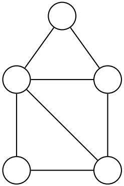
Polyspace slightly superexponential parameterized algorithm for Subgraph Isomorphism in proper-minor closed classes

Algorithmic application in Pilipczuk and Siebertz's paper on *p*-centered coloring

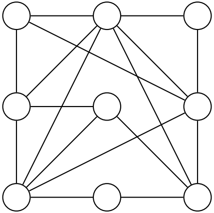
Édouard Bonnet

Virtual Meeting on Graph Theory, May 27th, 2020

Subgraph Isomorphism



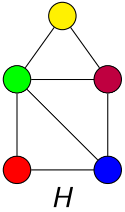
H

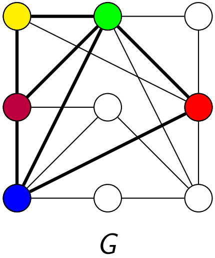


G

Is *H* a subgraph of *G*?

Subgraph Isomorphism





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Complexity of Subgraph Isomorphism

"Is H in G?" generalizes k-Clique

NP-complete, W[1]-complete parameterized by $p := |V(H)|$

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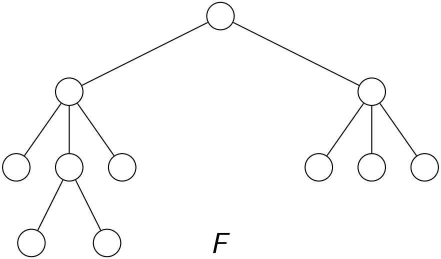
Theorem (Pilipczuk, Siebertz '19)

Subgraph Isomorphism *can be solved in time* $2^{O(p \log p)}n^{O(1)}$ *and* ***polynomial space****, when* G *is* K_t*-minor free.*

Treedepth

Treedepth of G**:** smallest height of a forest F such that G is a subgraph of the ancestor-descendant closure of F.



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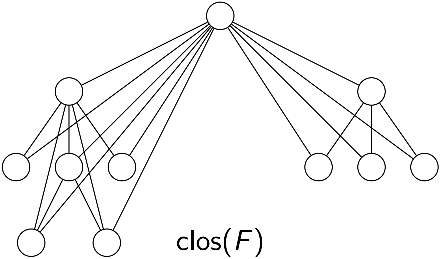

clos(F)

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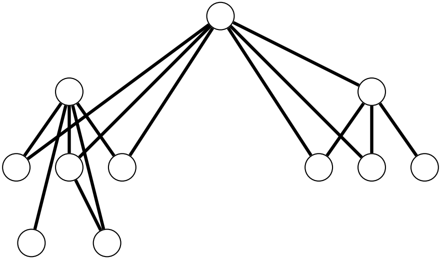

G has treedepth at most 4

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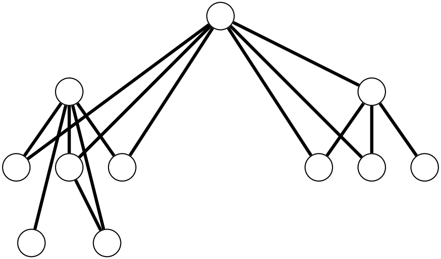

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The rest of the CC has at most $|X| - 1$ colors $\rightarrow$ recurse

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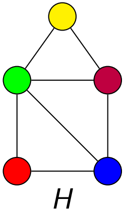
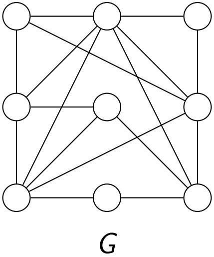
A solution cannot escape since it receives at most p colors

Color coding step





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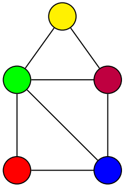

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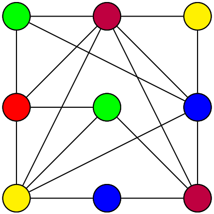

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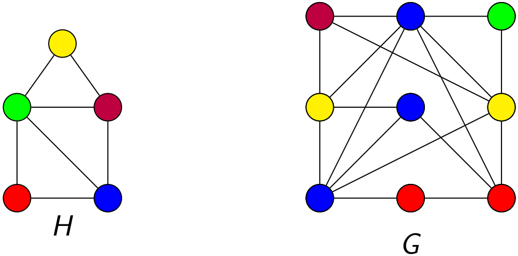

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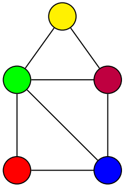

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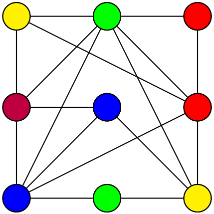

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Repeating this $100p^p$ times, well color a solution with prob. 0.999

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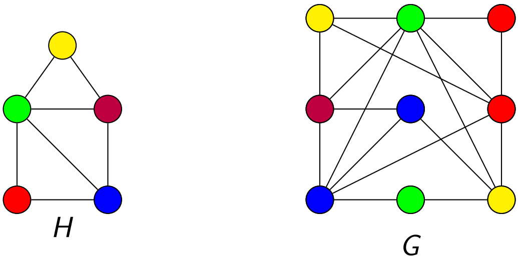

Repeating this $p^p n$ times, well color a solution a.a.s.

Derandomization

Theorem (Alon, Yuster, Zwick '95)

One can compute in polynomial-time a family $\mathcal{F}$ *of* $p^{O(1)} \log n$ *functions* $f : V(G) \to \{1, \ldots, p^2\}$ *such that for every set* $X \subseteq V(G)$ *of size* p *there exists* $f \in \mathcal{F}$ *injective on* X*.*

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Theorem (Schmidt, Siegal '90)

One can compute in polynomial-time a family $\mathcal{G}$ of $2^{O(p)}$ functions $f : \{1, \ldots, p^2\} \to \{1, \ldots, p\}$ such that for every set $X \subseteq \{1, \ldots, p^2\}$ of size p there exists $g \in \mathcal{G}$ injective on X.

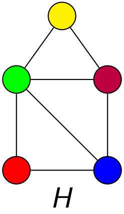
$\mathcal{F}' = \{\sigma \circ g \circ f \mid f \in \mathcal{F} \text{ and } g \in \mathcal{G} \text{ and } \sigma \in S_p\}$
$|\mathcal{F}'| = p! \cdot 2^{O(p)} \cdot p^{O(1)} \log n = 2^{O(p \log p)} \log n$

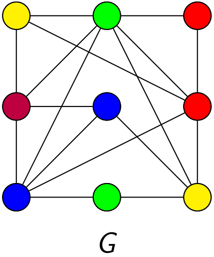
Colored Subgraph Isomorphism on bounded treedepth graphs

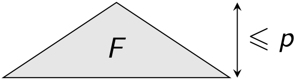
We are now left with proving:

Theorem (Pilipczuk, Siebertz '19)

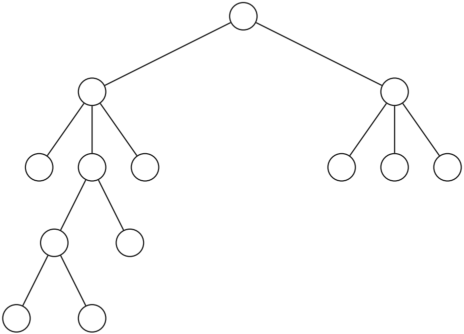
Colored Subgraph Isomorphism *can be solved in time* $2^{O(p \log p)} n^{O(1)}$ *and polynomial space, when* G *is given with a treedepth decomposition of depth at most* p*.*







Some notations for the upcoming dynamic-programming



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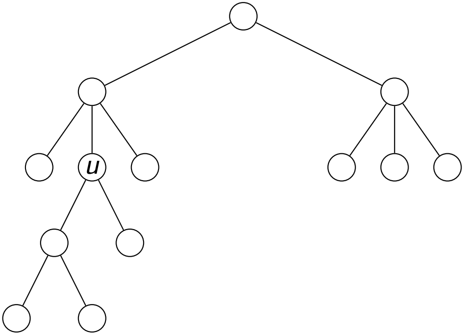

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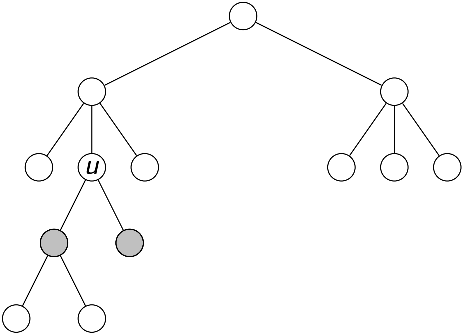

Chld(u): set of children of u

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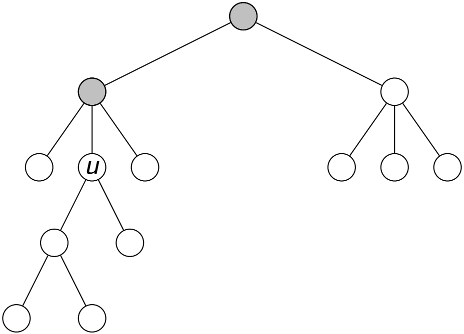

Tail(u): set of strict ancestors of u

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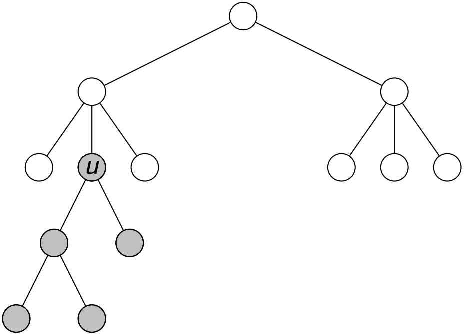
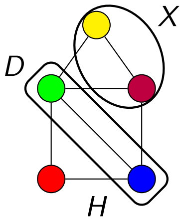


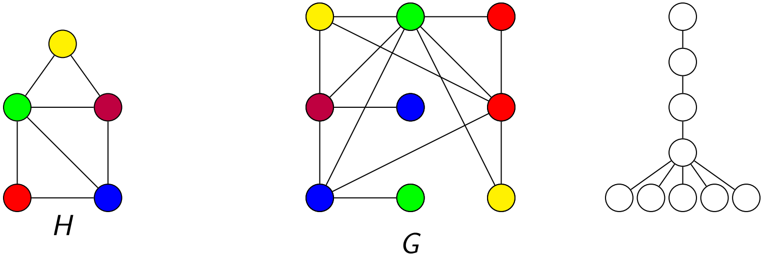
Desc(u): set of descendants of u, including u

Chunk



disjoint pair (X, D),
$H[X]$ connected, and
$N_H(X) \subseteq D$

Subproblems



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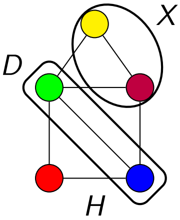
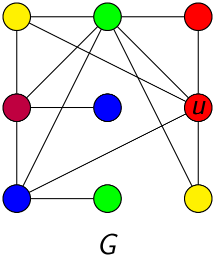
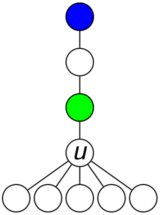






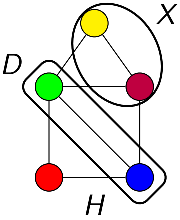
A tuple (u, X, D, γ)

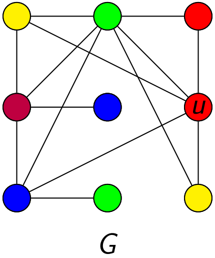
$u \in V(G)$

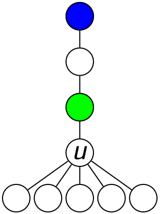
(X, D) is a chunk

$\gamma : D \to \text{Tail}(u)$ injective

Subproblems





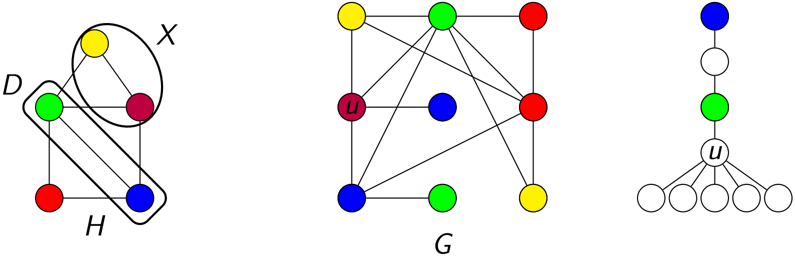


$\mathsf{Val}(u, X, D, \gamma) =$

Is there $\gamma' : X \to \mathsf{Desc}(u)$ such that

$\gamma \cup \gamma'$ is a (color-preserving) subgraph embedding?

Subproblems



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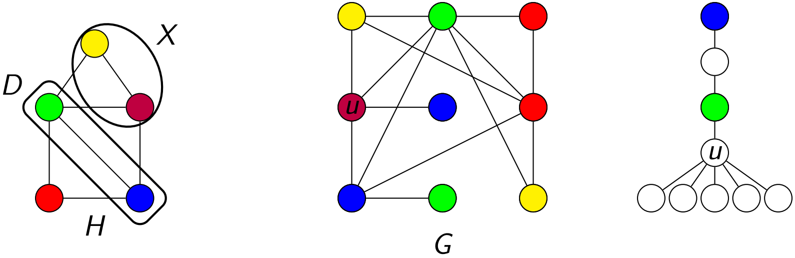

How many tuples (u, X, D, γ)?

$\leqslant n \cdot 3^p \cdot p^p = 2^{O(p \log p)} n$

Computing Val, u is a leaf of F

$\mathsf{Val}(u, \emptyset, D, \gamma) = [\gamma$ is a subgraph embedding$]$

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$\mathrm{Val}(u, \{w\}, D, \gamma) = [u$ is colored w and $\gamma \cup \{w \to u\}$ is a s.e.$]$

Computing Val, u is a *not* a leaf of F

If u has *not* a color of X:

$$\mathsf{Val}(u, X, D, \gamma) = \bigvee_{v \in \mathsf{Chld}(u)} \mathsf{Val}(v, X, D, \gamma)$$

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If u is colored $w \in X$:
$$\mathrm{Val}(u, X, D, \gamma) = \bigvee_{v \in \mathrm{Chld}(u)} \mathrm{Val}(v, X, D, \gamma) \vee$$
$$\bigwedge_{Z \in \mathrm{CC}(X - \{w\})} \bigvee_{v \in \mathrm{Chld}(u)} \mathrm{Val}(v, Z, D \cup \{w\}, \gamma \cup \{w \to u\})$$

Algorithm

Compute $\bigvee_{r \text{ root of } F} \mathsf{Val}(r, X, \emptyset, \emptyset)$ for every X CC of H

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If all positive answers $\rightarrow$ overall solution. Why?

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Disjointness. That was the point of color coding.

Complexity

Space: polynomial, calling stack bounded by treedepth

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Time: $2^{O(p \log p)} n^{O(1)}$ all recursive calls are different.
A non-root call defines a unique parent tuple.

Summary

p color classes of a p-centered coloring have treedepth p

Color coding for solution disjointness

Treedepth DP allows polyspace, as opposed to treewidth DP

An example of such an algorithm, notion of chunk

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Thank you for your attention!