text{ if and only if } tp_k^{\mathcal{L}}(\mathscr{A}) = tp_k^{\mathcal{L}}(\mathscr{B}).$$

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**Proof.**

Induction on $k$.

($\Rightarrow$) $\mathcal{L}[k+1]$ formulas are Boolean combinations of $\exists x \varphi$ or $\exists X \varphi$ where $\varphi \in \mathcal{L}[k]$. Use the answer of Duplicator to $x = a$ or $X = A$.

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($\Leftarrow$) If $\mathrm{tp}_{k+1}^{\mathcal{L}}(\mathcal{A}) = \mathrm{tp}_{k+1}^{\mathcal{L}}(\mathcal{B})$, then the type $\mathrm{tp}_k^{\mathcal{L}}(\mathcal{A}, a)$ is equal to some $\mathrm{tp}_k^{\mathcal{L}}(\mathcal{B}, b)$. Move $a$ can be answered by playing $b$. □

# MSO model checking for component twin-width $d$

**Partitioned sentences:** sentences on $(E, U_1, \ldots, U_d)$-structures, interpreted as a graph vertex partitioned in $d$ parts

Maintain for every red component $C$ of every trigraph $G_i$

$$\text{tp}_k^{\text{MSO}}(G, \mathcal{P}_i, C) = \{\varphi \in \text{MSO}_{E,U_1,\ldots,U_d}(k) : (G\langle C\rangle, \mathcal{P}_i\langle C\rangle) \models \varphi\}.$$

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For each $v \in V(G)$, $\text{tp}_k(G, \mathcal{P}_n, \{v\}) =$ type of $K_1$
$\text{tp}_k(G, \mathcal{P}_1, \{V(G)\}) =$ type of $G$

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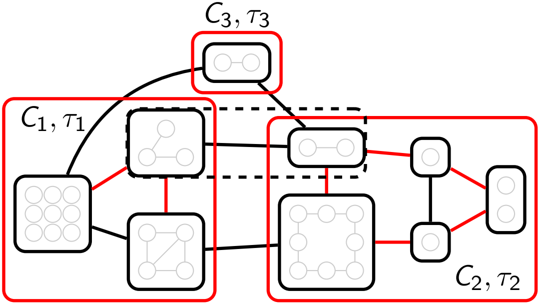


$\tau = \mathrm{tp}_k^{\mathrm{MSO}}(G, \mathcal{P}_i, C)$ based on the $\tau_j = \mathrm{tp}_k^{\mathrm{MSO}}(G, \mathcal{P}_{i+1}, C_j)$?

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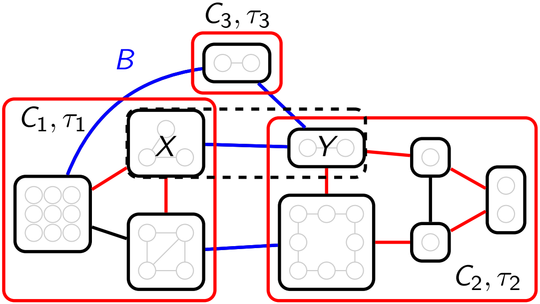
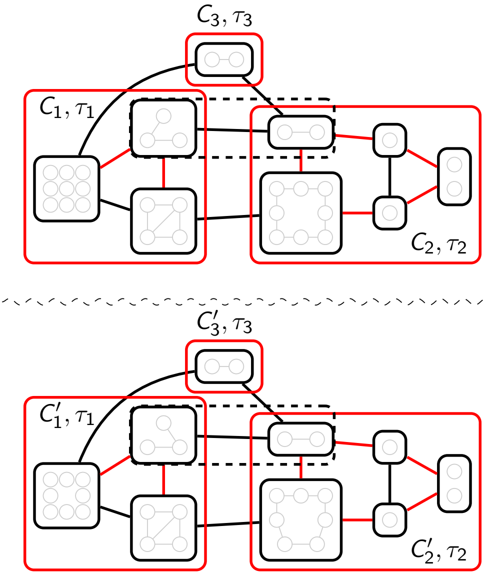



$C$ arises from $C_1, \ldots, C_{d'}$: $\tau = F(\tau_1, \ldots, \tau_{d'}, B, X, Y)$

# Showing $\tau = F(\tau_1, \ldots, \tau_{d'}, B, X, Y)$ via MSO EF game




Duplicator combines her strategies in the red components

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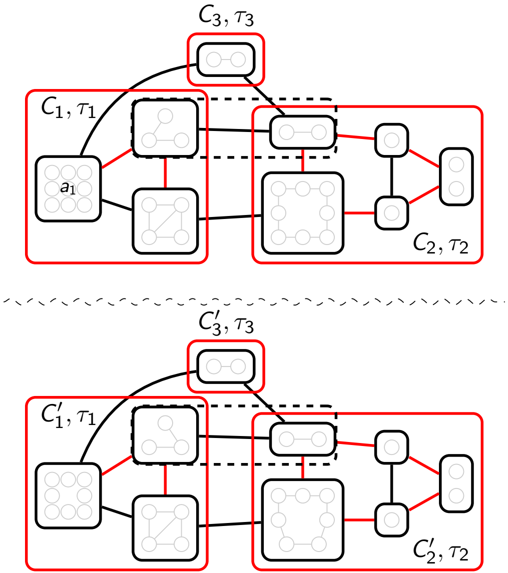


If Spoiler plays a vertex in the component of type $\tau_1$,

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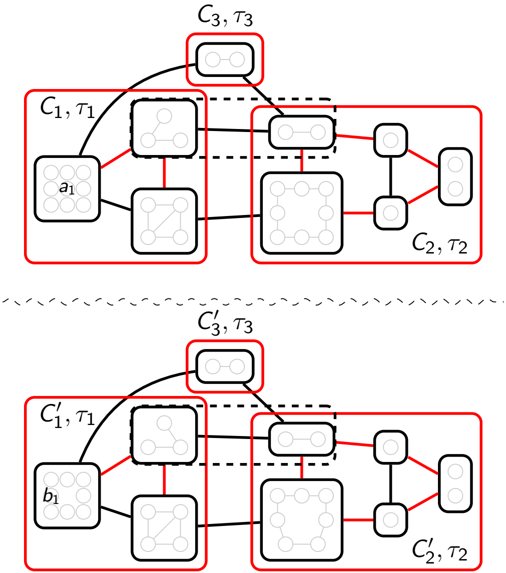


Duplicator answers the corresponding winning move

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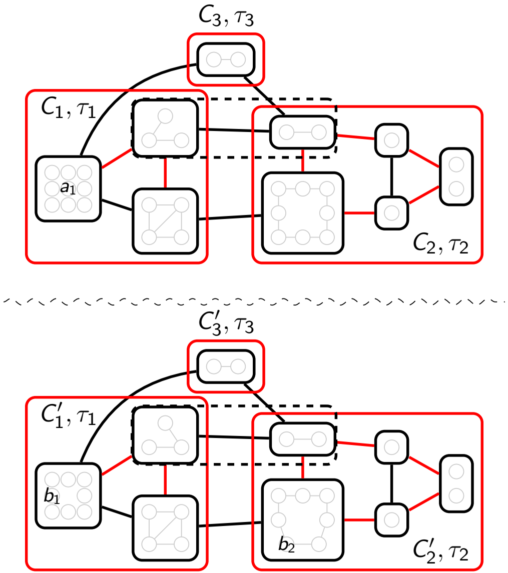


Same in the component of type $\tau_2$

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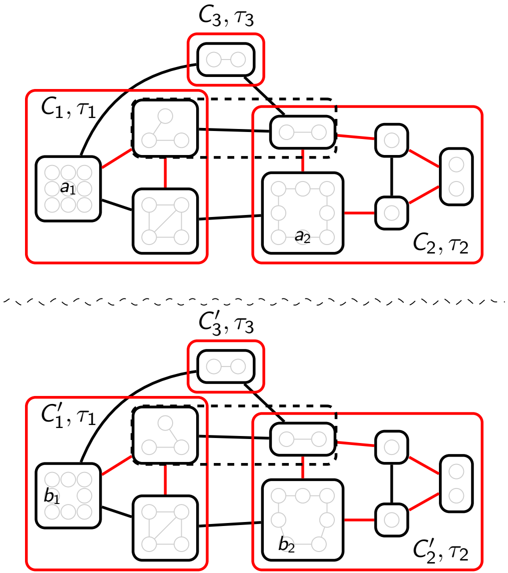


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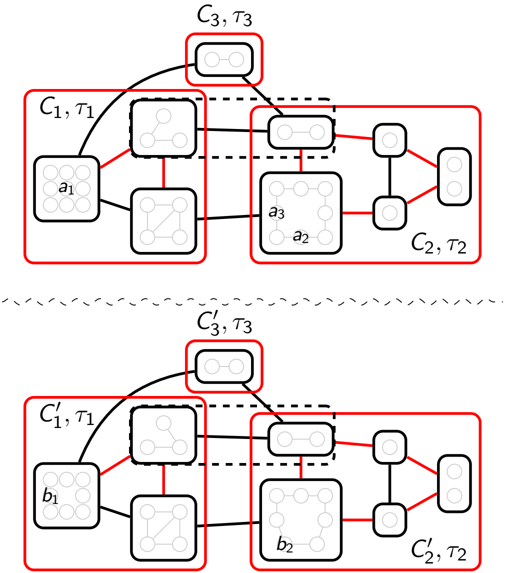


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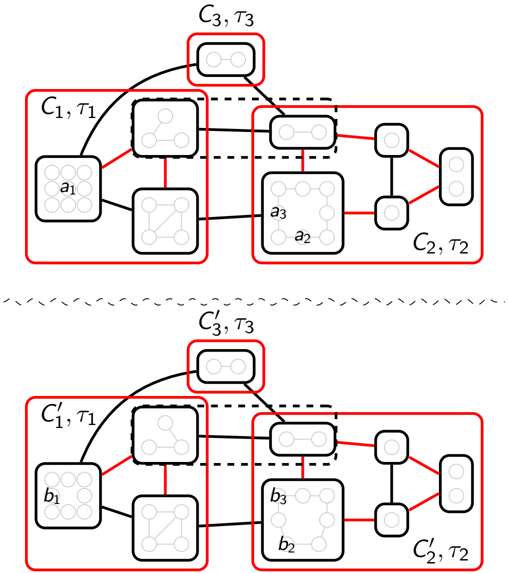


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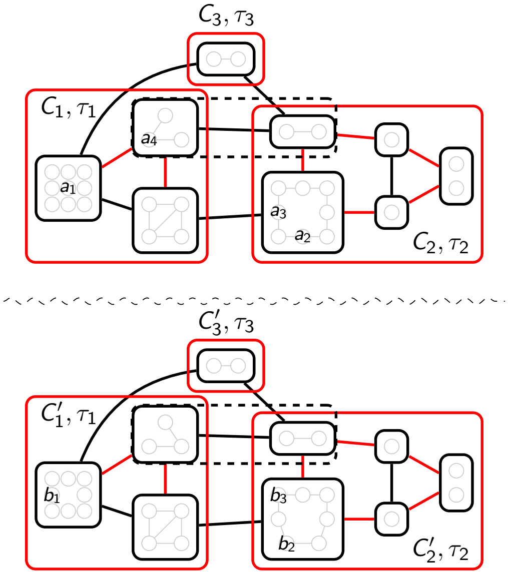


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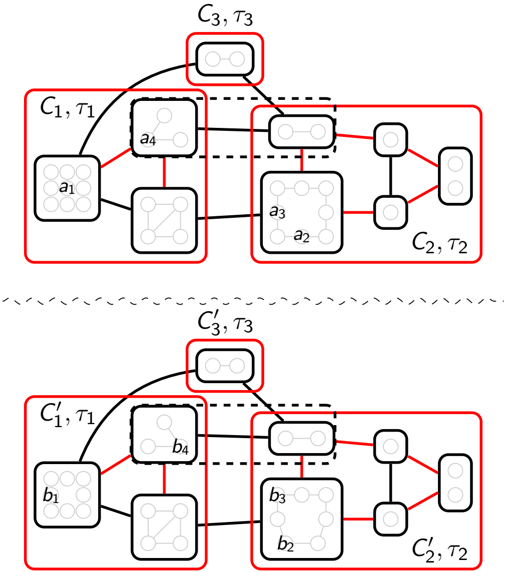


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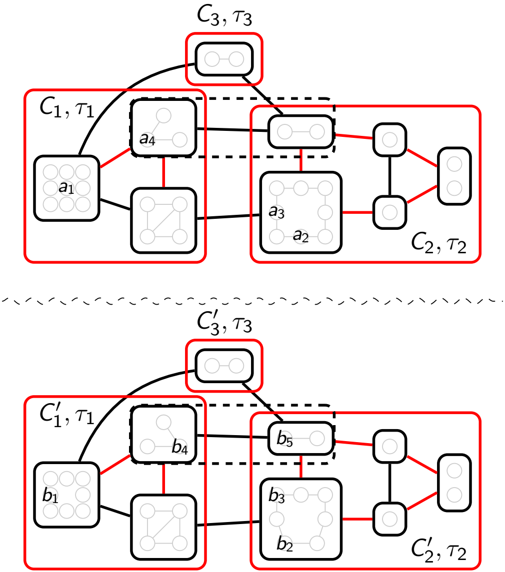


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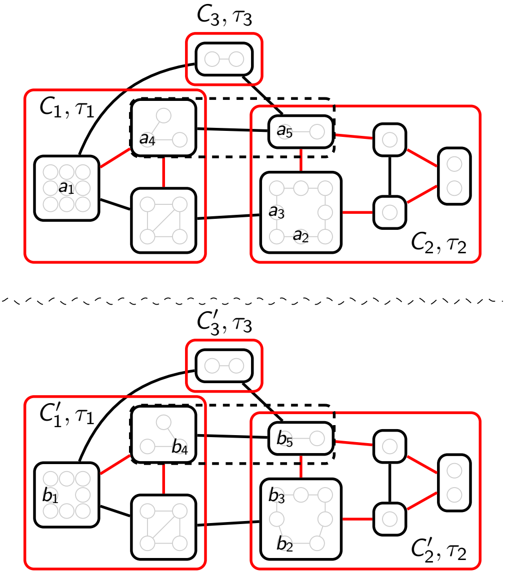


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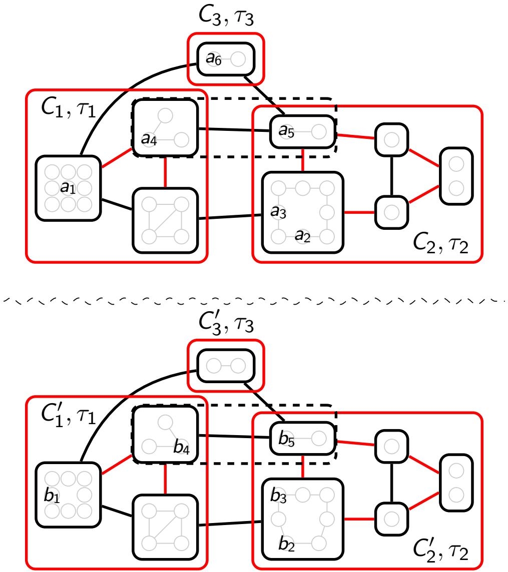


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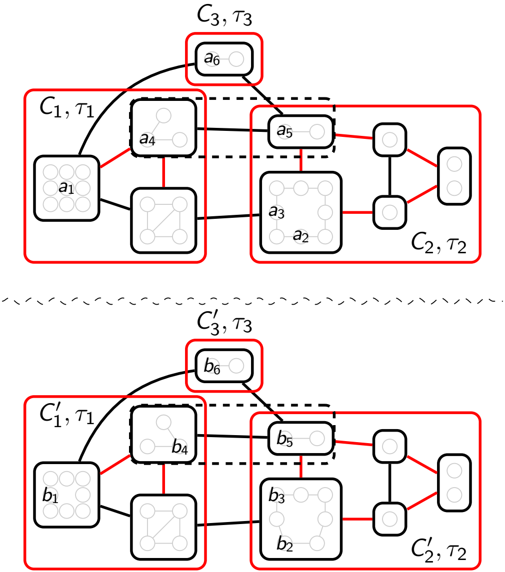


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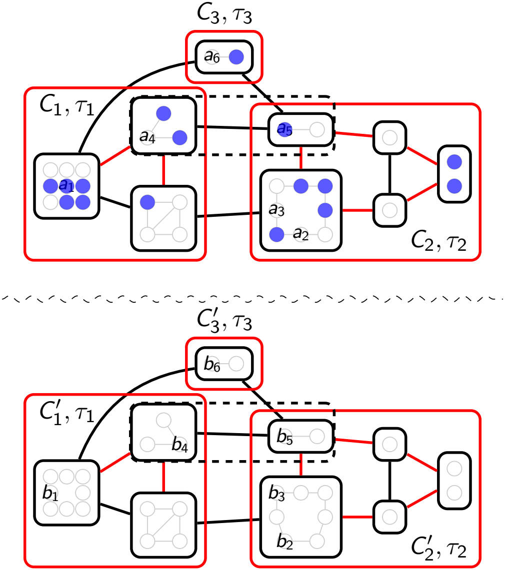


If Spoiler plays a set, Duplicator looks at the intersection with $C_1$,

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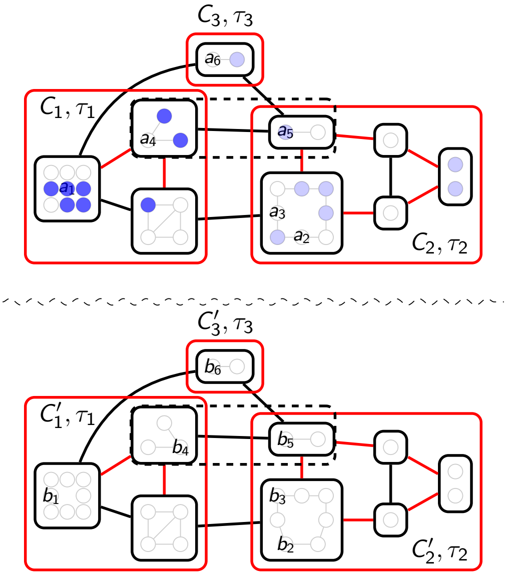


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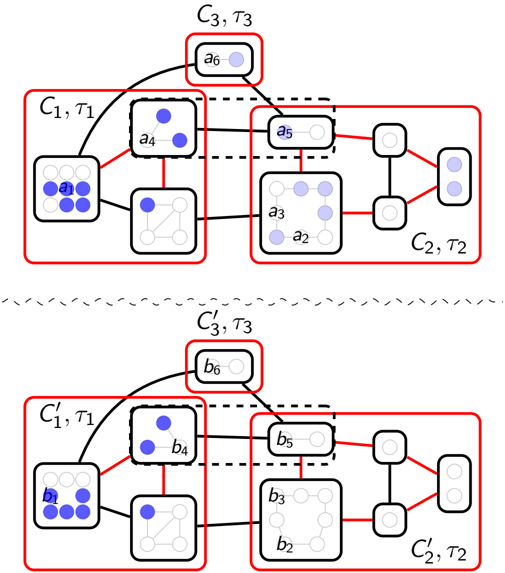


calls her winning strategy in $C'_1$

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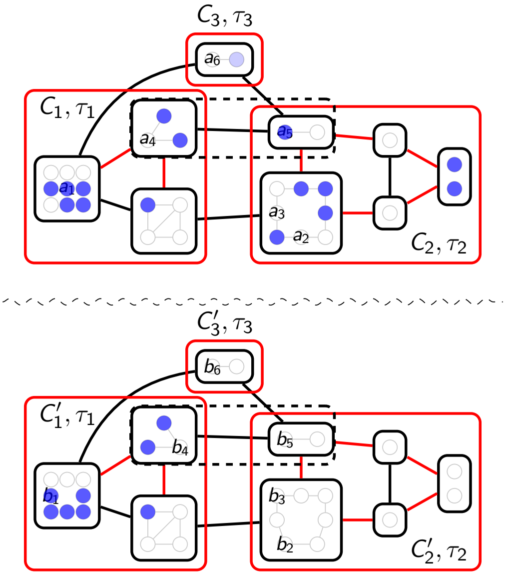


same for the other components

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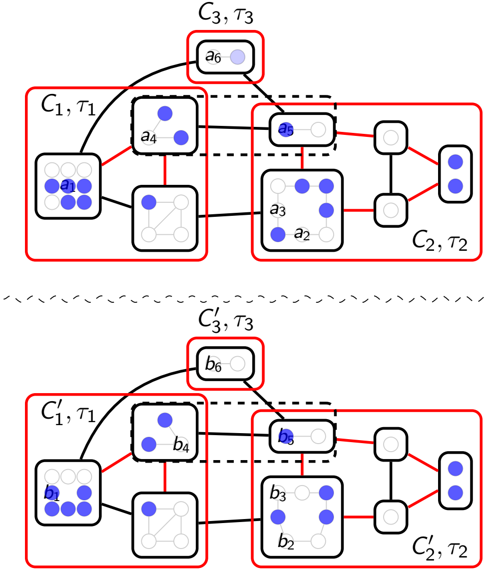


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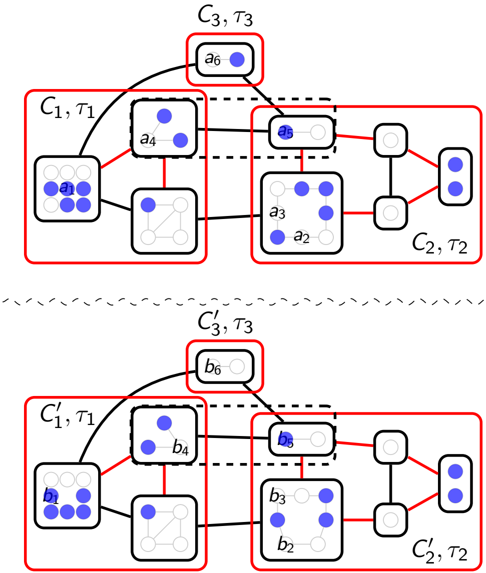


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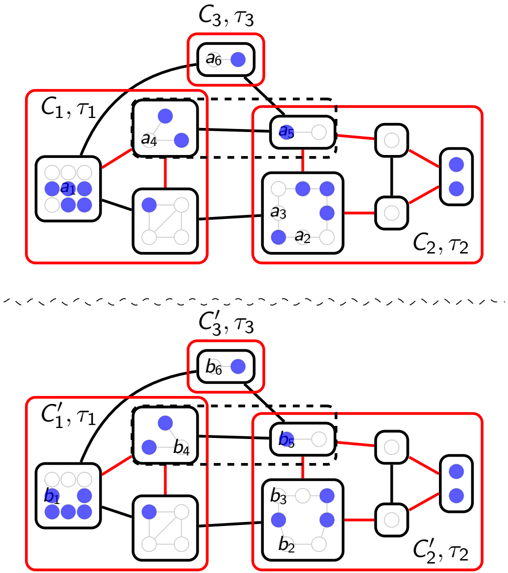


and plays the union

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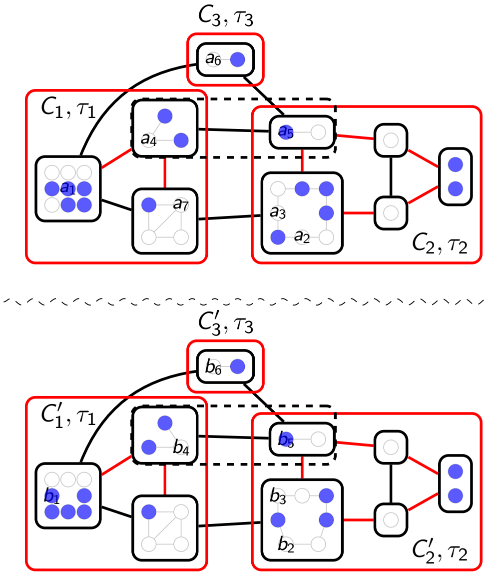


that fully defines the winning strategy of Duplicator

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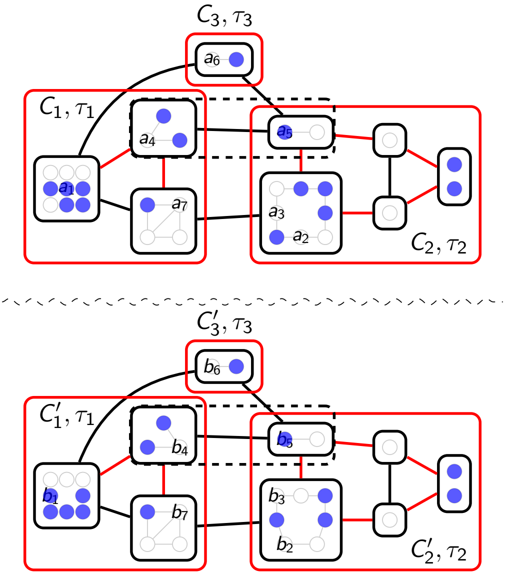


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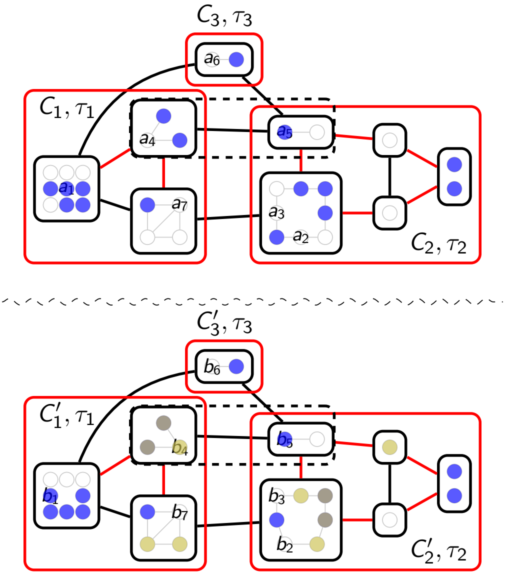


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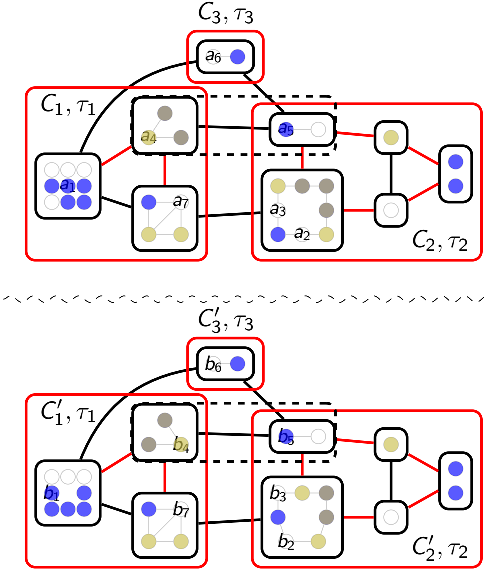


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# Turning it into a uniform algorithm

Reminder:

- #non-equivalent partitioned sentences of rank $k$: $f(d,k)$
- #rank-$k$ partitioned types bounded by $g(d,k) = 2^{f(d,k)}$

For each newly observed type $\tau$,

- keep a representative $(H, \mathcal{P})_\tau$ on at most $(d+1)^{g(d,k)}$ vertices
- determine the $0,1$-vector of satisfied sentences on $(H, \mathcal{P})_\tau$
- record the value of $F(\tau_1, \ldots, \tau_{d'}, B, X, Y)$ for future uses

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To decide $G \models \varphi$, look at position $\varphi$ in the $0,1$-vector of $\text{tp}_k^{\text{MSO}}(G)$

# $k$-Independent Set given a $d$-sequence

$d$-sequence: $G = G_n, G_{n-1}, \ldots, G_2, G_1 = K_1$

Algorithm: **For every connected subset $D$ of size at most $k$ of the red graph of every $G_i$, store in $T[D, i]$ one largest independent set in $G\langle D \rangle$ intersecting every vertex of $D$.**

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Initialization: $T[\{v\}, n] = \{v\}$

End: $T[\{V(G)\}, 1] =$ IS of size at least $k$ or largest IS in $G$

Running time: $d^{2k} n^2$ red connected subgraphs,
actually only $d^{2k} n = 2^{O_d(k)} n$ updates

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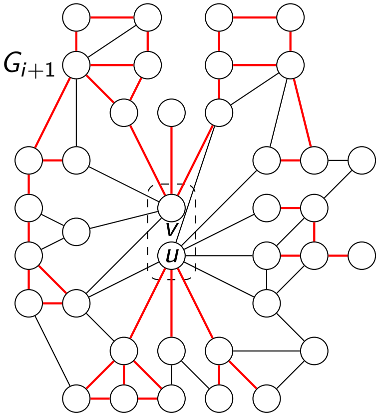
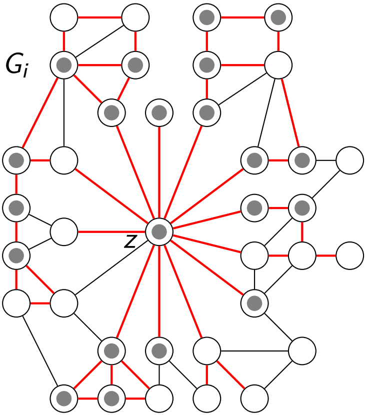
**How to compute $T[D, i]$ from all the $T[D', i+1]$?**

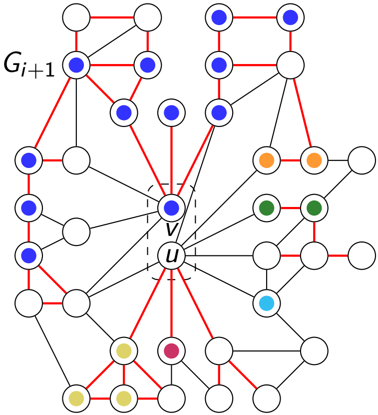
# $k$-INDEPENDENT SET: Update of partial solutions




Best partial solution inhabiting •?

# $k$-Independent Set: Update of partial solutions




3 unions of $\leqslant d + 2$ red connected subgraphs to consider in $G_{i+1}$ with $u$, or $v$, or both

# FO model checking on graphs of bounded twin-width

We will now generalize the previous algorithm to:

**Theorem (B., Kim, Thomassé, Watrigant '20)**

*FO model checking can be solved in time $f(|\varphi|, d) \cdot |V(G)|$ on graphs $G$ given with a $d$-sequence.*

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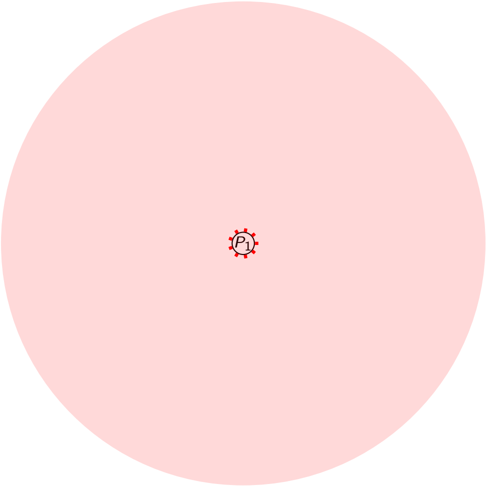
Add **Gaifman's locality** to our MSO model checking algorithm

Following [Gajarský, Pilipczuk, Przybyszewski, Toruńczyk '22]

# Local tuple of parts




$(P_1, P_2, \ldots, P_q)$ is *local around* $P_1$ if...

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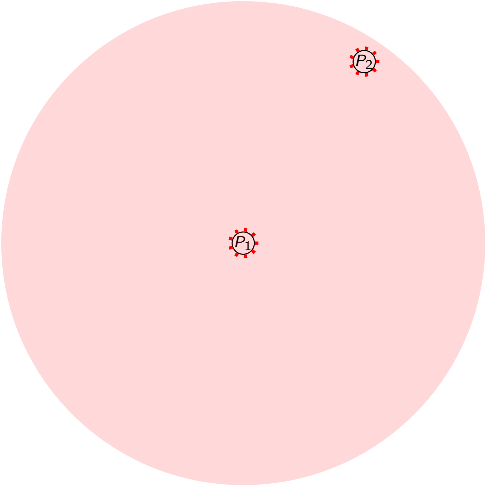


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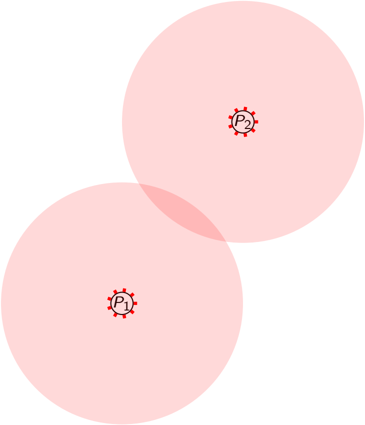


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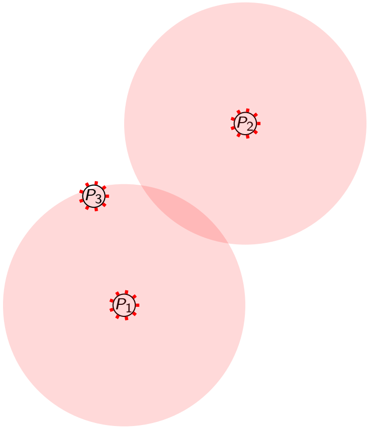


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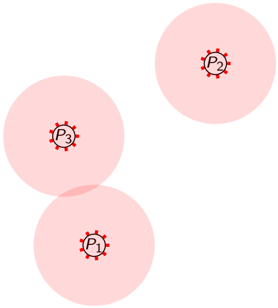


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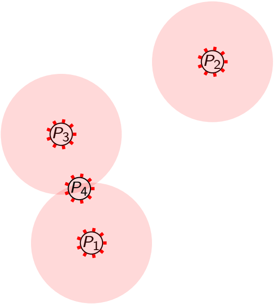
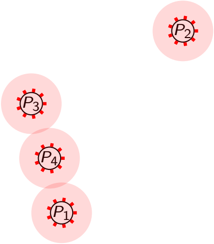



$(P_1, P_2, \ldots, P_q)$ is *local around* $P_1$ if...

$P_4$ is at distance at most $2^{k-4}$ from $\{P_1, P_2, P_3\}$ in $(G, \mathcal{P}_i)$

# Local tuple of parts

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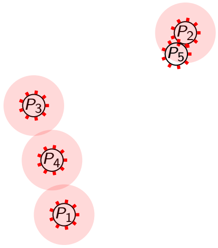


$(P_1, P_2, \ldots, P_q)$ is *local around* $P_1$ if...

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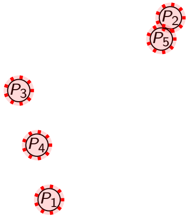


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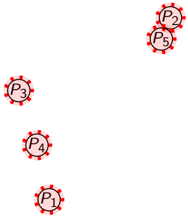


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# Partitioned local sentences and types

A prenex sentence is *partitioned local around* $X$ in $(G, \mathcal{P}_i)$ if of the form $Qx_1 \in X \; Qx_2 \in P_2 \; \dots \; Qx_k \in P_k \; \psi(x_1, \dots, x_k)$ with

- $\psi$ is quantifier-free, and
- $(X, P_2, \dots, P_k)$ local around $X$ in $(G, \mathcal{P}_i)$.

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- $\psi$ is quantifier-free, and
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And the corresponding types:

$$\mathrm{ltp}_k^{\mathrm{FO}}(G, \mathcal{P}_i, X) = \{\varphi : \mathrm{qr}(\varphi) \leqslant k,$$

$$\varphi \text{ is partitioned local around } X \text{ in } (G, \mathcal{P}_i),$$

$$(G, \mathcal{P}_i) \models \varphi\}.$$

# Partitioned local sentences/types in $(G, \mathcal{P}_n)$ and $(G, \mathcal{P}_1)$

**Initialization of the dynamic programming**

In $(G, \mathcal{P}_n = \{\{v\} : v \in V(G)\})$: for every $v \in V(G)$,
$Qx_1 \in \{v\}\ Qx_2 \in \{v\}\ \dots\ Qx_k \in \{v\}\ \psi \equiv \psi(v, v, \dots, v)$

Partitioned local types are easy to compute in $(G, \mathcal{P}_n)$

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**Output of the dynamic programming**

In $(G, \mathcal{P}_1 = \{V(G)\})$:
$Qx_1 \in V(G)\ Qx_2 \in V(G)\ \ldots\ Qx_k \in V(G)\ \psi \equiv$ classic sentences

The partitioned local type in $(G, \mathcal{P}_1)$ coincides with the type of $G$

# Partitioned local types give the partitioned types

Isom. $f : \mathcal{P}_i \to \mathcal{P}'_i$ with $\mathsf{ltp}_k^{\mathsf{FO}}(G, \mathcal{P}_i, X) = \mathsf{ltp}_k^{\mathsf{FO}}(G', \mathcal{P}'_i, f(X))$

$(G, \mathcal{P}_i)$

$(G', \mathcal{P}'_i)$

Local strategies win the global game

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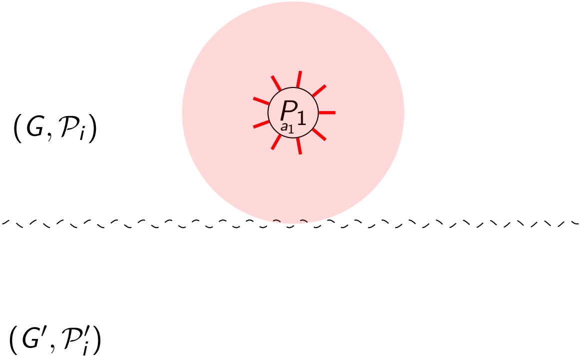


Say, Spoiler plays in $P_1$

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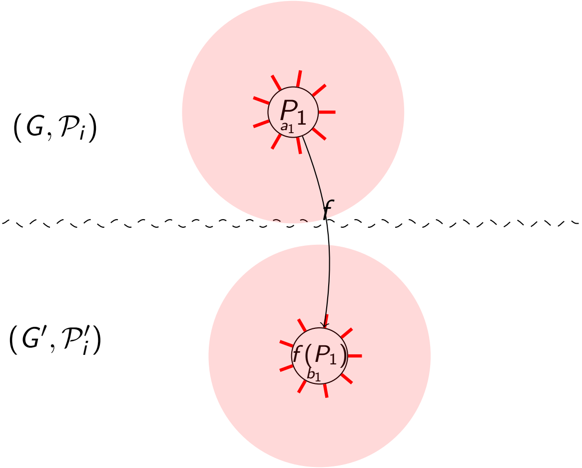


Duplicator answers in $f(P_1)$ following the local game around $P_1$

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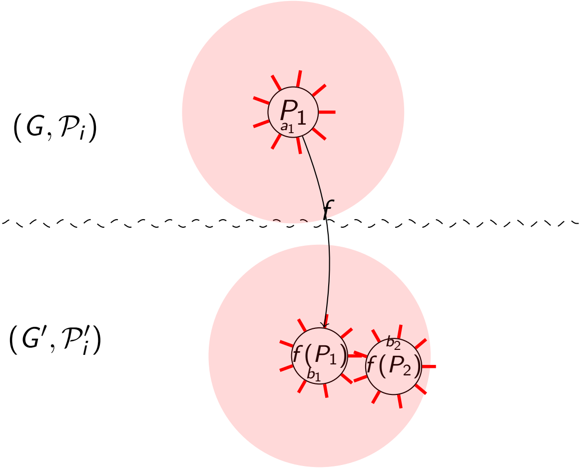


Now when Spoiler plays close to $P_1$ or $f(P_1)$

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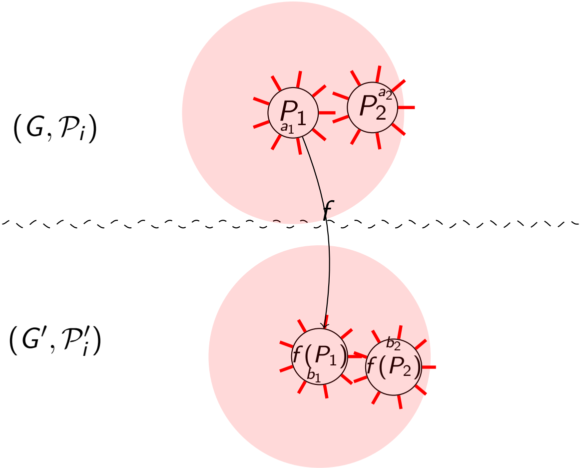


Duplicator follows the winning local strategy

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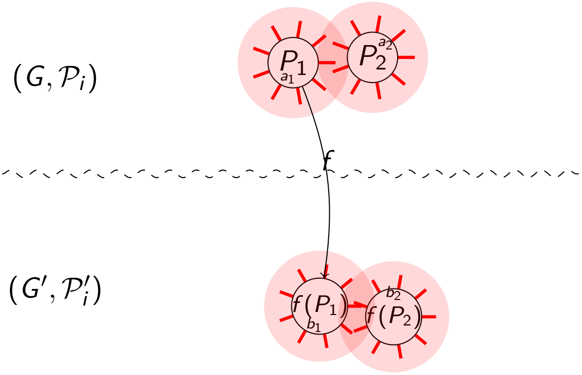


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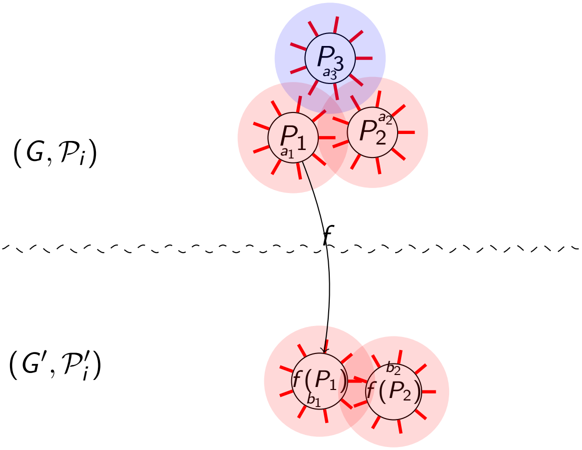


If Spoiler plays too far

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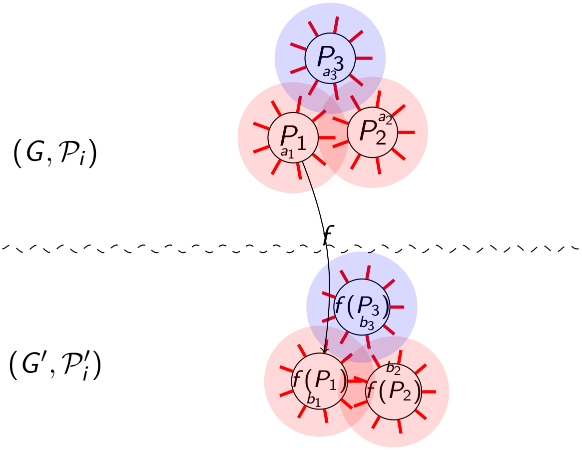


Duplicator starts a new local game around that new part

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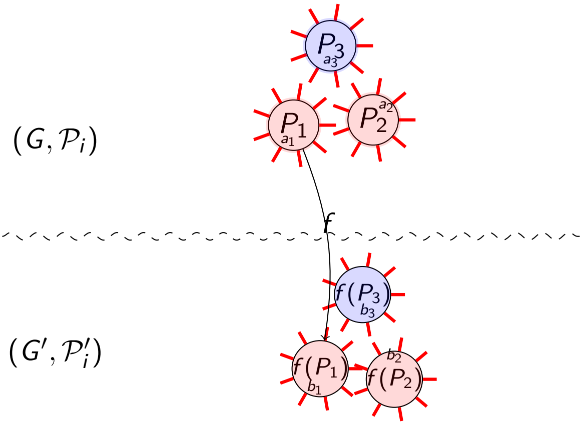


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# Concluding as in the MSO model checking algorithm

$(G, \mathcal{P}_{i+1}) \rightsquigarrow (G, \mathcal{P}_i)$ : $X$ and $Y$ are merged in $Z$

Partitioned local types around $P$

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Each contraction: $O_{d,k}(1) = O(d^{2^k})$ updates in $O_{d,k}(1) = f(d,k)$
Total time: $O_{d,k}(n)$

# First-order interpretations and transductions

**FO interpretation:** redefine the edges by a first-order formula
$\varphi(x, y) = \neg E(x, y)$ (complement)
$\varphi(x, y) = E(x, y) \vee \exists z E(x, z) \wedge E(z, y)$ (square)

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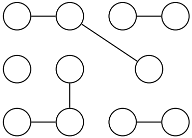

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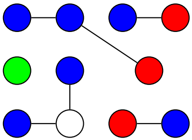

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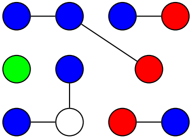

$$\varphi(x, y) = E(x, y) \vee (G(x) \wedge B(y) \wedge \neg \exists z R(z) \wedge E(y, z))$$
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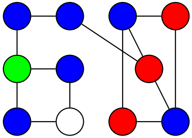

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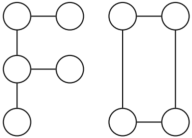
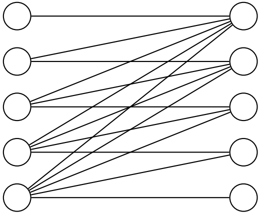


# Stable and NIP for hereditary classes

Due to [Baldwin, Shelah '85; Braunfeld, Laskowski '22]

**Stable class:** no transduction of the class contains all ladders
**NIP class:** no transduction of the class contains all graphs

ladder



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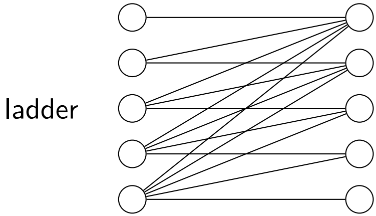


Bounded-degree graphs $\rightarrow$ stable
Unit interval graphs $\rightarrow$ NIP but not stable
Interval graphs $\rightarrow$ not NIP

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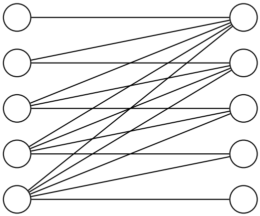
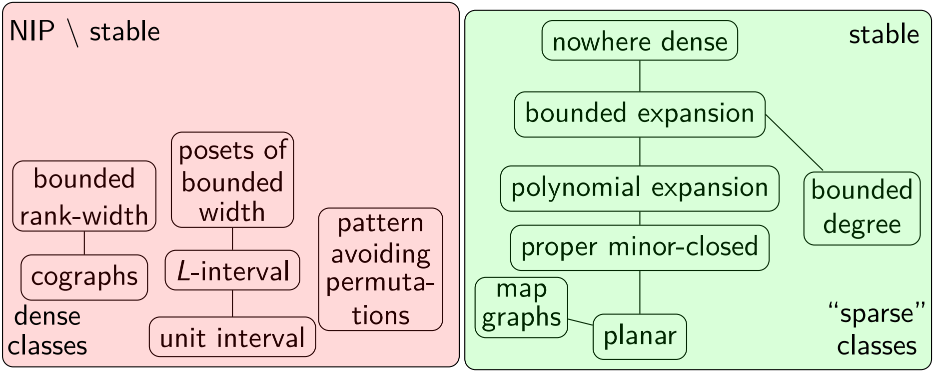


Bounded-degree graphs → stable
Unit interval graphs → NIP but not stable
Interval graphs → not NIP

**Bounded twin-width classes → NIP, but in general not stable**

# Classes with known tractable FO model checking



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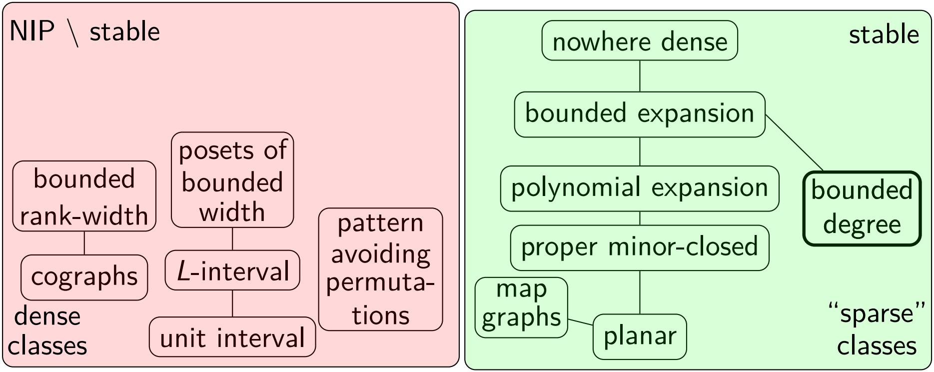
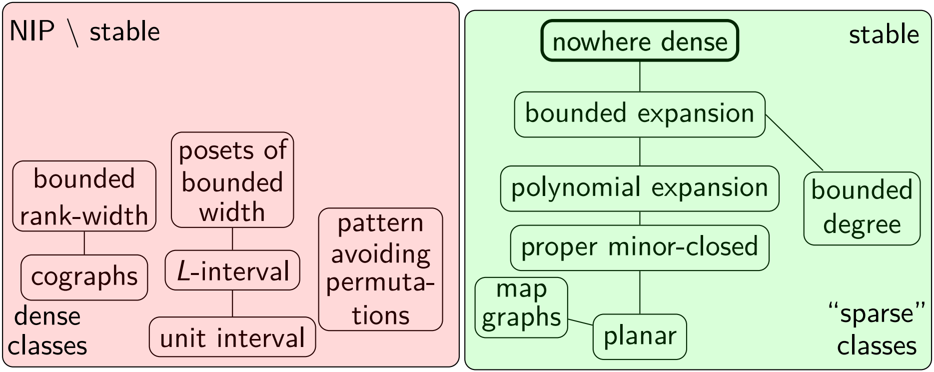


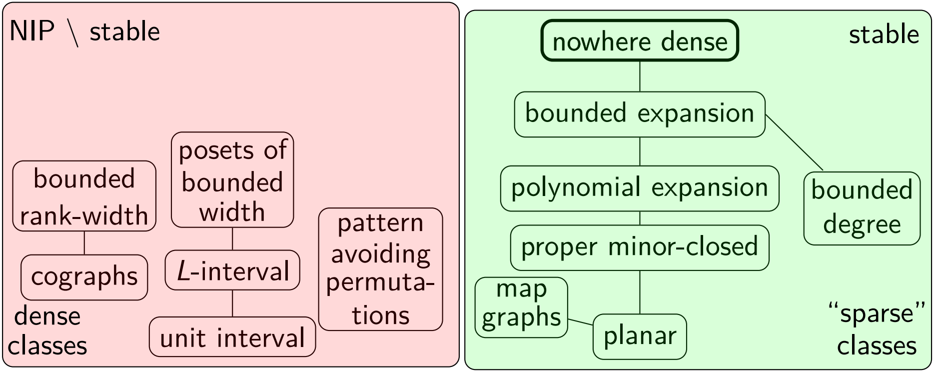

FO Model Checking solvable in $f(|\varphi|)n$ on bounded-degree graphs
[Seese ’96]

# Classes with known tractable FO model checking




FO MODEL CHECKING solvable in $f(|\varphi|)n^{1+\varepsilon}$ on any nowhere dense class
[Grohe, Kreutzer, Siebertz '14]

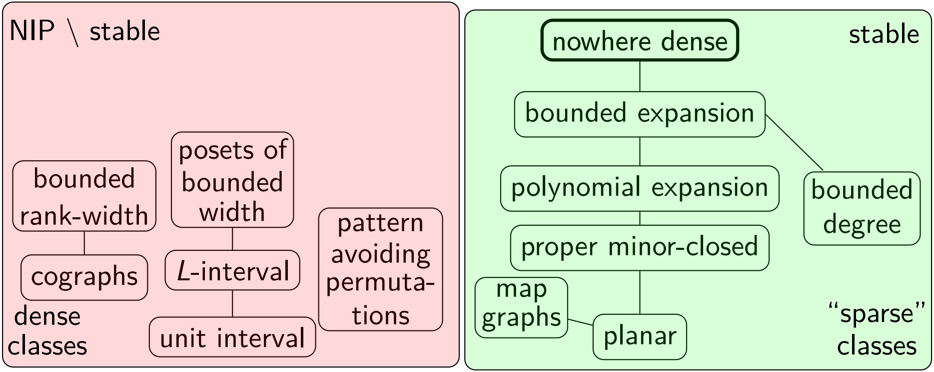
# Classes with known tractable FO model checking




End of the story for the subgraph-closed classes

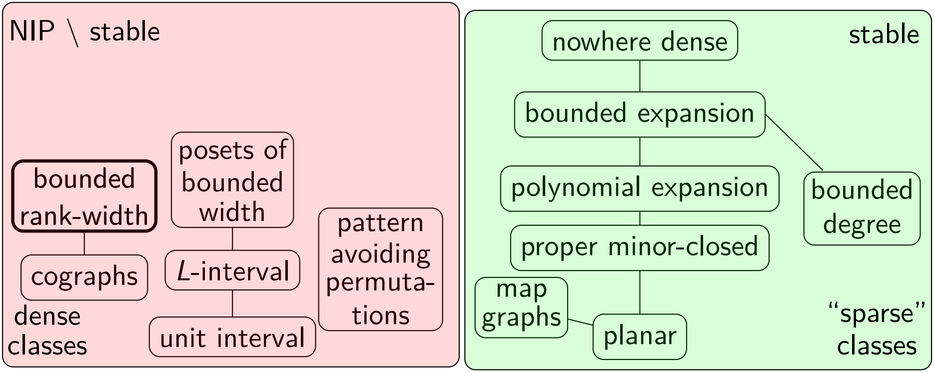
tractable FO Model Checking ⇔ nowhere dense ⇔ stable

# Classes with known tractable FO model checking



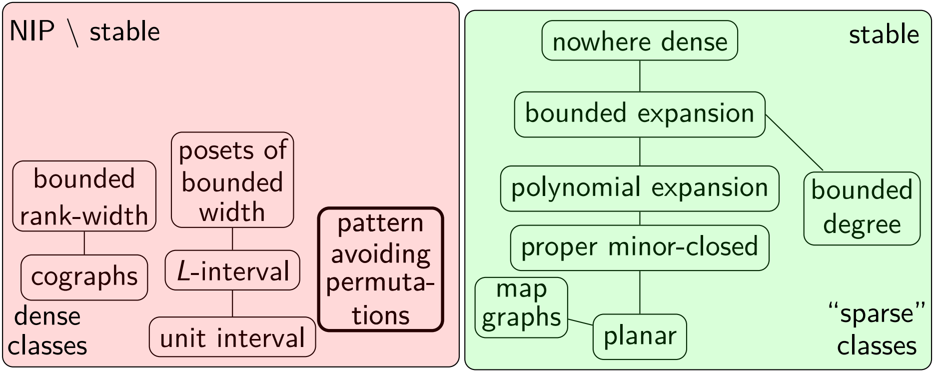

New program: transductions of nowhere dense classes
Not sparse anymore but still stable

# Classes with known tractable FO model checking



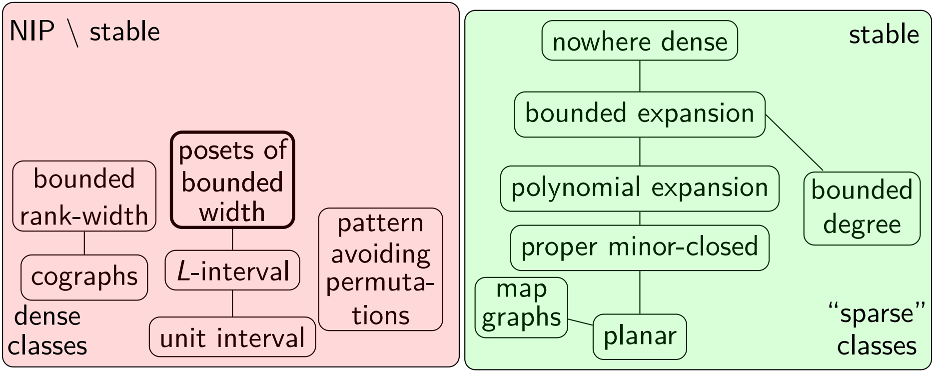

MSO$_1$ Model Checking solvable in $f(|\varphi|, w)n$ on graphs of rank-width $w$
[Courcelle, Makowsky, Rotics '00]

# Classes with known tractable FO model checking




*Is $\sigma$ a subpermutation of $\tau$?* solvable in $f(|\sigma|)|\tau|$
[Guillemot, Marx '14]

# Classes with known tractable FO model checking




FO MODEL CHECKING solvable in $f(|\varphi|, w)n^2$ on posets of width $w$
[GHLOORS '15]

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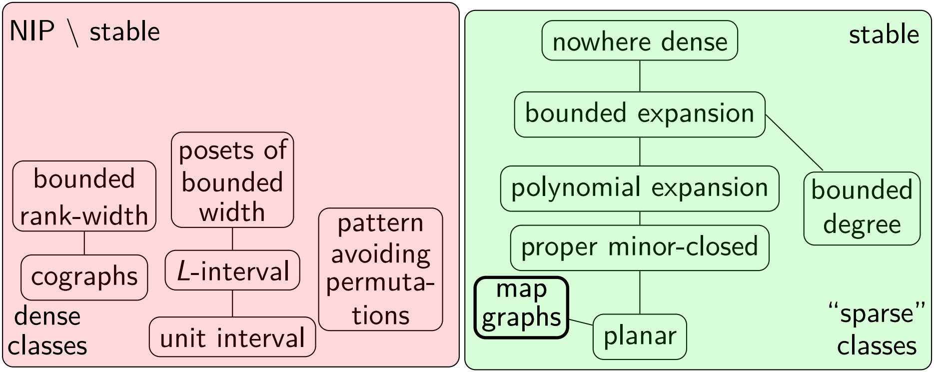


FO MODEL CHECKING solvable in $f(|\varphi|)n^{O(1)}$ on map graphs
[Eickmeyer, Kawarabayashi '17]

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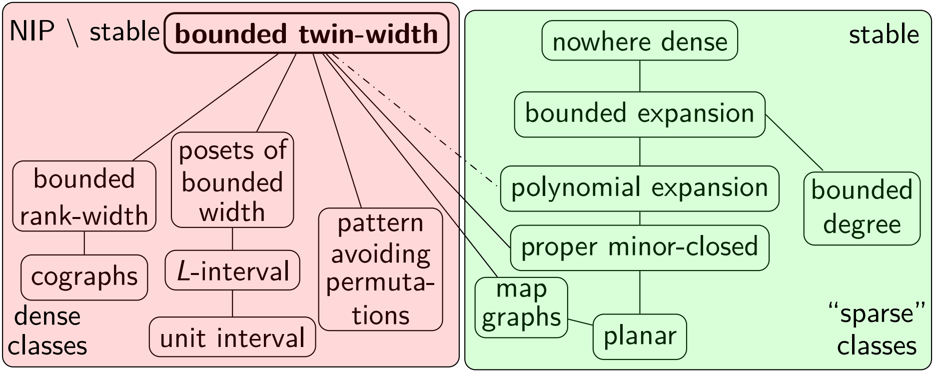


FO Model Checking solvable in $f(|\varphi|, d)n$ on graphs with a $d$-sequence
[B., Kim, Thomassé, Watrigant '20]

# First-order transductions preserve bounded twin-width

**Theorem (B., Kim, Thomassé, Watrigant '20)**

*For every class $\mathcal{C}$ of binary structures with bounded twin-width and transduction $\mathscr{T}$, the class $\mathscr{T}(\mathcal{C})$ has bounded twin-width.*

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- Refine parts of the partition sequence by partitioned local type

# More about FO model checking

- Extension to enumeration of FO model checking with contraction sequences [Gajarský, Pilipczuk, Przybyszewski, Toruńczyk '22]

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- Progress on stable/structurally sparse classes, e.g. [Dreier, Mählmann, Siebertz '23]
- Bounded *flip-width* [Toruńczyk '23] common generalization of bounded expansion and bounded twin-width
- Hope that structures from NIP classes decompose into an "ordered part" and a "stable part"

# Stable and structurally sparse classes

### Conjecture (Ossona de Mendez)

*Every monadically stable class is the FO transduction of a nowhere dense class.*

Morally: *Stability coincides with structural sparsity*

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Shown among classes of bounded linear cliquewidth, cliquewidth, and now twin-width:

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*Every stable class of bounded twin-width is the FO transduction of a class of bounded twin-width without arbitrarily large bicliques.*

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# The lens of contraction sequences

| Class of bounded | FO transduction of | constraint on red graphs | efficient MC |
|---|---|---|---|
| linear rank-width | linear order | bd #edges | MSO |
| rank-width | tree order | bd component | **MSO** |
| twin-width | **?** | bd degree | **FO** |

# Compiling bounded twin-width graphs as p-f permutations

Our next goal:

**Theorem** (B., Nešetřil, Ossona de Mendez, Siebertz, Thomassé '21)

*A class of binary structures has bounded twin-width if and only if it is a first-order transduction of a proper permutation class.*

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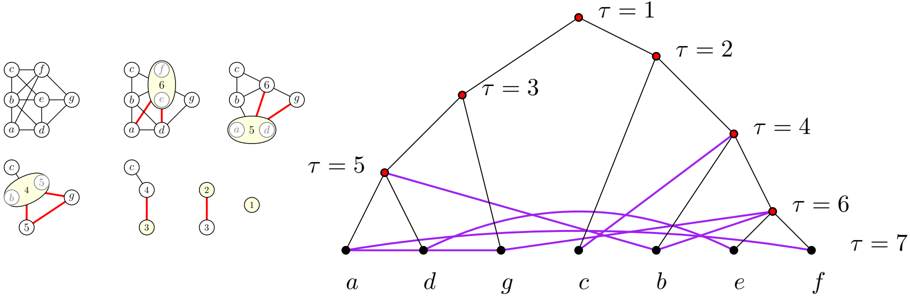
We already know the "if" part, thus we want to show:

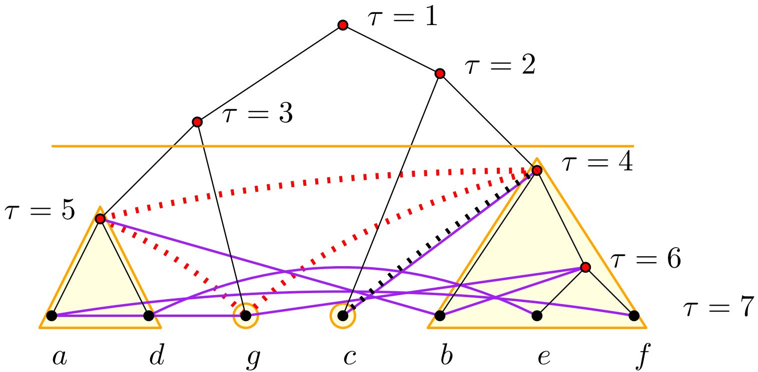
$\forall$ class $\mathcal{C}$ of bounded twin-width, $\exists$ permutation class $\mathcal{P}$ avoiding one permutation and an FO transduction $\mathscr{T}$ such that $\mathcal{C} \subseteq \mathscr{T}(\mathcal{P})$.

# Twin-decomposition

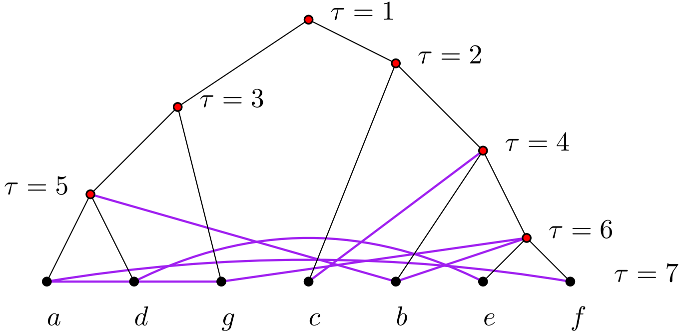



Contraction tree + transversal adjacencies (bicliques) + time $\tau$

# Reading out trigraphs from a twin-decomposition



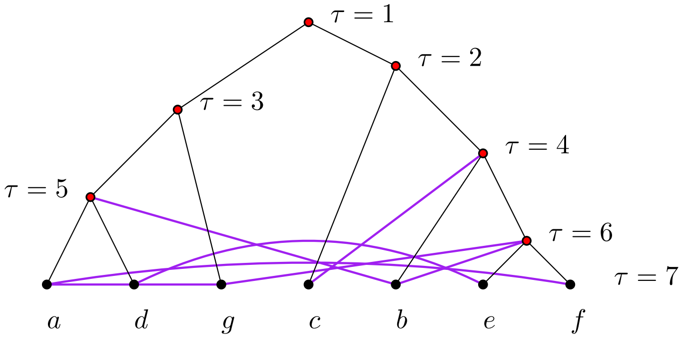
# Twin-models




Twin-model: tree edges $T$, transversal edges $V$

*Example:* $T(3, 5)$, $V(4, c)$

# Twin-models

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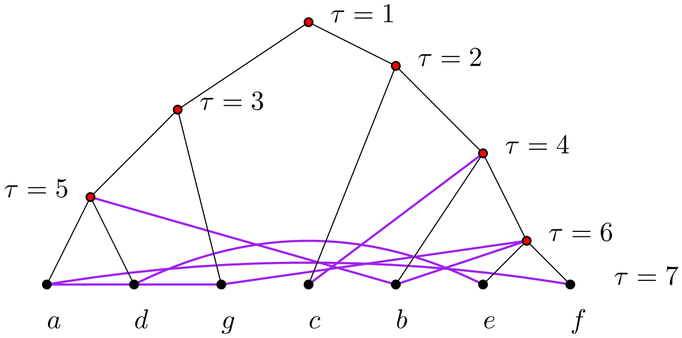
Full twin-model: ancestor–descendant relation $\prec$, $V$

*Example:* $2 \prec e$

# Twin-models

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$1 < 3 < 5 < a < d < g < 2 < c < 4 < b < 6 < e < f$

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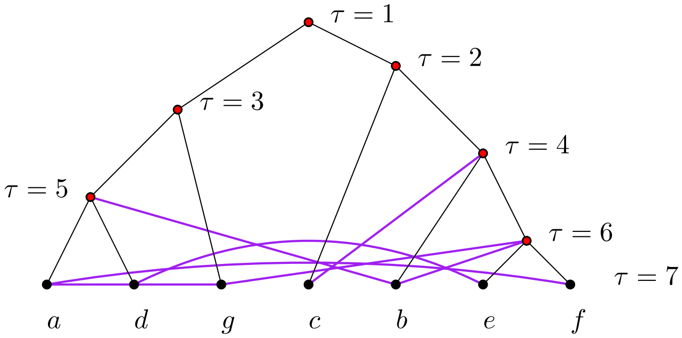


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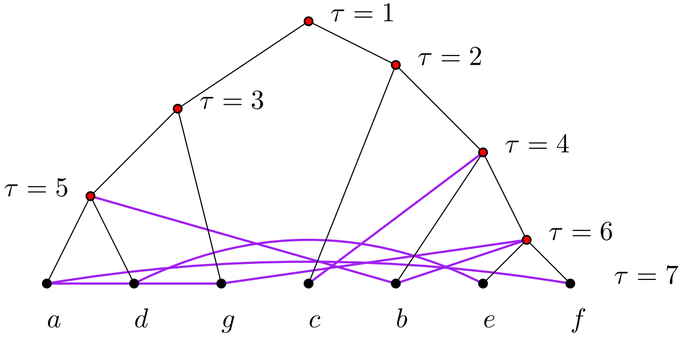
# Why full twin-models?




One can FO reconstruct the initial graph from a full twin-model

$$E(x, y) := \exists x' \exists y' \left( x' \preceq x \ \wedge \ y' \preceq y \ \wedge \ V(x', y') \right)$$

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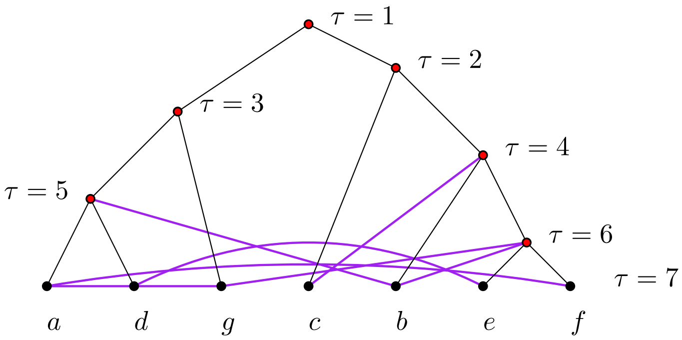


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*Example:* $E(c, f)$ *since* $c \preceq c$, $4 \preceq f$, $V(4, c)$

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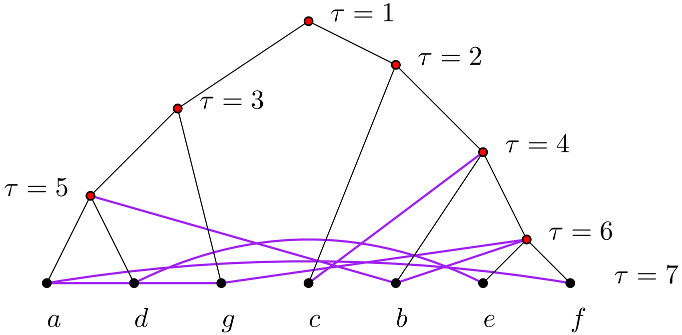


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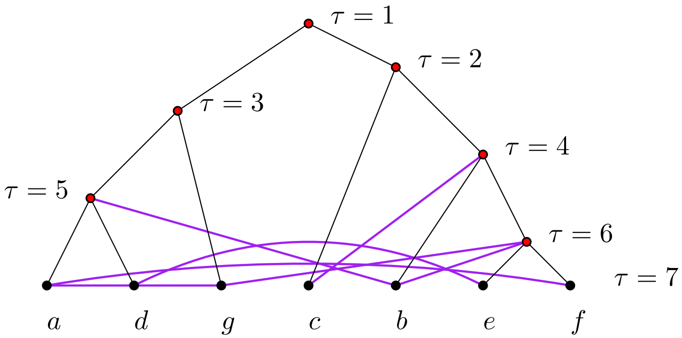
but *not* from a mere twin-model, in general

# Why ordered twin-models?

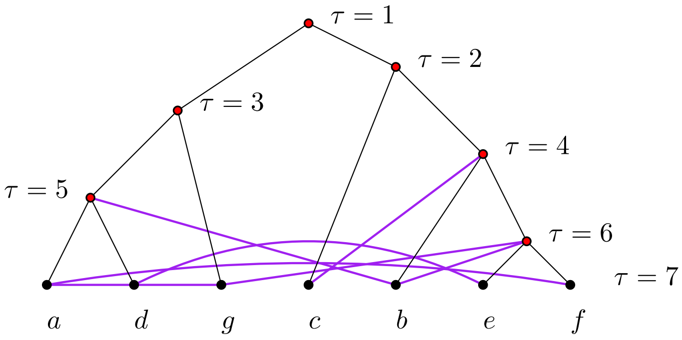



A linear order

$$1 < 3 < 5 < a < d < g < 2 < c < 4 < b < 6 < e < f$$

brings us closer to a permutation ($\equiv$ two linear orders)

# Full and ordered twin-models are transduction equivalent

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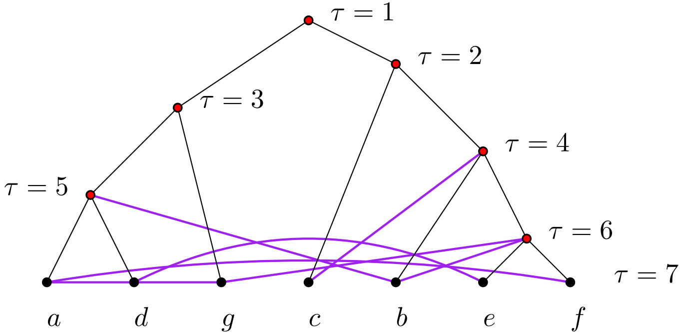
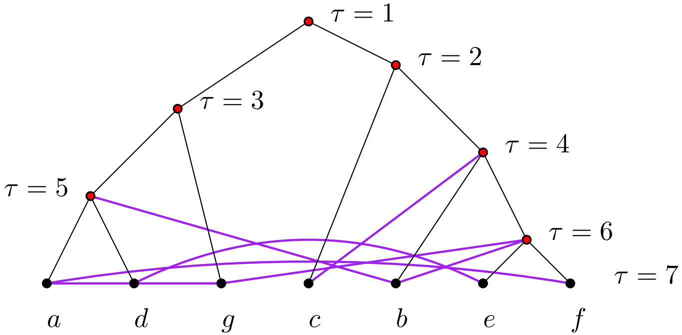



$$x \prec y := x < y \ \wedge \ \forall x < z \leq y \ \forall w \ T(z, w) \to x \leq w$$

*$y$ is a strict descendant of $x$ if it comes after in the pre-order, and every neighbor $w$ (in the tree) of any intermediate $z$ (possibly $y$) comes (non-strictly) after $x$*

# Full and ordered twin-models are transduction equivalent




To define $x < y$ from $\prec$, **mark each left child with one color**, and express that the before-last vertex on the path from $x$ to the least ancestor of $x$ and $y$ is marked (or simply $x \prec y$)

# Done and left to do

graphs $\twoheadleftarrow$ full twin-models $\twoheadleftarrow\!\!\twoheadrightarrow$ ordered twin-models

bounded twin-width

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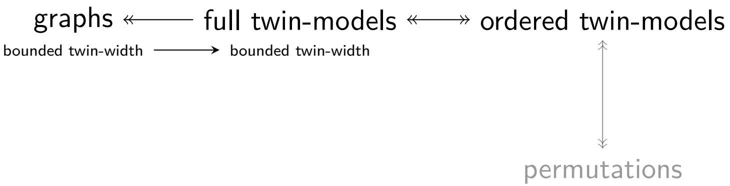
bounded twin-width $\longrightarrow$ bounded twin-width

Mimicking a good contraction sequence on a full twin-model yields a good contraction sequence

# Done and left to do




Past this point *bounded twin-width* is preserved by the FO transductions, and we just need to show that:

ordered twin-models and permutations are transduction equivalent

# Sparsity of the twin-model

Twin-models have bounded twin-width and degeneracy

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*Bounded twin-width and degeneracy ⇒ bounded expansion.*

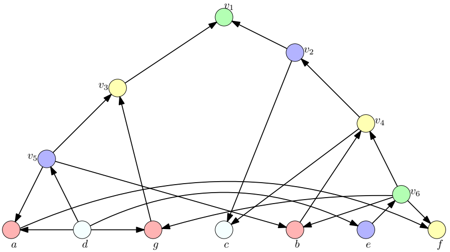
Theorem (Nešetřil, Ossona de Mendez '08)

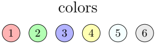
*Bounded expansion ⇒ bounded star chromatic number.*

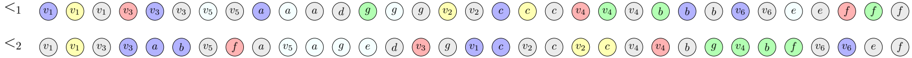
I.e., proper $O(1)$-coloring such that every two colors induce a disjoint union of stars

# Encoding: Ordered twin-models to permutations

Fix a star coloring and orient edges away from centers of stars
→ bounded in-degree



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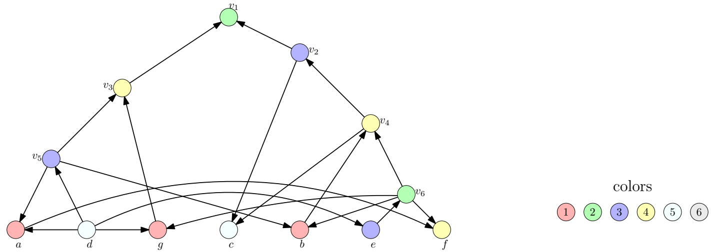
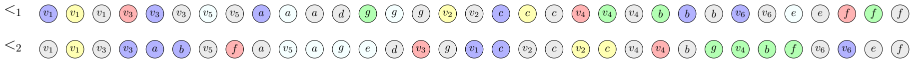


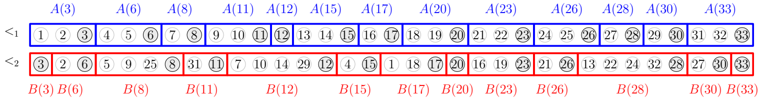

List in the pre-order traversal:

- $<_1$: the incoming arcs
- $<_2$: the outgoing arcs

where an arc is a copy of its out-vertex with color of its in-vertex

# Decoding: Permutations to ordered twin-models

Guess the block ends (color 1)




$3 < 6 < 8 < 11 < 12 < 15 < 17 < 20 < 23 < 26 < 28 < 30 < 33$

is the tree pre-order (on the domain of the image)

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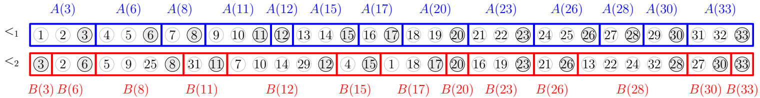


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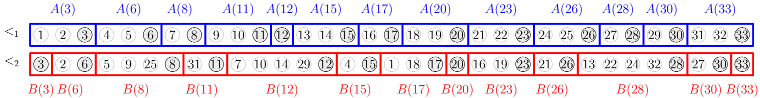


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Use an extra color for the transversal edges (color 2)

# Recent developments

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*Pattern-free permutations are bounded products of separable permutations.*

As a consequence of these two results,

**Corollary** (B., Bourneuf, Geniet, Thomassé '23)
*There is a proper permutation class $\mathcal{P}$ such that every class of binary structures has bounded twin-width if and only if it is a first-order transduction of $\mathcal{P}$.*

# The lens of contraction sequences

| Class of bounded | FO transduction of | constr. on red graphs | efficient MC |
| --- | --- | --- | --- |
| linear rank-width | linear order | bd #edges | MSO |
| rank-width | tree order | bd component | **MSO** |
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**Thank you for your attention!**