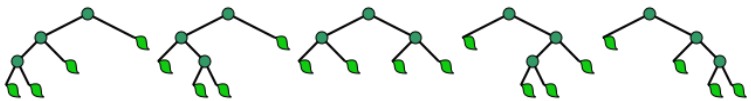
Introduction to Ramsey theory

Edwige Cyffers

Ramsey theory is a subfield of combinatorics

- Enumerative combinatorics



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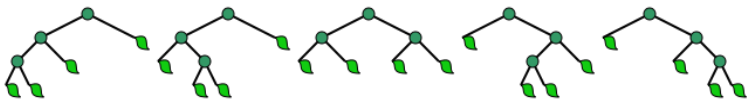
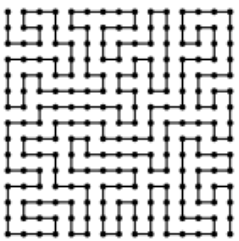


- Probabilistic combinatorics



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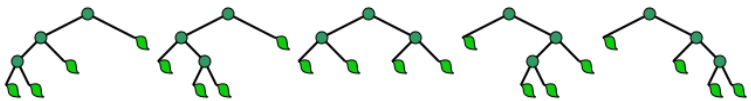

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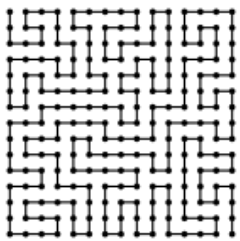
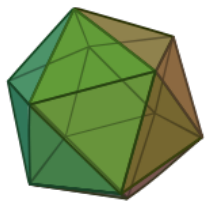


- Geometric combinatorics



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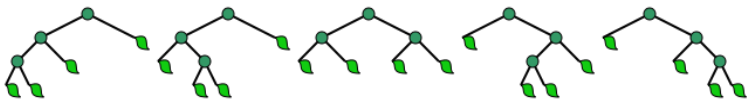
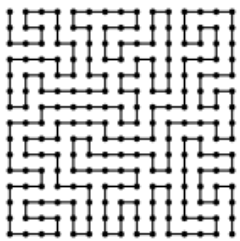


- Probabilistic combinatorics



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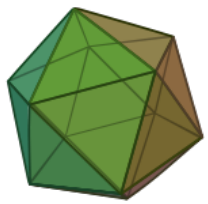


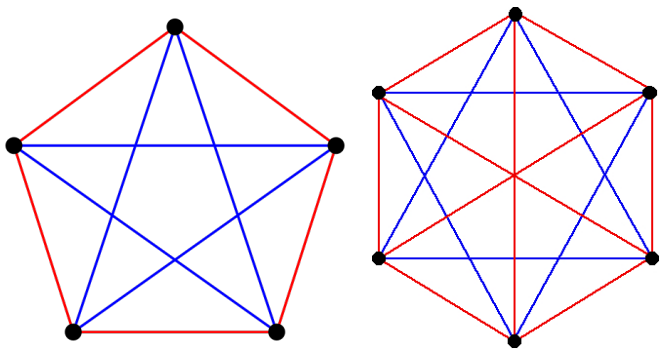
- Extremal combinatorics

1. Ramsey theory studies the emergence of order
2. Results come from different approaches
3. Ramsey has applications in various fields

Example

Emergence of order

How large must a structure be to guarantee that a particular property will hold?



We can generalize the result

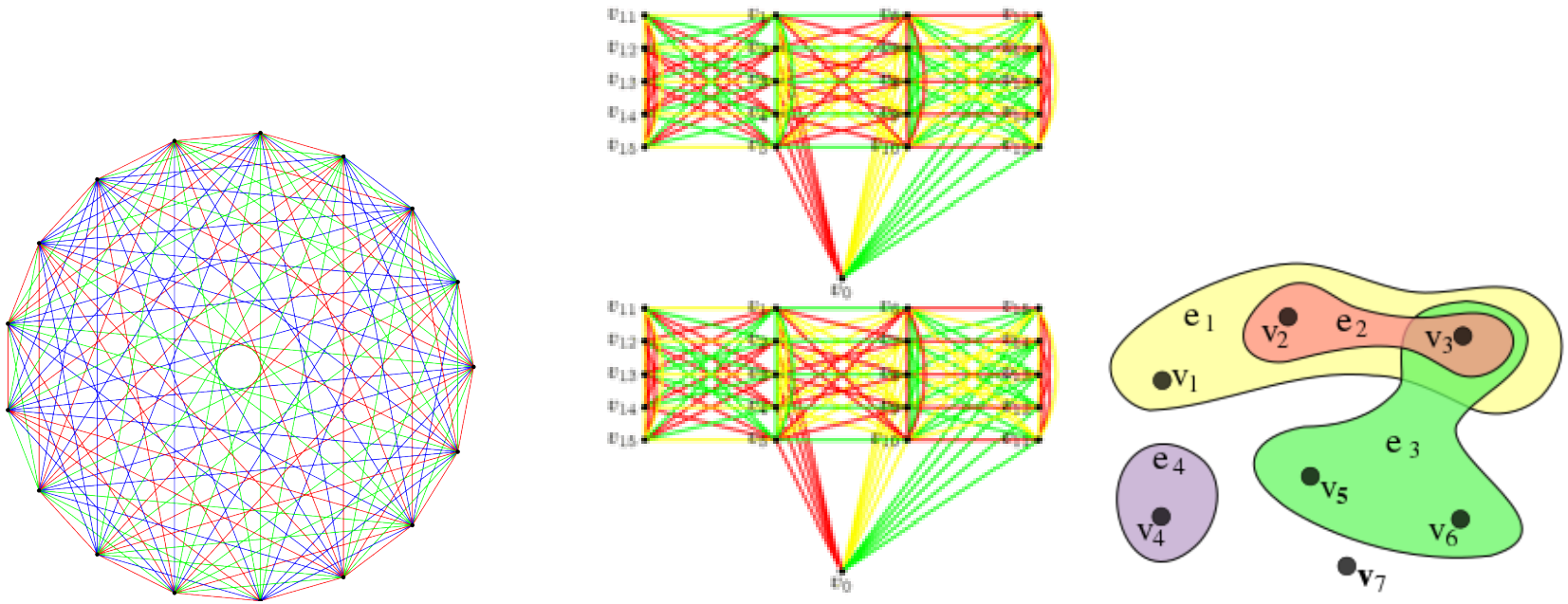


Figure: bigger subsets, more colors and hypergraphs

Ramsey theorem is a theorem of existence

Definition

Ramsey number $R(k, c, m)$ is the minimum integer for which, for each coloration with c colors of a k hypergraph with $R(k, c, m)$ vertices, there exists a complete monochromatic subgraph of m vertices.

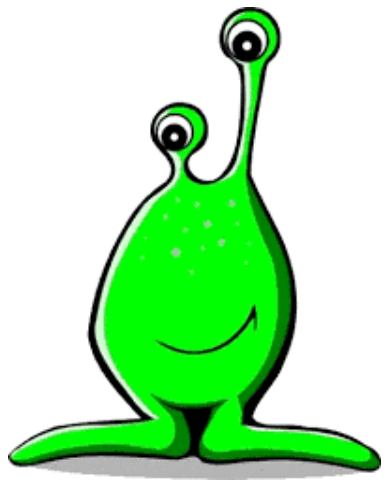
Theorem

For any integers k, r and m, $R(k, r, m)$ is finite.

Ramsey theorem is a theorem of existence

by Joel Spencer

Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of R(2,2,5) or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for R(2,2, 6). In that case, he believes, we should attempt to destroy the aliens.



The probabilistic method is a powerful tool

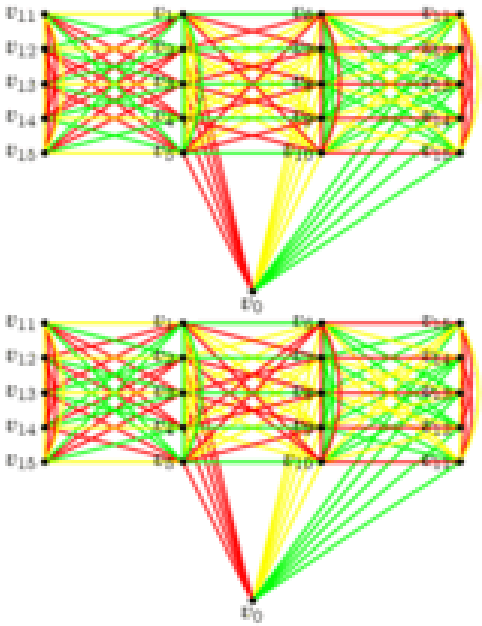


Figure: Very few exceptions

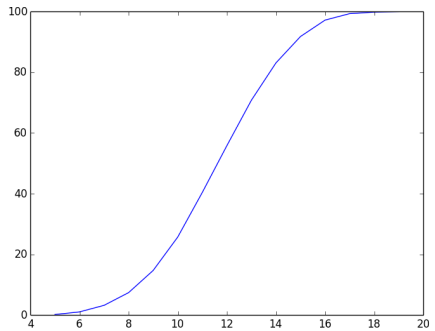


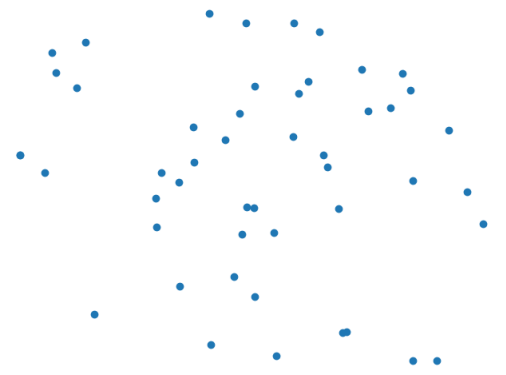
Figure: Proportion of counterexamples for a monochromatique pentagon

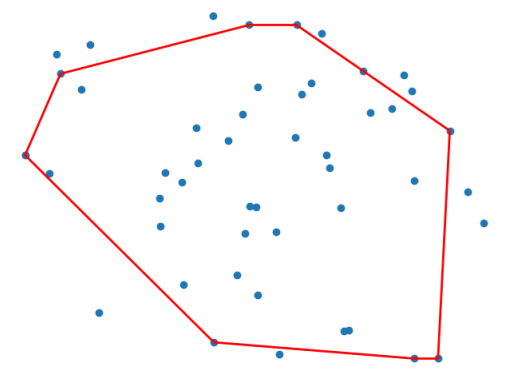
Topology can be used

1	2	3	4	5	6	7	8	9	10
1	**2**	3	4	5	6	7	8	9	10
1	2	**3**	4	5	6	7	8	9	10
1	2	3	**4**	5	6	7	8	9	10
1	2	3	4	**5**	6	7	8	9	10
1	2	3	4	5	**6**	7	8	9	10
1	2	3	4	5	6	**7**	8	9	10

Figure: Cantor's diagonal argument

Ramsey theorem has applications in Geometry

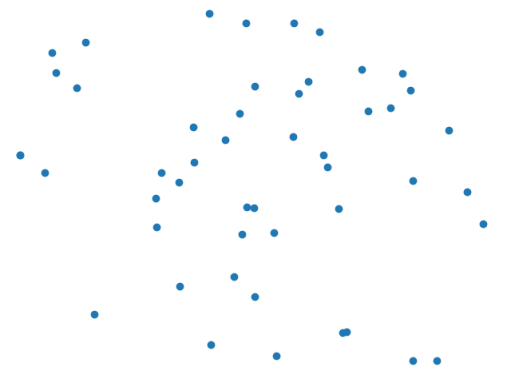


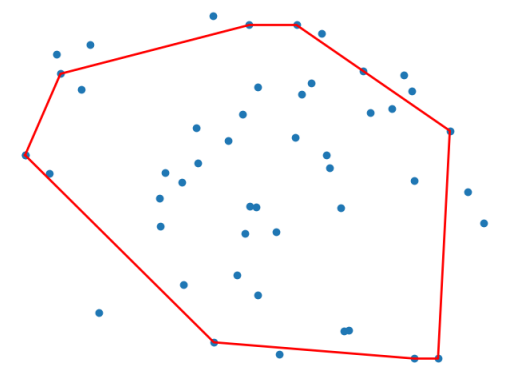


Ramsey theorem has applications in Geometry

The happy ending problem

For all n, there exists $\mathcal{N}(n)$ such as for any set of $\mathcal{N}(n)$ points, there is a subset of n points which form a convex n-gon.





Ramsey theorem has applications in Arithmetic

Schur theorem on Fermat equations

Let n be an integer. There exists p a prime such as

$$x^n + y^n \equiv z^n \, [p]$$

has non-trivial solutions.