Exploiting multiple resolutions to accelerate inverse problems in imaging

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Mathematics and Image Analysis (MIA'25)

Collaboration


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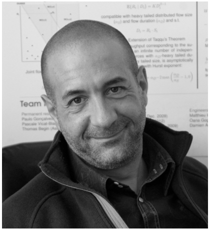
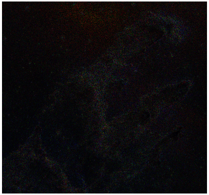


Paulo GONCALVES

The context: image restoration problems

$$\widehat{x} \in \arg\min_{x} \frac{1}{2}\|Ax - z\|_2^2 + \lambda \|Lx\|_\star$$

$\overline{x}$



z

$\widehat{x}$

The context: image restoration problems

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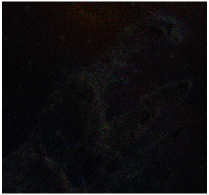


z

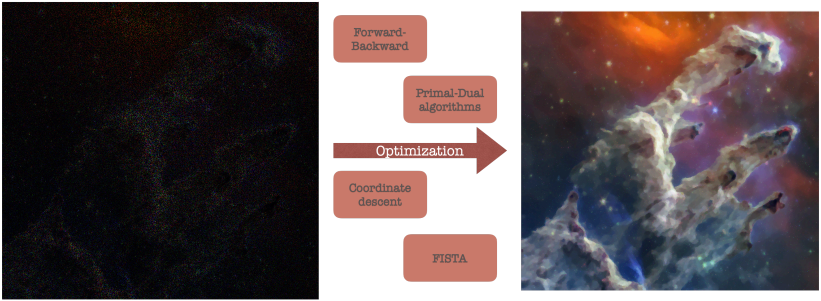
$\widehat{x}$

More generally

$$\min_x \; f(x) + g(x)$$

- f differentiable with Lipschitz gradient
- g possibly non-smooth and often non-proximable

Focus on large-scale problems



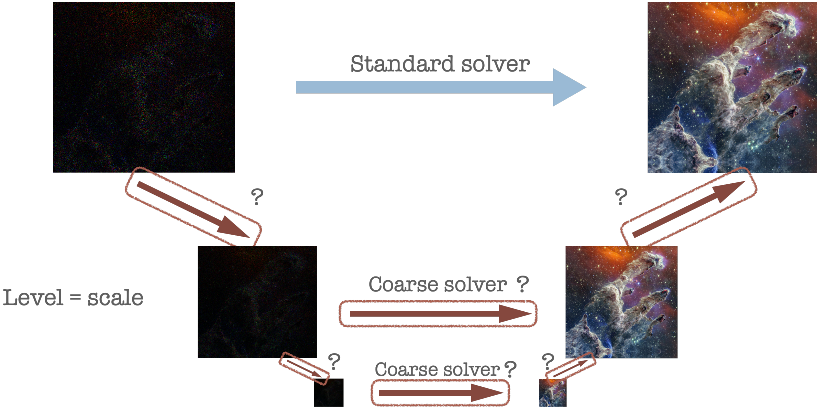
Prohibitive cost for large-scale problems: how to reduce this cost?

Existing acceleration strategies

- **FISTA** [Beck & Teboulle, 2009] [Chambolle & Dossal, 2015],
- **Preconditioning** [Donatelli, 2019][Repetti et al., 2014],
- **Blocks methods** [Liu, 1996] [Chouzenoux et al., 2016] [Salzo, Villa 2022],

Alternative: Exploit the problem structure with a multiresolution strategy

The multilevel paradigm



The multilevel literature

CONTEXT	MULTILEVEL STRATEGY	CONVERGENCE GUARANTEES	PRACTICAL PERFORMANCE	REFERENCES
Linear PDEs	✓	✓	✓	[Hackbush 1985], [McCormick 1987], [Briggs 2000]
Smooth non-linear PDEs	✓	✓	✓	[Nash 2000], [Gratton 2008], [Calandra 2020]
Smooth imaging optimization	✓	✓	✗	[Javaherian 2017], [Plier 2021], [Fung 2020]
Non-smooth optimization	✓	✗ No convergence of the iterates	✗	[Parpas 2016], [Parpas 2017], [Ang 2023]
Non-smooth & non-proximable	✗	✗	✗	

Our contribution

IML FISTA: inexact multilevel FISTA

- A general multilevel algorithm with state-of-the-art convergence guarantees for image restoration that handles state-of-the-art non-proximable a-priori (TV, NLTV)
- Adaptation of IML FISTA to multiple image restoration contexts with state-of-the art practical performance
- Adaptation of IML FISTA to radio-interferometric imaging with state-of-the art practical performance

Our context

The problem:

$$\min_x F(x) := f(x) + \varphi(Lx)$$

The method: inexact FISTA [Aujol, Dossal, 2015]

$$x_{k+1} \approx_{\epsilon_k} \text{prox}_{\tau\varphi\circ L}(y_k - \tau\nabla f(y_k))$$
$$y_{k+1} = x_{k+1} + \alpha_k(x_{k+1} - x_k)$$

where $\alpha_k = \frac{t_k - 1}{t_{k+1}}$ and $t_k = \left(\frac{k-1+a}{a}\right)^d$.

Contribution: update y_k through a multilevel step

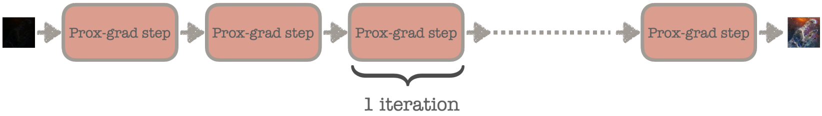
Outline

Outline

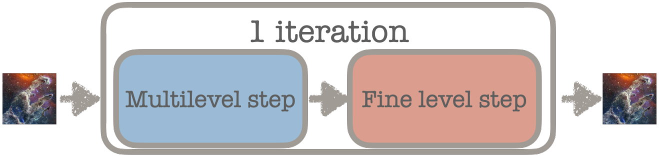
The multilevel step

- Classical proximal methods:

$$x_{k+1} = \text{prox}_{\tau\varphi\circ L}(x_k - \tau\nabla f(x_k))$$



- The multilevel paradigm:



Idea of the multilevel (ML) step

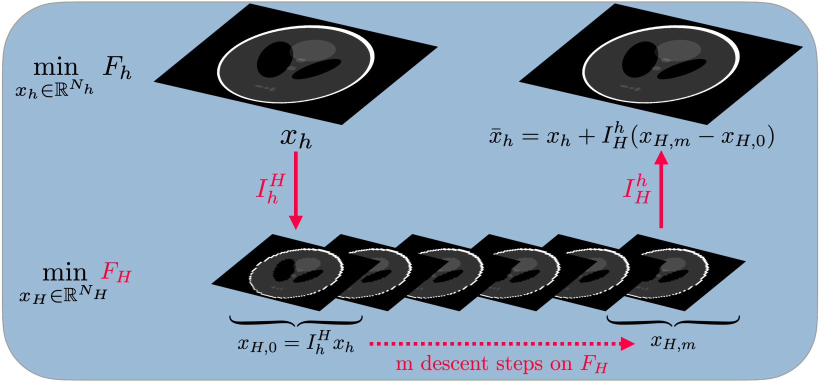
Exploit different resolutions of the problem and alternate iterations between fine and coarse levels.

Example: two-levels case - (h) fine level (H) coarse level

F_h 

F_H 

The multilevel step - two level case



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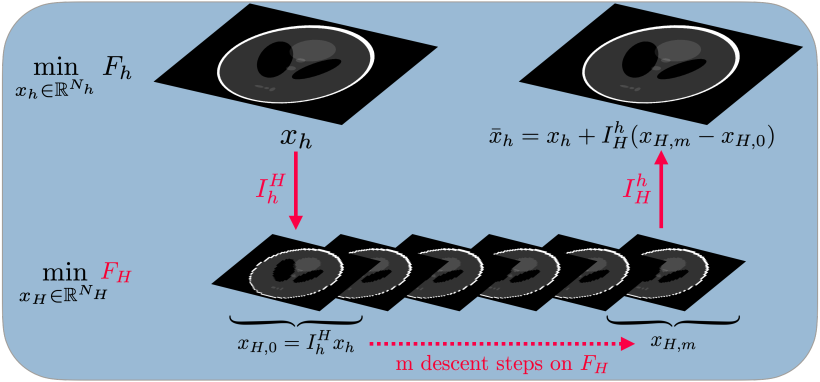
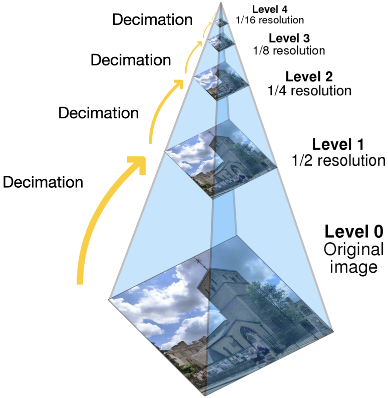
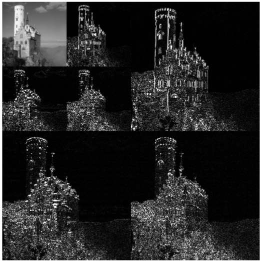


I_h^H, I_H^h, F_H?

Outline

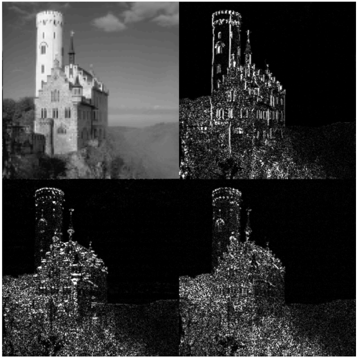
A hierarchy of images: I_h^H, I_H^h





Example: the wavelet transform

I_h^H



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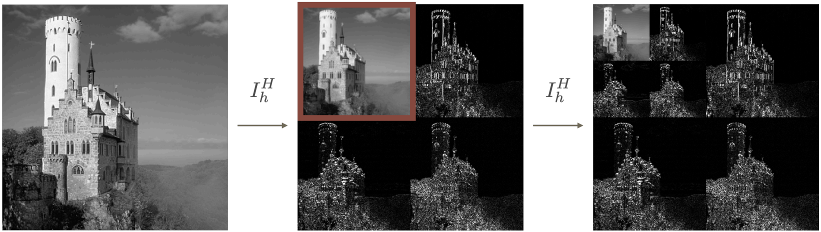

Example: the wavelet transform

$$I_H^h \longrightarrow$$

Coarse model definition F_H

$$F_h(x) = F(x) = \frac{1}{2}\|Ax - z\|_2^2 + \varphi(Lx)$$
$$F_H(x) = \frac{1}{2}\|A_H x_H - z\|_2^2 + \varphi(L_H x_H)$$

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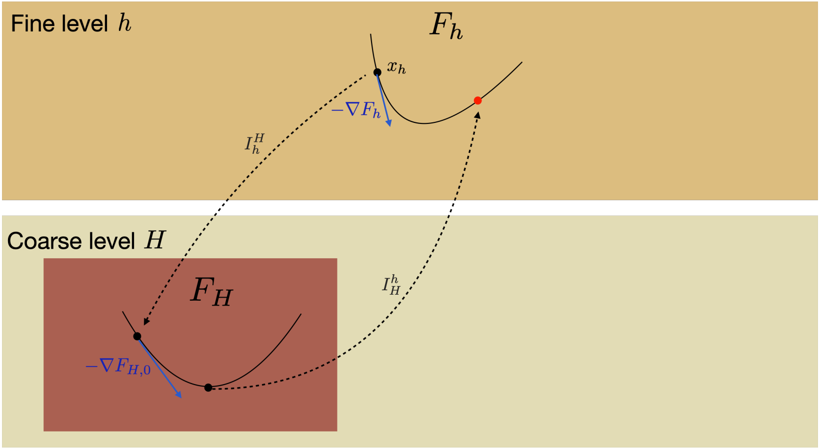
Is this model useful in minimizing F?

Design of F_H in smooth context: First order coherence



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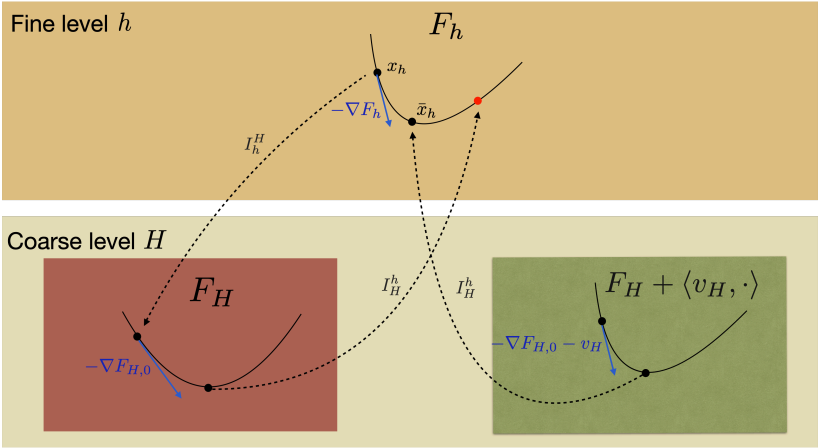

Coarse model definition

Assume $\varphi \circ L$ smooth. We define:

$$F(x) = \frac{1}{2}\|Ax - z\|_2^2 + \varphi(Lx)$$

$$F_H(x_H) = \underbrace{\frac{1}{2}\|A_H x_H - z\|_2^2 + \varphi(L_H x_H)}_{\tilde{F}_H} + < v_H, x_H >$$

where, given x_h iterate at fine level, we define:

$$v_H = I_h^H \nabla F(x_h) - \nabla \tilde{F}_H(I_h^H x_h)$$

This implies the first-order coherence:

$$I_h^H \nabla F(x_h) = \nabla F_H(I_h^H x_h)$$

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What to do in the non-smooth case?

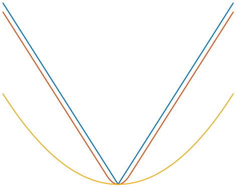
Smoothing

Nonsmooth case $\rightarrow$ smoothing!

The Moreau envelope:

$$^{\gamma}g = \inf_{y \in \mathcal{H}} g(y) + \frac{1}{2\gamma}\| \cdot - y\|^2$$

Illustration: Moreau envelope of l_1-norm for $\gamma = 0.1$ and $\gamma = 1$



Coarse model for the non-smooth case

Coarse model F_H for non-smooth functions

$$F_H = f_H + ({}^{\gamma_H}\varphi_H \circ \mathrm{L}_H) + \langle v_H, \cdot \rangle$$

where

$$\begin{aligned} v_H =& I_h^H \nabla^{\gamma_h} F(x_h) - \nabla F_H(I_h^H x_h) \\ =& \; I_h^H \left(\nabla f_h(x_h) + \nabla({}^{\gamma_h}\varphi_h \circ L_h)(x_h) \right) \\ & - \left(\nabla f_H(I_h^H x_h) + \nabla({}^{\gamma_H}\varphi_H \circ L_H)(I_h^H x_h) \right) \end{aligned}$$

Theoretical results

If $x_{H,m} - x_{H,0}$ is a descent direction for F_H, then

$$F_h(x_h + \bar{\tau} I_H^h(x_{m,0} - x_{H,0})) \leq F_h(x_h) + O(\gamma_h)$$

IML FISTA

- The steps:

$$x_{k+1} \approx_{\epsilon_{h,k}} \text{prox}_{\tau\varphi\circ\mathrm{L}}\left(\bar{y}_k - \tau\nabla f(\bar{y}_k)\right)$$
$$y_{k+1} = x_{k+1} + \alpha_k(x_{k+1} - x_k)$$

  - FISTA: $\bar{y}_k = y_k$
  - IML FISTA: $\bar{y}_k = ML(y_k)$ (i.e., min F_H)

- IML FISTA recovers state-of-the-art convergence guarantees:

$$x_{k+1} \approx_{\epsilon_{h,k}} \text{prox}_{\tau\varphi\circ\mathrm{L}}\left(\bar{y}_k - \tau\nabla f(\bar{y}_k) + e_{h,k}\right)$$

Multilevel steps= bounded errors on the gradient

If $\sum_{k=1}^{\infty} k^{2d}\epsilon_{h,k} < \infty$, then

- The sequence $k^{2d}(F_h(x_{h,k} - F(x^*)))_{k\in\mathbb{N}}$ belongs to $\ell_\infty(\mathbb{N})$
- The sequence $(x_{h,k})_{k\in\mathbb{N}}$ converges to a minimizer of F_h

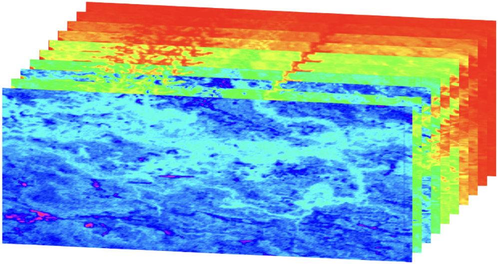
Outline

The multilevel step

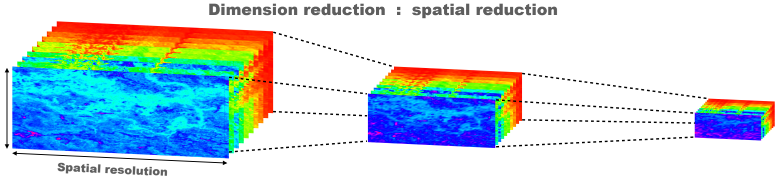
The ingredients of the multilevel scheme
- The transfer operators
- The coarse model

Numerical experiments
- Hyperspectral images
- Radio-interferometric imaging

Hyperspectral images



How to build the coarse approximations?



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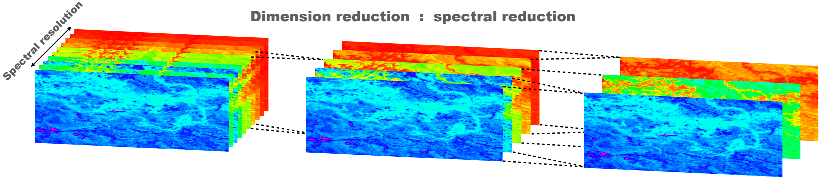
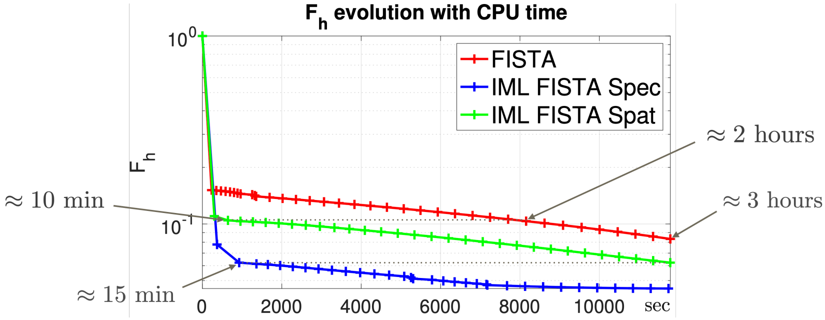
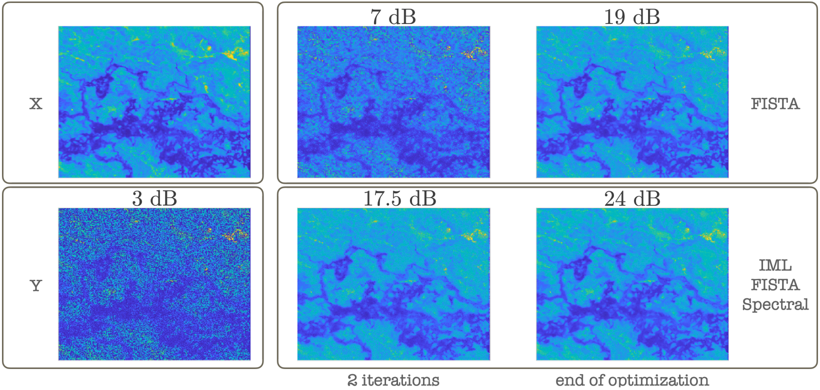


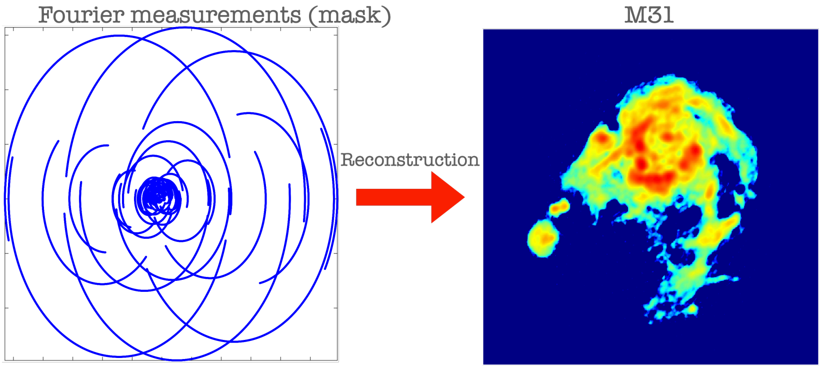
Objective function evolution



Reconstruction



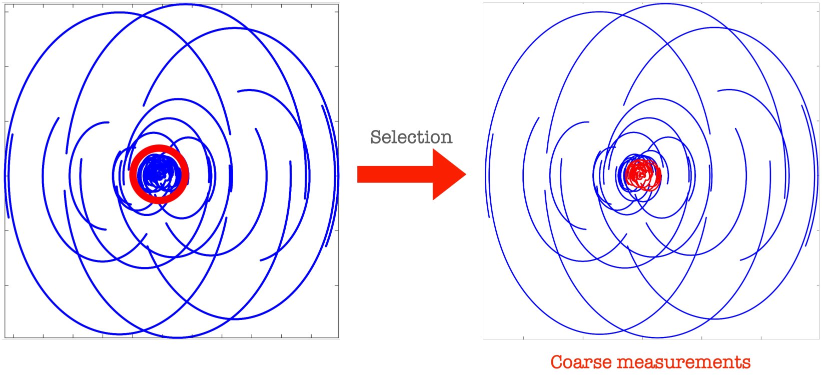
Radio-interferometric imaging



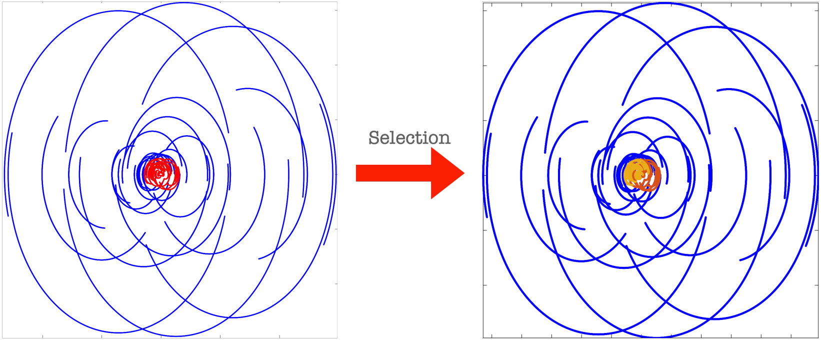
Dimension bottleneck: number of observations

Collaboration with Audrey Repetti and Yves Wiaux

Radio-interferometric imaging

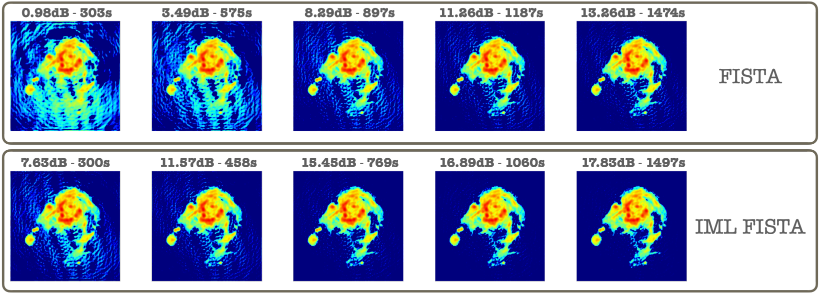


Radio-interferometric imaging



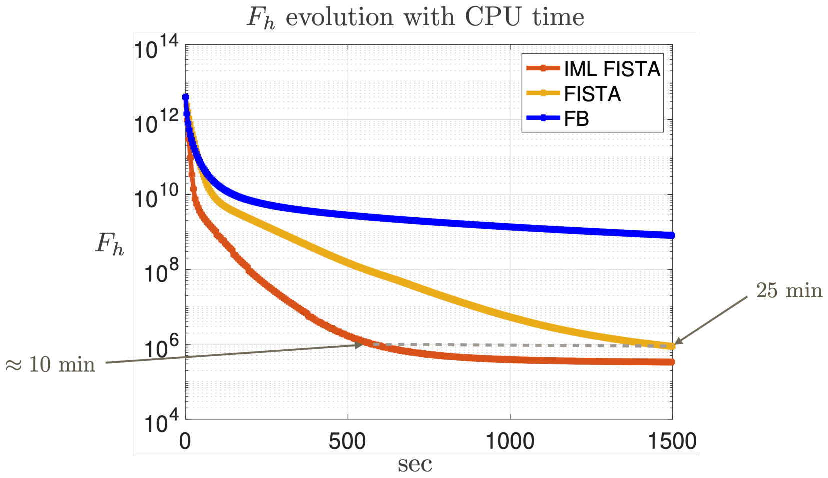
Coarser measurements

Radio-interferometric imaging



Reconstruction in log-scale of a region of the M31 galaxy

Radio-interferometric imaging



Conclusions and perspectives

Conclusions:

- We have proposed a general multilevel framework for image restoration with state-of-the-art convergence guarantees
- We have specialised the method in various contexts with really good practical performance

Perspectives:

- Extend the multilevel framework to scientific machine learning: multilevel Plug-and-Play methods, unrolled multilevel methods
- Extend the multilevel framework to second-oder methods: proximal Gauss-Newton methods

Thank you for your attention!

A few references

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