Twin-width

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ENS Lyon, LIP

Utrecht Algorithms seminar,
September 29th 2020

Cograph generalization attempt

Iteratively identify **near** twins

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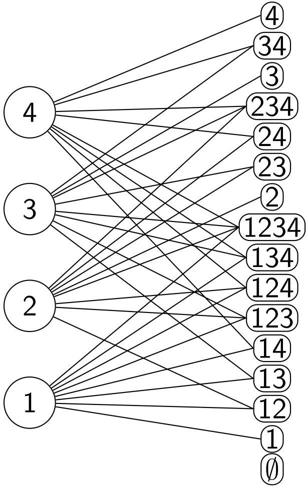

This complicated graph passes the test

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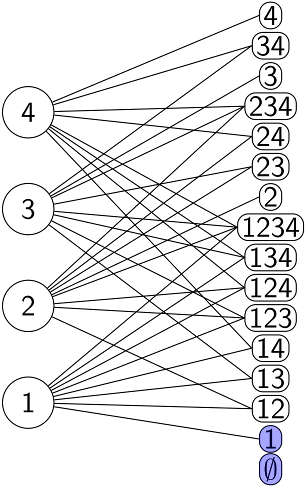

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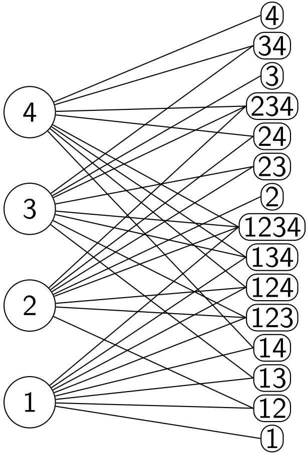

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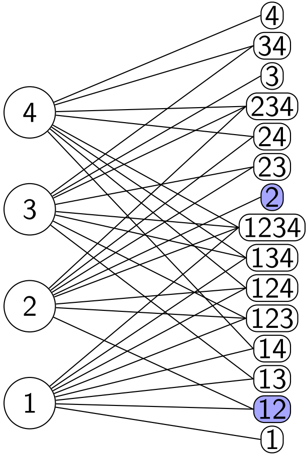

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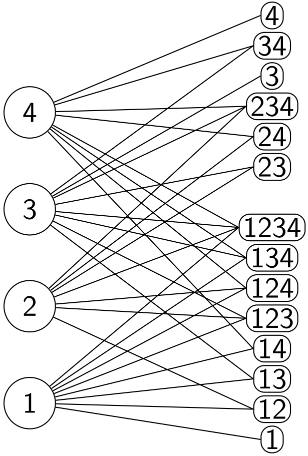

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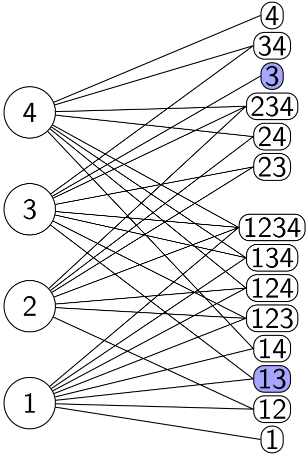

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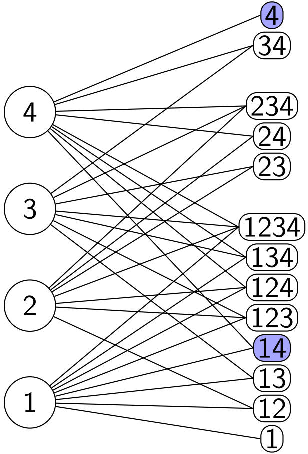

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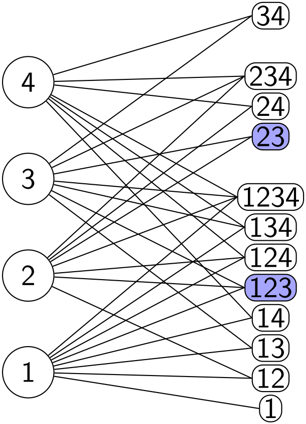

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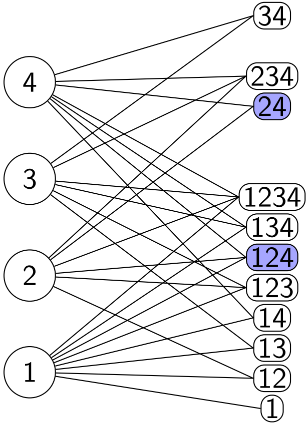

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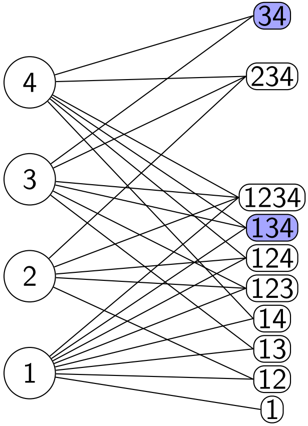

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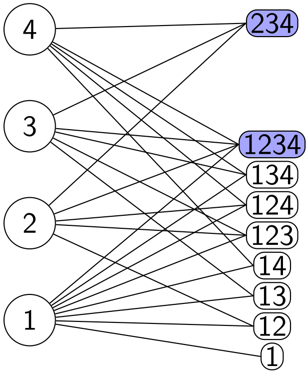

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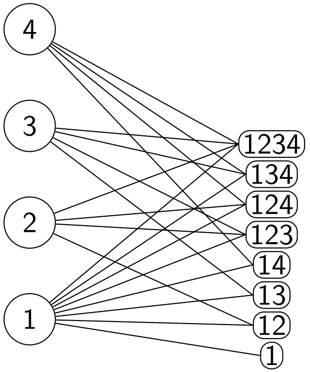
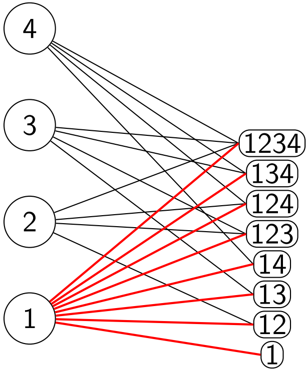


This complicated graph passes the test

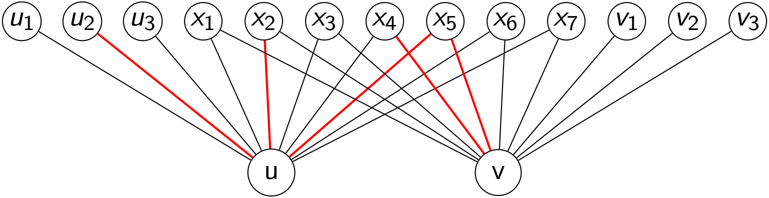
Cograph generalization

Iteratively identify **near** twins and **keep the error degree small**



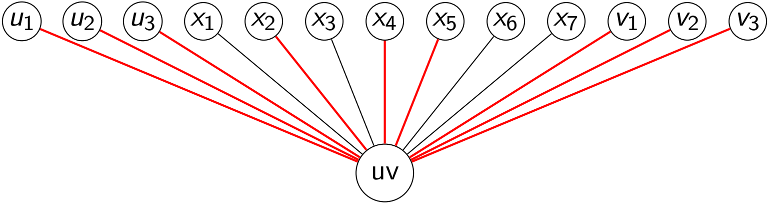
It would not with that further restriction

Contraction and trigraph



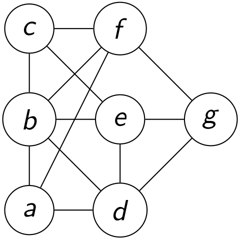
Trigraph: non-edges, edges, and red edges (error)

Contraction and trigraph



edges to $N(u) \triangle N(v)$ turn red, for $N(u) \cap N(v)$ red is absorbing

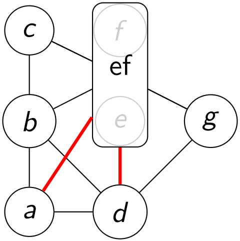
Contraction sequence and twin-width



Maximum red degree $= 0$

overall maximum red degree = 0

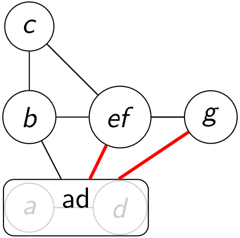
Contraction sequence and twin-width



Maximum red degree = 2

overall maximum red degree = 2

Contraction sequence and twin-width



Maximum red degree = 2

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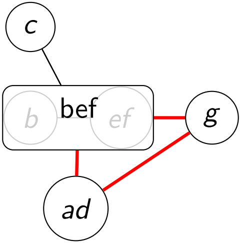

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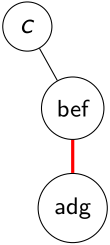
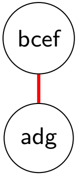


Maximum red degree = 1
overall maximum red degree = 2

Contraction sequence and twin-width



Maximum red degree = 1

overall maximum red degree = 2

Contraction sequence and twin-width

Maximum red degree = 0

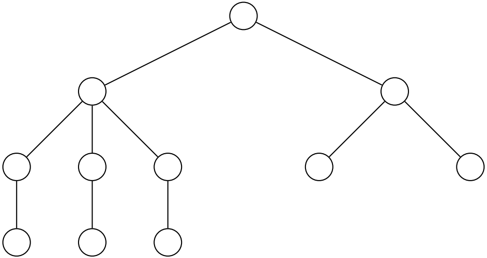
overall maximum red degree = 2

Contraction sequence and twin-width

Sequence of 2-contractions or 2-sequence, twin-width at most 2

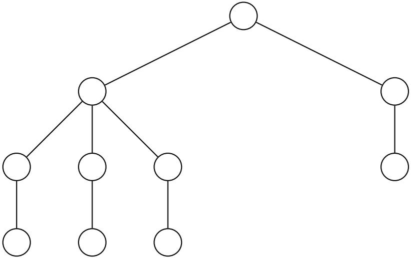
Maximum red degree = 0
overall maximum red degree = 2

Graphs with bounded twin-width – trees



If possible, contract two twin leaves

Graphs with bounded twin-width – trees



If not, contract a deepest leaf with its parent

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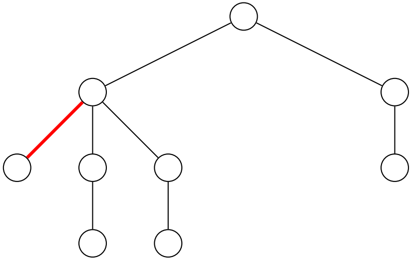

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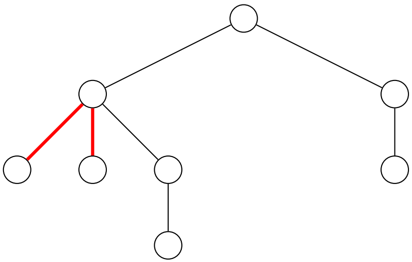

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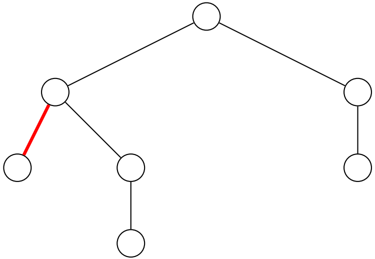

Cannot create a red degree-3 vertex

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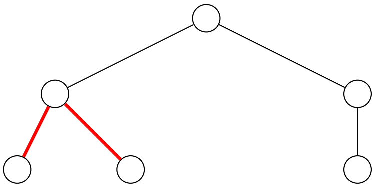

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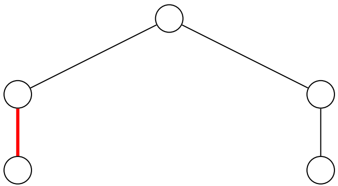

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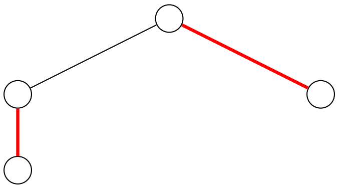

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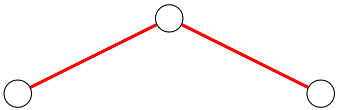

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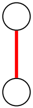
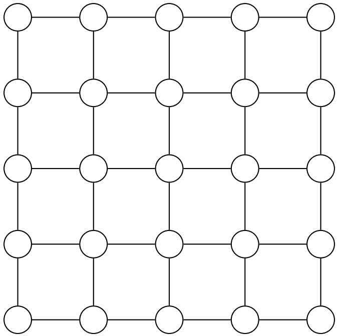


Cannot create a red degree-3 vertex

Graphs with bounded twin-width – trees

Generalization to bounded treewidth and even bounded rank-width

Graphs with bounded twin-width – grids



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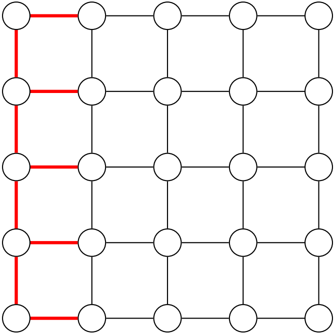

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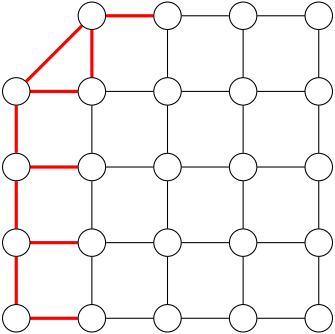

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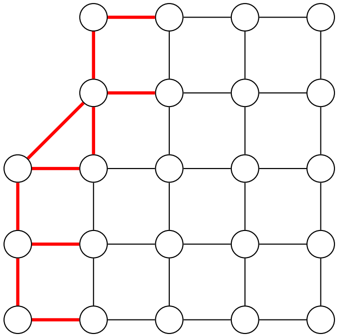

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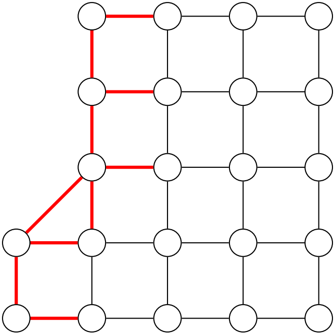

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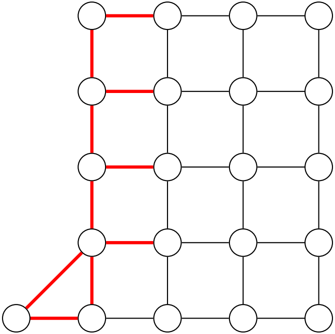

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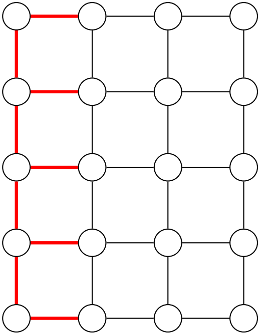
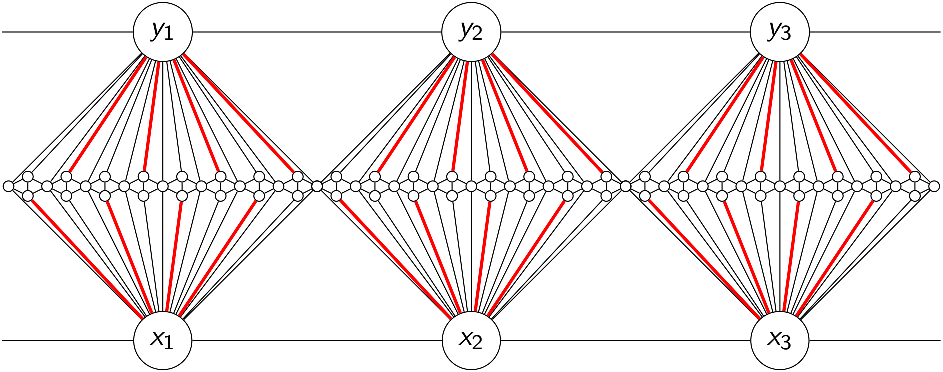


4-sequence for planar grids, 3d-sequence for d-dimensional grids

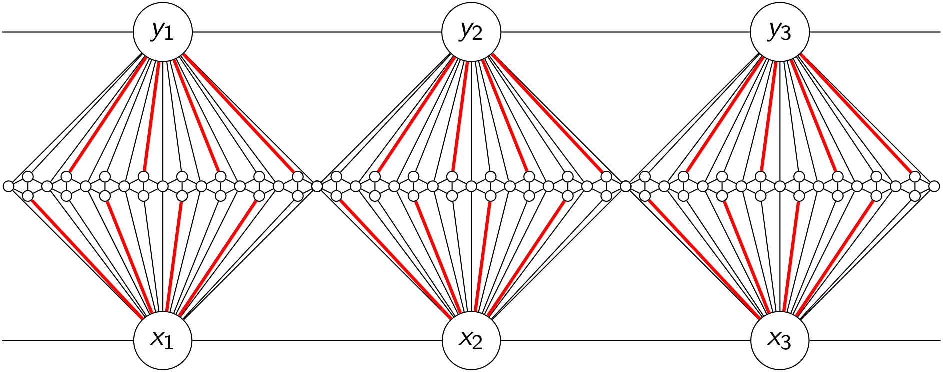
Graphs with bounded twin-width – planar graphs?

Graphs with bounded twin-width – planar graphs?



For every d, a planar trigraph without planar d-contraction

Graphs with bounded twin-width – planar graphs?



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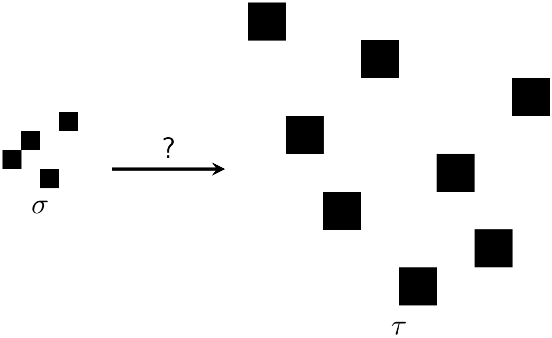
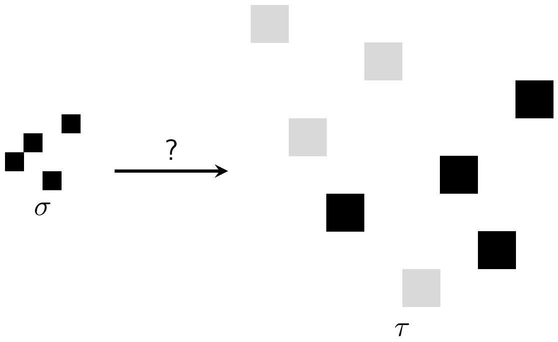
More powerfool tool needed

The origin: PERMUTATION PATTERN



The origin: Permutation Pattern



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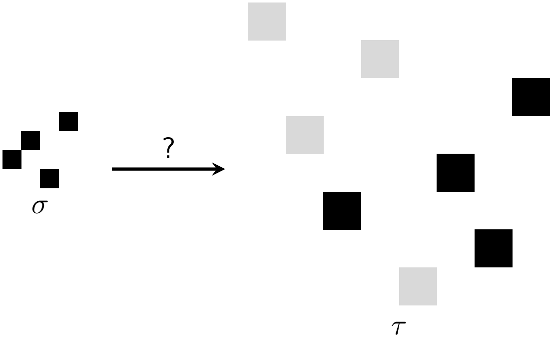

Theorem (Guillemot, Marx '14)

PERMUTATION PATTERN *can be solved in time* $2^{|\sigma|^2}|\tau|$.

Guillemot and Marx's win-win algorithm

Theorem (Marcus, Tardos '04)

$\forall t$, $\exists c_t$ $\forall$ $n \times n$ 0,1-matrix with $\geqslant c_t n$ entries 1 has a t-grid minor.

4-grid minor
$$\left[\begin{array}{cc|cc|cc|cc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

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A) $\geqslant c_{|\sigma|} n$ entries 1 $\rightarrow$ YES from the $|\sigma|$-grid minor.
B) $< c_{|\sigma|} n$ entries 1 $\rightarrow$ merge of two "similar" rectangles

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If B) always happens $\rightarrow$ DP on this merge sequence

Our generalization to the dense case – mixed minor

Mixed zone: not horizontal nor vertical

$$\left[\begin{array}{cc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

3-mixed minor

Our generalization to the dense case – mixed minor

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$$\left[\begin{array}{cc|ccc|ccc} 1&1&1&1&1&1&1&0\\ 0&1&1&0&0&1&0&1\\ \hline 0&0&0&0&0&0&0&1\\ 0&1&0&0&1&0&1&0\\ \hline 1&0&0&1&1&0&1&0\\ 0&1&1&1&1&1&0&0\\ 1&0&1&1&1&0&0&1 \end{array}\right]$$

3-mixed minor

A matrix is said t-**mixed free** if it does not have a t-mixed minor

Grid minor theorem for twin-width

Theorem (B, Kim, Thomassé, Watrigant 20)

If $\exists\sigma$ *s.t.* $Adj_\sigma(G)$ *is t-mixed free, then* $tww(G) = 2^{2^{O(t)}}$.

Grid minor theorem for twin-width

Theorem (B, Kim, Thomassé, Watrigant 20)

If $\exists \sigma$ *s.t.* $Adj_\sigma(G)$ *is* t*-mixed free, then* $tww(G) = 2^{2^{O(t)}}$.

Now to bound the twin-width of a class $\mathcal{C}$:

1) Find a *good* vertex-ordering procedure

2) Argue that, in this order, a t-mixed minor would conflict with $\mathcal{C}$

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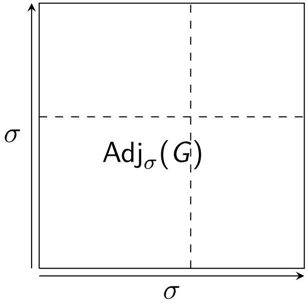

Cutting after the $t/2$-th division of the t-grid minor

Grid minor theorem for twin-width

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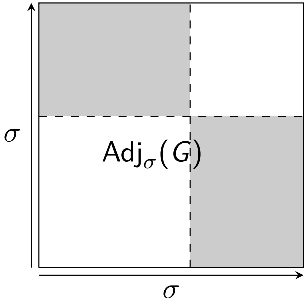
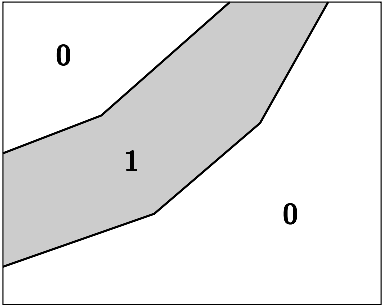


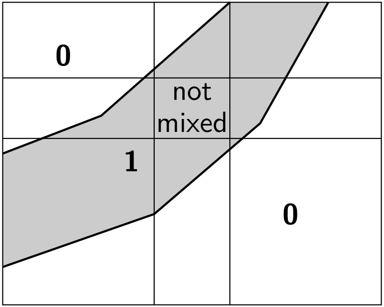
One of the shaded areas contains a $t/2$-grid minor on disjoint sets

Bounded twin-width – unit interval graphs



order by left endpoints

Bounded twin-width – unit interval graphs

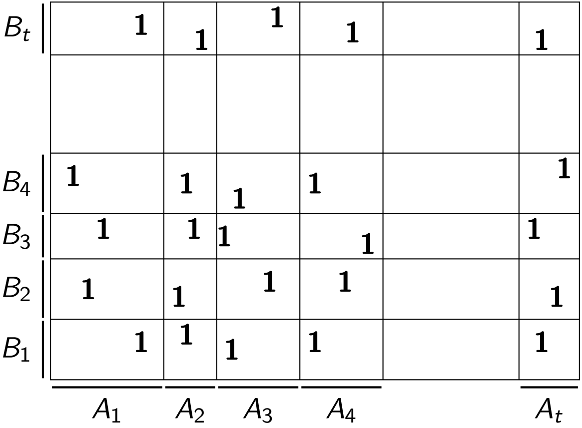


No 3-by-3 grid has all 9 cells crossed by two non-decreasing curves

Bounded twin-width – K_t-minor free graphs

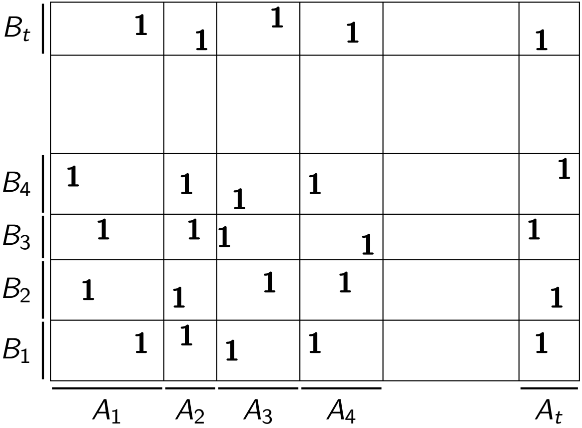
Given a hamiltonian path, we would just use this order

Bounded twin-width – K_t-minor free graphs



Contracting the $2t$ subpaths yields a $K_{t,t}$-minor, hence a K_t-minor

Bounded twin-width – K_t-minor free graphs



Instead we use a specially crafted lex-DFS discovery order

Theorem

The following classes have bounded twin-width, and $O(1)$-sequences can be computed in polynomial time.

- *Bounded rank-width, and even, boolean-width graphs,*
- *every hereditary proper subclass of permutation graphs,*
- *posets of bounded antichain size (seen as digraphs),*
- *unit interval graphs,*
- K_t*-minor free graphs,*
- *map graphs,*
- *subgraphs of* d*-dimensional grids,*
- K_t*-free unit* d*-dimensional ball graphs,*
- $\Omega(\log n)$*-subdivisions of all the* n*-vertex graphs,*
- *cubic expanders defined by iterative random 2-lifts from* K_4*,*
- *strong products of two bounded twin-width classes, one with bounded degree, etc.*

Theorem

The following classes have bounded twin-width, and $O(1)$-sequences can be computed in polynomial time.

- *Bounded rank-width, and even, boolean-width graphs,*
- *every hereditary proper subclass of permutation graphs,*
- *posets of bounded antichain size (seen as digraphs),*
- *unit interval graphs,*
- *K_t-minor free graphs,*
- *map graphs,*
- *subgraphs of d-dimensional grids,*
- *K_t-free unit d-dimensional ball graphs,*
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Can we solve problems faster, given an $O(1)$-sequence?

Example of k-Independent Set

d-sequence: $G = G_n, G_{n-1}, \ldots, G_2, G_1 = K_1$

Algorithm: **Compute by dynamic programming a best partial solution in each red connected subgraph of size at most k.**

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In G_1: If solution of size at least k, global solution.

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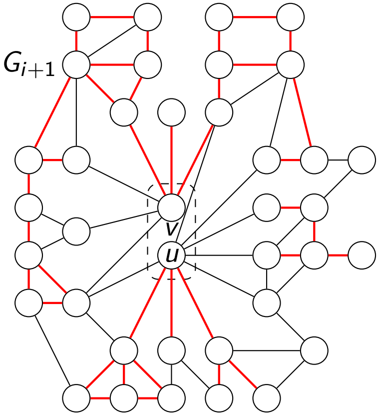
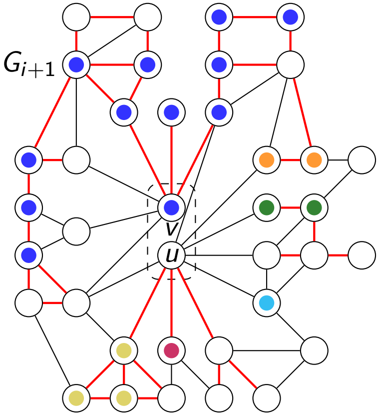
How to go from the partial solutions of G_{i+1} to those of G_i?



Best partial solution inhabiting •?



3 unions of $\leqslant d+2$ red connected subgraphs to consider in G_{i+1} with u, or v, or both

Other (almost) single-exponential parameterized algorithms

Theorem

Given a d-sequence $G = G_n, \ldots, G_1 = K_1$,

- k-Independent Set,
- k-Clique,
- (r, k)-Scattered Set,
- k-Dominating Set, *and*
- (r, k)-Dominating Set

can be solved in time $2^{O_d(k)} n$,
whereas Subgraph Isomorphism *and* Induced Subgraph Isomorphism *can be solved in time* $2^{O_d(k \log k)} n$.

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A more general FPT algorithm?

First-order model checking on graphs

GRAPH FO MODEL CHECKING **Parameter:** $|\varphi|$
Input: A graph G and a first-order sentence $\varphi \in FO(\{E_2, =_2\})$
Question: $G \models \varphi$?

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Example:

$$\varphi = \exists x_1 \exists x_2 \cdots \exists x_k \forall x \bigvee_{1 \leqslant i \leqslant k} x = x_i \vee \bigvee_{1 \leqslant i \leqslant k} E(x, x_i) \vee E(x_i, x)$$

$G \models \varphi$? $\Leftrightarrow$

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First-order model checking on graphs

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Example:

$$\varphi = \exists x_1 \exists x_2 \cdots \exists x_k \bigwedge_{1 \leqslant i < j \leqslant k} \neg(x_i = x_j) \wedge \neg E(x_i, x_j) \wedge \neg E(x_j, x_i)$$

$G \models \varphi$? $\Leftrightarrow$

First-order model checking on graphs

GRAPH FO MODEL CHECKING	**Parameter:** $\|\varphi\|$
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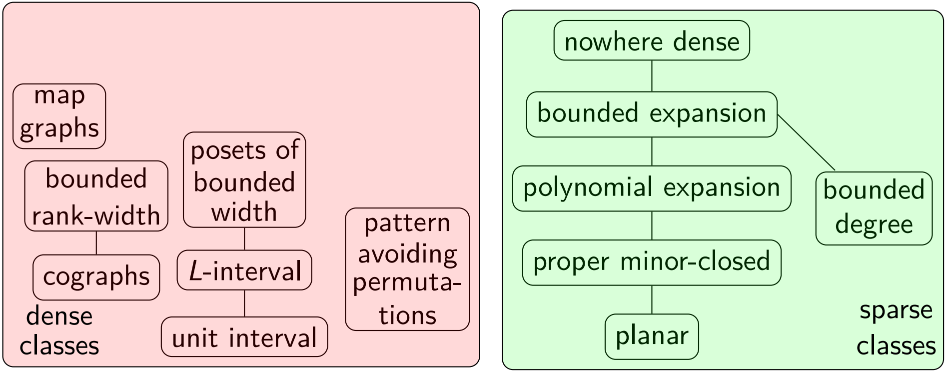
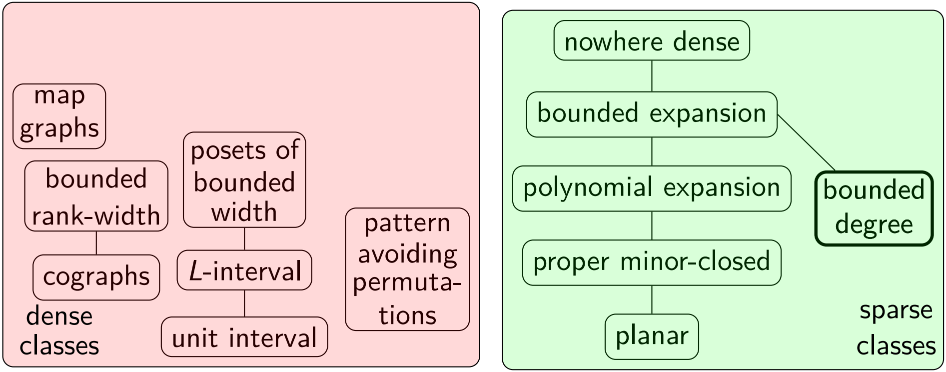
$G \models \varphi$? $\Leftrightarrow$ k-INDEPENDENT SET

Classes with known tractable FO model checking

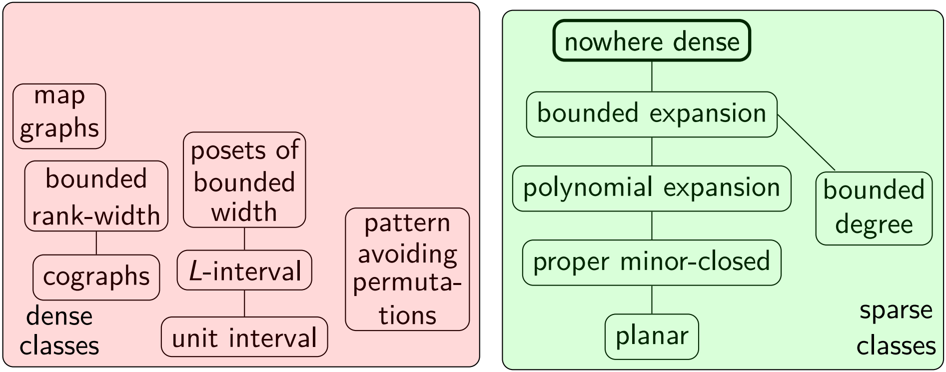


Classes with known tractable FO model checking



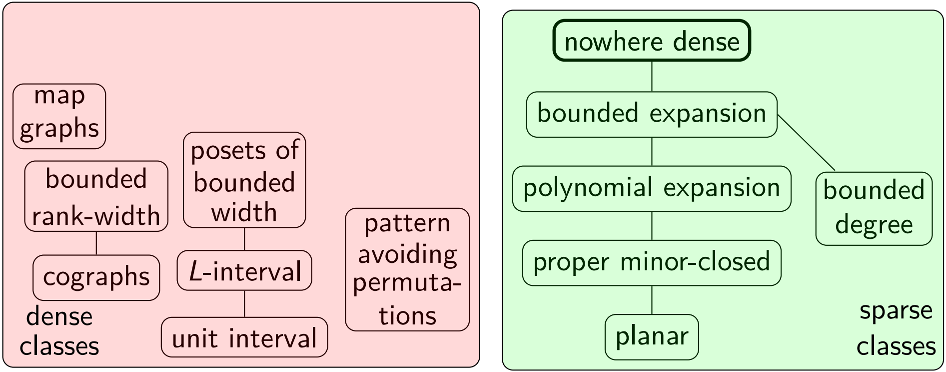
FO MODEL CHECKING solvable in $f(|\varphi|)n$ on bounded-degree graphs [Seese '96]

Classes with known tractable FO model checking



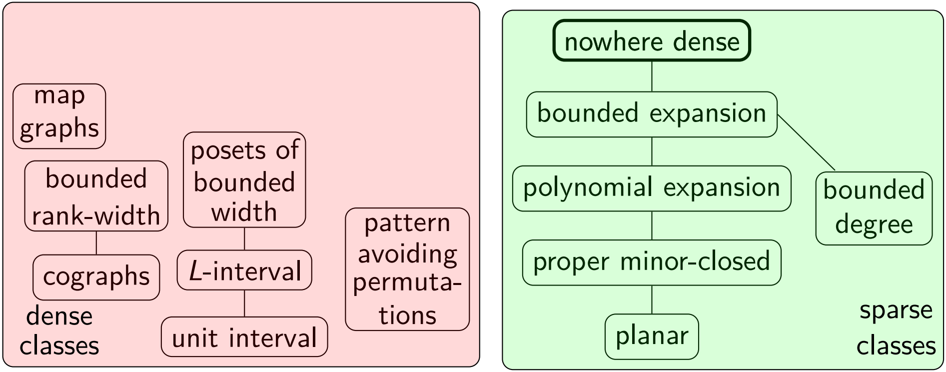
FO MODEL CHECKING solvable in $f(|\varphi|)n^{1+\varepsilon}$ on any nowhere dense class [Grohe, Kreutzer, Siebertz '14]

Classes with known tractable FO model checking



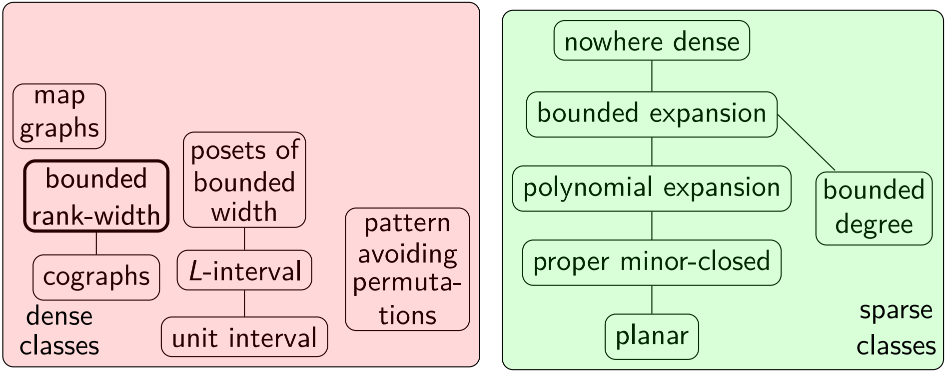
End of the story for the classes closed by taking subgraphs
tractable FO MODEL CHECKING $\Leftrightarrow$ nowhere dense

Classes with known tractable FO model checking



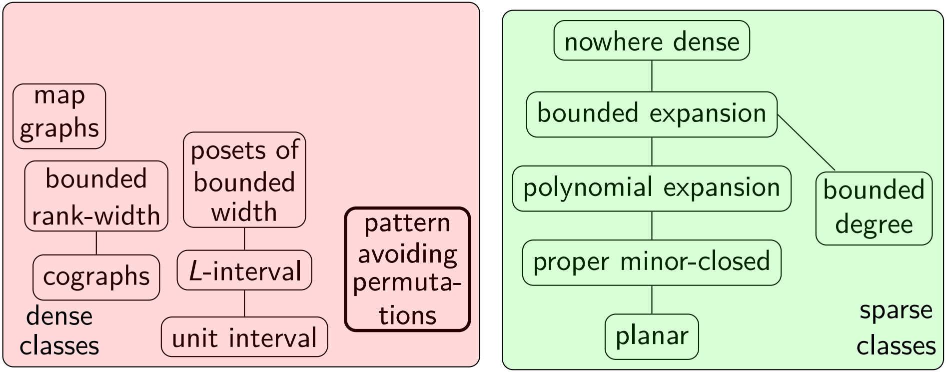
New program: dense (hence *not* subgraph-closed) classes

Classes with known tractable FO model checking



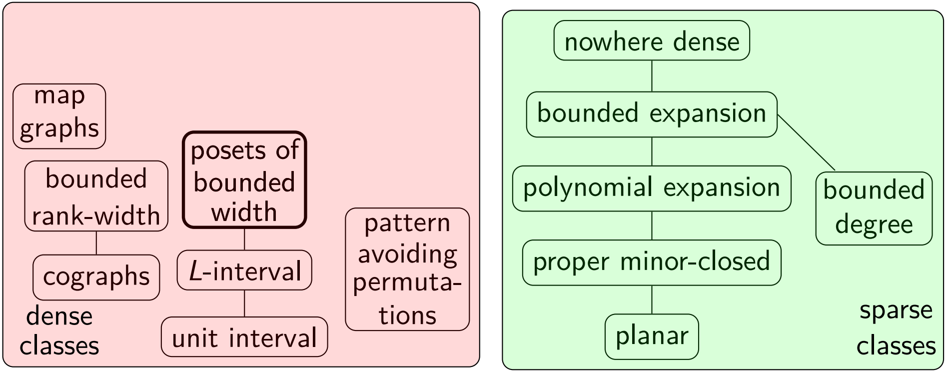
MSO_1 Model Checking solvable in $f(|\varphi|, w)n$ on graphs of rank-width w
[Courcelle, Makowsky, Rotics '00]

Classes with known tractable FO model checking



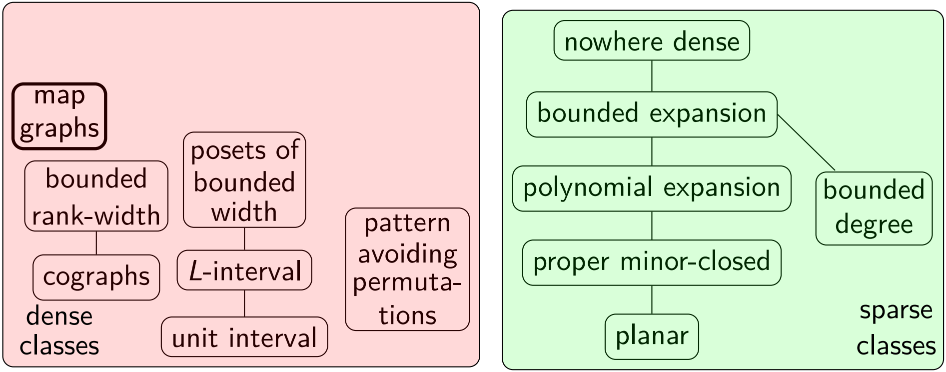
Is σ a subpermutation of τ? solvable in $f(|\sigma|)|\tau|$
[Guillemot, Marx '14]

Classes with known tractable FO model checking



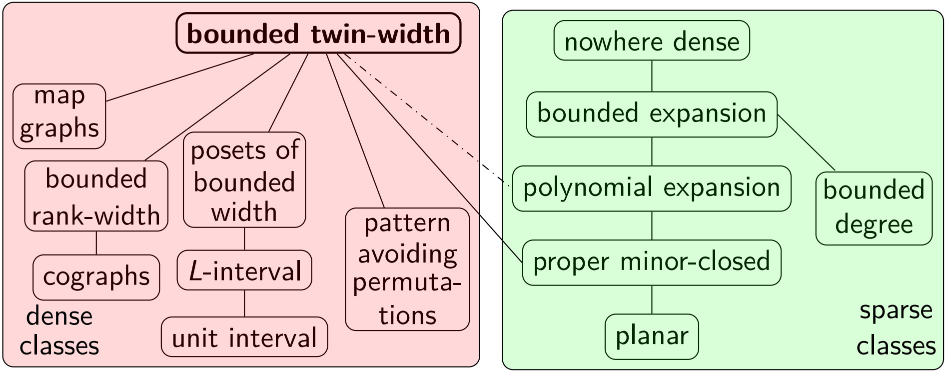
FO MODEL CHECKING solvable in $f(|\varphi|, w)n^2$ on posets of width w
[GHLOORS '15]

Classes with known tractable FO model checking



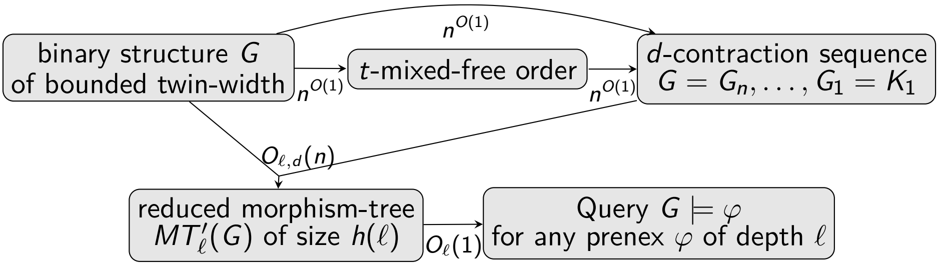
FO Model Checking solvable in $f(|\varphi|)n^{O(1)}$ on map graphs
[Eickmeyer, Kawarabayashi '17]

Classes with known tractable FO model checking



FO MODEL CHECKING solvable in $f(|\varphi|, d)n$ on graphs with a d-sequence [B, Kim, Thomassé, Watrigant '20+]

Workflow of the FO model checking algorithm



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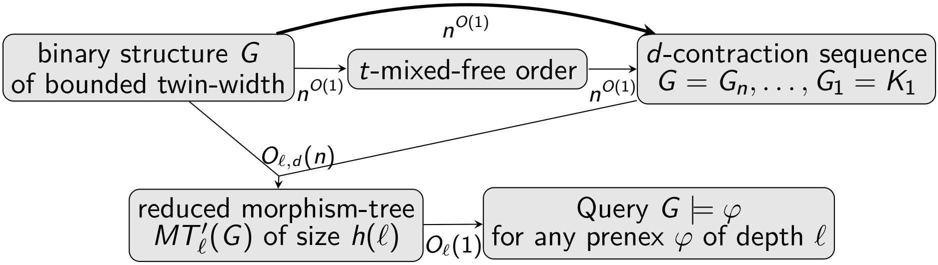
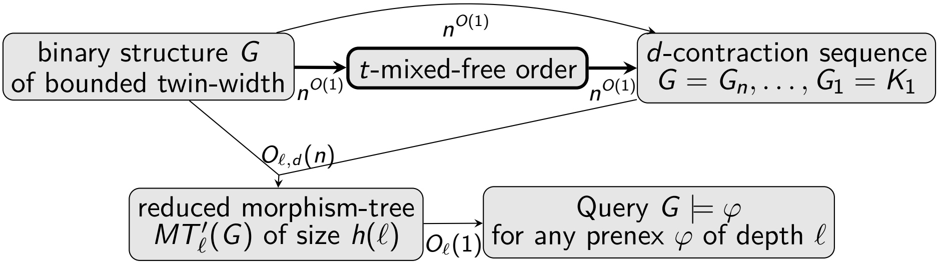


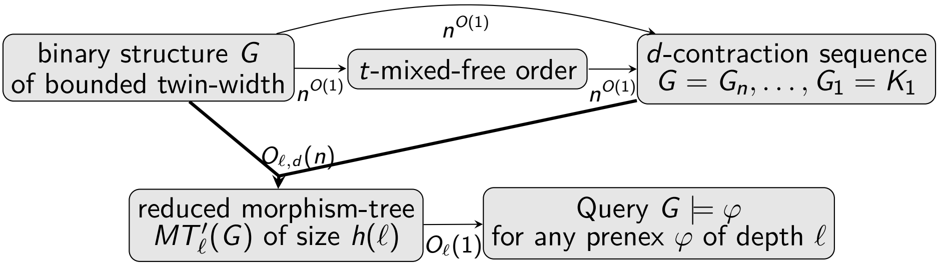
Direct examples: **trees**, bounded rank-width, **grids**, d-dimensional grids, unit interval graphs, K_t-free unit ball graphs

Workflow of the FO model checking algorithm



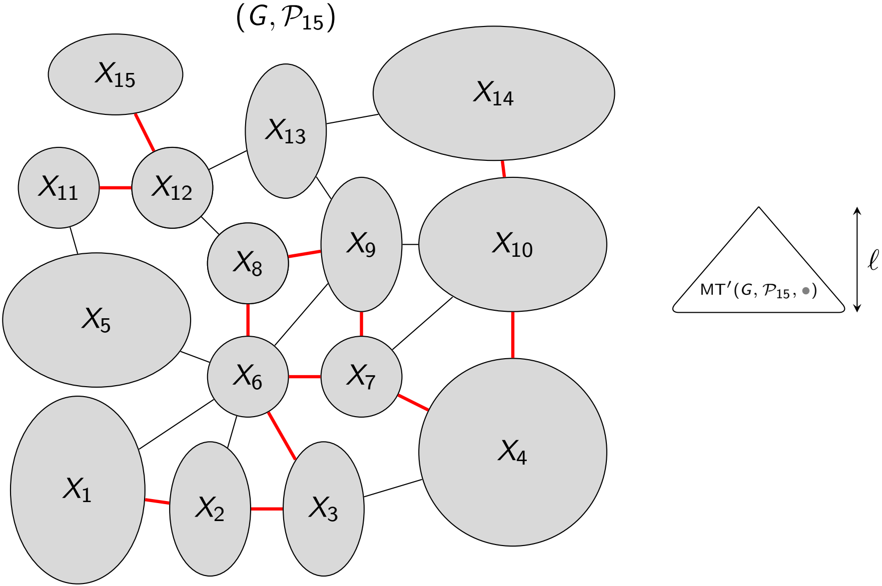
Detour via mixed minor for: pattern-avoiding permutations, **bounded width posets**, K_t-minor free graphs

Workflow of the FO model checking algorithm



Let us see a snapshot of the FO model checking

DP for FO model checking with d-sequence



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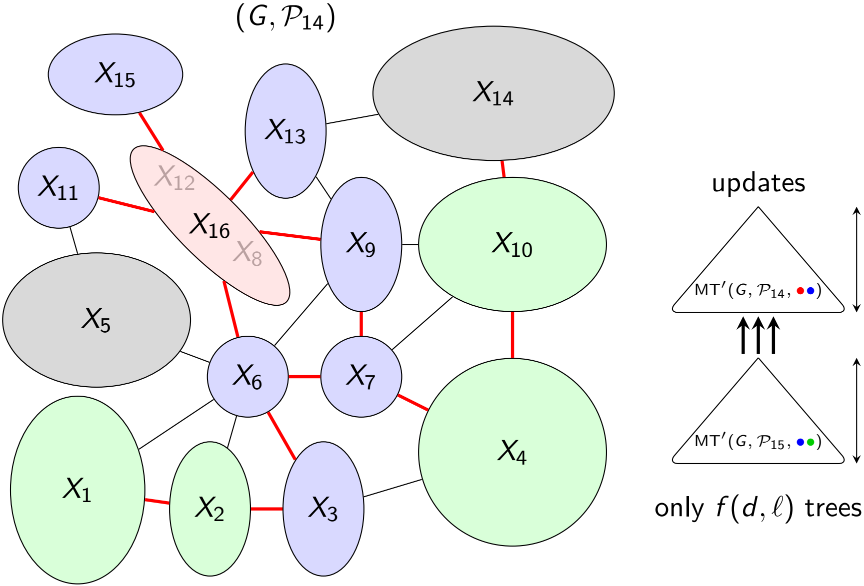

Small classes

Small: class with at most $n!c^n$ labeled graphs on $[n]$.

Theorem (B, Geniet, Kim, Thomassé, Watrigant 20+)

Bounded twin-width classes are small.

Unifies and extends the same result for:
σ-free permutations [Marcus, Tardos '04]
K_t-minor free graphs [Norine, Seymour, Thomas, Wollan '06]

Small classes

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Theorem (B, Geniet, Kim, Thomassé, Watrigant 20+)

Bounded twin-width classes are small.

Subcubic graphs, interval graphs, triangle-free unit segment graphs have **unbounded** twin-width

Small classes

Small: class with at most $n!c^n$ labeled graphs on $[n]$.

Theorem (B, Geniet, Kim, Thomassé, Watrigant 20+)

Bounded twin-width classes are small.

Is the converse true for hereditary classes?

Conjecture (small conjecture)

A hereditary class has bounded twin-width if and only if it is small.

Future directions

Obvious questions:
Algorithm to compute/approximate twin-width in general
Fully classify classes with tractable FO model checking
Small conjecture, polynomial expansion

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Better approximation algorithms on bounded twin-width classes
Twin-width of Cayley graphs of finitely generated groups
⋮

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On arxiv
Twin-width I: tractable FO model checking [BKTW '20]
Twin-width II: small classes [BGKTW '20]
Twin-width III: Max Independent Set and Coloring [BGKTW '20]