Twin-width — Part 2

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ENS Lyon, LIP

10th October 2024, New Tools in Parameterized Complexity: Paths, Cuts, and Decomposition, Dagstuhl

Theorem (B., Geniet, Kim, Thomassé, Watrigant '20 & '21)

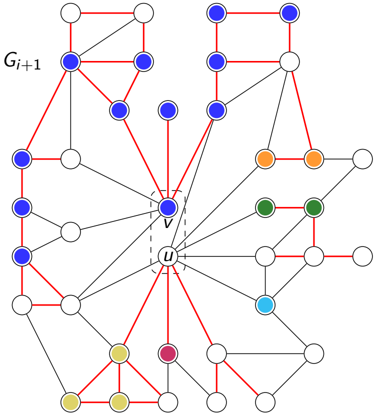
The following classes have bounded twin-width, and $O(1)$-sequences can be computed in polynomial time.

- *Bounded rank-width or clique-width graphs,*
- *every hereditary proper subclass of permutation graphs,*
- *posets of bounded antichain size,*
- *unit interval graphs,*
- *K_t-minor free graphs,*
- *map graphs,*
- *subgraphs of d-dimensional grids,*
- *K_t-free unit d-dimensional ball graphs,*
- *$\Omega(\log n)$-subdivisions of all the n-vertex graphs,*
- *strong products of two bounded twin-width classes, one with bounded degree,*
- *(given) first-order transductions of the above.*

Given a d-sequence, one can solve

any problem definable with a FO sentence φ in time $f(d, \varphi)|V(G)|$.

special cases like k-Independent Set in time $2^{O_d(k)}|V(G)|$



How hard is computing twin-width?

Theorem (Bergé, B., Déprés '22)

It is NP-complete to decide if the twin-width is at most 4.

Question

Given a graph G and an integer d, is it possible to either provide an $f(d)$-sequence of G or correctly conclude that tww$(G) > d$, in time $g(d)|V(G)|^{O(1)}$ or $|V(G)|^{g(d)}$?

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Theorem (Balabán, Ganian, Rocton '24)

There is an FPT algorithm that computes (2-approximates) twin-width parameterized by feedback edge number (vertex integrity).

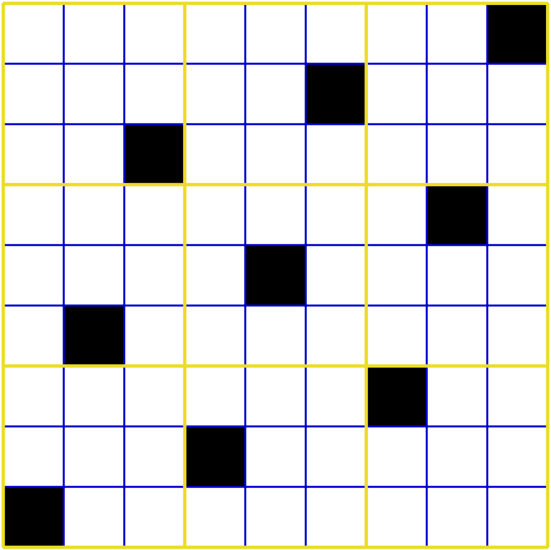
Question

Is there an FPT (XP) $f(OPT)$-approximation algorithm for twin-width parameterized by pathwidth, treewidth, rank-width?

Unconditional parameterized algorithms

(À la Guillemot-Marx)

k-grid permutation

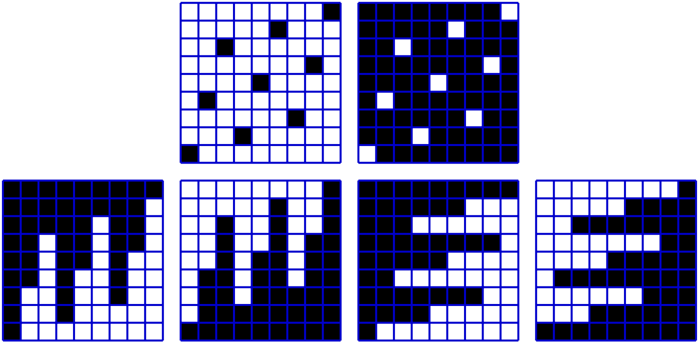


Here with $k = 3$, it has every 3-permutation as subpermutation

The 6 minimal families of unbounded twin-width

Theorem (B., Giocanti, Ossona de Mendez, Simon, Thomassé, Toruńczyk '22)

$\exists f, g$ s.t., given an $n \times n$ adjacency matrix $Adj_{\prec}(G)$, in time $g(k)n^{O(1)}$ one can find an $f(k)$-sequence of $(G, \prec)$ or one of the six following encodings of a k-grid permutation submatrix:



Semi-induced matching/antimatching, and 4 half-graphs or ladders

Ordered binary structures

Theorem (B., Giocanti, Ossona de Mendez, Simon, Thomassé, Toruńczyk '22)

Let $\mathscr{C}$ be a hereditary class of ordered graphs. There is an FPT $f(OPT)$-approximation for twin-width on $\mathscr{C}$, and the following are equivalent.

(1) *$\mathscr{C}$ has bounded twin-width.*

(2) *$\mathscr{C}$ is dependent.*

(3) *$\mathscr{C}$ contains $2^{O(n)}$ ordered n-vertex graphs.*

(4) *$\mathscr{C}$ contains less than $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} k!$ ordered n-vertex graphs, for some n.*

(5) *$\mathscr{C}$ does not include one of 25 hereditary ordered graph classes with unbounded twin-width.*

(6) *FO-model checking is fixed-parameter tractable on $\mathscr{C}$ (assuming $FPT \neq AW[*]$).*

Twin-width win-win

Goal: compute FO-definable parameter p in FPT time in $\mathcal{C}$.

Show that $\exists f$ non-decreasing, such that $\forall G \in \mathcal{C}$ an $f(p(G))$-sequence of G can be computed in FPT time

- Width $> f(k)$: report $p(G) > k$
- Width $\leqslant f(k)$: use FO model checking algorithm

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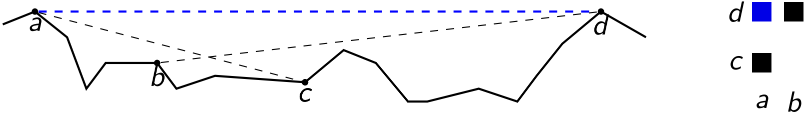
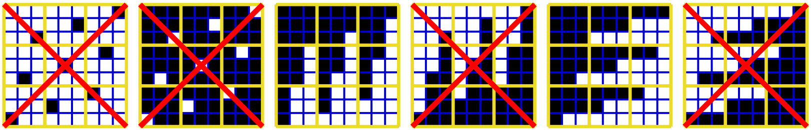
$\rightarrow$ k-BICLIQUE in visibility graphs of 1.5D terrains
$\rightarrow$ k-INDEPENDENT SET in visibility graphs of simple polygons

[B., Chakraborty, Kim, Köhler, Lopes, Thomassé '22]

Visibility graphs of 1.5D terrains

Order along x-coordinates





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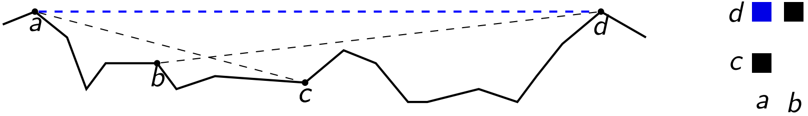
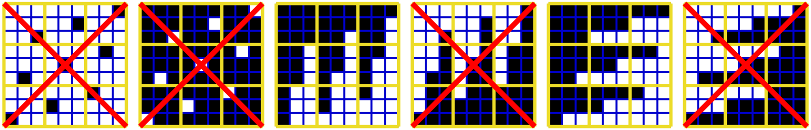
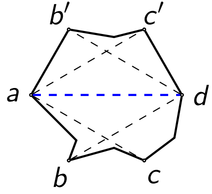
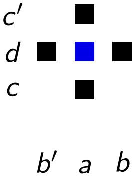




k-BICLIQUE and k-LADDER are FPT in this class

Ordering along the boundary of the polygon





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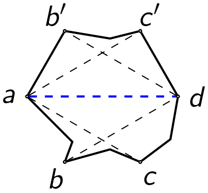
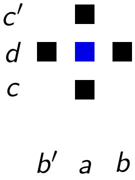
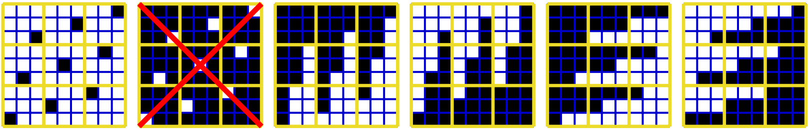
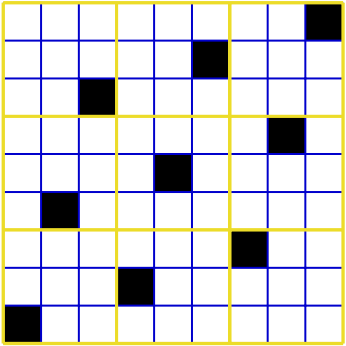


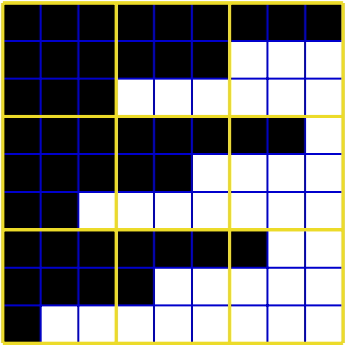




Extractions

Here we only need a decreasing pattern





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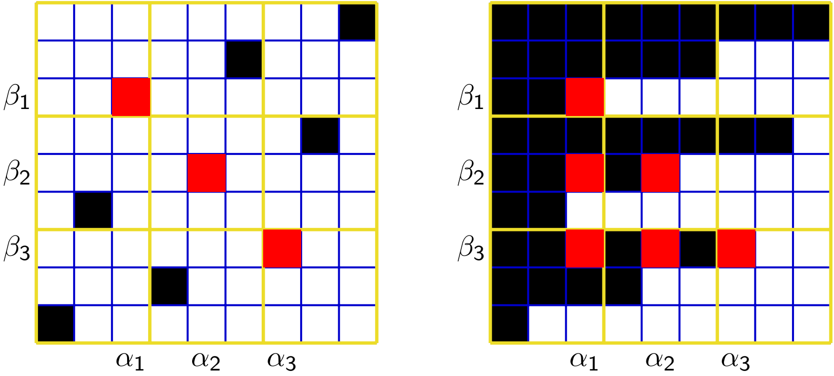

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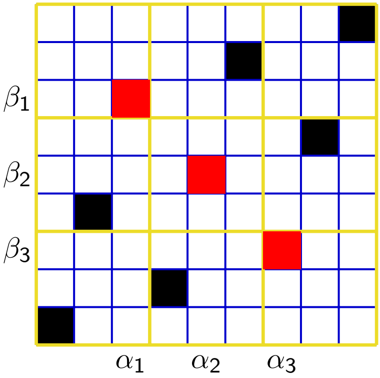
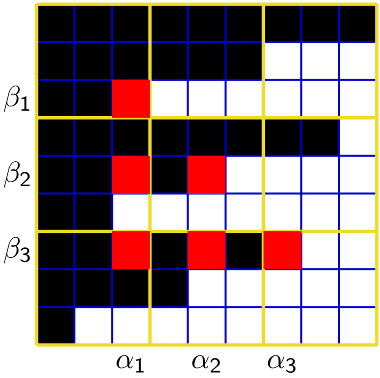
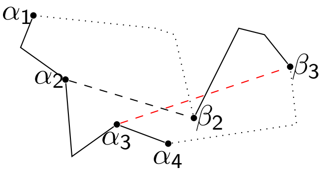




By Ramsey's theorem, we can assume that the α_is and the β_is both induce a clique.

Geometric arguments



Quadrangle $\alpha_2\alpha_3\beta_3\beta_2$ is not self-crossing

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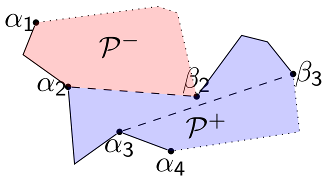
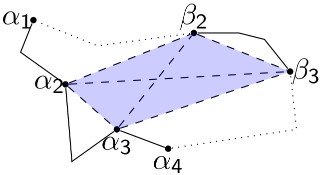


Quadrangle $\alpha_2\alpha_3\beta_3\beta_2$ has to be convex

Geometric arguments



Then $\alpha_2, \alpha_3, \beta_3, \beta_2$ induce K_4, a contradiction

Matrix Multiplication

Theorem (B., Giocanti, Ossona de Mendez, Thomassé '23)

Given two $n \times n$ $\mathbb{F}_q$-matrices A, B of twin-width at most d, one can compute AB in time $O_{d,q}(n^2 \log n)$.

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Consequence of:

- $O_{d,q}(n^2 \log n)$ time $f(\mathrm{OPT})$-approximation for twin-width of ordered binary structures,
- FO+MOD interpretations preserve bounded twin-width,
- squaring is an FO+MOD interpretation, and

Theorem (Gajarský, Pilipczuk, Przybyszewski, Toruńczyk '22)

Given an FO(+MOD) interpretation $\varphi(x_1, \ldots, x_k)$ and a binary structure G with a d-sequence, a data structure can be computed in time $O_{d,\varphi}(n^{1+\varepsilon})$ that answers queries "does $\varphi(v_1, \ldots, v_k)$ hold in G?" in time $O_{d,\varphi}(1/\varepsilon)$.

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$\varphi_1(r, c)$ holds in $\widehat{M}$ $\equiv$ there is a 1-entry at row r, column c of M^2.

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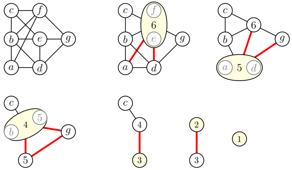
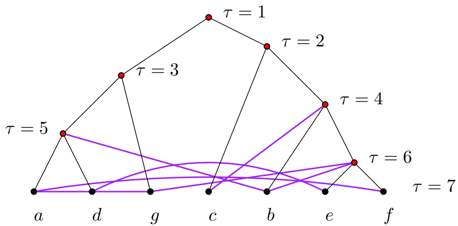
- $O_{d,q}(n^2 \log n)$ time $f(\text{OPT})$-approximation for twin-width of ordered binary structures,
- a d-sequence can be turned into a twin-decomposition of width d in time $O_d(n^2)$, and
- $q^{O(d)}n$-time algorithm for the twin-decomposition of M^2.





Approximation algorithms

Balanced contraction sequences

Theorem (B., Geniet, Kim, Thomassé, Watrigant '21)

For every d, there is a D such that every n-vertex graph with twin-width at most d iteratively admits $\frac{n}{D}$ disjoint pairs that can be contracted in a D-sequence.

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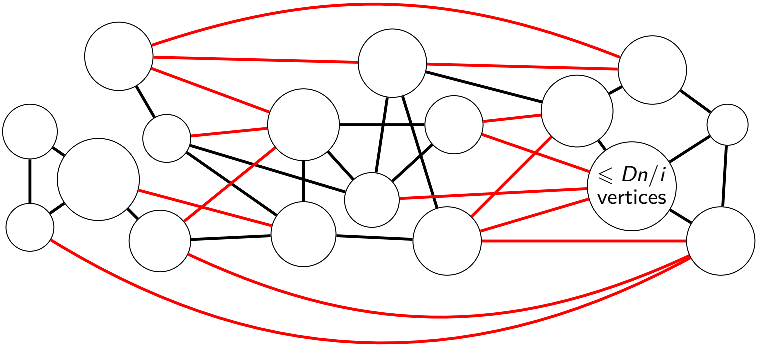
Consequence: We can turn a d-sequence into a *balanced* D-sequence $\mathcal{S}$, i.e., such that $\forall \mathcal{P}_i \in \mathcal{S}, \forall P \in \mathcal{P}_i$, $|P| \leqslant D\frac{n}{i}$



Approximating Max Independent Set

In general graphs: an $n^{1-\varepsilon}$-approximation or $r(n)$-approximation in time $\exp(\frac{n^{1-\varepsilon}}{r(n)^{1+\varepsilon}})$ are unlikely

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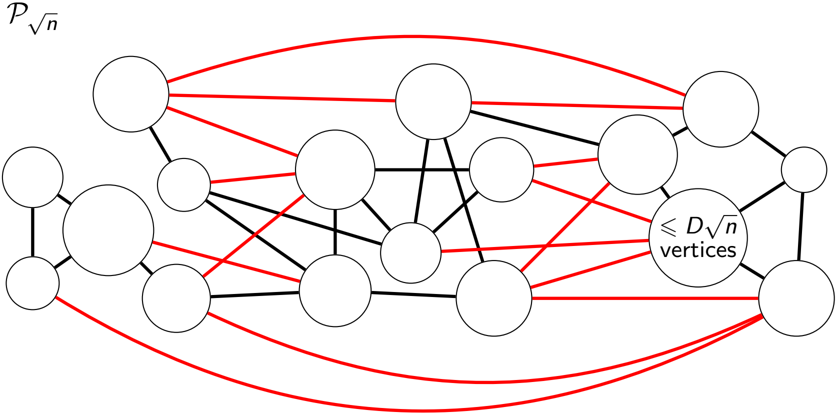

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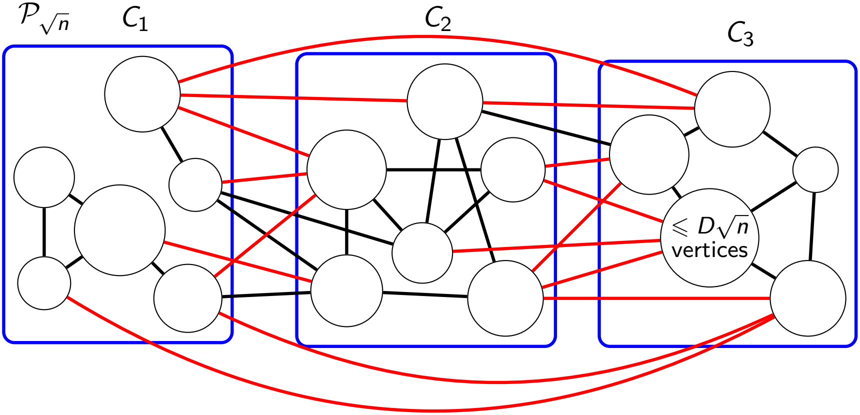

$D+1$-color the red graph of $G/\mathcal{P}_{\sqrt{n}}$ in polynomial time

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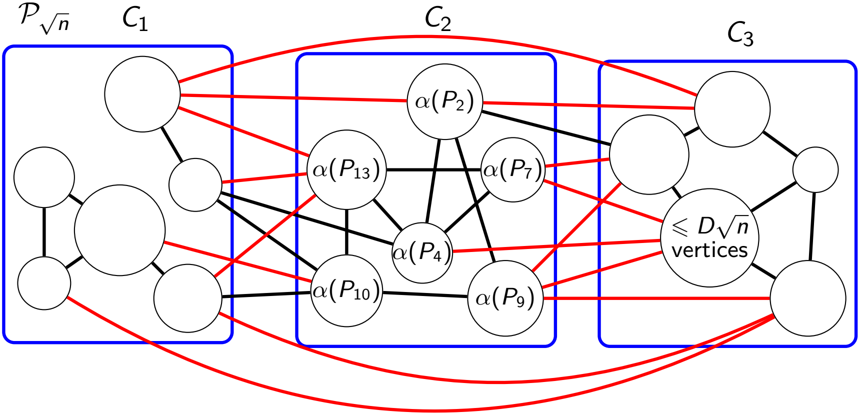

Solve MIS in $G[P_j]$ for every $P_j \in \mathcal{P}_{\sqrt{n}}$ in $2^{O(D\sqrt{n})}$ time

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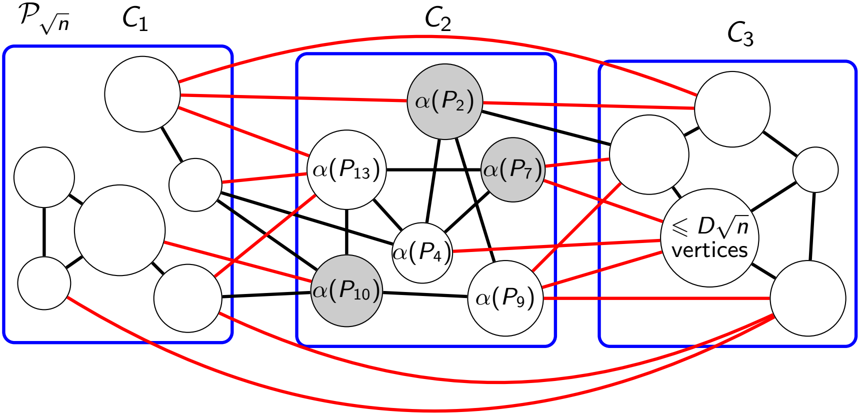

Solve weighted MIS in $G/\mathcal{P}_{\sqrt{n}}[C_i]$, $\forall i \in [D+1]$ in $2^{O(\sqrt{n})}$ time

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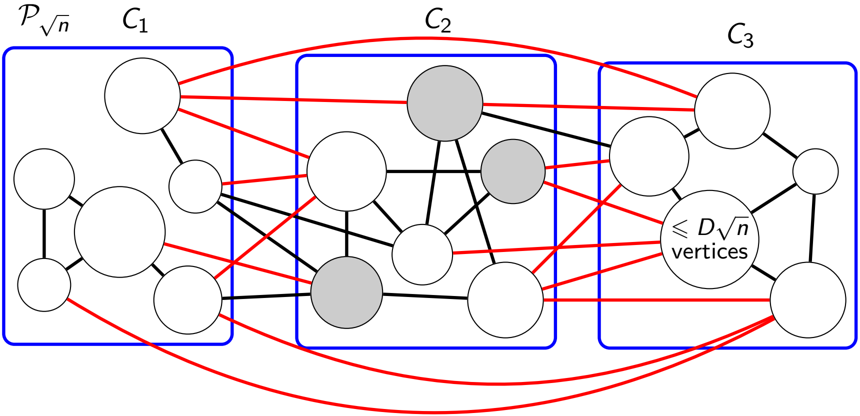

A heaviest such solution is a $(D+1)$-approximation

Approximating MIS given a d-sequence

Theorem (Bergé, B., Déprés, Watrigant '23)

MIS *can be* $O_d(1)$*-approximated in time* $2^{O_d(\sqrt{n})}$.

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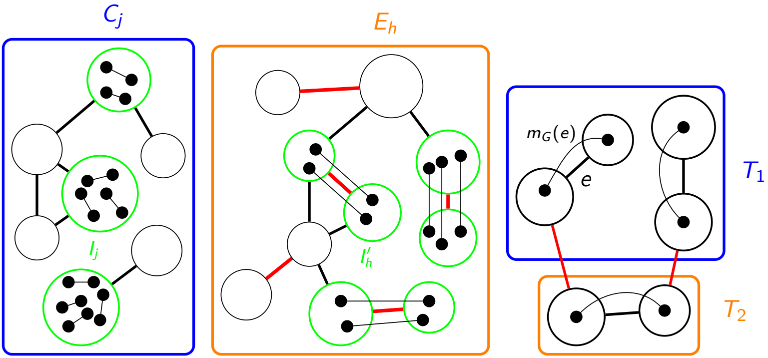
Setting $q := \log \frac{\varepsilon \log n}{O_d(1)}$

Theorem (Bergé, B., Déprés, Watrigant '23)

MIS *can be* n^ε*-approximated in polynomial time.*

Coloring, Max Induced Matching

Similar results for these problems



Open questions

FPT/XP approximation of twin-width (parameterized by larger parameters)

Find an explicit family of bounded-degree graphs with unbounded twin-width (counting-free argument)

Practical FPT algorithms (for the problems on polygons/terrains)

Better than n^{ε}-approximation for MIS given $O(1)$-sequences? (every such approximation would then self-improve)

More unexpected uses of the FO model checking algorithm on bounded twin-width (like [HJLMPSS '23] for Directed Multicut with three terminal pairs parameterized by cutset size)

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Thank you for your attention!