Twin-width and Sparsity

Édouard Bonnet

ENS Lyon, LIP

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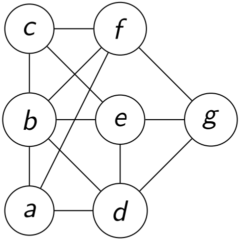
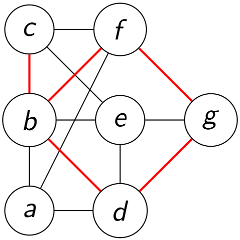
Graphs



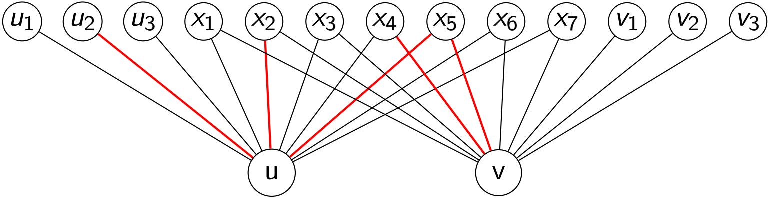
Two outcomes between a pair of vertices:
edge or non-edge

Trigraphs



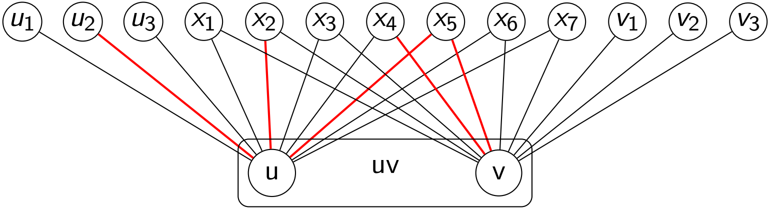
Three outcomes between a pair of vertices:
edge, or non-edge, or red edge (error edge)

Contractions in trigraphs



Identification of two non-necessarily adjacent vertices

Contractions in trigraphs



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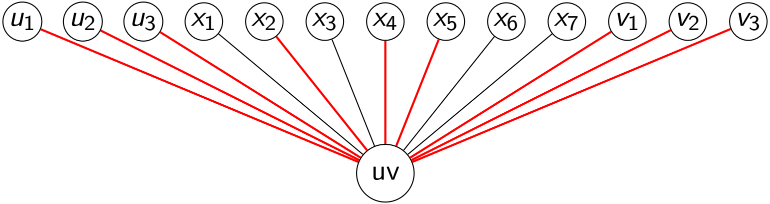
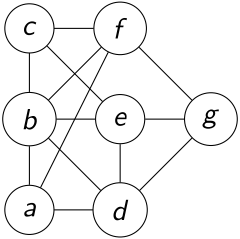


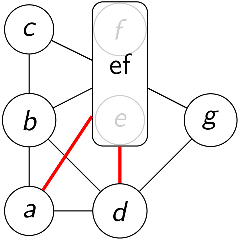
edges to $N(u) \triangle N(v)$ turn red, for $N(u) \cap N(v)$ red is absorbing

Contraction sequence



A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that G_i is obtained by performing one contraction in G_{i+1}.

Contraction sequence



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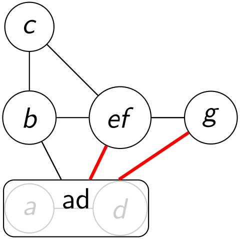

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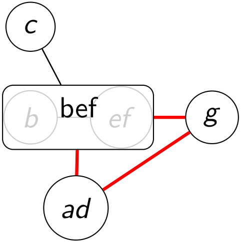

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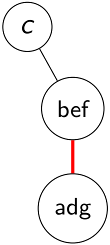

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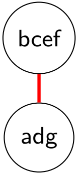

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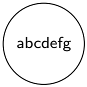

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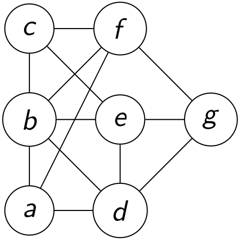
Twin-width

tww(G): Least integer d such that G admits a contraction sequence where all trigraphs have *maximum red degree* at most d.



Maximum red degree $= 0$

overall maximum red degree = 0

Twin-width

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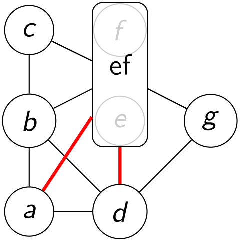

Maximum red degree = 2

overall maximum red degree = 2

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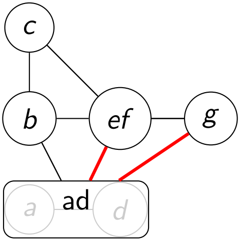

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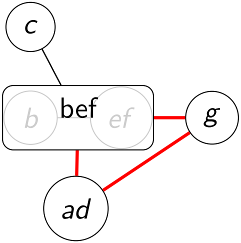

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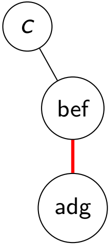

Maximum red degree $= 1$

overall maximum red degree = 2

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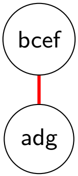

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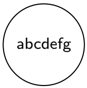


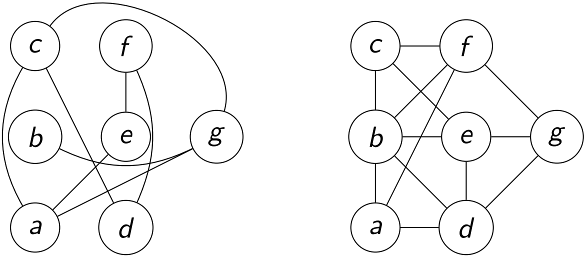
Maximum red degree $= 0$

overall maximum red degree = 2

Simple operations preserving small twin-width

- complementation: remains the same
- taking induced subgraphs: may only decrease
- adding one vertex linked arbitrarily: at most "doubles"
- substitution, lexicographic product: max of the twin-widths

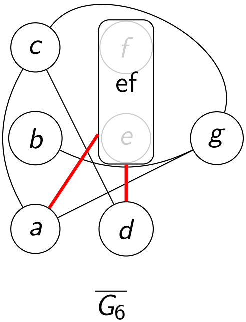
Complementation



$\overline{G}$ $\qquad\qquad$ G

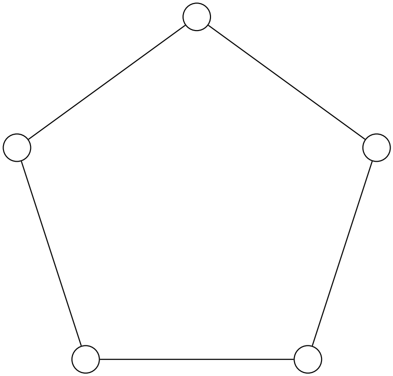
$$\text{tww}(\overline{G}) = \text{tww}(G)$$

Complementation



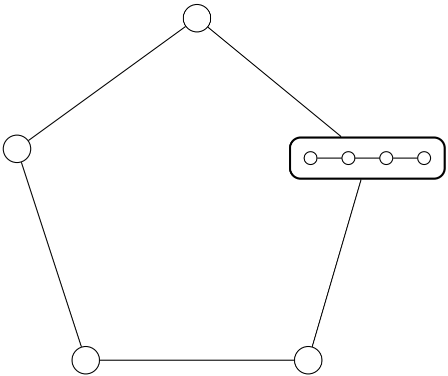
$$\text{tww}(\overline{G}) = \text{tww}(G)$$

Substitution and lexicographic product



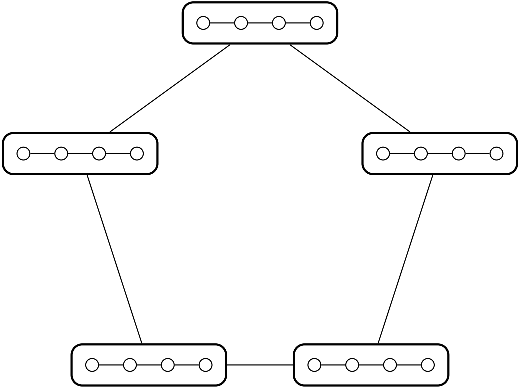
$G = C_5$

Substitution and lexicographic product



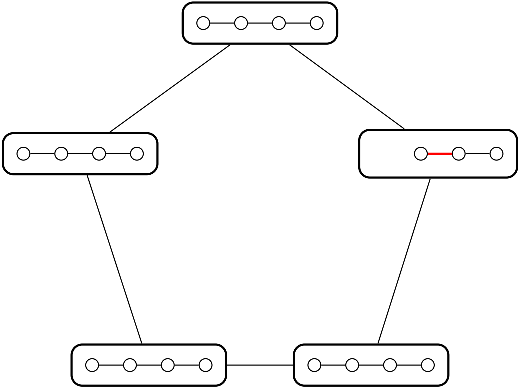
$G = C_5$, $H = P_4$, substitution $G[v \leftarrow H]$

Substitution and lexicographic product



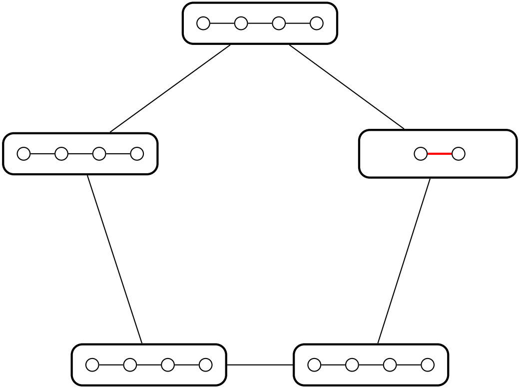
$G = C_5$, $H = P_4$, lexicographic product $G[H]$

Substitution and lexicographic product



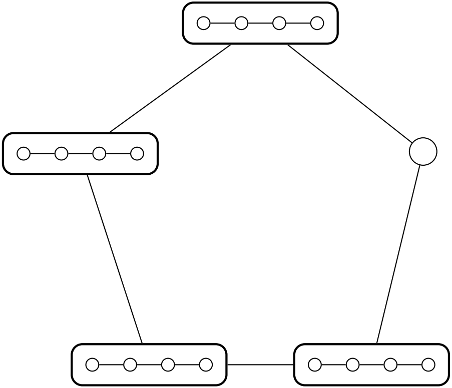
More generally any modular decomposition

Substitution and lexicographic product



More generally any modular decomposition

Substitution and lexicographic product

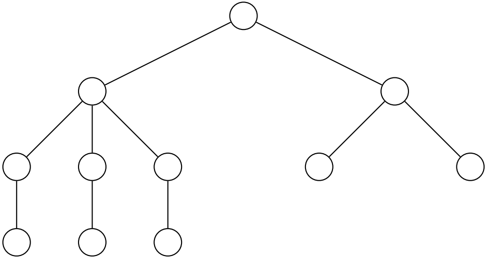


$$\text{tww}(G[H]) = \max(\text{tww}(G), \text{tww}(H))$$

Classes with bounded twin-width

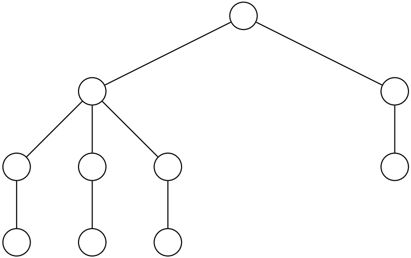
- cographs = twin-width 0
- trees, bounded treewidth, clique-width/rank-width
- grids
- ...

Trees



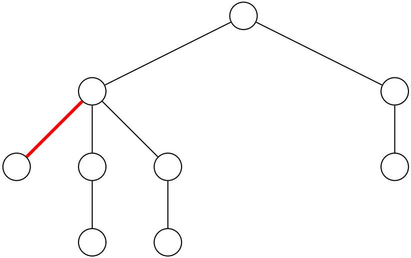
If possible, contract two twin leaves

Trees



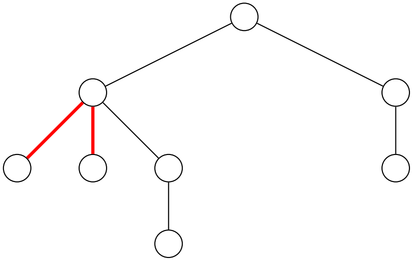
If not, contract a deepest leaf with its parent

Trees



If not, contract a deepest leaf with its parent

Trees



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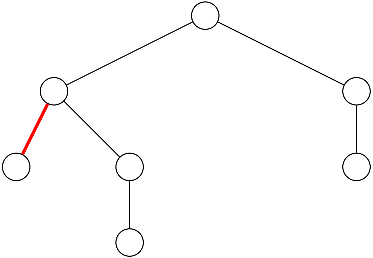
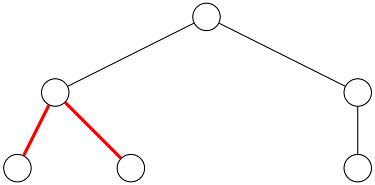


Cannot create a red degree-3 vertex

Trees



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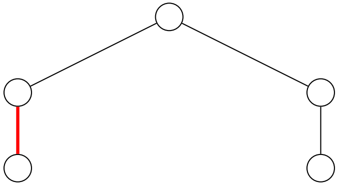

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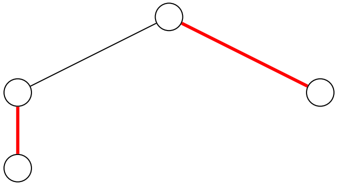

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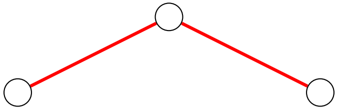

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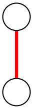

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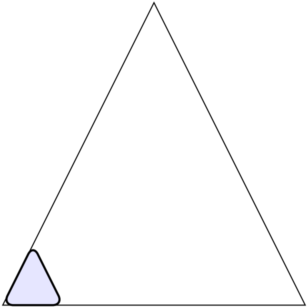
Trees

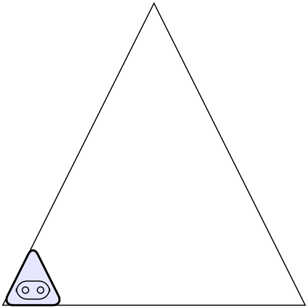
Cannot create a red degree-3 vertex

Bounded rank-width graphs



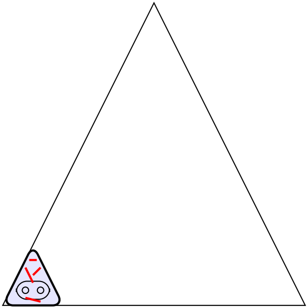
Generalization to bounded *rank-width*

Bounded rank-width graphs



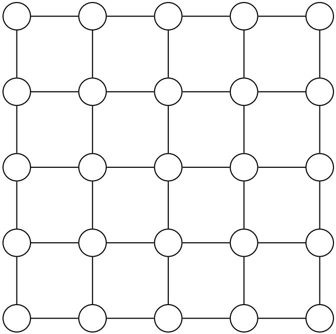
Two twins with respect to the exterior in a small subtree $\rightarrow$ contraction

Bounded rank-width graphs



Red edges cluster in bounded size components

Grids



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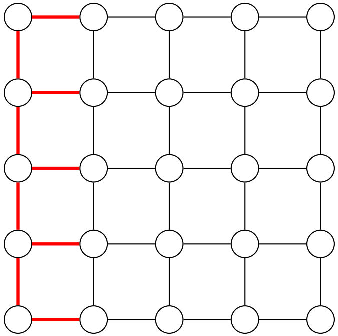

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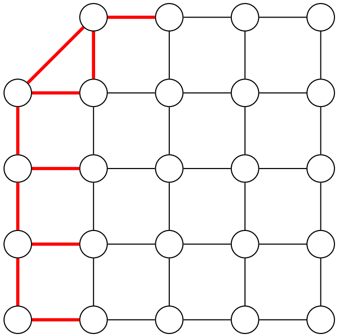

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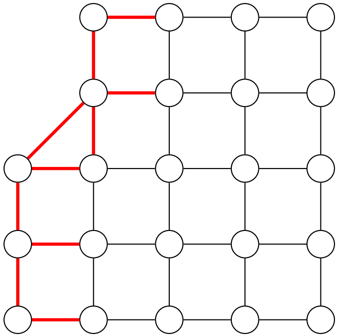

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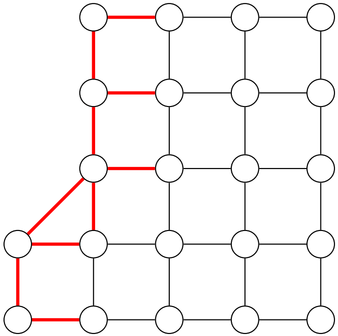

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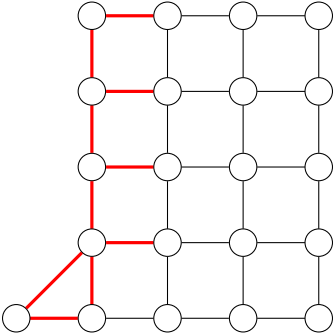

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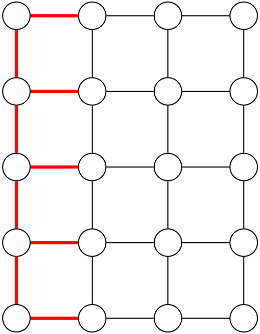
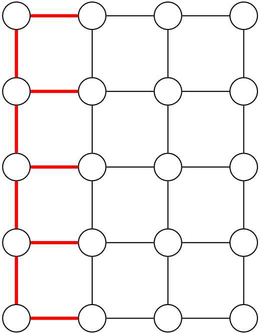


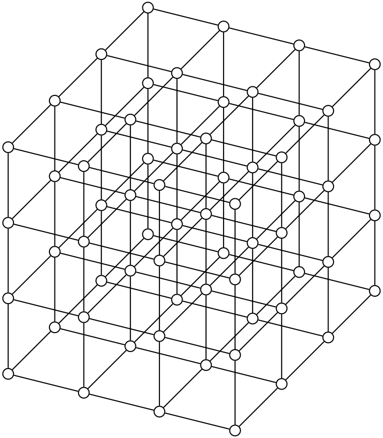
4-sequence for planar grids

Grids



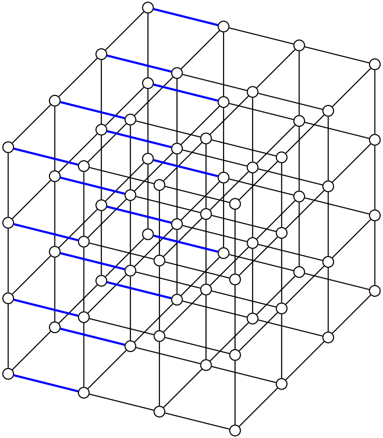
More generally: if a "parallel" contraction of disjoint vertex pairs go from *red degree d* to *red degree d*, then any sequentialization has red degree at most $2d$

3-dimensional grids



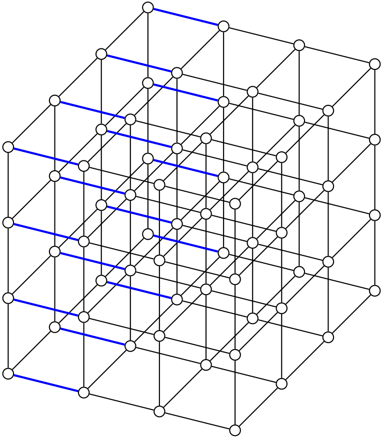
Still bounded degree but contains arbitrary large clique minors

3-dimensional grids



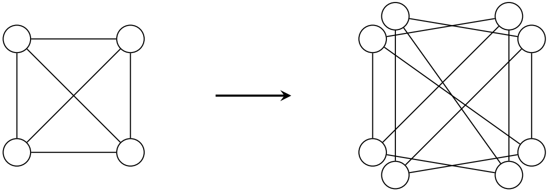
Contract the blue edges in any order $\rightarrow$ 12-sequence

3-dimensional grids



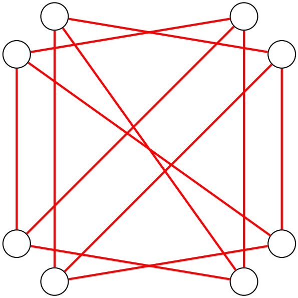
The d-dimensional grid has twin-width $\leqslant 4d$ (even $3d$)

2-lifts, expanders with bounded twin-width



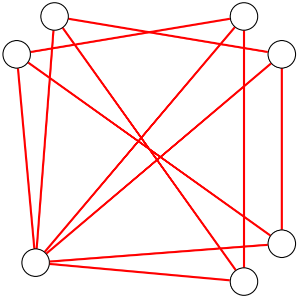
split each vertex in 2, replace each edge by 1 of the 2 matchings

2-lifts, expanders with bounded twin-width



Iterated 2-lifts of K_4 have twin-width at most 6

2-lifts, expanders with bounded twin-width



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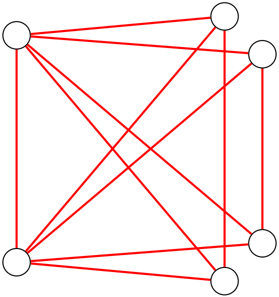

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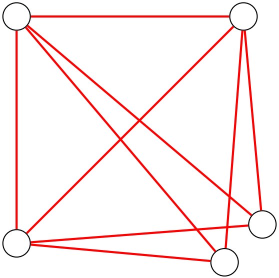

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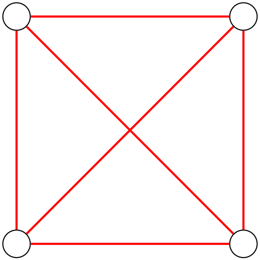
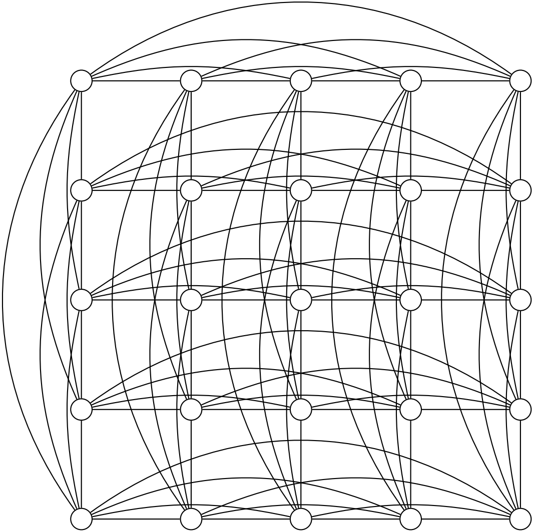
2-lifts, expanders with bounded twin-width



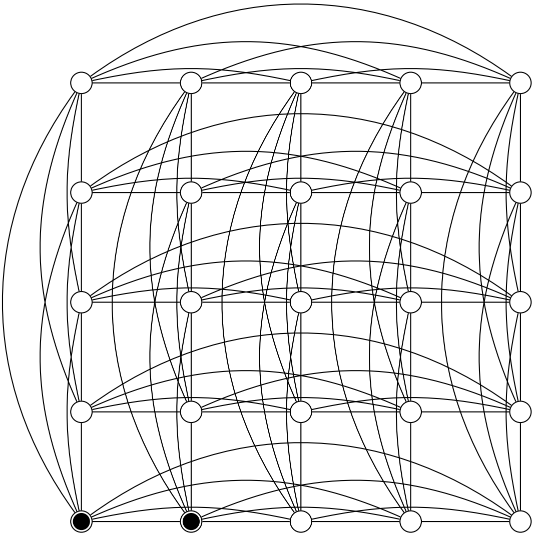
Iterated 2-lifts of K_4 have twin-width at most 6
but no balanced separators of size $o(n)$

First example of unbounded twin-width



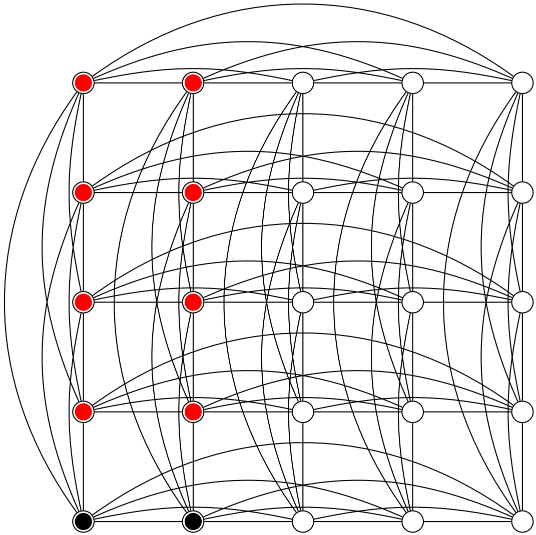
Line graph of a biclique a.k.a. rook graph

First example of unbounded twin-width



No pair of near twins

First example of unbounded twin-width



No pair of near twins

Universal bipartite graph

No $O(1)$-contraction sequence:
twin-width is *not* an iterated identification of near twins.

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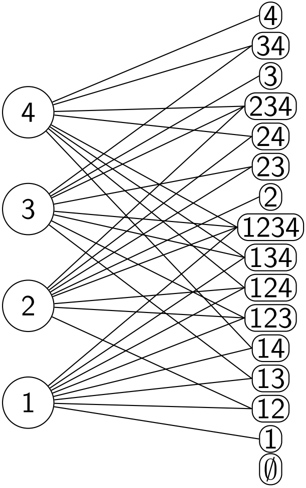

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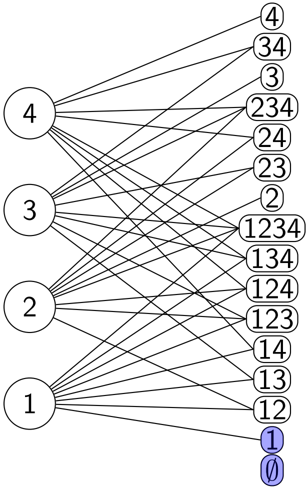

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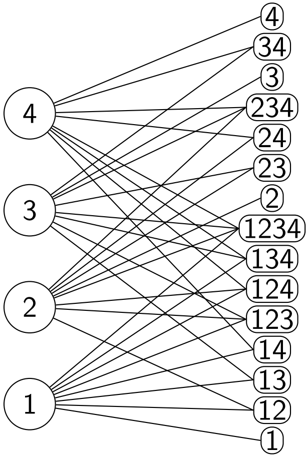

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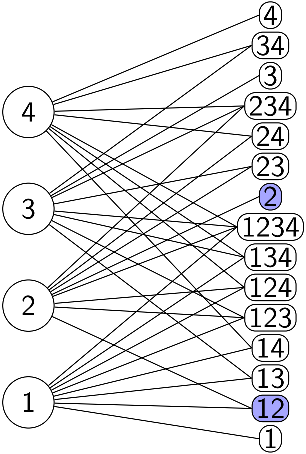

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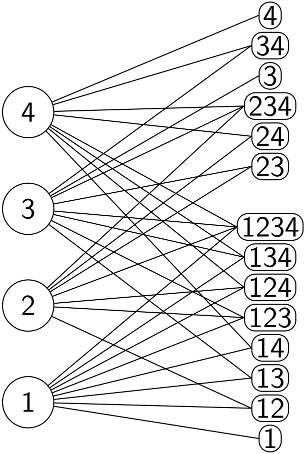

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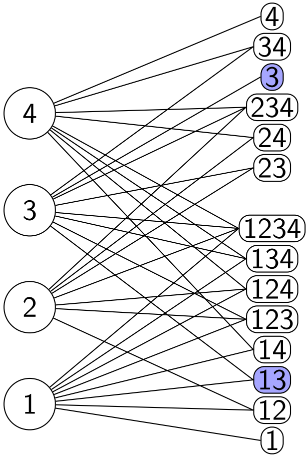

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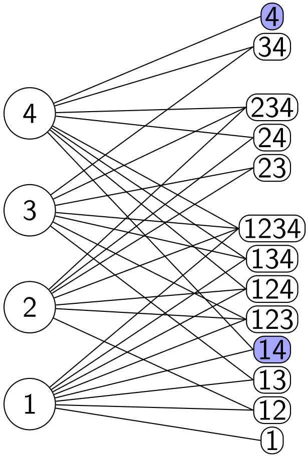

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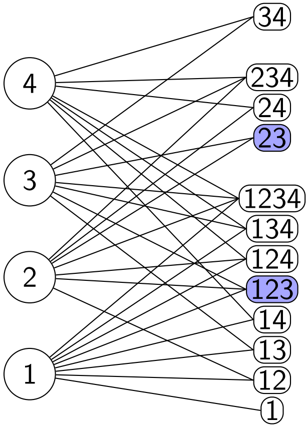

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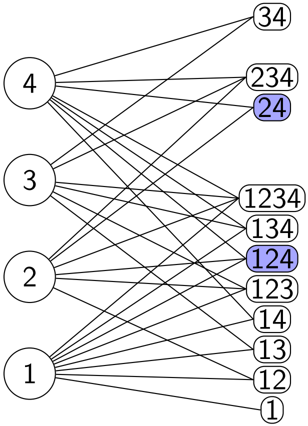

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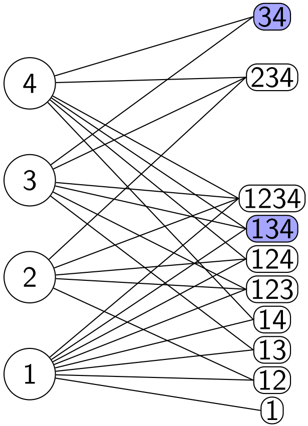

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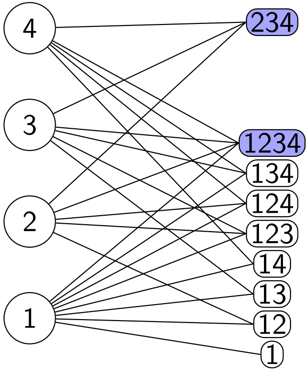

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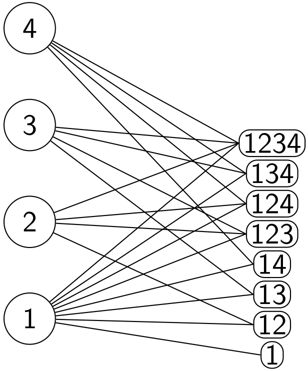

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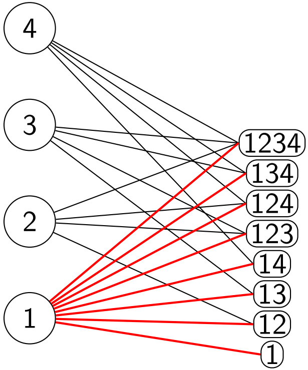
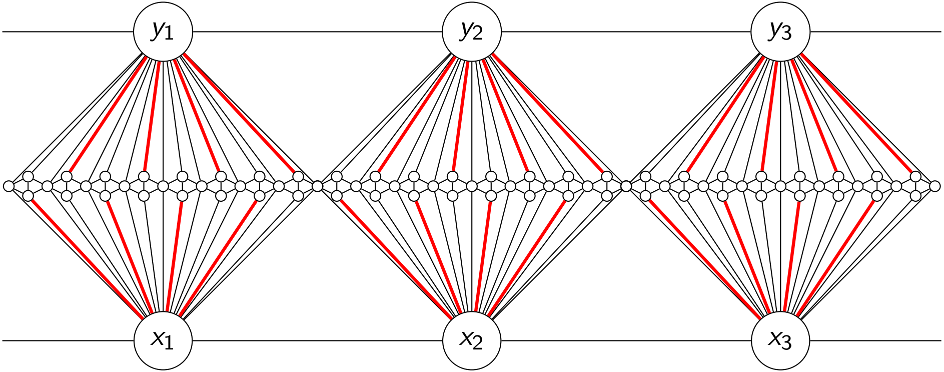


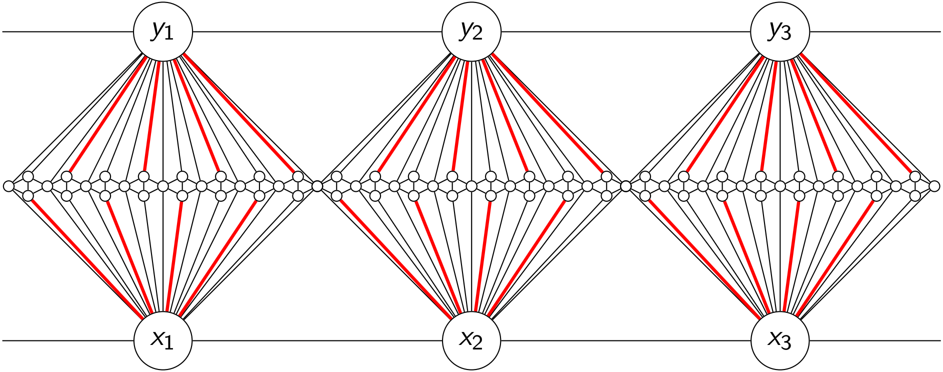
Planar graphs?

Planar graphs?



For every d, a planar trigraph without planar d-contraction

Planar graphs?



For every d, a planar trigraph without planar d-contraction

More powerfool tool needed

Twin-width in the language of matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Encode a bipartite graph (or, if symmetric, any graph)

Twin-width in the language of matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Contraction of two columns (similar with two rows)

Twin-width in the language of matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & r & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & r & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & r & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

How is the twin-width (re)defined?

Twin-width in the language of matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & r & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & r & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & r & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

How to tune it for non-bipartite graph?

Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

$$\left[\begin{array}{c|c|c|c|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}$$

Maximum number of non-constant zones per column or row part
= **error value**

Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

1	1	1	1	1	1	1	0
0	1	1	0	0	1	0	1
0	0	0	0	0	0	0	1
0	1	0	0	1	0	1	0
1	0	0	1	1	0	1	0
0	1	1	1	1	1	0	0
1	0	1	1	1	0	0	1

Maximum number of non-constant zones per column or row part
. . . until there are a single row part and column part

Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

1	1	1	1	1	1	1	0
0	1	1	0	0	1	0	1
0	0	0	0	0	0	0	1
0	1	0	0	1	0	1	0
1	0	0	1	1	0	1	0
0	1	1	1	1	1	0	0
1	0	1	1	1	0	0	1

Twin-width as maximum error value of a contraction sequence

Grid minor

t-grid minor: $t \times t$-division where every cell is non-empty
Non-empty cell: contains at least one 1 entry

$$\left[\begin{array}{cc|cc|cc|cc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

4-grid minor

Grid minor

t-grid minor: $t \times t$-division where every cell is non-empty
Non-empty cell: contains at least one 1 entry

$$\left[\begin{array}{cc|cc|cc|cc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

4-grid minor

A matrix is said t-**grid free** if it does not have a t-grid minor

Mixed minor

Mixed cell: not horizontal nor vertical

$$\left[\begin{array}{cc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

3-mixed minor

Mixed minor

Mixed cell: not horizontal nor vertical

$$\left[\begin{array}{cc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

3-mixed minor

Every mixed cell is witnessed by a 2×2 square = **corner**

Mixed minor

Mixed cell: not horizontal nor vertical

$$\left[\begin{array}{cc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

3-mixed minor

A matrix is said t-**mixed free** if it does not have a t-mixed minor

Mixed value

R_4	1	1	1	0	0	1	1	0
R_3	1	0	1	0	0	1	0	1
	1	0	1	0	0	0	0	1
R_2	0	1	0	0	1	0	1	0
	1	1	0	0	1	0	1	0
R_1	0	1	1	1	0	1	0	0
	1	0	1	0	1	0	0	1
				C_2				

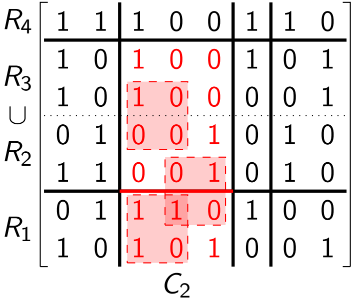
$\approx$ (maximum) number of cells with a corner per row/column part

Mixed value

R_4	1	1	1	0	0	1	1	0
R_3	1	0	1	0	0	1	0	1
	1	0	1	0	0	0	0	1
R_2	0	1	0	0	1	0	1	0
	1	1	0	0	1	0	1	0
R_1	0	1	1	1	0	1	0	0
	1	0	1	0	1	0	0	1
				C_2				

But we add the number of *boundaries* containing a corner

Mixed value



∴ merging row parts do not increase mixed value of column part

Twin-width and mixed freeness

Theorem (B., Kim, Thomassé, Watrigant '20)

If G admits ***a*** *t-mixed free adjacency matrix, then* $\text{tww}(G) = 2^{2^{O(t)}}$.

Twin-width and mixed freeness

Theorem (B., Kim, Thomassé, Watrigant '20)

If $\exists \sigma$ s.t. $Adj_\sigma(G)$ is t-mixed free, then $tww(G) = 2^{2^{O(t)}}$.

Twin-width and mixed freeness

Theorem (B., Kim, Thomassé, Watrigant '20)

If $\exists \sigma$ *s.t.* $Adj_\sigma(G)$ *is* t*-mixed free, then* $tww(G) = 2^{2^{O(t)}}$.

Step 1: find a division sequence $(\mathcal{D}_i)_i$ **with mixed value** $f(t)$

1	1	1	1	1	1	1	0
0	1	1	0	0	1	0	1
0	0	0	0	0	0	0	1
0	1	0	0	1	0	1	0
1	0	0	1	1	0	1	0
0	1	1	1	1	1	0	0
1	0	1	1	1	0	0	1

Merge consecutive parts greedily

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1	0	0	1	1	0	1	0
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0	1	1	0	0	1	0	1
0	0	0	0	0	0	0	1
0	1	0	0	1	0	1	0
1	0	0	1	1	0	1	0
0	1	1	1	1	1	0	0
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0	1	1	0	0	1	0	1
0	0	0	0	0	0	0	1
0	1	0	0	1	0	1	0
1	0	0	1	1	0	1	0
0	1	1	1	1	1	0	0
1	0	1	1	1	0	0	1

Stuck, removing every other separation $\rightarrow \frac{f(t)}{2}$ mixed cells per part

Marcus-Tardos theorem

Theorem (Marcus and Tardos '04, Stanley-Wilf conjecture)

For every k, there is a c_k such that every $n \times m$ $0,1$-matrix with at least $c_k \max(n, m)$ 1 entries admits a k-grid minor.

Marcus-Tardos theorem

Theorem (Marcus and Tardos '04, Stanley-Wilf conjecture)
For every k, there is a c_k such that every $n \times m$ $0,1$-matrix with at least $c_k \max(n, m)$ 1 entries admits a k-grid minor.

Auxiliary 0,1-matrix with one entry per cell: a 1 iff the cell is mixed

Twin-width and mixed freeness

Theorem (B., Kim, Thomassé, Watrigant '20)

If $\exists\sigma$ *s.t.* $Adj_\sigma(G)$ *is* t*-mixed free, then* $tww(G) = 2^{2^{O(t)}}$.

Step 1: find a division sequence $(\mathcal{D}_i)_i$ **with mixed value** $f(t)$

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}$$

Stuck, removing every other separation $\rightarrow \frac{f(t)}{2}$ mixed cells per part

Twin-width and mixed freeness

Theorem (B., Kim, Thomassé, Watrigant '20)

If $\exists \sigma$ s.t. $Adj_\sigma(G)$ is t-mixed free, then $tww(G) = 2^{2^{O(t)}}$.

Step 1: find a division sequence $(\mathcal{D}_i)_i$ with mixed value $f(t)$

1	1	1	1	1	1	1	0
0	1	1	0	0	1	0	1
0	0	0	0	0	0	0	1
0	1	0	0	1	0	1	0
1	0	0	1	1	0	1	0
0	1	1	1	1	1	0	0
1	0	1	1	1	0	0	1

Stuck, removing every other separation $\rightarrow \frac{f(t)}{2}$ mixed cells per part

Impossible!

Twin-width and mixed freeness

Theorem (B., Kim, Thomassé, Watrigant '20)

If $\exists \sigma$ s.t. $Adj_\sigma(G)$ is t-mixed free, then $tww(G) = 2^{2^{O(t)}}$.

Step 1: find a division sequence $(\mathcal{D}_i)_i$ with mixed value $f(t)$
Step 2: find a contraction sequence with error value $g(t)$

1	1	1	1	1	1	1	0
0	1	1	0	0	1	0	1
0	0	0	0	0	0	0	1
0	1	0	0	1	0	1	0
1	0	0	1	1	0	1	0
0	1	1	1	1	1	0	0
1	0	1	1	1	0	0	1

Refinement of $\mathcal{D}_i$ where each part coincides on the non-mixed cells

Twin-width and mixed freeness

Theorem (B., Kim, Thomassé, Watrigant '20)

If $\exists \sigma$ *s.t.* $Adj_\sigma(G)$ *is* t*-mixed free, then* $tww(G) = 2^{2^{O(t)}}$.

Twin-width and mixed freeness

Theorem (B., Kim, Thomassé, Watrigant '20)

If $\exists\sigma$ *s.t.* $Adj_\sigma(G)$ *is* t*-mixed free, then* $tww(G) = 2^{2^{O(t)}}$.

Now to bound the twin-width of a class $\mathcal{C}$:

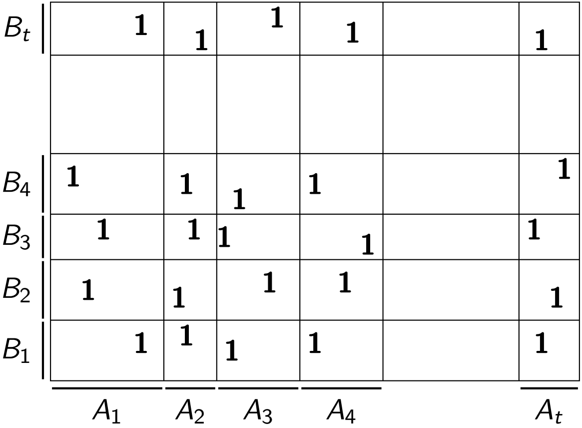
1) Find a *good* vertex-ordering procedure

2) Argue that, in this order, a t-mixed minor would conflict with $\mathcal{C}$

Bounded twin-width – K_t-minor free graphs

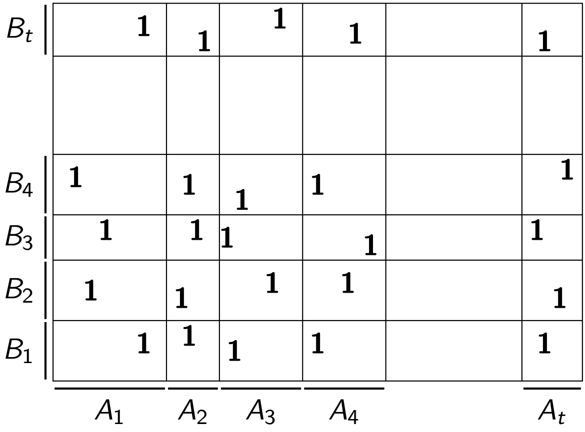
Given a hamiltonian path, we would just use this order

Bounded twin-width – K_t-minor free graphs



Contracting the $2t$ subpaths yields a $K_{t,t}$-minor, hence a K_t-minor

Bounded twin-width – K_t-minor free graphs

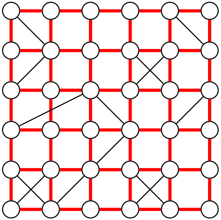


Instead we use a specially crafted lex-DFS discovery order

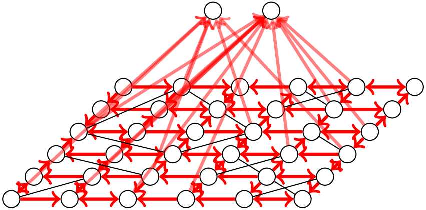
A surprising and convenient equivalent

Theorem (B., Kim, Reinald, Thomassé '21+)

Twin-width and oriented twin-width are functionally equivalent.



red degree



red out-degree
(red arcs oriented from the contraction)

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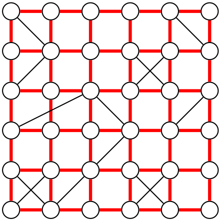
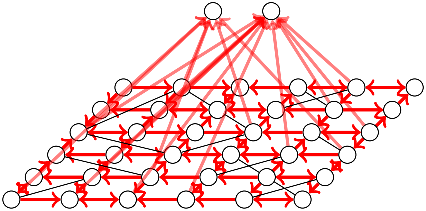


red degree



red out-degree
(red arcs oriented from the contraction)

Theorem (Kotzig's theorem '55)

Planar graphs have oriented twin-width at most 9.

Theorem (B., Geniet, Kim, Thomassé, Watrigant '20 & '21)

The following classes have bounded twin-width, and $O(1)$-sequences can be computed in polynomial time.

- *Bounded rank-width, and even, boolean-width graphs,*
- *every hereditary proper subclass of permutation graphs,*
- *posets of bounded antichain size (seen as digraphs),*
- *unit interval graphs,*
- *K_t-minor free graphs,*
- *map graphs,*
- *subgraphs of d-dimensional grids,*
- *K_t-free unit d-dimensional ball graphs,*
- *$\Omega(\log n)$-subdivisions of all the n-vertex graphs,*
- *cubic expanders defined by iterative random 2-lifts from K_4,*
- *strong products of two bounded twin-width classes, one with bounded degree, etc.*

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Can we solve problems faster, given an $O(1)$-sequence?

One cograph definition

Cographs form the unique *maximal hereditary* class in which every[1] graph has two *twins*

[1] provided it has at least two vertices

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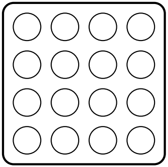

Is there another algorithmic scheme based on this definition?

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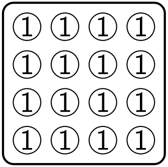

Let's try with $\alpha(G)$, and store in a vertex its inner max solution

[1] provided it has at least two vertices

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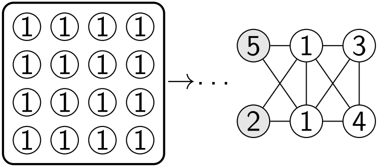

We can find a pair of false/true twins

[1] provided it has at least two vertices

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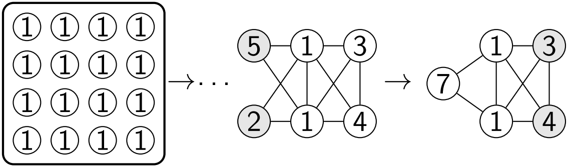

Sum them if they are false twins

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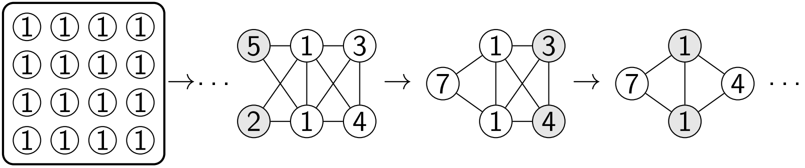

Max them if they are true twins

[1] provided it has at least two vertices

Example of k-Independent Set

d-sequence: $G = G_n, G_{n-1}, \ldots, G_2, G_1 = K_1$

Algorithm: **Compute by dynamic programming a best partial solution in each red connected subgraph of size at most k.**

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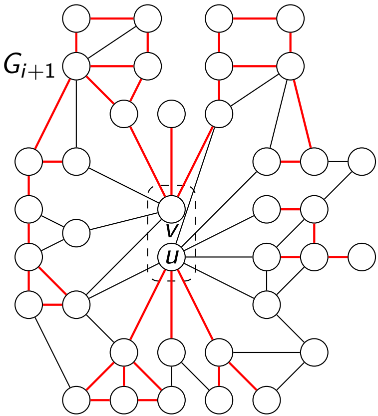
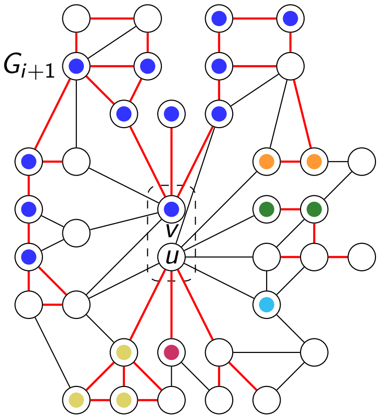
How to go from the partial solutions of G_{i+1} to those of G_i?



Best partial solution inhabiting •?



3 unions of $\leqslant d+2$ red connected subgraphs to consider in G_{i+1} with u, or v, or both

Other (almost) single-exponential parameterized algorithms

Theorem (B., Geniet, Kim, Thomassé, Watrigant '21)

Given a d-sequence $G = G_n, \dots, G_1 = K_1$,

- k-Independent Set,
- k-Clique,
- (r, k)-Scattered Set,
- k-Dominating Set, *and*
- (r, k)-Dominating Set

can be solved in time $2^{O(k)}n$,
whereas Subgraph Isomorphism *and* Induced Subgraph Isomorphism *can be solved in time $2^{O(k \log k)}n$.*

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A more general FPT algorithm?

First-order model checking on graphs

GRAPH FO MODEL CHECKING	**Parameter:** $\|\varphi\|$

Input: A graph G and a first-order sentence $\varphi \in FO(\{E\})$
Question: $G \models \varphi$?

First-order model checking on graphs

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Example:

$$\varphi = \exists x_1 \exists x_2 \cdots \exists x_k \forall x \bigvee_{1 \leqslant i \leqslant k} x = x_i \vee \bigvee_{1 \leqslant i \leqslant k} E(x, x_i) \vee E(x_i, x)$$

$G \models \varphi$? $\Leftrightarrow$

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$G \models \varphi$? $\Leftrightarrow$ k-DOMINATING SET

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Example:

$$\varphi = \exists x_1 \exists x_2 \cdots \exists x_k \bigwedge_{1 \leqslant i < j \leqslant k} \neg(x_i = x_j) \wedge \neg E(x_i, x_j) \wedge \neg E(x_j, x_i)$$

$G \models \varphi$? $\Leftrightarrow$

First-order model checking on graphs

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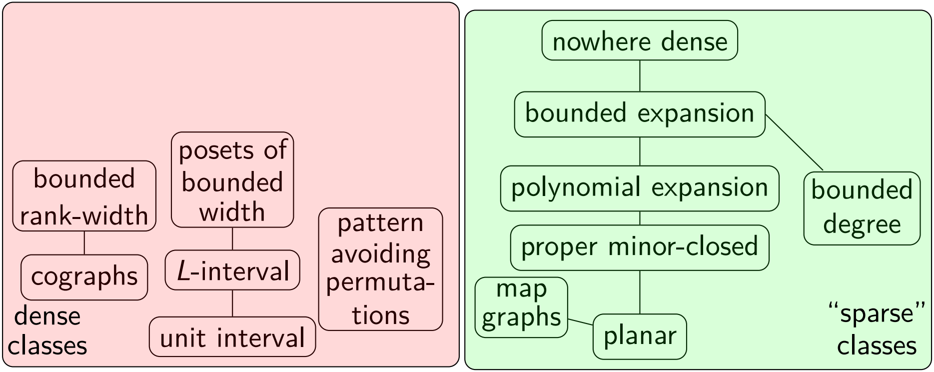
$G \models \varphi$? $\Leftrightarrow$ k-INDEPENDENT SET

Classes with known tractable FO model checking



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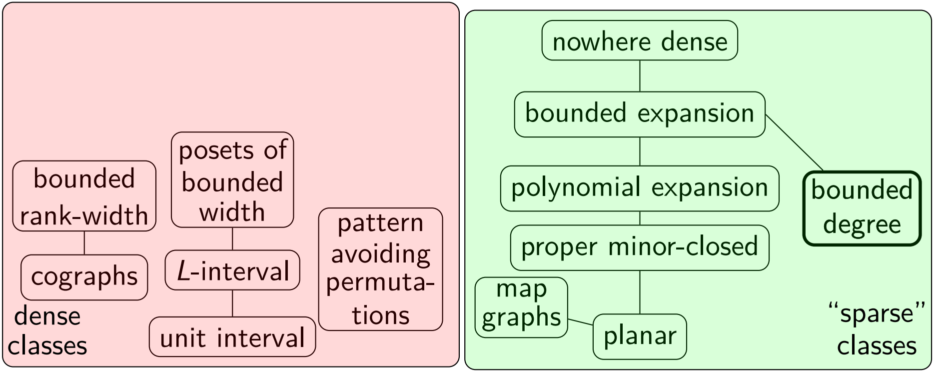
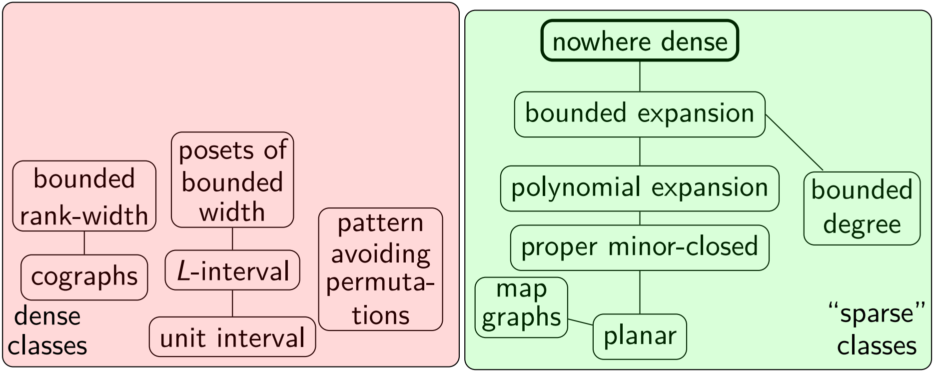


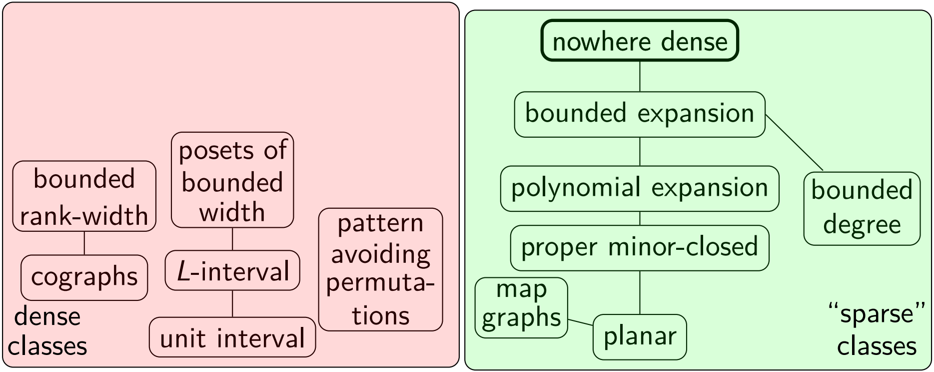
FO MODEL CHECKING solvable in $f(|\varphi|)n$ on bounded-degree graphs [Seese '96]

Classes with known tractable FO model checking



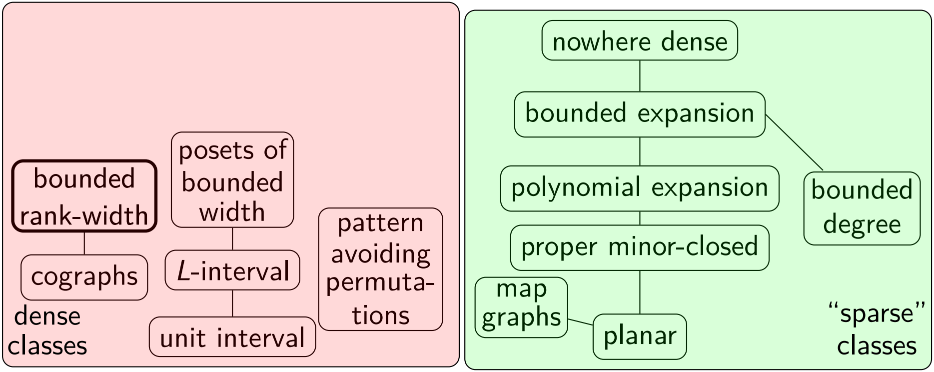
FO MODEL CHECKING solvable in $f(|\varphi|)n^{1+\varepsilon}$ on any nowhere dense class
[Grohe, Kreutzer, Siebertz '14]

Classes with known tractable FO model checking



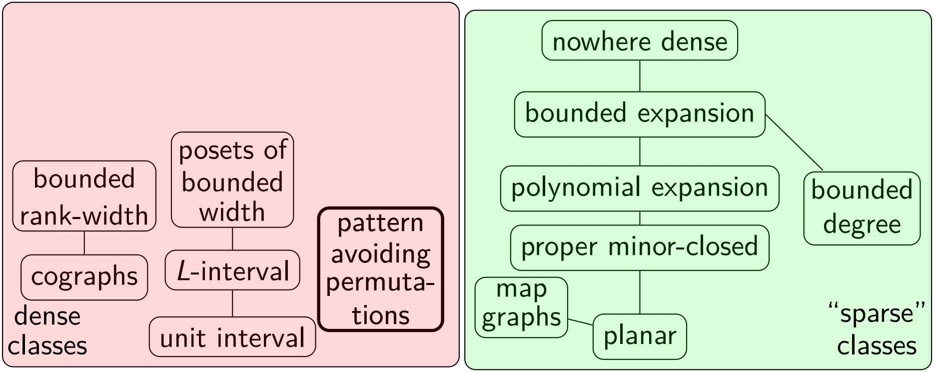
End of the story for the subgraph-closed classes
tractable FO Model Checking ⇔ nowhere dense

Classes with known tractable FO model checking



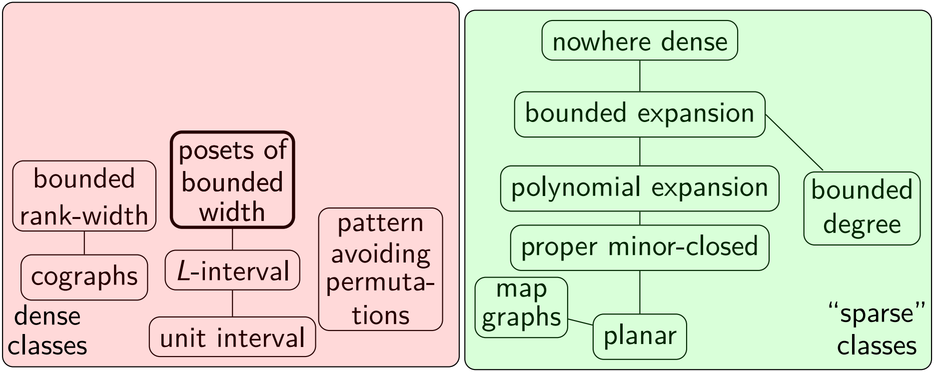
MSO$_1$ Model Checking solvable in $f(|\varphi|, w)n$ on graphs of rank-width w
[Courcelle, Makowsky, Rotics '00]

Classes with known tractable FO model checking



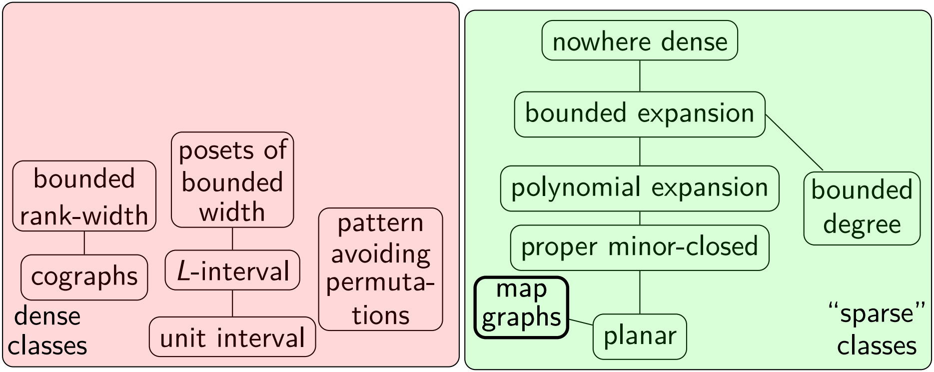
Is σ a subpermutation of τ? solvable in $f(|\sigma|)|\tau|$
[Guillemot, Marx '14]

Classes with known tractable FO model checking



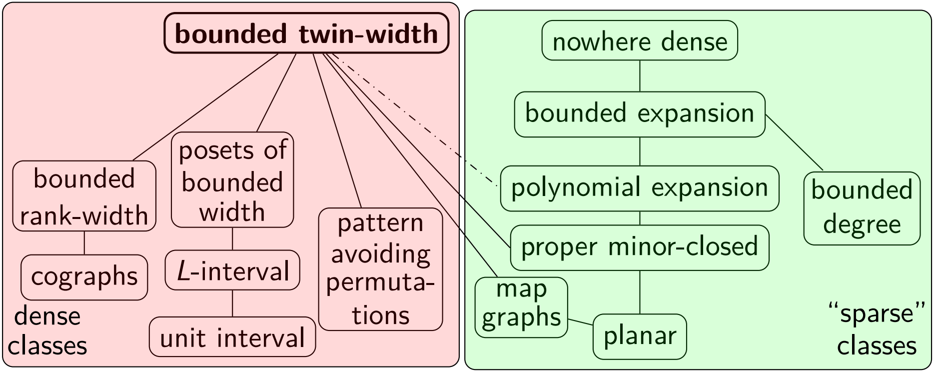
FO MODEL CHECKING solvable in $f(|\varphi|, w)n^2$ on posets of width w [GHLOORS '15]

Classes with known tractable FO model checking



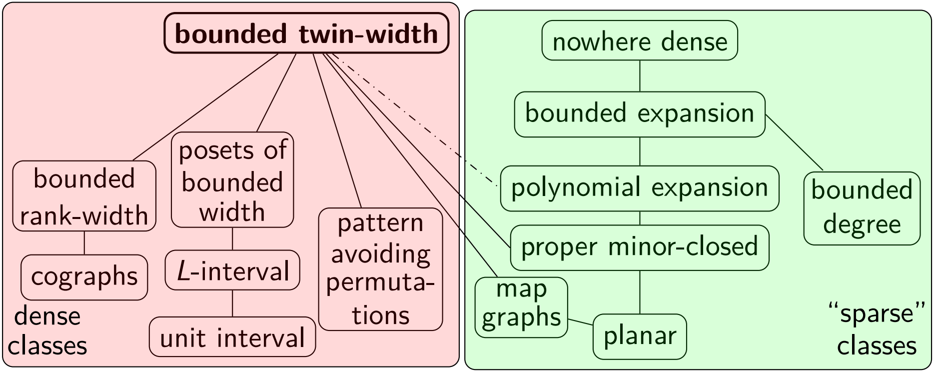
FO Model Checking solvable in $f(|\varphi|)n^{O(1)}$ on map graphs
[Eickmeyer, Kawarabayashi '17]

Classes with known tractable FO model checking



FO MODEL CHECKING solvable in $f(|\varphi|, d)n$ on graphs with a d-sequence [B., Kim, Thomassé, Watrigant '20]

Classes with known tractable FO model checking

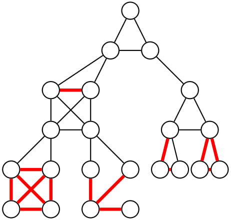


Every transduction of a bounded twin-width class has bounded twin-width [B., Kim, Thomassé, Watrigant '20]

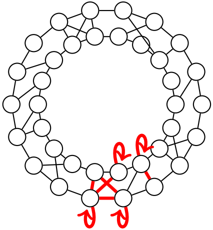
Classic width-measures via contraction sequences

Theorem (B., Kim, Reinald, Thomassé '21+)

Component twin-width is functionally equivalent to rank-width.
Total twin-width is functionally equivalent to linear rank-width.



Component twin-width:
max red component size



Total twin-width:
max number of red edges

The sparse regime captures treewidth and pathwidth

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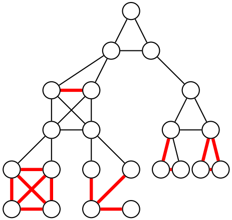

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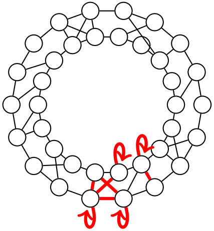

Total twin-width:
max number of red edges

Alternative proof of Courcelle, Makowsky, Rotics's theorem:
FO model checking approach using Feferman-Vaught instead of Gaifman's theorem

Sparse classes with bounded twin-width

Theorem (B., Geniet, Kim, Thomassé, Watrigant '21)

Let $\mathcal{C}$ a hereditary class of bounded twin-width. TFAE:

- *graphs in $\mathcal{C}$ have d-grid free adjacency matrices;*
- *graphs in $\mathcal{C}$ are $K_{t,t}$-free;*
- *graphs in $\mathcal{C}$ have linearly many edges;*
- *The subgraph-closure of $\mathcal{C}$ has bounded twin-width;*
- *$\mathcal{C}$ has bounded expansion.*

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- *$\mathcal{C}$ has bounded expansion.*

Still an interesting family of classes including bounded queue/stack number, K_t-minor free, and some expander classes

Does polynomial expansion imply bounded twin-width?

χ-boundedness

$\mathcal{C}$ χ-bounded: $\exists f, \forall G \in \mathcal{C}$, $\chi(G) \leqslant f(\omega(G))$

Theorem (B., Geniet, Kim, Thomassé, Watrigant '21)

Every twin-width class is χ-bounded.
More precisely, every graph G of twin-width at most d admits a proper $(d+2)^{\omega(G)-1}$-coloring.

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More precisely, every graph G of twin-width at most d admits a proper $(d+2)^{\omega(G)-1}$-coloring.

Are they polynomially χ-bounded? i.e., $\chi(G) = O(\omega(G)^d)$

Bounded twin-width graphs do satisfy strong Erdős-Hajnal

$d + 2$-coloring in the triangle-free case

Algorithm: **Start from $G_1 = K_1$, color its unique vertex 1, and rewind the d-sequence. A contraction seen backward is a split and we shall find colors for the two new vertices.**

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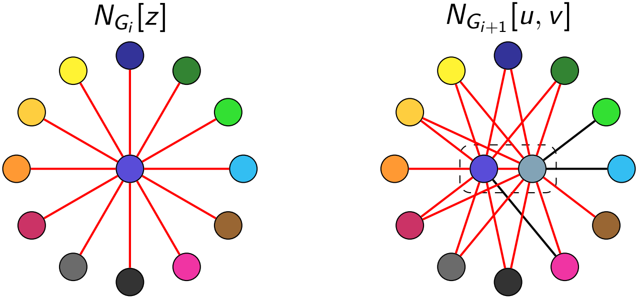

z has only red incident edges $\rightarrow$ $d + 2$-nd color available to v

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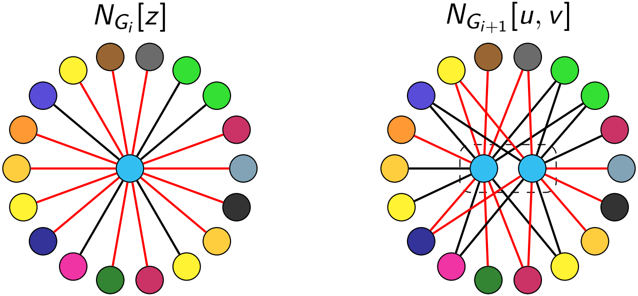
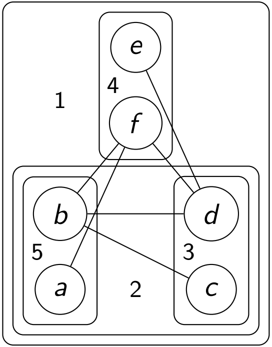
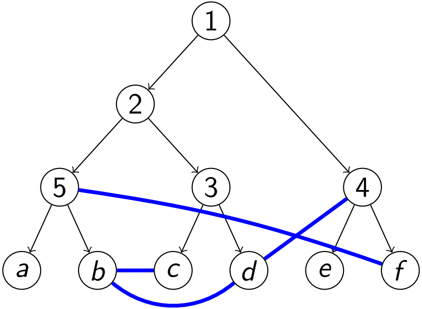


z incident to at least one black edge $\rightarrow$ non-edge between u and v

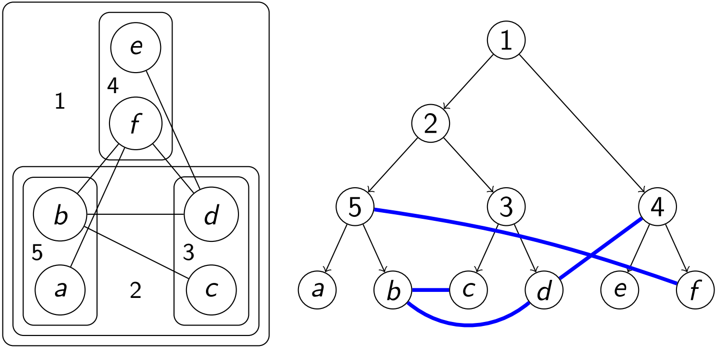
Twin-decomposition





Sparse model for bounded twin-width graphs
(degeneracy of the blue graph by orienting)

Twin-decomposition



Theorem (B., Nešetřil, Ossona de Mendez, Siebertz, Thomassé '21+)

A class of binary structures has bounded twin-width if and only if it is an FO transduction of a proper permutation class.

Small classes

Small: class with at most $n!2^{O(n)}$ labeled graphs on $[n]$.

Theorem (B., Geniet, Kim, Thomassé, Watrigant '21)

Bounded twin-width classes are small.

Theorem (B., Nešetřil, Ossona de Mendez, Siebertz, Thomassé '21+)

...even at most $2^{O(n)}$ graphs up to isomorphism.

Unifies and extends the same result for:
σ-free permutations [Marcus, Tardos '04]
K_t-minor free graphs [Norine, Seymour, Thomas, Wollan '06]

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Subcubic graphs, interval graphs, triangle-free unit segment graphs have **unbounded** twin-width

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The converse for hereditary classes does not hold

Theorem (B., Geniet, Tessera, Thomassé '21+)

There is a randomized construction of a finitely-generated group whose hereditary class of finite restrictions of the Cayley graph has unbounded twin-width (and yet is small).

The case of ordered binary structures

Theorem (B., Giocanti, Ossona de Mendez, Simon, Thomassé, Toruńczyk '21+)

Let $\mathscr{C}$ be a hereditary class of ordered graphs. TFAE:

(1) $\mathscr{C}$ *has bounded twin-width;*

(2) $\mathscr{C}$ *is monadically dependent;*

(3) $\mathscr{C}$ *is dependent;*

(4) $\mathscr{C}$ *contains* $2^{O(n)}$ *ordered n-vertex graphs;*

(5) $\mathscr{C}$ *contains less than* $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} k!$ *ordered n-vertex graphs;*

(6) $\mathscr{C}$ *does not include one of 24 minimal hereditary classes of ordered graphs with unbounded twin-width.*

(7) *FO-model checking is fixed-parameter tractable on* $\mathscr{C}$.

Open questions

Algorithm to compute/approximate twin-width in general

Explicit examples of bounded-degree graphs of unbounded twin-width

Fully classify classes with tractable FO model checking

Some more classes could have bounded twin-width: polynomial expansion, $K_{t,t}$-free string graphs, etc.

Could smallness alone be algorithmically exploitable?