Twin-width

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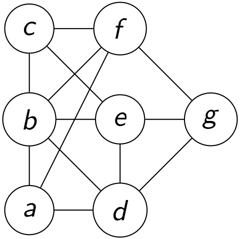
based on joint works with Colin Geniet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant

ENS Lyon, LIP

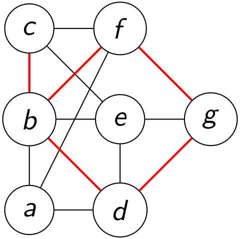
January 20th, 2021, tutorial at Universität Bremen

Graphs



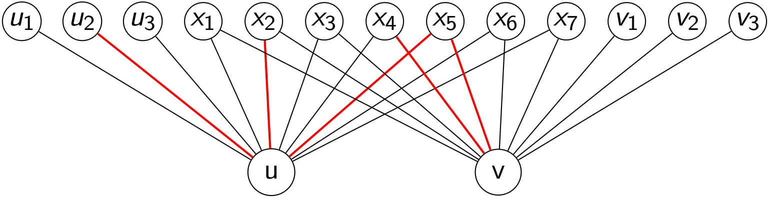
Two outcomes between a pair of vertices:
edge or non-edge

Trigraphs



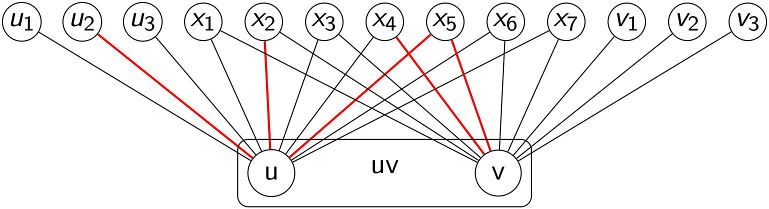
Three outcomes between a pair of vertices:
edge, or non-edge, or red edge (error edge)

Contractions in trigraphs



Identification of two non-necessarily adjacent vertices

Contractions in trigraphs



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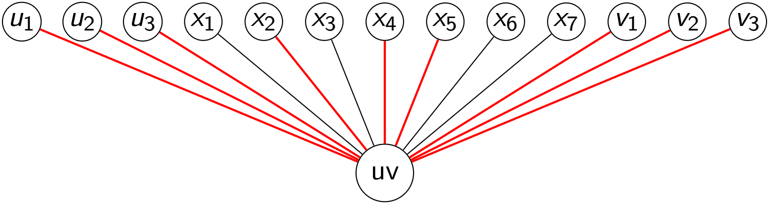
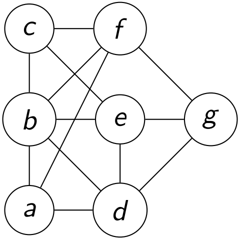


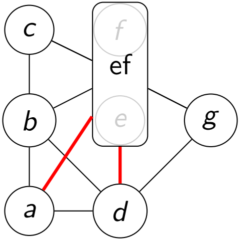
edges to $N(u) \triangle N(v)$ turn red, for $N(u) \cap N(v)$ red is absorbing

Contraction sequence



A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that
G_i is obtained by performing one contraction in G_{i+1}.

Contraction sequence



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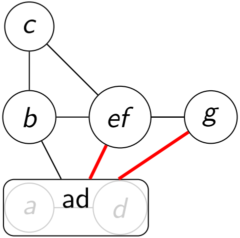

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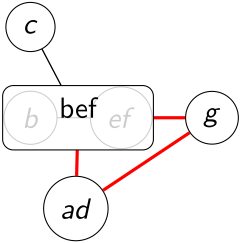

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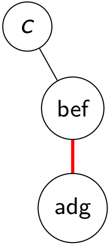

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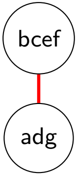

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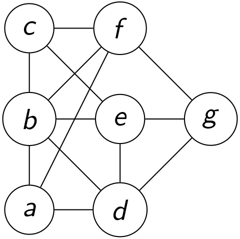
Twin-width

tww(G): Least integer d such that G admits a contraction sequence where all trigraphs have *maximum red degree* at most d.



Maximum red degree $= 0$

overall maximum red degree $= 0$

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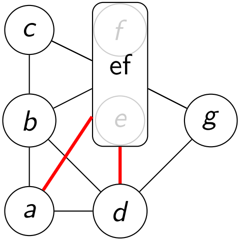

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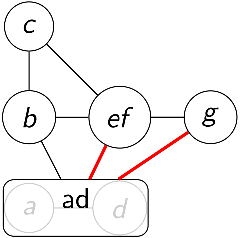

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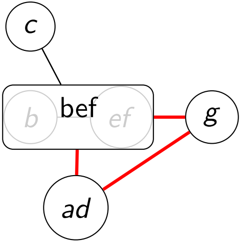

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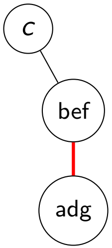

Maximum red degree $= 1$

overall maximum red degree = 2

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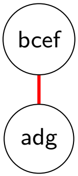

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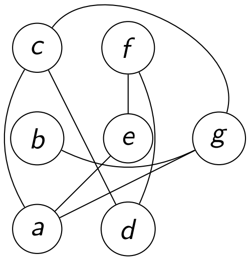
Maximum red degree $= 0$

overall maximum red degree = 2

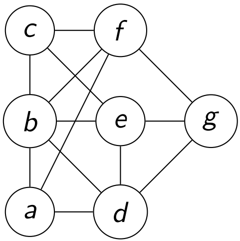
Simple operations preserving small twin-width

- complementation: remains the same
- taking induced subgraphs: may only decrease
- adding one vertex linked arbitrarily: at most "doubles"

Complementation



$\overline{G}$



G

$$\mathrm{tww}(\overline{G}) = \mathrm{tww}(G)$$

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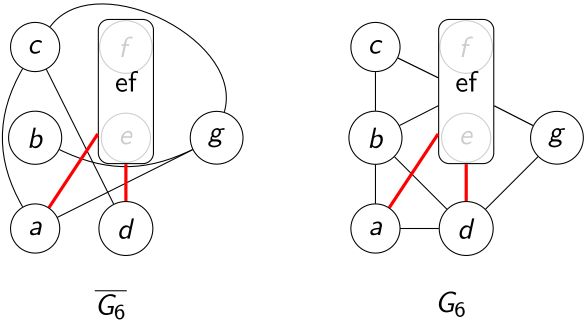

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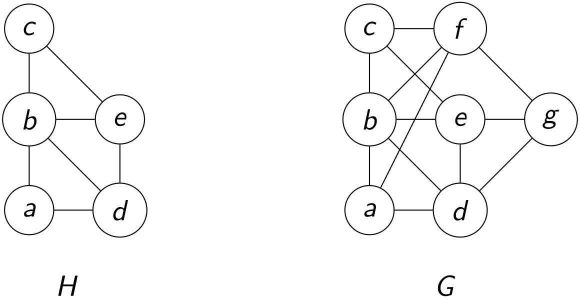
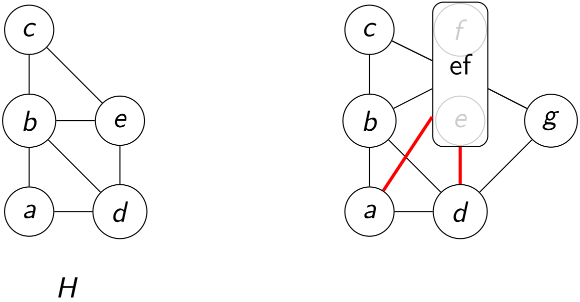
Induced subgraph



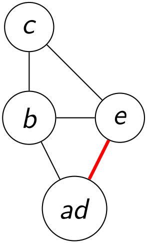
$$\mathrm{tww}(H) \leqslant \mathrm{tww}(G)$$

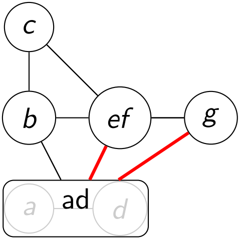
Induced subgraph



Ignore absent vertices

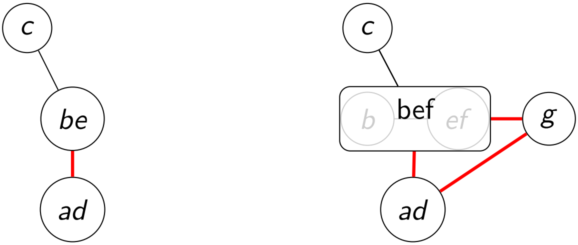
Induced subgraph





Mimic the contractions otherwise

Induced subgraph



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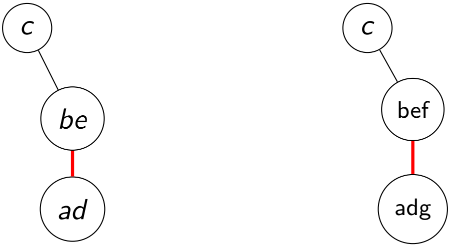

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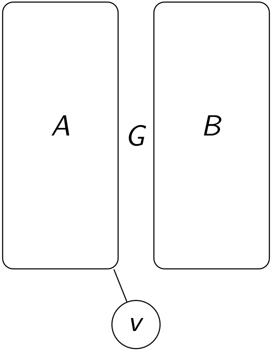
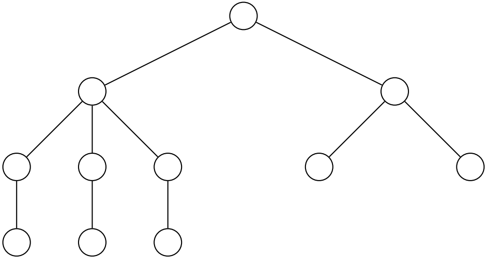
Adding one vertex v

Left as an exercise



Hint: Up until the very end, v shall have no incident red edge

Graphs with bounded twin-width – trees



If possible, contract two twin leaves

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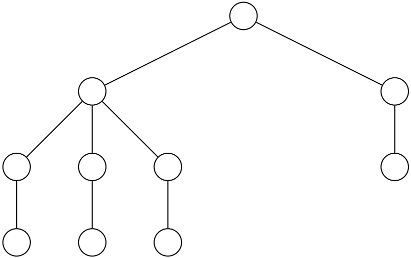

If not, contract a deepest leaf with its parent

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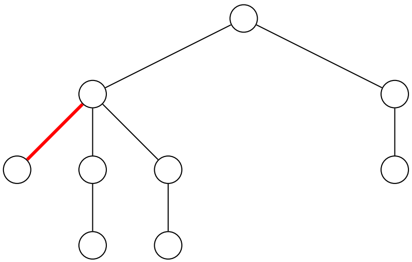

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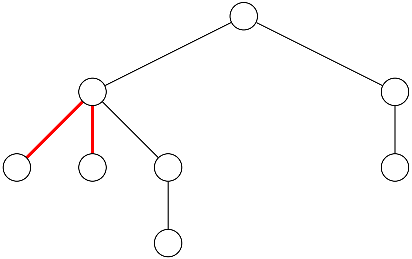

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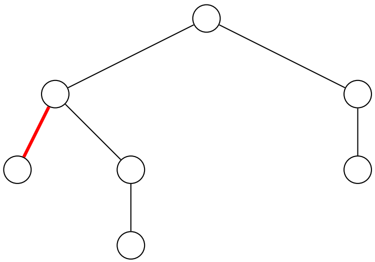

Cannot create a red degree-3 vertex

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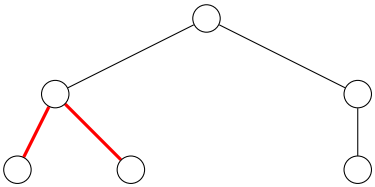

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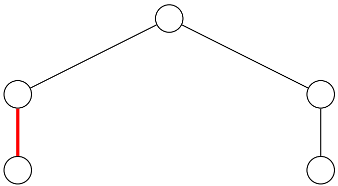

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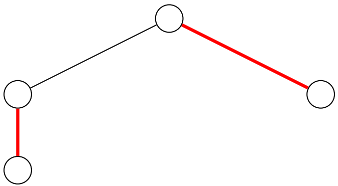

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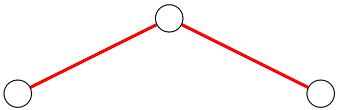

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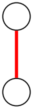

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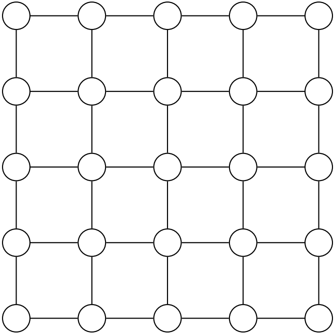
Graphs with bounded twin-width – trees

Generalization to bounded *treewidth* and even bounded *rank-width*

Graphs with bounded twin-width – grids



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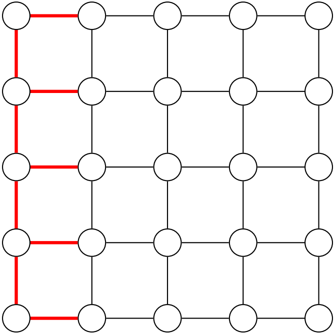

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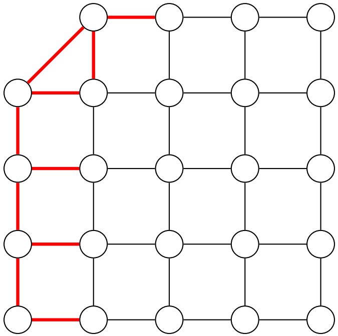

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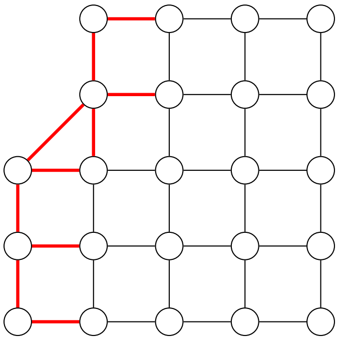

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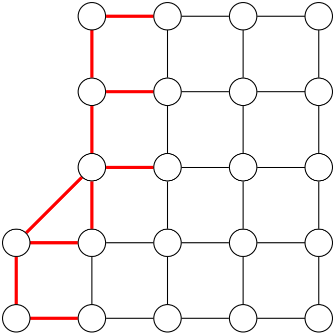

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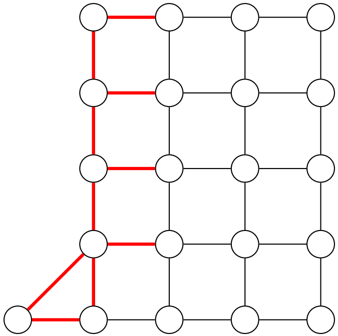

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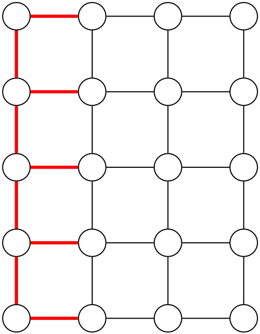

4-sequence for planar grids, $3d$-sequence for d-dimensional grids

Universal bipartite graph

No $O(1)$-contraction sequence:
twin-width is *not* an iterated identification of near twins.

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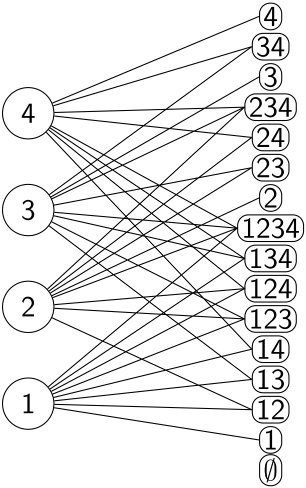

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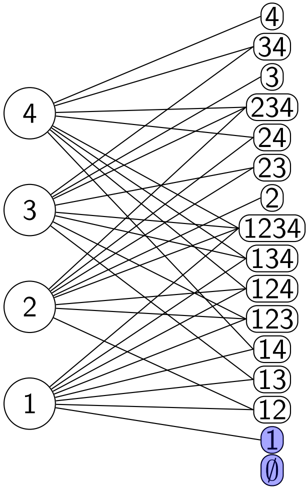

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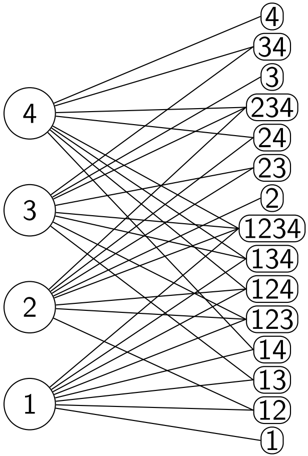

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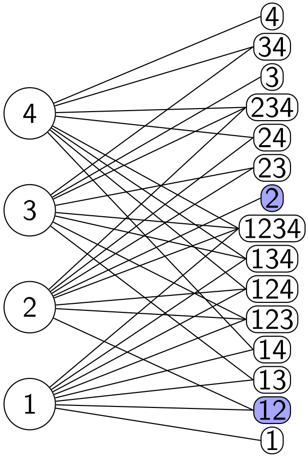

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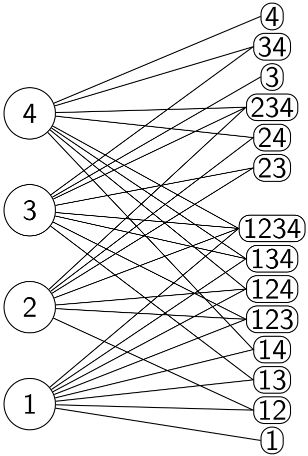

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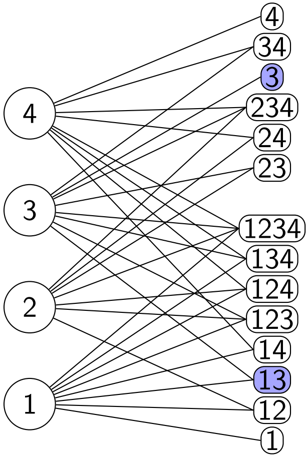

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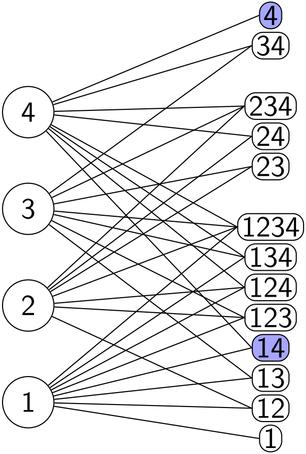

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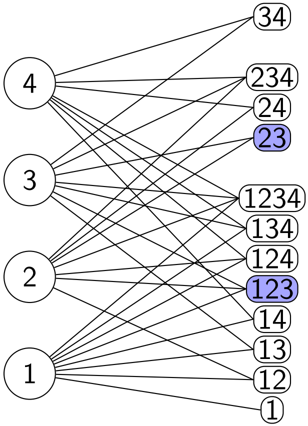

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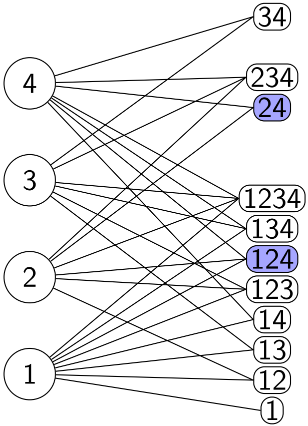

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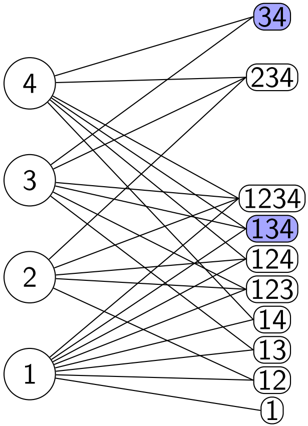

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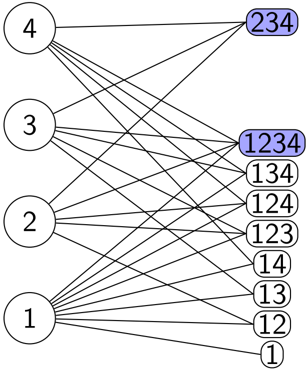

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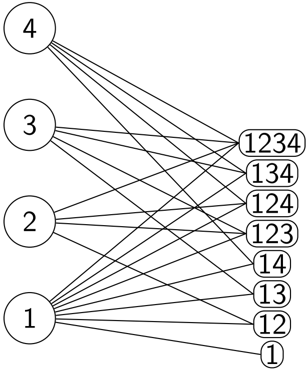

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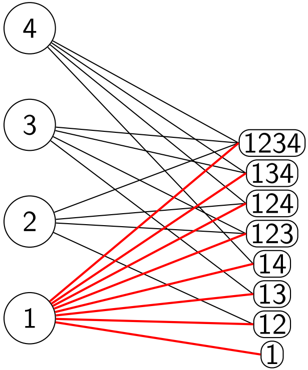

Graphs with bounded twin-width – planar graphs?

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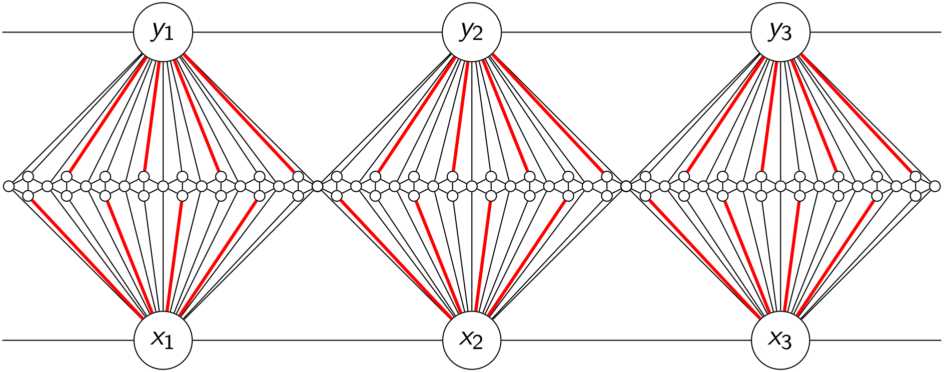
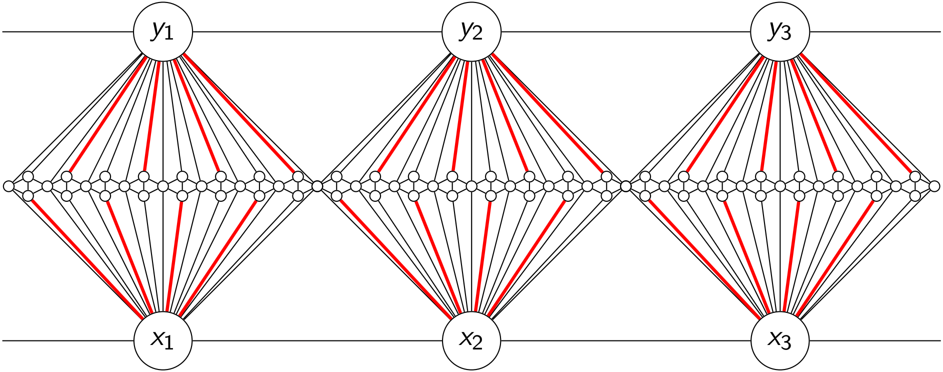


For every d, a planar trigraph without planar d-contraction

Graphs with bounded twin-width – planar graphs?



For every d, a planar trigraph without planar d-contraction

More powerfool tool needed

Twin-width in the language of matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Encode a bipartite graph (or, if symmetric, any graph)

Twin-width in the language of matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Contraction of two columns (similar with two rows)

Twin-width in the language of matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & r & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & r & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & r & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

How is the twin-width (re)defined?

Twin-width in the language of matrices

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & r & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & r & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & r & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
$$

How to tune it for non-bipartite graph?

Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

$$\left[\begin{array}{c|c|c|c|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

$$\left[\begin{array}{c|c|cc|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

Maximum number of non-constant zones per column or row part
$=$ **error value**

Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

$$\left[\begin{array}{c|c|cc|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

Maximum number of non-constant zones per column or row part
. . . until there are a single row part and column part

Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

$$\left[\begin{array}{c|c|cc|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

Twin-width as maximum error value of a contraction/division sequence

Grid minor

t-grid minor: $t \times t$-division where every cell is non-empty

Non-empty cell: contains at least one 1 entry

$$\left[\begin{array}{cc|cc|cc|cc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

4-grid minor

Grid minor

t-grid minor: $t \times t$-division where every cell is non-empty
Non-empty cell: contains at least one 1 entry

$$\left[\begin{array}{cc|cc|cc|cc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

4-grid minor

A matrix is said t-**grid free** if it does not have a t-grid minor

Mixed minor

Mixed cell: not horizontal nor vertical

$$\left[\begin{array}{cc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

3-mixed minor

Mixed minor

Mixed cell: not horizontal nor vertical

$$\left[\begin{array}{cc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

3-mixed minor

Every mixed cell is witnessed by a 2×2 square = **corner**

Mixed minor

Mixed cell: not horizontal nor vertical

$$\left[\begin{array}{cc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

3-mixed minor

A matrix is said t-**mixed free** if it does not have a t-mixed minor

Mixed value

$$
\begin{array}{cc}
\begin{array}{c} R_4 \\ \hline R_3 \\ \\ \hline R_2 \\ \\ \hline R_1 \\ \\ \end{array}
&
\left[\begin{array}{cc|ccc|c|cc}
1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
0 & 1 & \boxed{1} & \boxed{1} & 0 & 1 & 0 & 0 \\
1 & 0 & \boxed{1} & \boxed{0} & 1 & 0 & 0 & 1
\end{array}\right] \\
 & \qquad\qquad C_2
\end{array}
$$

≈ (maximum) number of cells with a corner per row/column part

Mixed value

$$\begin{array}{cc}
\begin{array}{c} R_4 \\ \hline R_3 \\ \\ \hline R_2 \\ \\ \hline R_1 \\ \\ \end{array} &
\left[\begin{array}{cc|ccc|c|cc}
1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1
\end{array}\right] \\
 & \qquad\qquad C_2
\end{array}$$

But we add the number of *boundaries* containing a corner

Mixed value

$$
\begin{array}{cc}
\begin{array}{c} R_4 \\ \\ R_3 \\ \cup \\ R_2 \\ \\ R_1 \\ \\ \end{array} &
\left[\begin{array}{cc|ccc|c|cc}
1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1
\end{array}\right] \\
& C_2
\end{array}
$$

$\therefore$ merging row parts do not increase mixed value of column part

Twin-width and mixed freeness

Theorem
*If G admits **a** t-mixed free adjacency matrix, then* $tww(G) = 2^{2^{O(t)}}$.

Twin-width and mixed freeness

Theorem

If $\exists \sigma$ s.t. $Adj_\sigma(G)$ is t-mixed free, then $tww(G) = 2^{2^{O(t)}}$.

Twin-width and mixed freeness

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Step 1: find a division sequence $(\mathcal{D}_i)_i$ with mixed value $f(t)$

$$\left[\begin{array}{c|c|c|c|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

Merge consecutive parts greedily

Twin-width and mixed freeness

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1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
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$$\left[\begin{array}{c|c|cc|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

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$$\left[\begin{array}{c|c|cc|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

Stuck, removing every other separation $\rightarrow \frac{f(t)}{2}$ mixed cells per part

Stanley-Wilf conjecture / Marcus-Tardos theorem

Auxiliary 0,1-matrix with one entry per cell: a 1 iff the cell is mixed

Question

For every k, is there a c_k such that every $n \times m$ $0,1$-matrix with at least c_k 1 per row and column admits a k-grid minor?

Stanley-Wilf conjecture / Marcus-Tardos theorem

Auxiliary 0,1-matrix with one entry per cell: a 1 iff the cell is mixed

Conjecture (reformulation of Füredi-Hajnal conjecture '92)

For every k, there is a c_k such that every $n \times m$ $0,1$-matrix with at least $c_k \max(n, m)$ 1 entries admits a k-grid minor.

Stanley-Wilf conjecture / Marcus-Tardos theorem

Auxiliary 0,1-matrix with one entry per cell: a 1 iff the cell is mixed

Conjecture (reformulation of Füredi-Hajnal conjecture '92)

For every k, there is a c_k such that every $n \times m$ $0,1$-matrix with at least $c_k \max(n,m)$ 1 entries admits a k-grid minor.

Conjecture (Stanley-Wilf conjecture '80s)

Any proper permutation class contains only $2^{O(n)}$ n-permutations.

Klazar showed Füredi-Hajnal $\Rightarrow$ Stanley-Wilf in 2000

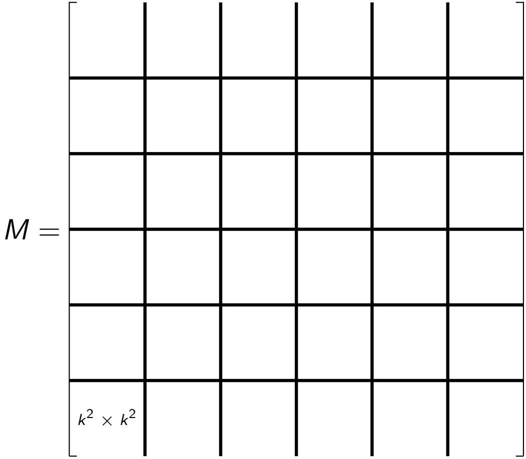
Marcus and Tardos showed Füredi-Hajnal in 2004

Marcus-Tardos one-page inductive proof

$$M = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

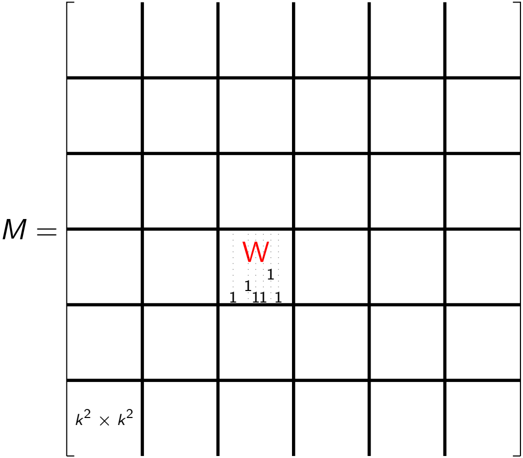
Let M be an $n \times n$ 0, 1-matrix without k-grid minor

Marcus-Tardos one-page inductive proof



Draw a regular $\frac{n}{k^2} \times \frac{n}{k^2}$ division on top of M

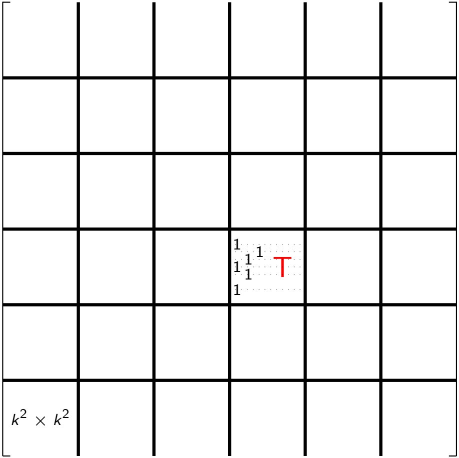
Marcus-Tardos one-page inductive proof



A cell is *wide* if it has at least k columns with a 1

Marcus-Tardos one-page inductive proof

$M =$



A cell is *tall* if it has at least k rows with a 1

Marcus-Tardos one-page inductive proof

$$M = \begin{bmatrix} & & W & & & \\ & & W & & & \\ & & & & & \\ & & W & & & \\ & & & & & \\ k^2 \times k^2 & & & & & \end{bmatrix}$$

There are less than $k\binom{k^2}{k}$ wide cells per column part. Why?

Marcus-Tardos one-page inductive proof

$$M = \left[\begin{array}{c|c|c|c|c|c} & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & \color{red}{\mathsf{T}} & & \color{red}{\mathsf{T}} & & \color{red}{\mathsf{T}} \\ \hline & & & & & \\ \hline k^2 \times k^2 & & & & & \end{array}\right]$$

There are less than $k\binom{k^2}{k}$ tall cells per row part

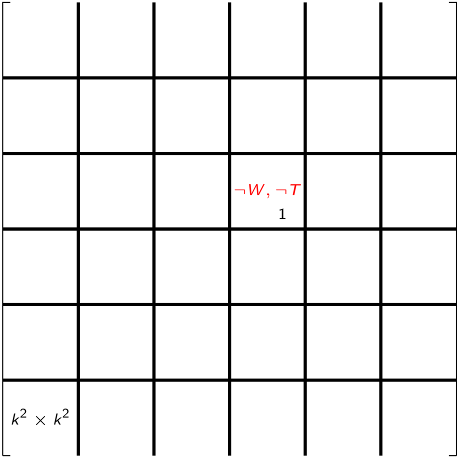
Marcus-Tardos one-page inductive proof

$$M = \left[\begin{array}{c|c|c|c|c|c}
 & & W & & & \\ \hline
 & W & W & & & T \\ \hline
 & & & & & \\ \hline
 & T & W & T & & T \\ \hline
 & & T & & & \\ \hline
k^2 \times k^2 & & & & & W
\end{array}\right]$$

In W and T, at most $2 \cdot \frac{n}{k^2} \cdot k\binom{k^2}{k} \cdot k^4 = 2k^3\binom{k^2}{k} n$ entries 1

Marcus-Tardos one-page inductive proof

$M =$



There are at most $(k-1)^2 c_k \frac{n}{k^2}$ remaining 1. Why?

Marcus-Tardos one-page inductive proof

$$M = \left[\begin{array}{c|c|c|c|c|c}
 & & W & & & \\ \hline
 & W & W & & & T \\ \hline
 & & & \begin{matrix}\neg W, \neg T \\ 1\end{matrix} & & \\ \hline
 & T & W & T & & T \\ \hline
 & & T & & & \\ \hline
k^2 \times k^2 & & & & & W
\end{array}\right]$$

Choose $c_k = 2k^4\binom{k^2}{k}$ so that $(k-1)^2 c_k \frac{n}{k^2} + 2k^3\binom{k^2}{k} n \leqslant c_k n$

Twin-width and mixed freeness

Theorem

If $\exists\sigma$ *s.t.* $Adj_\sigma(G)$ *is* t*-mixed free, then* $tww(G) = 2^{2^{O(t)}}$.

Step 1: find a division sequence $(\mathcal{D}_i)_i$ **with mixed value** $f(t)$

$$\left[\begin{array}{c|c|cc|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

Stuck, removing every other separation $\rightarrow \frac{f(t)}{2}$ mixed cells per part

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Step 1: find a division sequence $(\mathcal{D}_i)_i$ **with mixed value** $f(t)$

$$\left[\begin{array}{c|c|cc|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

Stuck, removing every other separation $\rightarrow \frac{f(t)}{2}$ mixed cells per part

Impossible!

Twin-width and mixed freeness

Theorem

If $\exists\sigma$ s.t. $Adj_\sigma(G)$ is t-mixed free, then $tww(G) = 2^{2^{O(t)}}$.

Step 1: find a division sequence $(\mathcal{D}_i)_i$ with mixed value $f(t)$
Step 2: find a contraction sequence with error value $g(t)$

$$\left[\begin{array}{c|c|cc|c|c|c|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
\hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

Refinement of $\mathcal{D}_i$ where each part coincides on the non-mixed cells

Twin-width and mixed freeness

Theorem

If $\exists\sigma$ s.t. $Adj_\sigma(G)$ is t-mixed free, then $tww(G) = 2^{2^{O(t)}}$.

Twin-width and mixed freeness

Theorem

If $\exists \sigma$ s.t. $Adj_\sigma(G)$ is t-mixed free, then $tww(G) = 2^{2^{O(t)}}$.

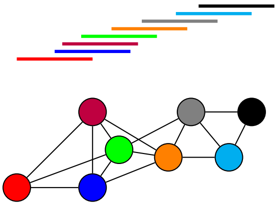
Now to bound the twin-width of a class $\mathcal{C}$:

1) Find a *good* vertex-ordering procedure

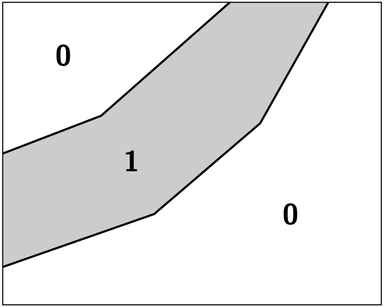
2) Argue that, in this order, a t-mixed minor would conflict with $\mathcal{C}$

Unit interval graphs

Intersection graph of unit segments on the real line

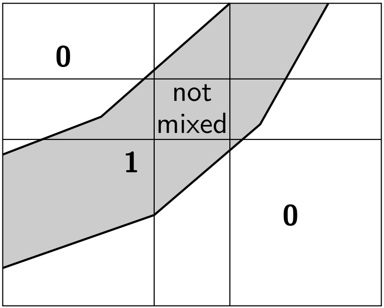


Bounded twin-width – unit interval graphs



order by left endpoints

Bounded twin-width – unit interval graphs



No 3-by-3 grid has all 9 cells crossed by two non-decreasing curves

Graph minors

Formed by **vertex deletion**, **edge deletion**, and **edge contraction**

A graph G is *H-minor free* if H is not a minor of G

A graph class is *H-minor free* if all its graphs are

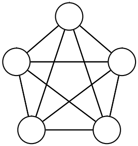
Graph minors

Formed by **vertex deletion**, **edge deletion**, and **edge contraction**

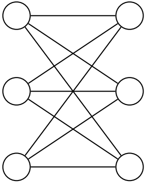
A graph G is *H-minor free* if H is not a minor of G

A graph class is *H-minor free* if all its graphs are

Planar graphs are exactly the graphs without K_5 or $K_{3,3}$ as a minor



K_5

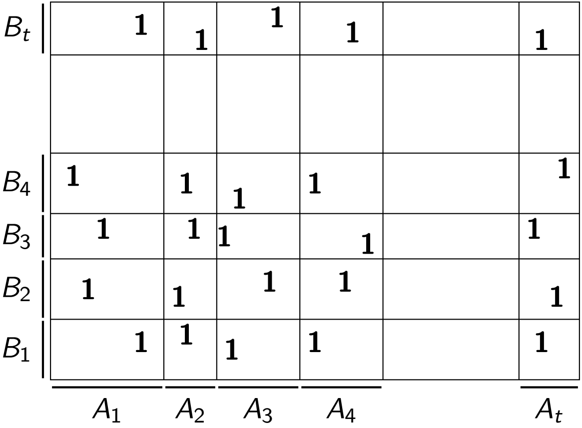


$K_{3,3}$

Bounded twin-width – K_t-minor free graphs

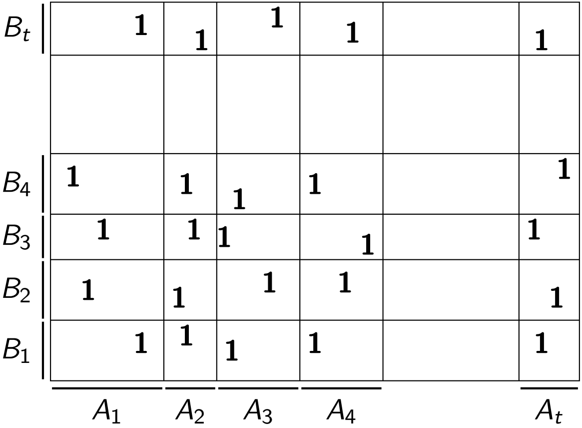
Given a hamiltonian path, we would just use this order

Bounded twin-width – K_t-minor free graphs



Contracting the $2t$ subpaths yields a $K_{t,t}$-minor, hence a K_t-minor

Bounded twin-width – K_t-minor free graphs



Instead we use a specially crafted lex-DFS discovery order

Theorem

The following classes have bounded twin-width, and $O(1)$-sequences can be computed in polynomial time.

- *Bounded rank-width, and even, boolean-width graphs,*
- *every hereditary proper subclass of permutation graphs,*
- *posets of bounded antichain size (seen as digraphs),*
- *unit interval graphs,*
- *K_t-minor free graphs,*
- *map graphs,*
- *subgraphs of d-dimensional grids,*
- *K_t-free unit d-dimensional ball graphs,*
- *$\Omega(\log n)$-subdivisions of all the n-vertex graphs,*
- *cubic expanders defined by iterative random 2-lifts from K_4,*
- *strong products of two bounded twin-width classes, one with bounded degree, etc.*

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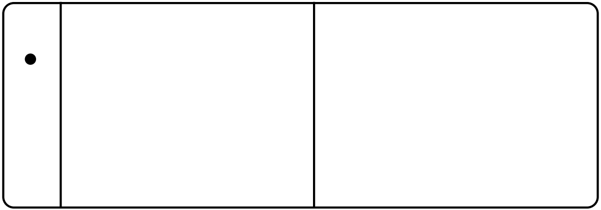
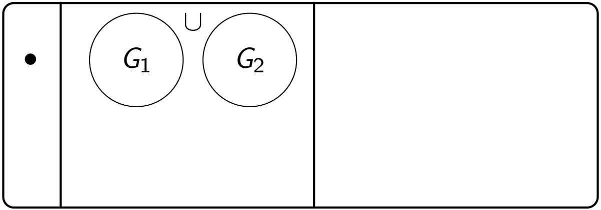
Can we solve problems faster, given an $O(1)$-sequence?

Cographs



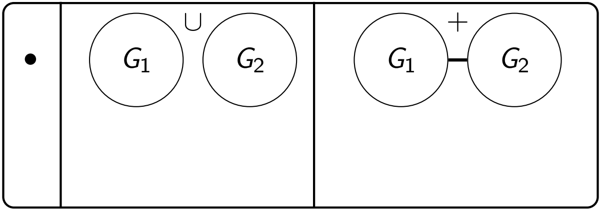
A single vertex is a cograph,

Cographs



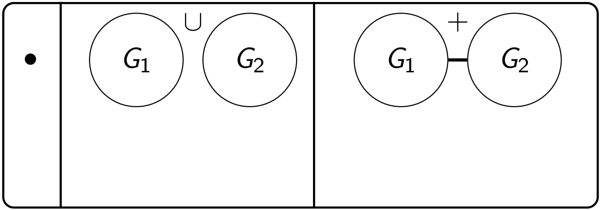
as well as the union of two cographs,

Cographs

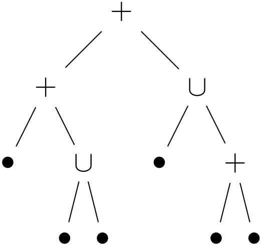


and the complete join of two cographs.

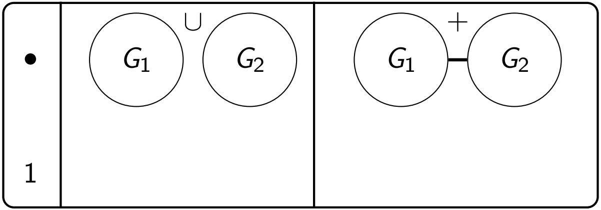
Cographs



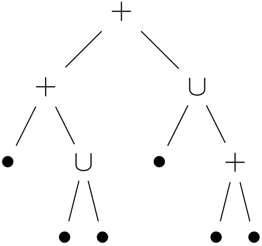
Many NP-hard problems are polytime solvable on cographs



Cographs



Let's try to compute the NP-hard $\alpha(G)$, independence number



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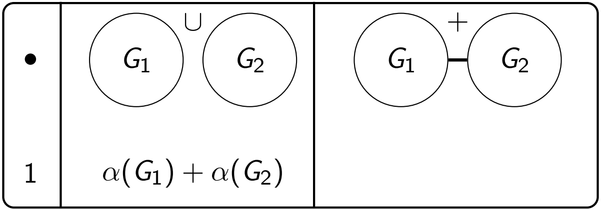
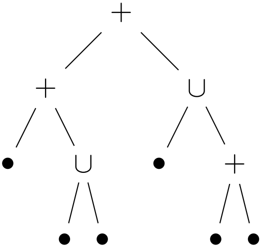


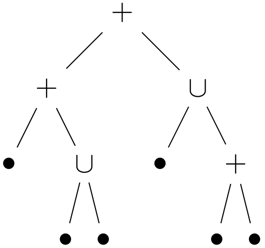
In case of a disjoint union: combine the solutions



Cographs

$\bullet$	$G_1 \cup G_2$	$G_1 + G_2$
1	$\alpha(G_1) + \alpha(G_2)$	$\max\{\alpha(G_1), \alpha(G_2)\}$

In case of a complete join: pick the larger one



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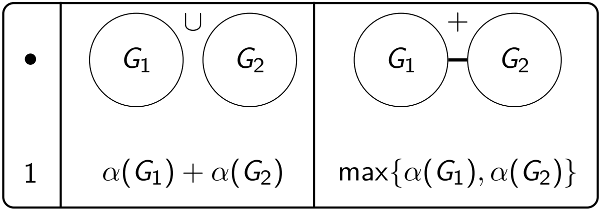
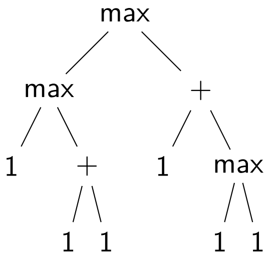


$\bullet$
$\cup$
G_1
G_2
$+$
G_1
G_2
1
$\alpha(G_1) + \alpha(G_2)$
$\max\{\alpha(G_1), \alpha(G_2)\}$



max
max
+
1
+
1
max
1 1
1 1

Equivalent cograph definition

Cographs form the unique *maximal hereditary* class in which every[1] graph has two *twins*

[1] provided it has at least two vertices

Equivalent cograph definition

Cographs form the unique *maximal hereditary* class in which every[1] graph has two *twins* ...wait a minute

[1] provided it has at least two vertices

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Cographs form the unique *maximal hereditary* class in which every[1] graph has two *twins* …yes, they coincide with **twin-width 0**

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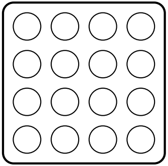
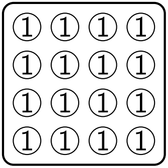


Is there another algorithmic scheme based on this definition?

[1] provided it has at least two vertices

Equivalent cograph definition

Cographs form the unique *maximal hereditary* class in which every[1] graph has two *twins* ...yes, they coincide with **twin-width 0**



Let's try with $\alpha(G)$, and store in a vertex its inner max solution

[1] provided it has at least two vertices

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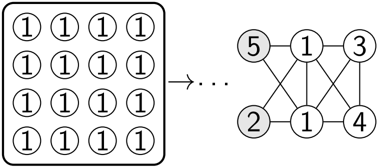

We can find a pair of false/true twins

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Equivalent cograph definition

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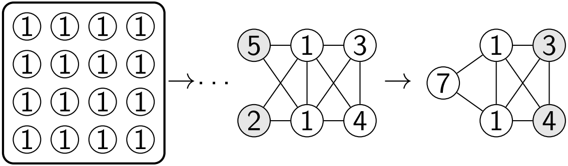

Sum them if they are false twins

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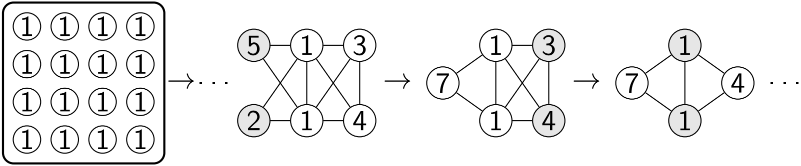

Max them if they are true twins

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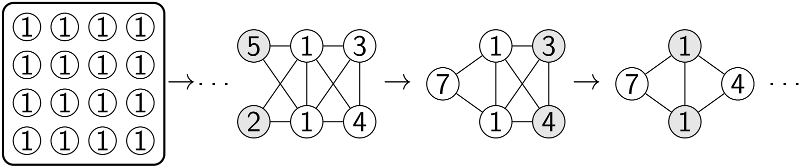

Why does it eventually compute $\alpha(G)$?

[1] provided it has at least two vertices

Example of k-Independent Set

d-sequence: $G = G_n, G_{n-1}, \ldots, G_2, G_1 = K_1$

Algorithm: **Compute by dynamic programming a best partial solution in each red connected subgraph of size at most k.**

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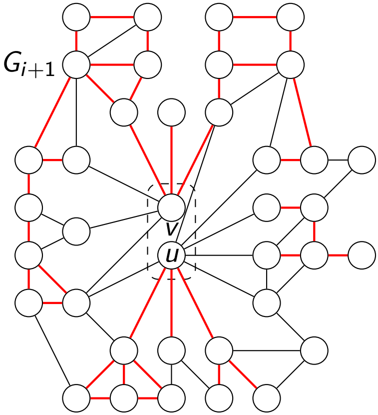
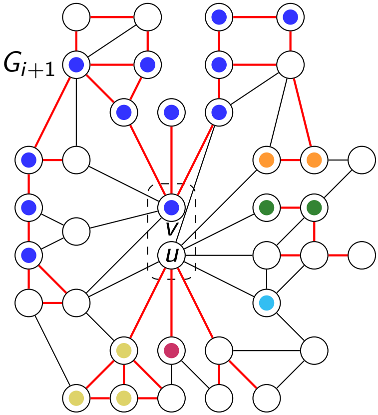
How to go from the partial solutions of G_{i+1} to those of G_i?



Best partial solution inhabiting •?



3 unions of $\leqslant d+2$ red connected subgraphs to consider in G_{i+1}
with u, or v, or both

Other (almost) single-exponential parameterized algorithms

Theorem

Given a d-sequence $G = G_n, \ldots, G_1 = K_1$,

- k-Independent Set,
- k-Clique,
- (r, k)-Scattered Set,
- k-Dominating Set, *and*
- (r, k)-Dominating Set

can be solved in time $2^{O(k)} n$,
whereas Subgraph Isomorphism *and* Induced Subgraph Isomorphism *can be solved in time* $2^{O(k \log k)} n$.

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A more general FPT algorithm?

First-order model checking on graphs

GRAPH FO MODEL CHECKING	**Parameter:** $\lvert\varphi\rvert$

Input: A graph G and a first-order sentence $\varphi \in FO(\{E_2, =_2\})$
Question: $G \models \varphi$?

First-order model checking on graphs

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Example:

$$\varphi = \exists x_1 \exists x_2 \cdots \exists x_k \forall x \bigvee_{1 \leqslant i \leqslant k} x = x_i \vee \bigvee_{1 \leqslant i \leqslant k} E(x, x_i) \vee E(x_i, x)$$

$G \models \varphi$? $\Leftrightarrow$

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$G \models \varphi$? $\Leftrightarrow$ k-DOMINATING SET

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$G \models \varphi$? $\Leftrightarrow$ k-INDEPENDENT SET

FO interpretations and transductions

FO interpretation: redefine the edges by a first-order formula

$\varphi(x, y) = \neg E(x, y)$ (complement)

$\varphi(x, y) = E(x, y) \vee \exists z E(x, z) \wedge E(z, y)$ (square)

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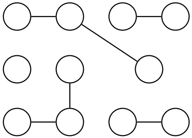
FO transduction: color by $O(1)$ unary relations, interpret, delete



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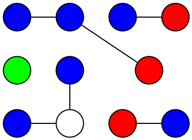

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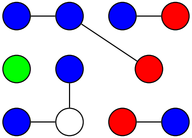

$$\varphi(x, y) = E(x, y) \vee (G(x) \wedge B(y) \wedge \neg\exists z R(z) \wedge E(y, z))$$
$$\vee (R(x) \wedge B(y) \wedge \exists z R(z) \wedge E(y, z) \wedge \neg\exists z B(z) \wedge E(y, z))$$

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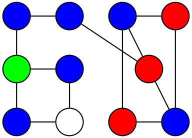

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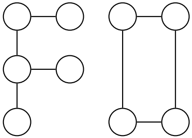

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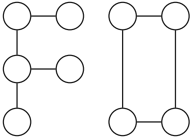
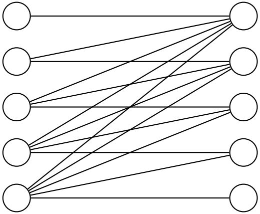


Theorem
Bounded twin-width is preserved by transduction.

Monadically Stable and NIP

Stable class: no transduction of the class contains all ladders
NIP class: no transduction of the class contains all graphs

ladder



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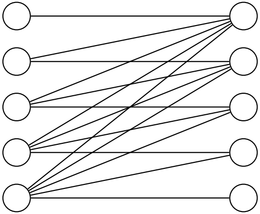

Bounded-degree graphs $\rightarrow$ stable
Unit interval graphs $\rightarrow$ NIP but not stable
Interval graphs $\rightarrow$ not NIP

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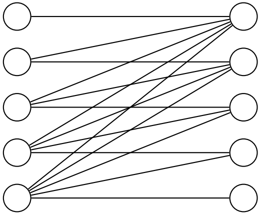
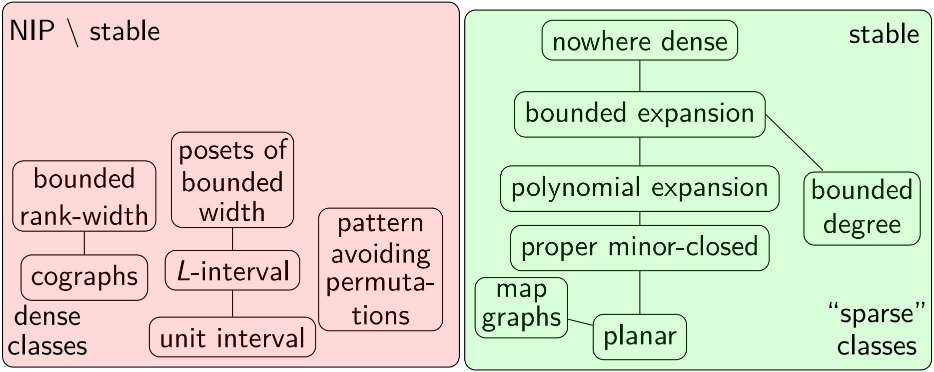


Bounded-degree graphs $\rightarrow$ stable
Unit interval graphs $\rightarrow$ NIP but not stable
Interval graphs $\rightarrow$ not NIP

Bounded twin-width classes $\rightarrow$ NIP but not stable in general

Classes with known tractable FO model checking



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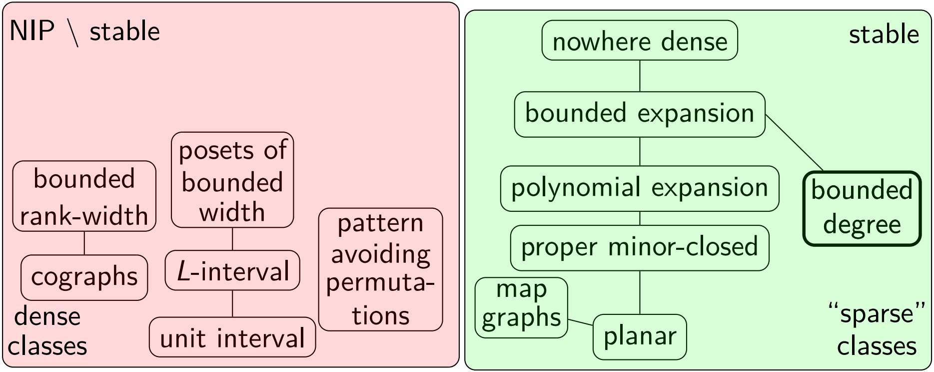
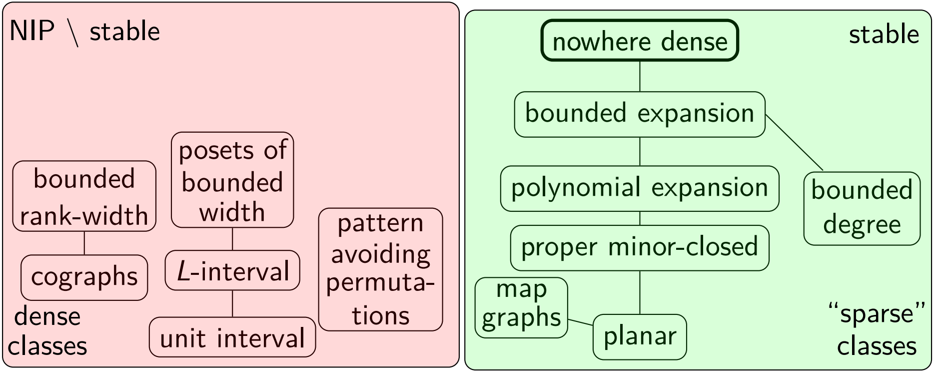


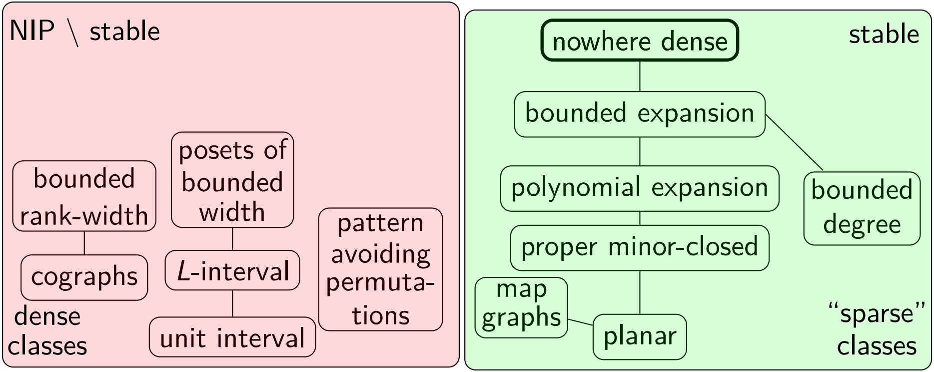
FO MODEL CHECKING solvable in $f(|\varphi|)n$ on bounded-degree graphs [Seese '96]

Classes with known tractable FO model checking



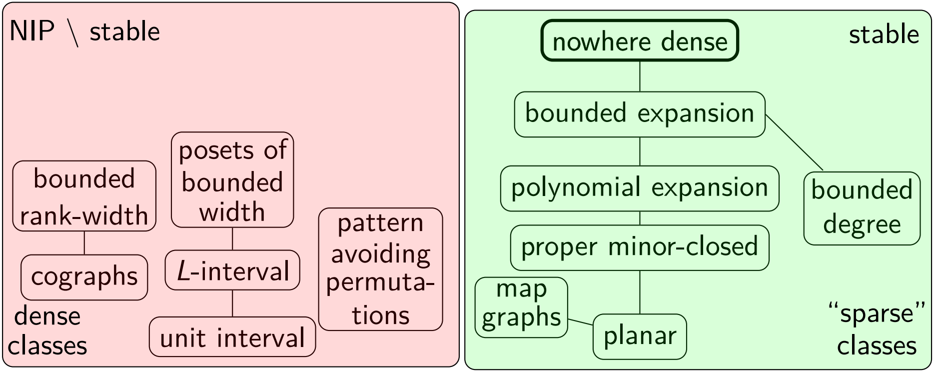
FO MODEL CHECKING solvable in $f(|\varphi|)n^{1+\varepsilon}$ on any nowhere dense class [Grohe, Kreutzer, Siebertz '14]

Classes with known tractable FO model checking



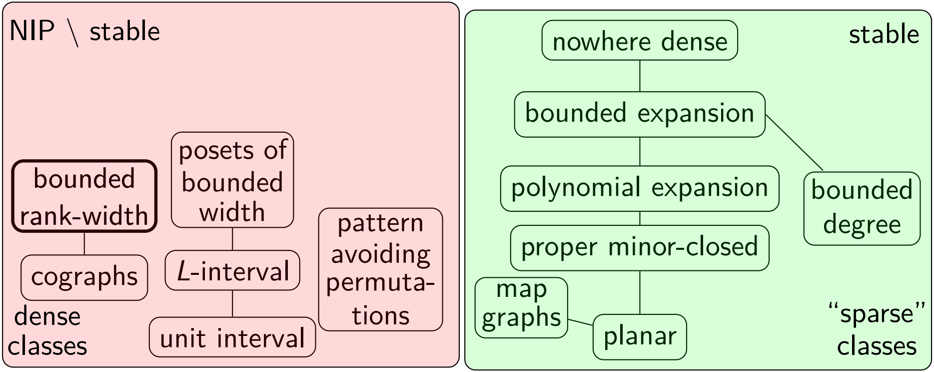
End of the story for the subgraph-closed classes
tractable FO MODEL CHECKING ⇔ nowhere dense ⇔ stable

Classes with known tractable FO model checking



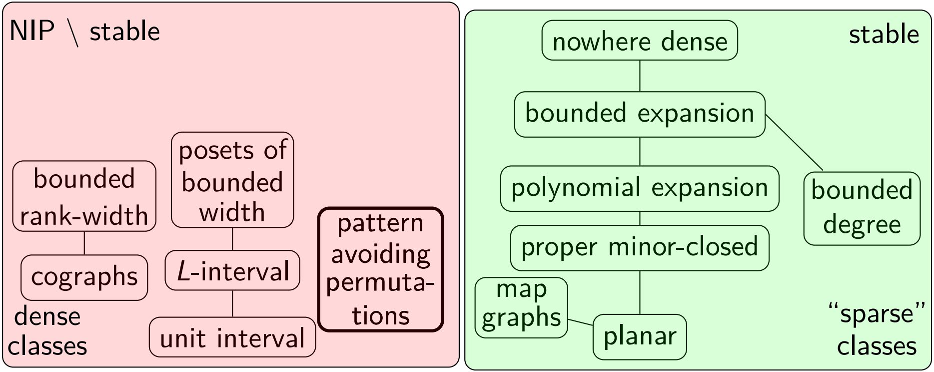
New program: transductions of nowhere dense classes
Not sparse anymore but still stable

Classes with known tractable FO model checking



MSO_1 Model Checking solvable in $f(|\varphi|, w)n$ on graphs of rank-width w
[Courcelle, Makowsky, Rotics '00]

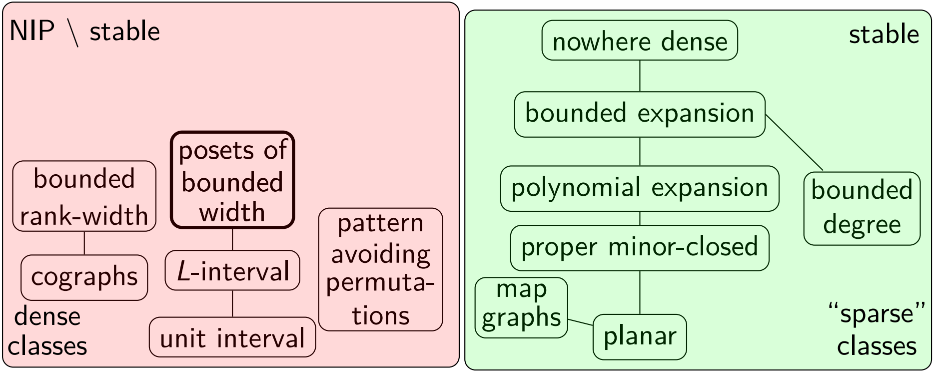
Classes with known tractable FO model checking



Is σ a subpermutation of τ? solvable in $f(|\sigma|)|\tau|$

[Guillemot, Marx '14]

Classes with known tractable FO model checking



FO Model Checking solvable in $f(|\varphi|, w)n^2$ on posets of width w [GHLOORS '15]

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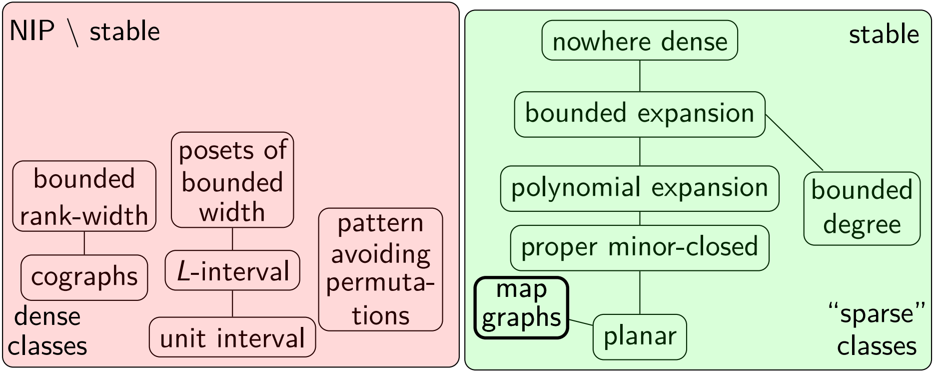

FO MODEL CHECKING solvable in $f(|\varphi|)n^{O(1)}$ on map graphs
[Eickmeyer, Kawarabayashi '17]

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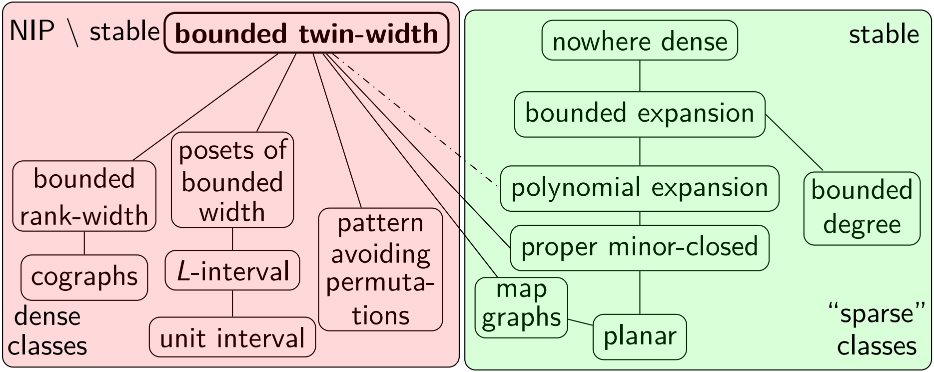
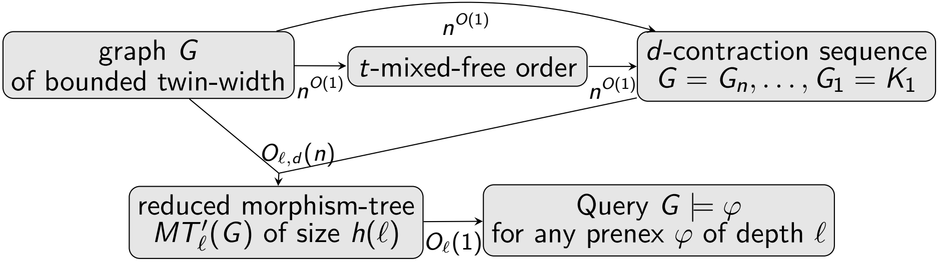


FO Model Checking solvable in $f(|\varphi|, d)n$ on graphs with a d-sequence

Workflow of the FO model checking algorithm



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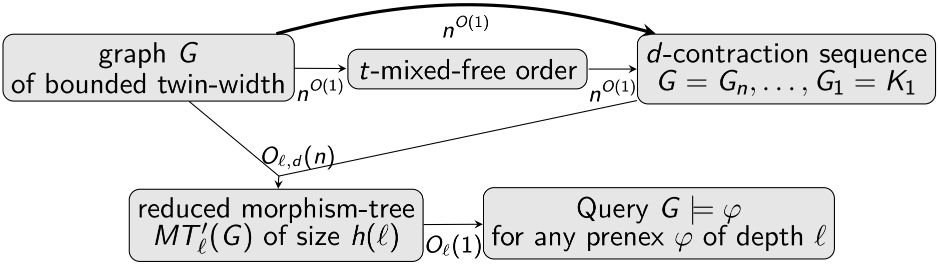
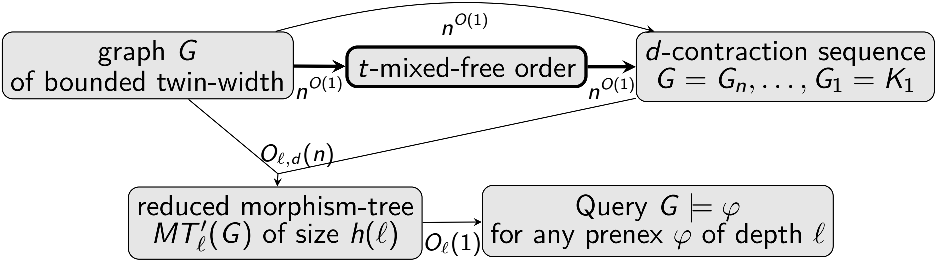


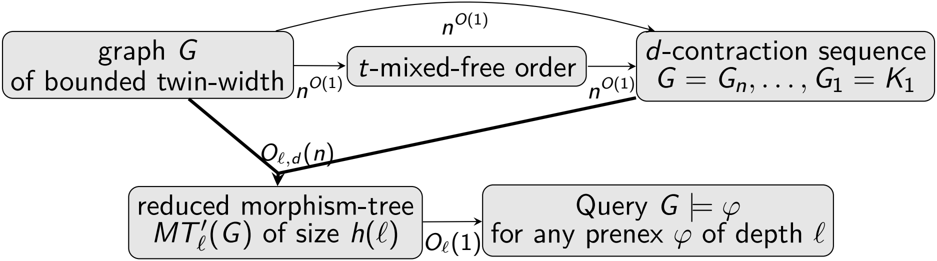
Direct examples: **trees**, bounded rank-width, **grids**, d-dimensional grids, K_t-free unit ball graphs

Workflow of the FO model checking algorithm



Detour via mixed minor for: pattern-avoiding permutations,
unit intervals, bounded width posets, K_t-**minor free graphs**

Workflow of the FO model checking algorithm



Generalization of what we saw for k-Independent Set

Small classes

Small: class with at most $n!c^n$ labeled graphs on $[n]$.

Theorem

Bounded twin-width classes are small.

Unifies and extends the same result for:

σ-free permutations [Marcus, Tardos '04]

K_t-minor free graphs [Norine, Seymour, Thomas, Wollan '06]

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Theorem
Bounded twin-width classes are small.

Subcubic graphs, interval graphs, triangle-free unit segment graphs have **unbounded** twin-width

Small classes

Small: class with at most $n!c^n$ labeled graphs on $[n]$.

Theorem

Bounded twin-width classes are small.

Is the converse true for hereditary classes?

Conjecture (small conjecture)

A hereditary class has bounded twin-width if and only if it is small.

χ-boundedness

$\mathcal{C}$ χ-bounded: $\exists f, \forall G \in \mathcal{C}$, $\chi(G) \leqslant f(\omega(G))$

Theorem

Every twin-width class is χ-bounded.
More precisely, every graph G of twin-width at most d admits a proper $(d+2)^{\omega(G)-1}$-coloring.

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Polynomially χ-bounded? i.e., $\chi(G) = O(\omega(G)^d)$

$d+2$-coloring in the triangle-free case

Algorithm: **Start from $G_1 = K_1$, color its unique vertex 1, and rewind the d-sequence. A contraction seen backward is a split and we shall find colors for the two new vertices.**

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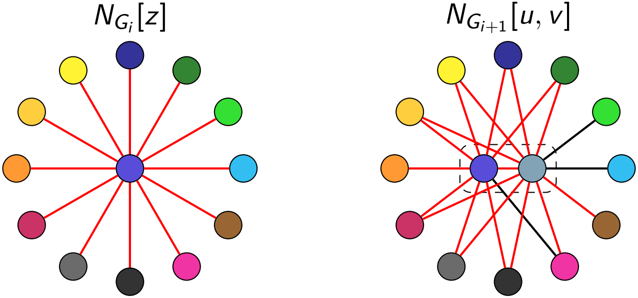

z has only red incident edges $\rightarrow$ $d + 2$-nd color available to v

$d + 2$-coloring in the triangle-free case

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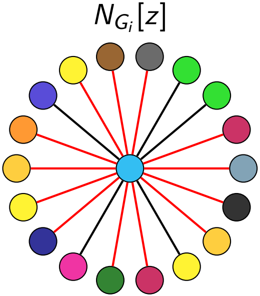

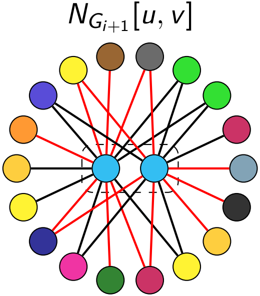

z incident to at least one black edge $\rightarrow$ non-edge between u and v

Future directions

Main questions:
Algorithm to compute/approximate twin-width in general
Fully classify classes with tractable FO model checking
Small conjecture
Better approximation algorithms on bounded twin-width classes
Twin-width of Cayley graphs of finitely generated groups...

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On arxiv

Twin-width I: tractable FO model checking	[BKTW '20]
Twin-width II: small classes	[BGKTW '20]
Twin-width III: Max Independent Set and Coloring	[BGKTW '20]