The geometry of twin-width

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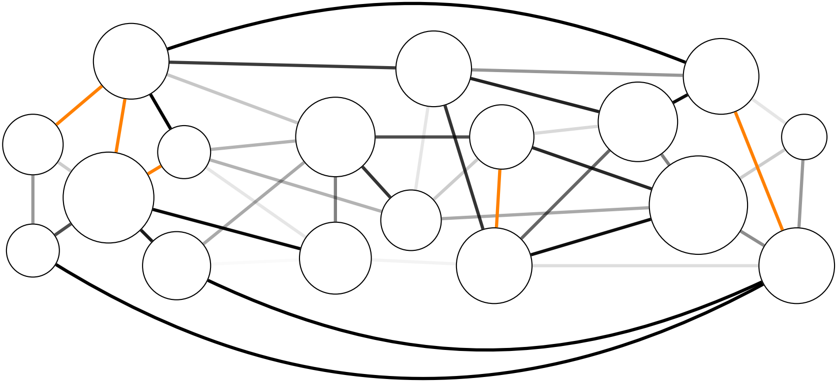
ENS Lyon, LIP

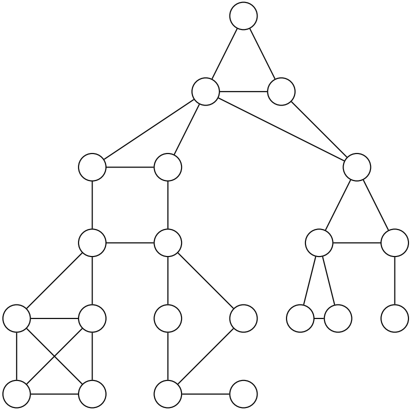
June 30th, 2022, Datashape seminar

Sketching a graph - Szemerédi's Regularity Lemma



Every graph can be vertex-partitioned into a constant number of (balanced) parts such that there is a random-like edge set between every but an arbitrarily small fraction of pairs of parts.

Sketching a graph - Tree decompositions



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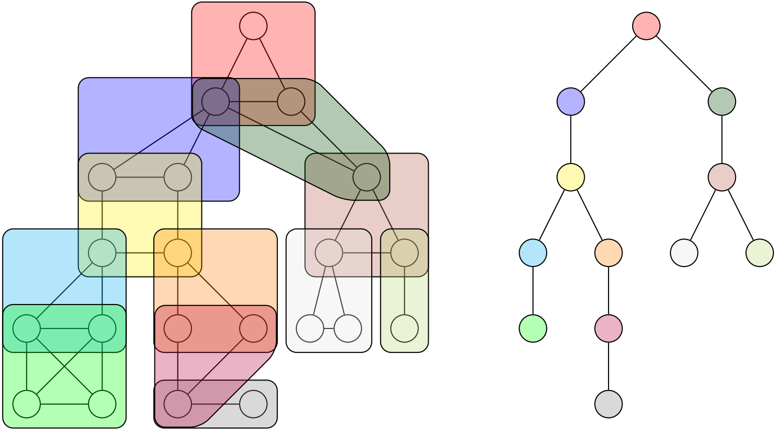
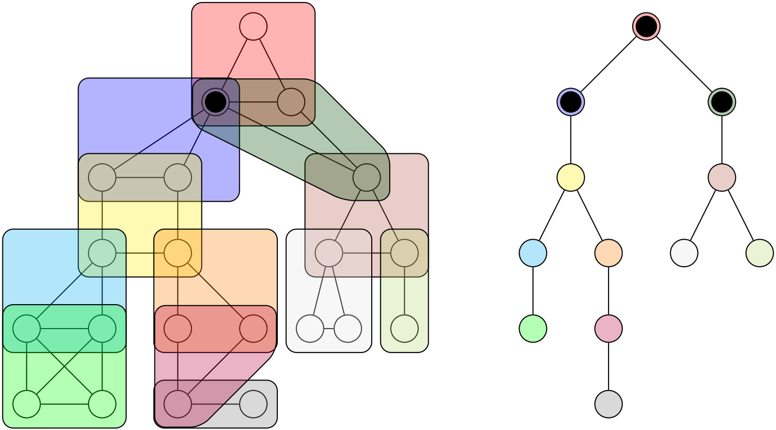


Edge cover by vertex subsets (called bags) mapping to a tree such that the bags containing any fixed vertex map to a subtree

Sketching a graph - Tree decompositions



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Limits

Two immense successes in combinatorics, algorithms, etc. but:

- the former is only meaningful in dense graphs ($m = \Omega(n^2)$)
- the latter is most helpful in sparse graphs ($m = O(n)$)

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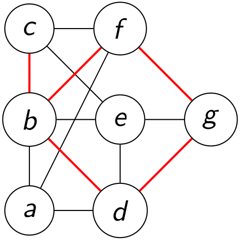
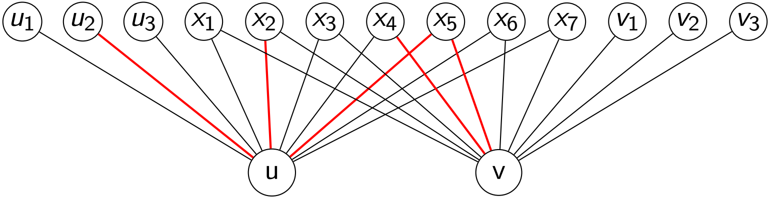
Other useful ways to approximate a graph?

Trigraphs



Three outcomes between a pair of vertices:
edge, or non-edge, or red edge (error edge)

Contractions in trigraphs



Identification of two non-necessarily adjacent vertices

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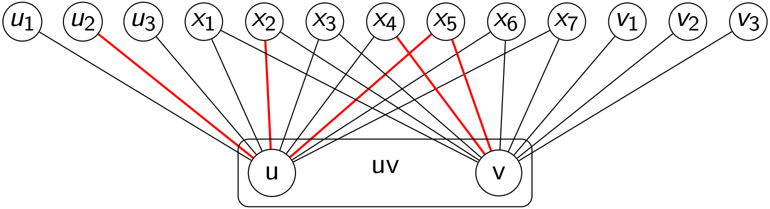

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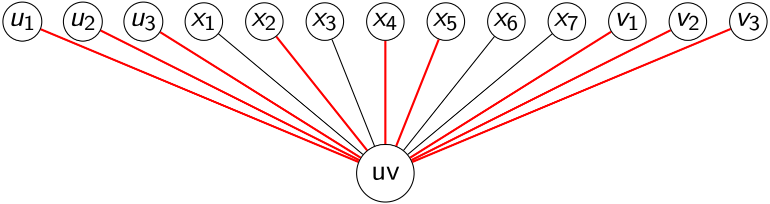
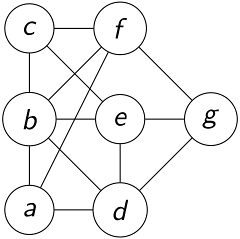


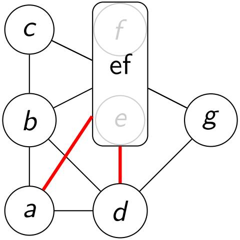
edges to $N(u) \triangle N(v)$ turn red, for $N(u) \cap N(v)$ red is absorbing

Contraction sequence



A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that G_i is obtained by performing one contraction in G_{i+1}.

Contraction sequence



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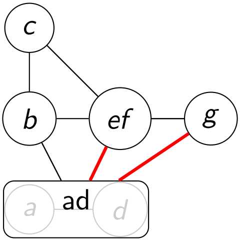

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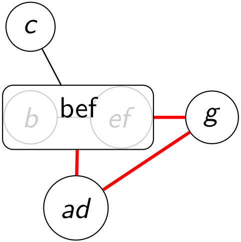

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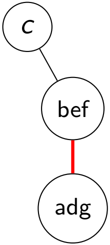

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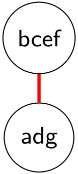

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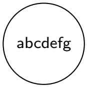

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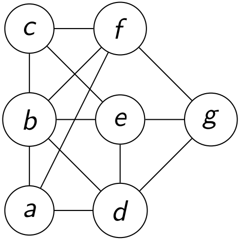
Twin-width

tww(G): Least integer d such that G admits a contraction sequence where all trigraphs have *maximum red degree* at most d.



Maximum red degree $= 0$

overall maximum red degree = 0

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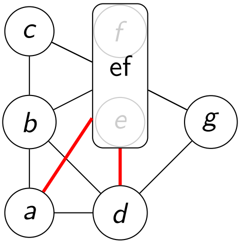

Maximum red degree $= 2$

overall maximum red degree = 2

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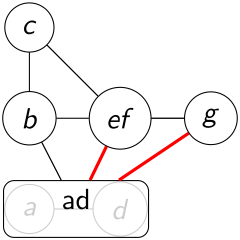

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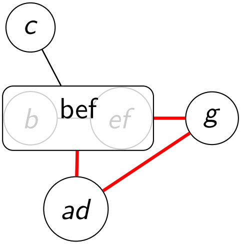

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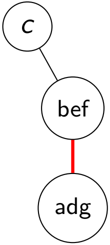

Maximum red degree $= 1$

overall maximum red degree = 2

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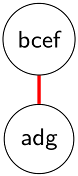

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tww(G): Least integer d such that G admits a contraction sequence where all trigraphs have *maximum red degree* at most d.

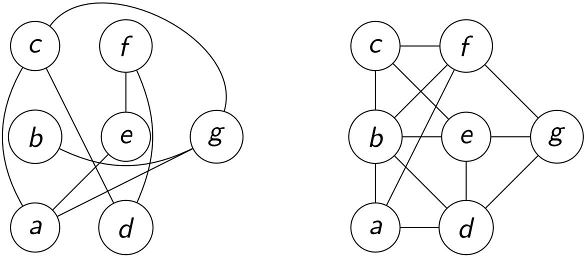
Maximum red degree $= 0$

overall maximum red degree = 2

Simple operations preserving small twin-width

- complementation: remains the same
- taking induced subgraphs: may only decrease
- adding one vertex linked arbitrarily: at most “doubles”
- substitution and lexicographic product

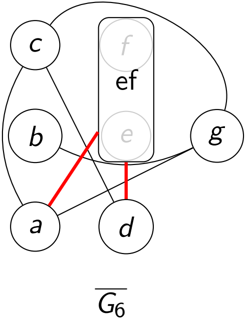
Complementation

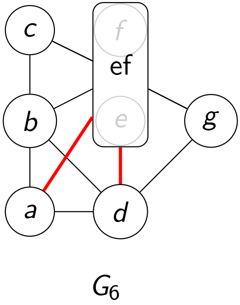


$\overline{G}$ $\quad$ G

$$\text{tww}(\overline{G}) = \text{tww}(G)$$

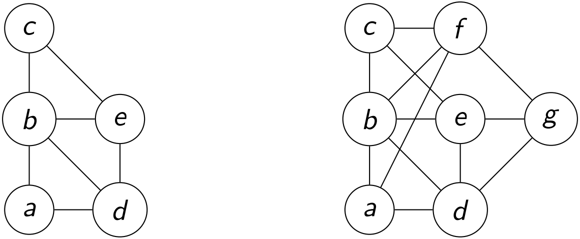
Complementation





$$\text{tww}(\overline{G}) = \text{tww}(G)$$

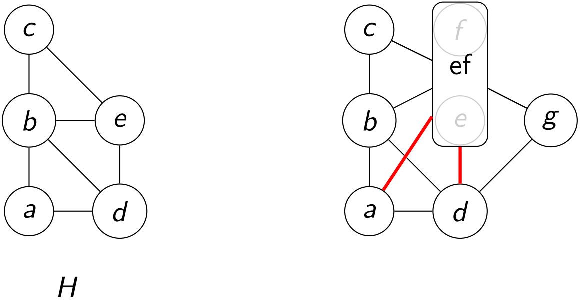
Induced subgraph



H G

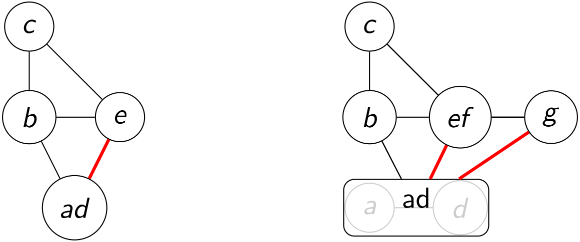
$$\text{tww}(H) \leqslant \text{tww}(G)$$

Induced subgraph



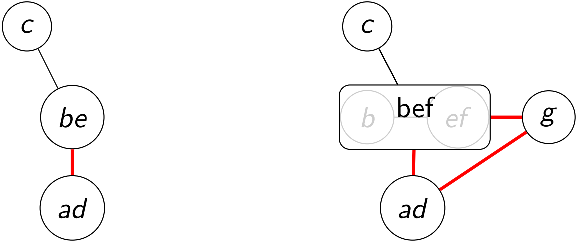
Ignore absent vertices

Induced subgraph



Mimic the contractions otherwise

Induced subgraph



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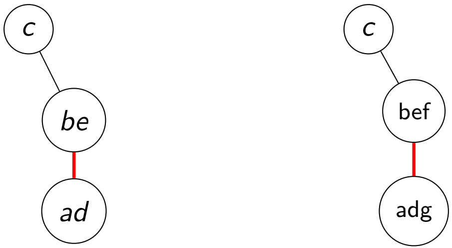

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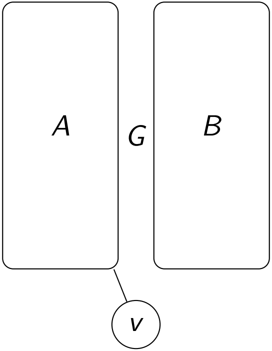
Adding one vertex v arbitrarily linked



Split every part into their part in A and in B until the very end

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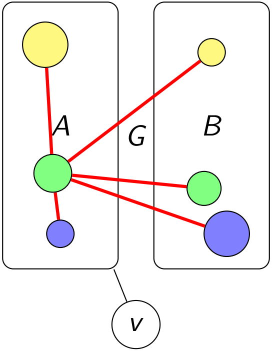
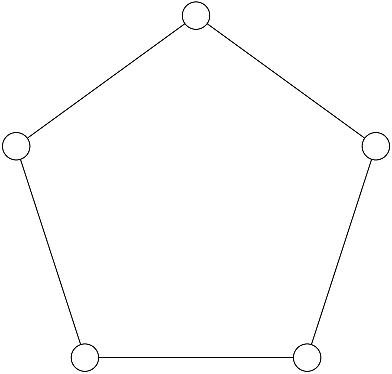


Split every part into their part in A and in B until the very end

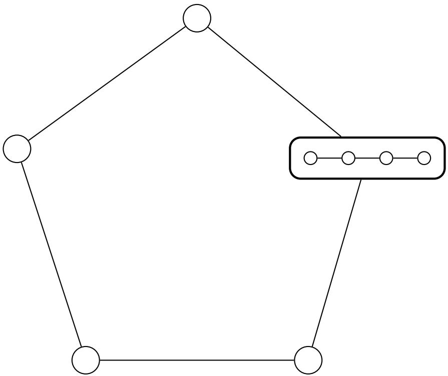
$$\text{tww}(G + v) \leqslant 2 \cdot \text{tww}(G) + 1$$

Substitution and lexicographic product



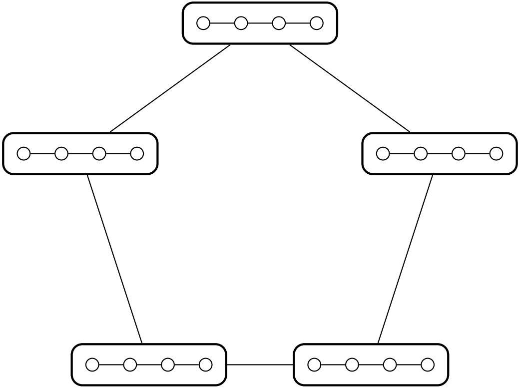
$G = C_5$

Substitution and lexicographic product



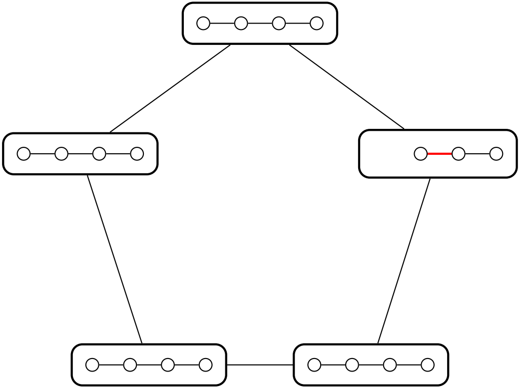
$G = C_5$, $H = P_4$, substitution $G[v \leftarrow H]$

Substitution and lexicographic product



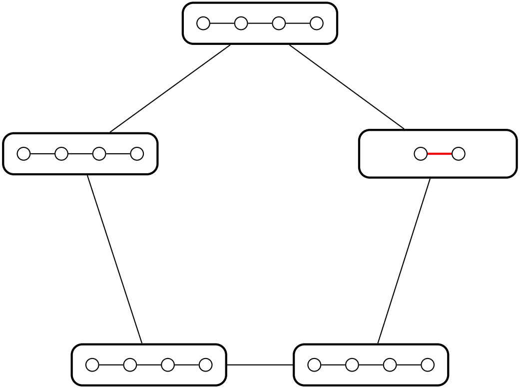
$G = C_5$, $H = P_4$, lexicographic product $G[H]$

Substitution and lexicographic product



More generally any modular decomposition

Substitution and lexicographic product



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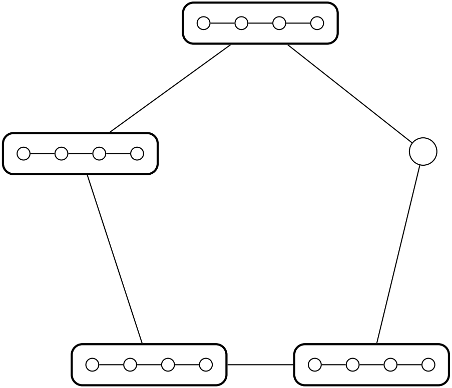
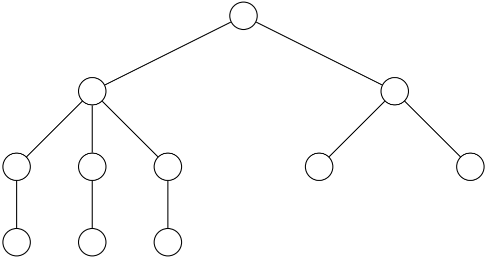


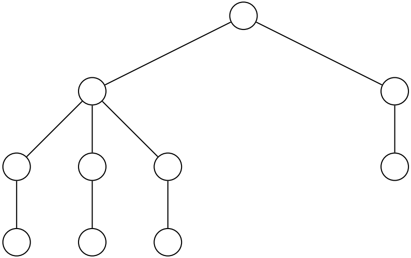
$$\text{tww}(G[H]) = \max(\text{tww}(G), \text{tww}(H))$$

Graphs with bounded twin-width – trees



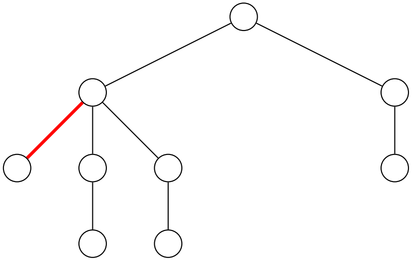
If possible, contract two twin leaves

Graphs with bounded twin-width – trees



If not, contract a deepest leaf with its parent

Graphs with bounded twin-width – trees



If not, contract a deepest leaf with its parent

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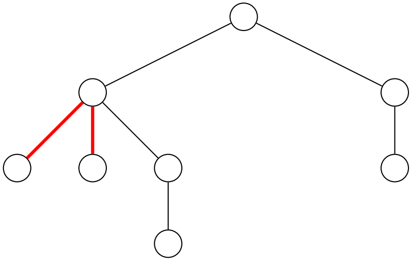

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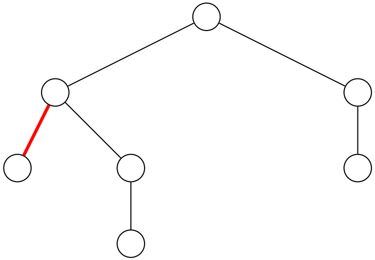

Cannot create a red degree-3 vertex

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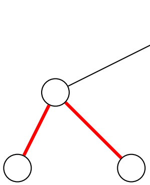

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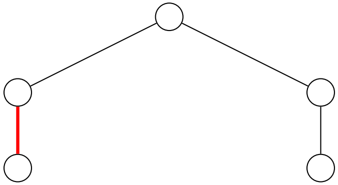

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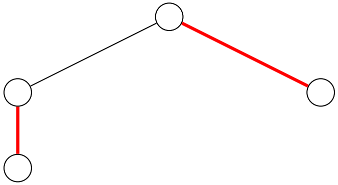

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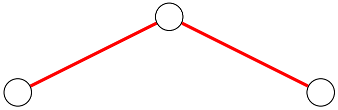

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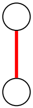

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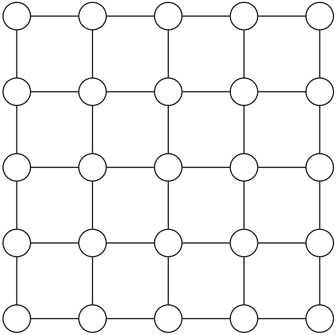
Graphs with bounded twin-width – trees

Generalization to bounded *treewidth* and even bounded *rank-width*

Graphs with bounded twin-width – grids



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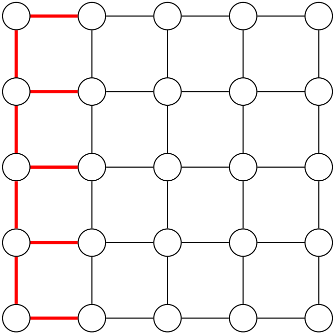

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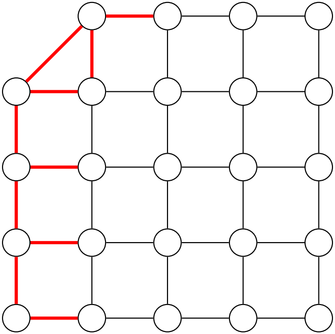

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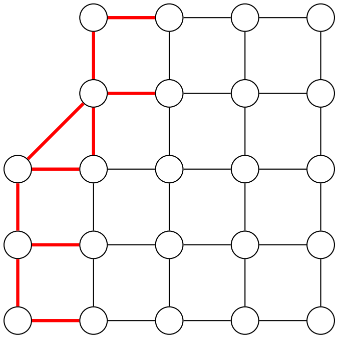

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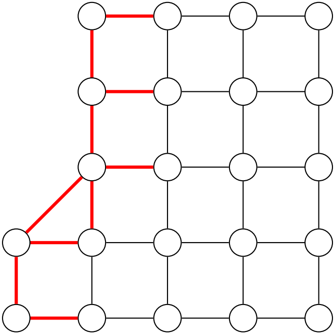

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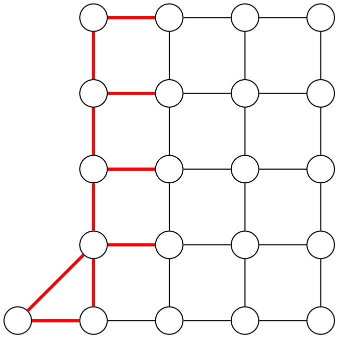

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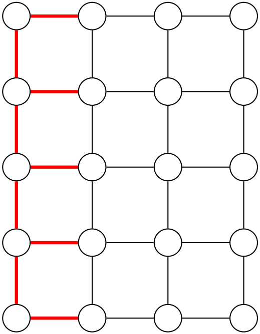
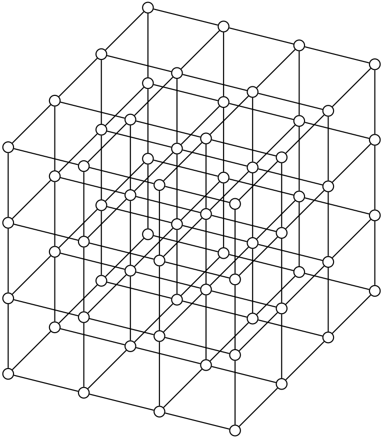


4-sequence for planar grids

3-dimensional grids



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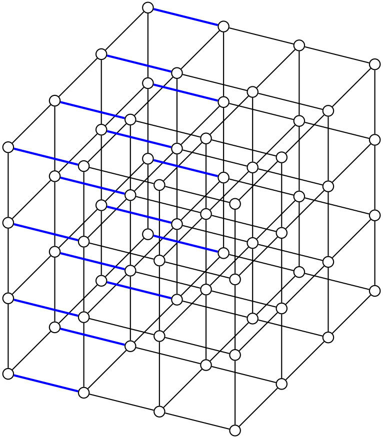
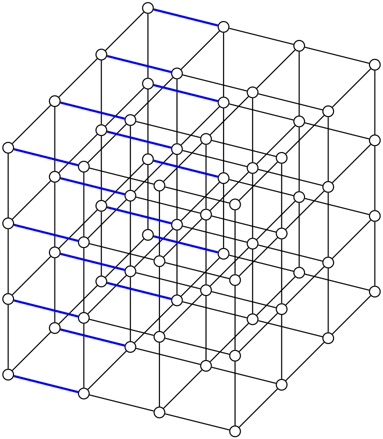


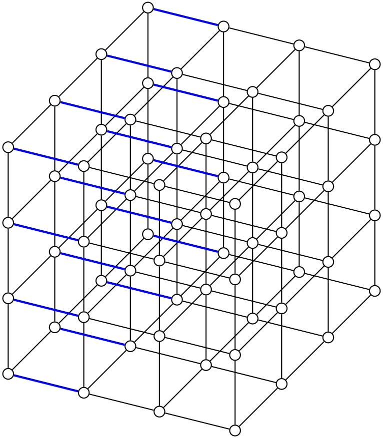
Contract the blue edges in any order $\rightarrow$ 12-sequence

3-dimensional grids



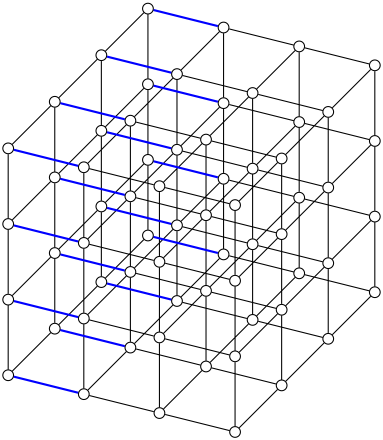
The d-dimensional grid has twin-width $\leqslant 4d$ (even $3d$)

3-dimensional grids



K_t-free unit d-dimensional disk graphs

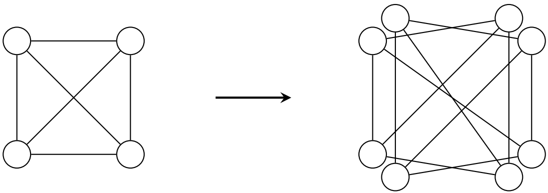
3-dimensional grids



~~K_t-free unit d-dimensional disk graphs~~

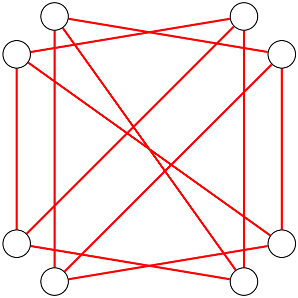
1-skeletons of Rips complexes of point sets in $\mathbb{R}^d$ with dimension less than t have bounded twin-width

2-lifts, expanders with bounded twin-width



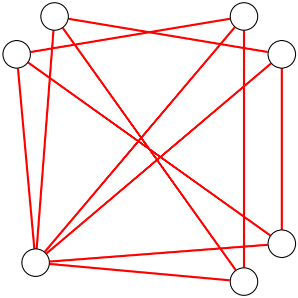
split each vertex in 2, replace each edge by 1 of the 2 matchings

2-lifts, expanders with bounded twin-width



Iterated 2-lifts of K_4 have twin-width at most 6

2-lifts, expanders with bounded twin-width



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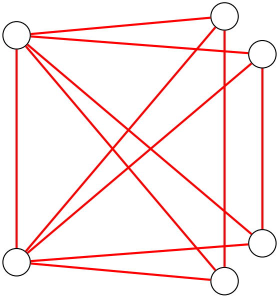

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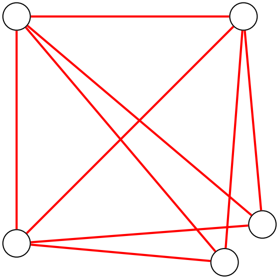

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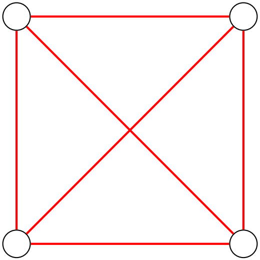
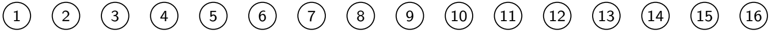


Iterated 2-lifts of K_4 have twin-width at most 6
treewidth $\Omega(n)$

($\geqslant 2\log n$)-subdivisions have twin-width at most 4



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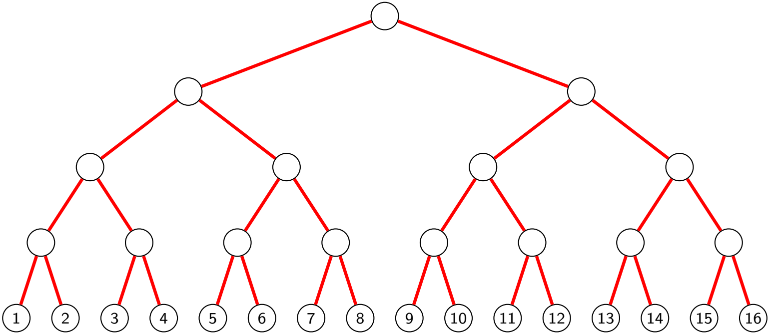
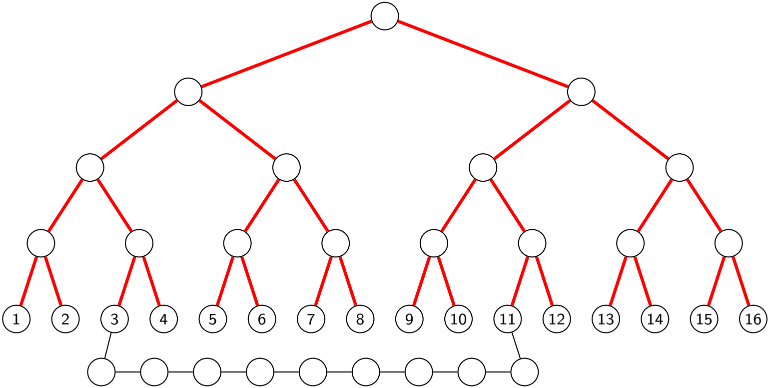


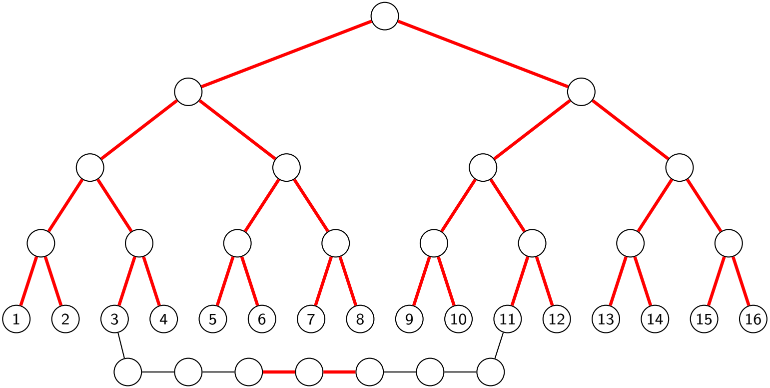
Add a red full binary tree whose leaves are the vertex set

($\geqslant 2 \log n$)-subdivisions have twin-width at most 4



Take any subdivided edge

($\geqslant 2 \log n$)-subdivisions have twin-width at most 4



Shorten it to the length of the path in the red tree

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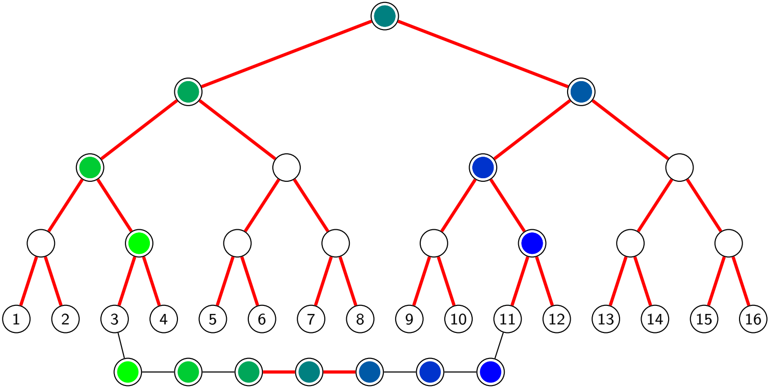
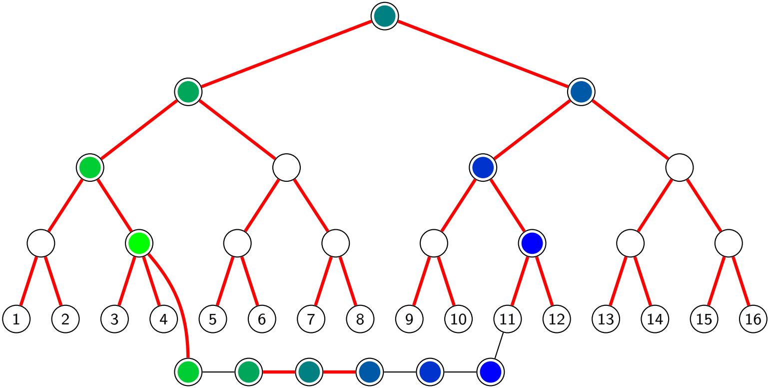


Zip the subdivided edge in the tree

$(\geqslant 2 \log n)$-subdivisions have twin-width at most 4



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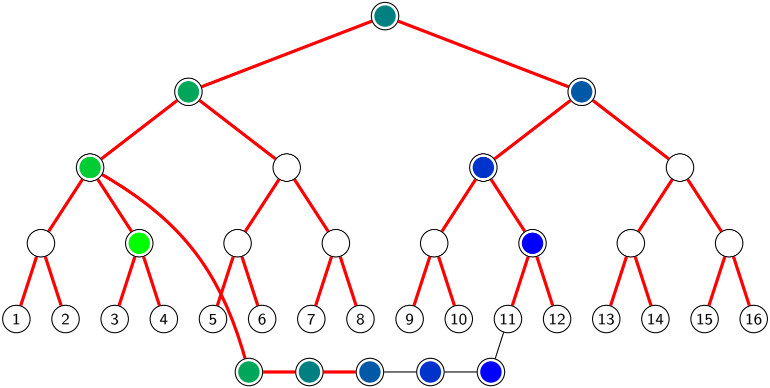

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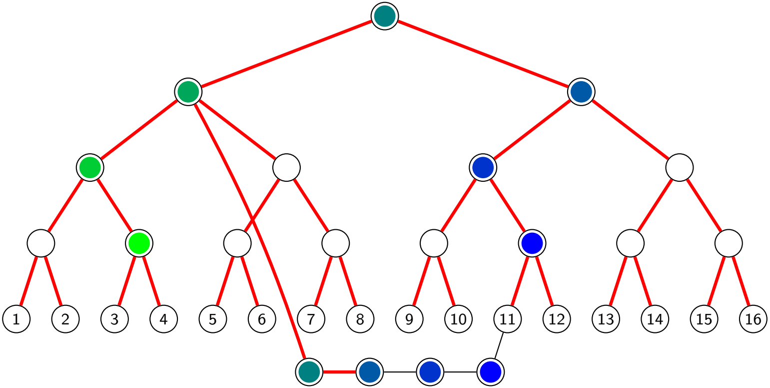

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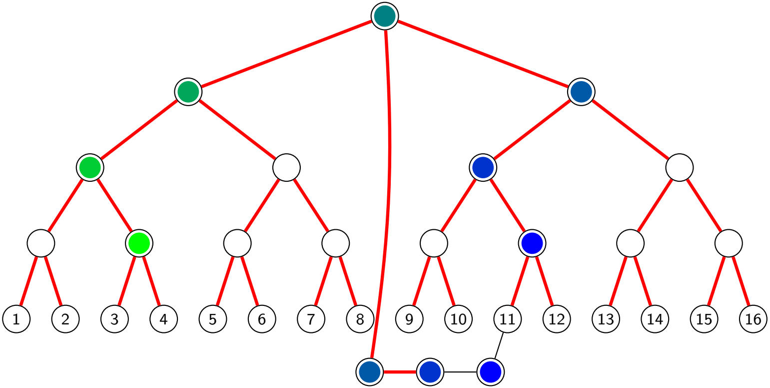

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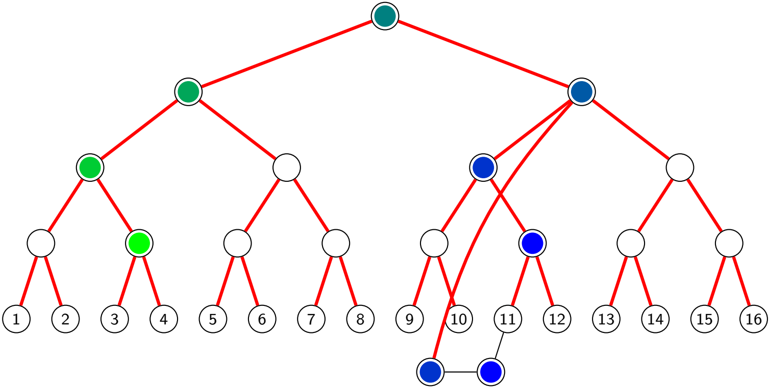

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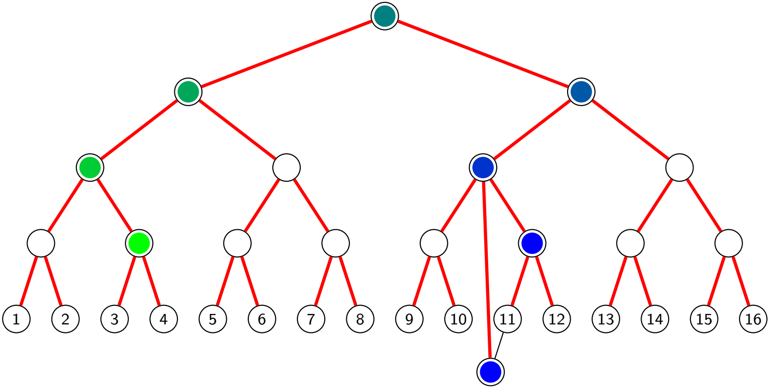

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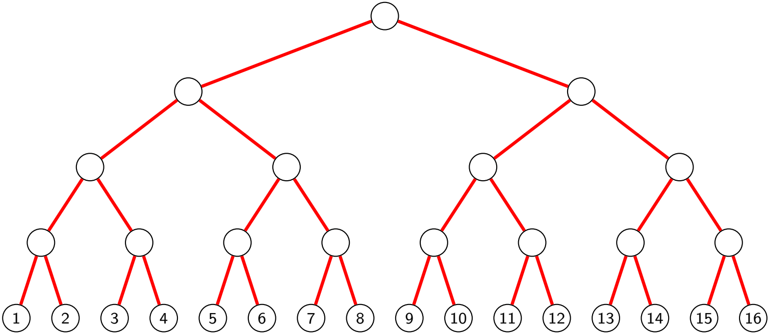
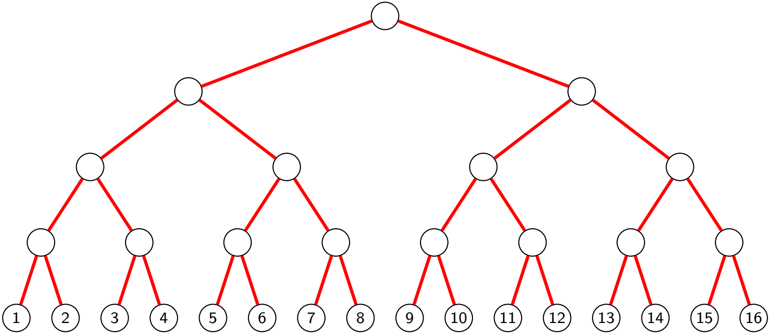


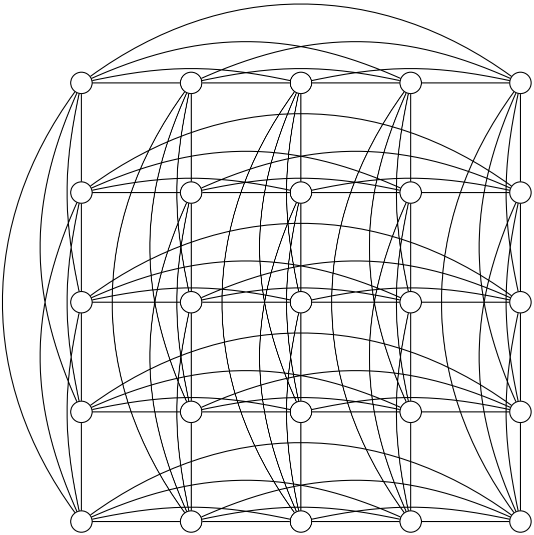
Move to the next subdivided edge also of unbounded cliquewidth

($\geqslant 2 \log n$)-subdivisions have twin-width at most 4



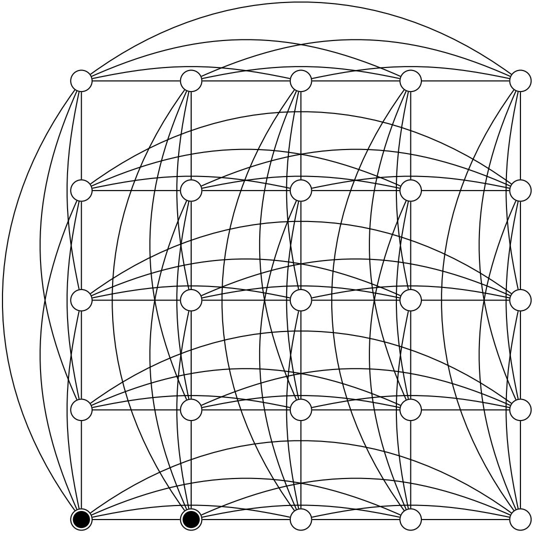
Twin-width is not topological (in the graph-theoretic sense)

First example of unbounded twin-width



Line graph of a biclique a.k.a. rook graph

First example of unbounded twin-width



No pair of near twins

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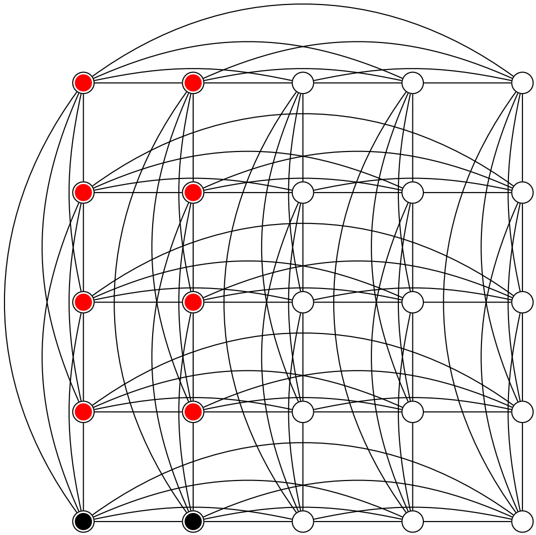

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Universal bipartite graph

No $O(1)$-contraction sequence:
twin-width is *not* an iterated identification of near twins.

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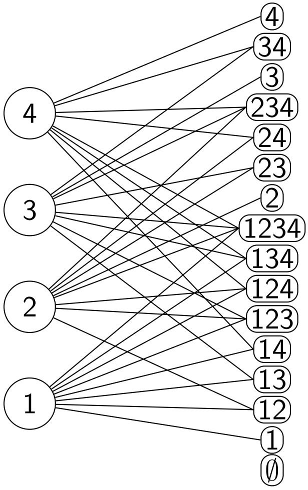

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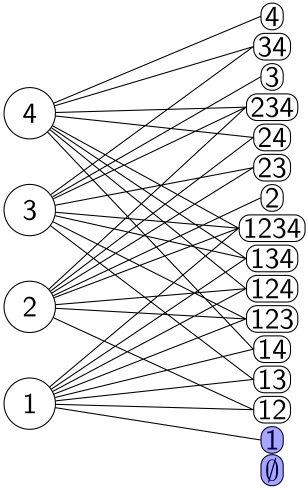

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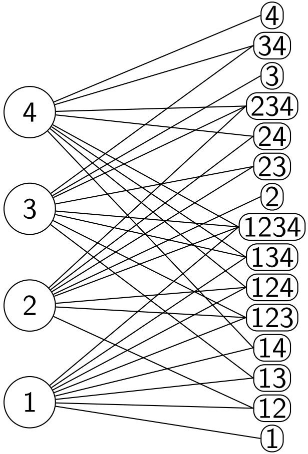

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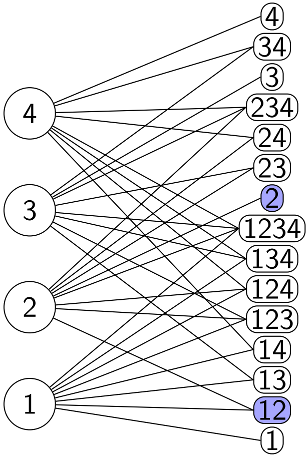

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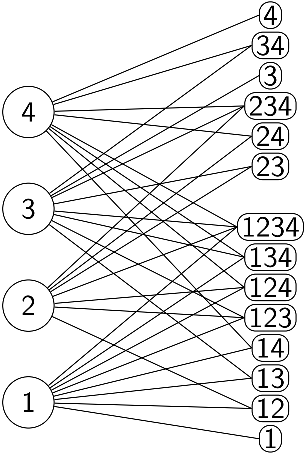

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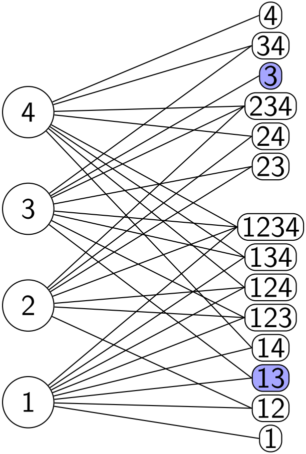

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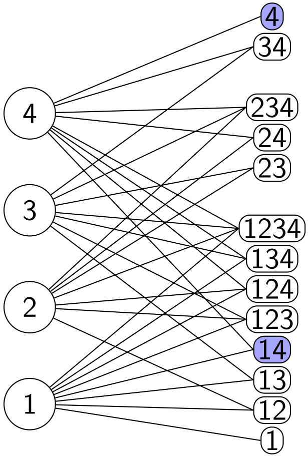

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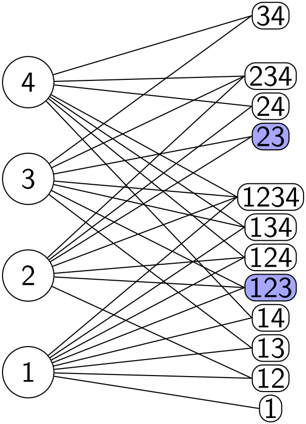

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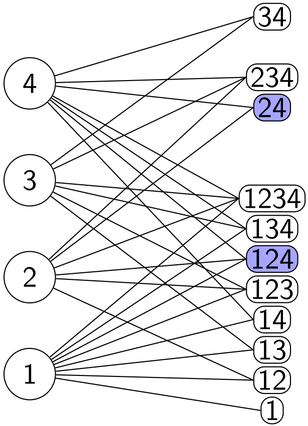

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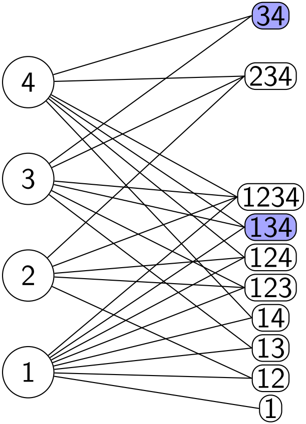

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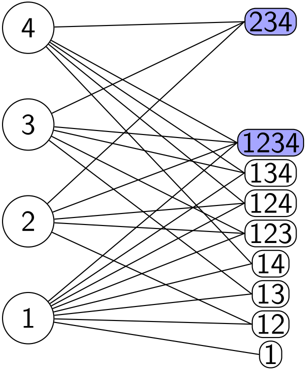

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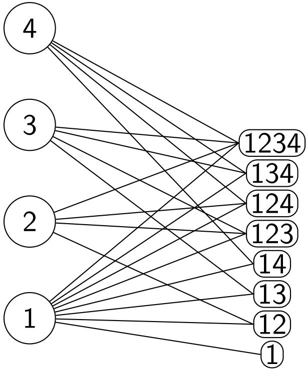

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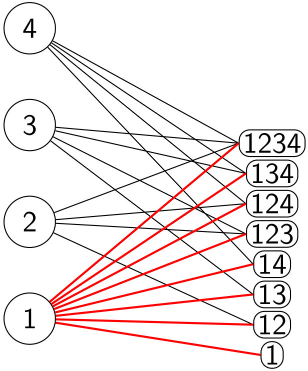

Graphs with bounded twin-width – planar graphs?

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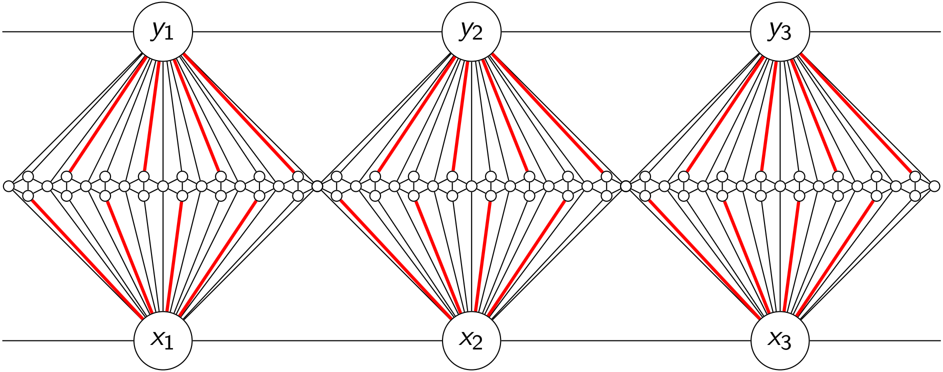
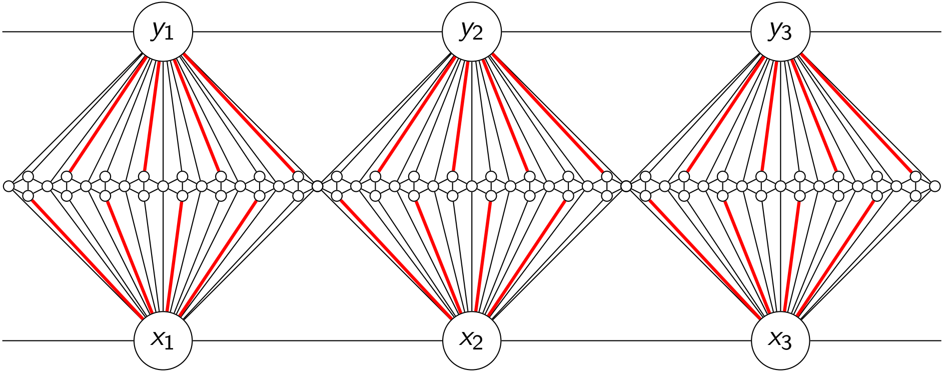


For every d, a planar trigraph without planar d-contraction

Graphs with bounded twin-width – planar graphs?



For every d, a planar trigraph without planar d-contraction

More powerfool tool needed

Bounded twin-width via adjacency matrices

A matrix is t-**mixed free** if it does not have a $t \times t$ division where every cell has two distinct rows and two distinct columns

$$\left[\begin{array}{cc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
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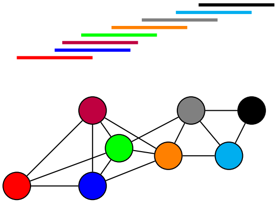
Theorem (B., Kim, Thomassé, Watrigant '20)

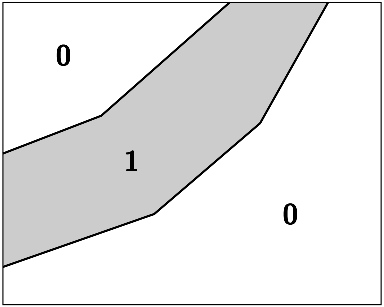
A graph class has bounded twin-width if and only if all its graphs admit an $O(1)$-mixed free adjacency matrix.

Unit interval graphs

Intersection graph of unit segments on the real line

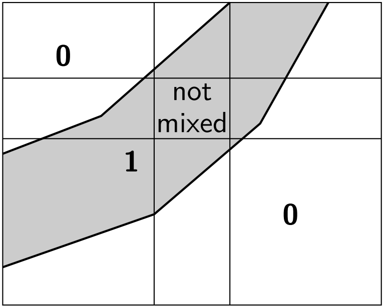


Unit interval graphs



order by left endpoints

Unit interval graphs



No 3-by-3 grid has all 9 cells crossed by two non-decreasing curves

Graph minors

Formed by **vertex deletion**, **edge deletion**, and **edge contraction**

A graph G is *H-minor free* if H is not a minor of G

A graph class is *H-minor free* if all its graphs are

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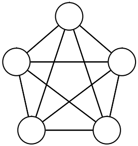
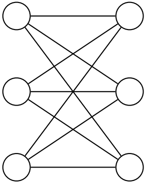
Planar graphs are exactly the graphs without K_5 or $K_{3,3}$ as a minor



K_5

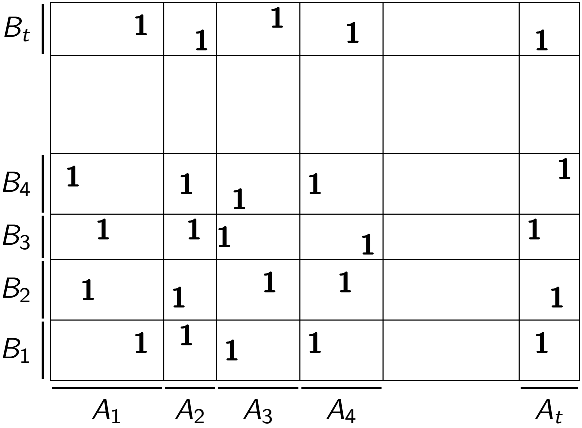


$K_{3,3}$

Bounded twin-width – K_t-minor free graphs

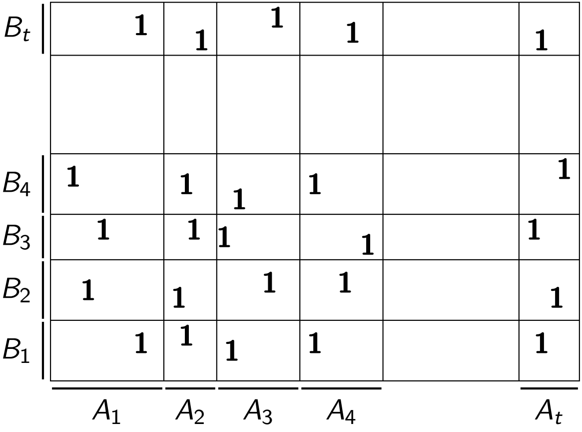
Given a hamiltonian path, we would just use this order

Bounded twin-width – K_t-minor free graphs



Contracting the $2t$ subpaths yields a $K_{t,t}$-minor, hence a K_t-minor

Bounded twin-width – K_t-minor free graphs



Instead we use a specially crafted lex-DFS discovery order

Better upper bounds

Theorem (Hlinený '22)

Planar graphs have twin-width at most 11.

Theorem (B., Kwon, Wood '22)

Graphs of genus g have twin-width $O(g)$.

Better upper bounds

Theorem (Hlinený '22)

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Graphs of genus g have twin-width $O(g)$.

Inspired and based on the recent graph product structure theorem:

Theorem (Dujmovic, Joret, Micek, Morin, Ueckerdt, Wood '19)

Every planar graph is the subgraph of the strong product of a path and a graph of treewidth at most 8.

Theorem (B., Geniet, Kim, Thomassé, Watrigant '20, '21)

The following classes have bounded twin-width, and $O(1)$-sequences can be computed in polynomial time.

- *Bounded rank-width, and even, boolean-width graphs,*
- *every hereditary proper subclass of permutation graphs,*
- *posets of bounded antichain size (seen as digraphs),*
- *unit interval graphs,*
- *K_t-minor free graphs,*
- *map graphs,*
- *subgraphs of d-dimensional grids,*
- *K_t-free unit d-dimensional ball graphs,*
- *$\Omega(\log n)$-subdivisions of all the n-vertex graphs,*
- *cubic expanders defined by iterative random 2-lifts from K_4,*
- *strong products of two bounded twin-width classes, one with bounded degree, etc.*

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Is this filtration any useful?

First-order model checking on graphs

GRAPH FO MODEL CHECKING	**Parameter:** $\lvert\varphi\rvert$

Input: A graph G and a first-order sentence $\varphi \in FO(\{E\})$
Question: $G \models \varphi$?

First-order model checking on graphs

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Example:

$$\varphi = \exists x_1 \exists x_2 \cdots \exists x_k \forall x \forall y \ (E(x,y) \Rightarrow \bigvee_{1 \leqslant i \leqslant k} x = x_i \vee y = x_i)$$

$G \models \varphi? \Leftrightarrow k$-VERTEX COVER

First-order model checking on graphs

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$$\wedge\ E(x, y) \Leftrightarrow \bigvee_{1 \leqslant i \leqslant k} (x = x_i \wedge y = y_i) \vee (x = y_i \wedge y = x_i)$$

$G \models \varphi? \Leftrightarrow$

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$$\wedge\ E(x,y) \Leftrightarrow \bigvee_{1 \leqslant i \leqslant k} (x = x_i \wedge y = y_i) \vee (x = y_i \wedge y = x_i)$$

$G \models \varphi$? $\Leftrightarrow$ k-INDUCED MATCHING

First-order model checking on graphs

GRAPH FO MODEL CHECKING	**Parameter:** $\lvert\varphi\rvert$
Input: A graph G and a first-order sentence $\varphi \in FO(\{E\})$	
Question: $G \models \varphi$?	

Example:

$$\varphi = \bigvee_{1 \leqslant q \leqslant k,\ q \text{ is odd}} \exists x_1 \notin \{s\}\ E(s, x_1) \wedge (\forall x_2 \notin \{s, x_1\}\ \neg E(x_1, x_2) \vee$$

$$(\exists x_3 \notin \{s, x_1, x_2\}\ E(x_2, x_3) \wedge (\forall x_4 \cdots (\exists x_q \notin \{s, x_1, \ldots, x_{q-1}\} E(x_{q-1}, x_q)$$

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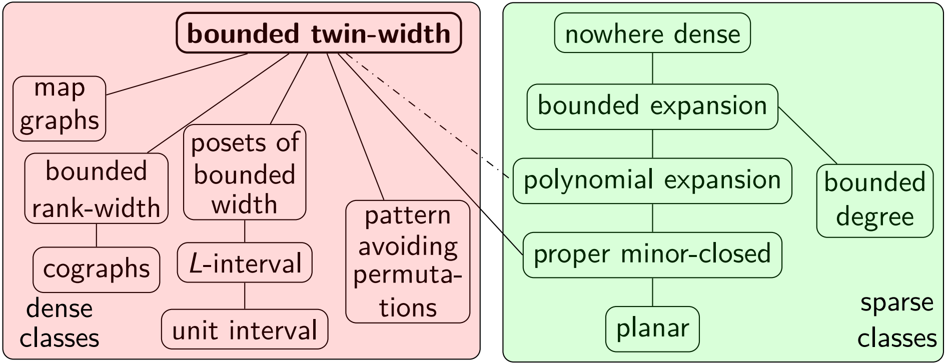
$G \models \varphi$? $\Leftrightarrow$ SHORT GENERALIZED GEOGRAPHY

Classes with known tractable FO model checking



Theorem (B., Kim, Thomassé, Watrigant '20)

FO Model Checking *solvable in* $f(|\varphi|, d)n$ *on graphs with a* d*-sequence.*

FO interpretations and transductions

FO simple interpretation: redefine the edges by a first-order formula
$\varphi(x, y) = \neg E(x, y)$ (complement)
$\varphi(x, y) = E(x, y) \vee \exists z E(x, z) \wedge E(z, y)$ (square)

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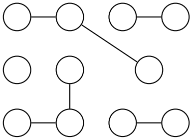
FO transduction: color by $O(1)$ unary relations, interpret, delete



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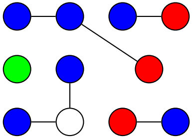

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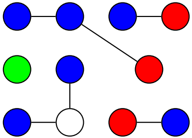

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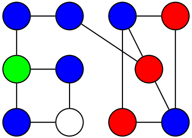

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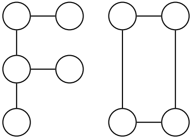

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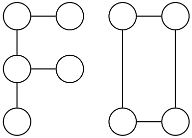

Theorem (B., Kim, Thomassé, Watrigant '20)

Transductions of bounded twin-width classes have bounded twin-width.

Small classes

Small: class with at most $n!c^n$ labeled graphs on $[n]$.

Theorem (B., Geniet, Kim, Thomassé, Watrigant '21)

Bounded twin-width classes are small.

Unifies and extends the same result for:
σ-free permutations [Marcus, Tardos '04]
K_t-minor free graphs [Norine, Seymour, Thomas, Wollan '06]

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Is the converse true for hereditary classes?

Conjecture (small conjecture)

A hereditary class has bounded twin-width if and only if it is small.

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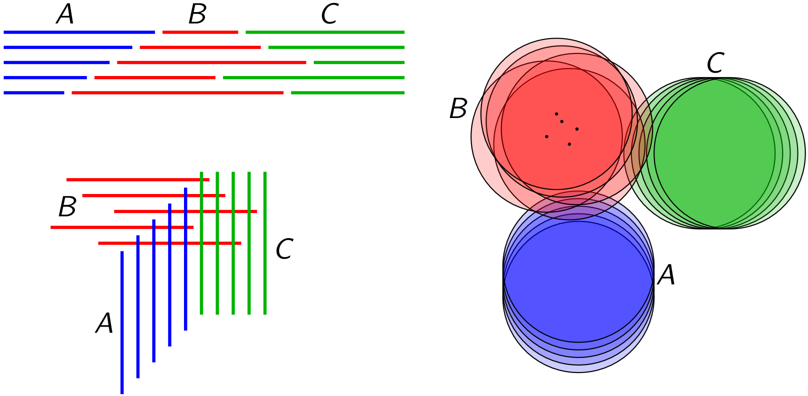
Is the converse true for hereditary classes?

Conjecture (refuted: B., Geniet, Tessera, Thomassé '22)

A hereditary class has bounded twin-width if and only if it is small.

Interval graphs, unit disk graphs, triangle-free unit segment graphs have unbounded twin-width



since there are too many such graphs

The transduction tool

Our stability

- adding noise keeps twin-width tamed
- robust notion for finite model theory
- indeed bounded twin-width is the right limit for tractability of FO model checking in ordered graphs (Twin-width IV)
- concrete example: biclique-free segment graphs have bounded twin-width by FO transducing from planar graphs

Reduced parameters

A graph class has bounded reduced X if all its members admit a contraction sequence whose red graphs have bounded X

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red graphs have bounded ...	characterize bounded ...
outdegree	(oriented) twin-width
degree	**twin-width**
degree + treewidth	reduced (degree + treewidth)
cutwidth	reduced cutwidth
bandwidth	reduced bandwidth
bandwidth with fixed order	stretch-width
component size	**cliquewidth** (sparse: treewidth)
number of edges*	linear cliquewidth (sparse: pathwidth)

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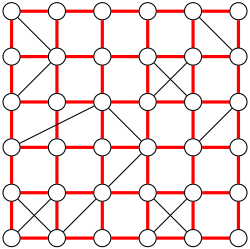
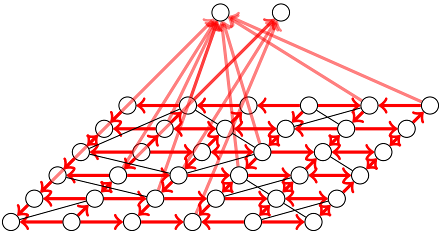
Long subdivisions of general graphs have unbounded stretch-width

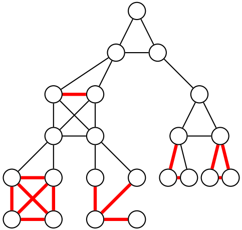
Different conditions imposed on the red graphs



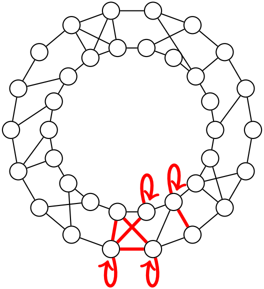
bd degree: defines bd twin-width



bd outdegree: defines bd oriented twin-width



bd component: redefines bd cliquewidth



bd #edges: redefines bd linear cliquewidth

Some directions and questions

- Twin-width of higher arity structures? Simplicial complexes?
- Do compact 3-manifolds have uniformly bounded twin-width?
  (shamelessly borrowed from Kristóf's research project)
- Draw uniformly at random n points in $[0,1]^d$ (or from a different distribution), grow balls from radius 0 to $\sqrt{d}/2$, and track the largest twin-width of the intersection graph. Asymptotics of this function of n?
- Bound the twin-width of some geometric graph classes when half-graphs are forbidden.