# *Artificial Neural Networks : scalable optimization methods for large-scale training problems*

E. Riccietti (IRIT-INP, Toulouse)

Joint work with: H. Calandra (Total)
S. Gratton (IRIT-INP, Toulouse)
X. Vasseur (ISAE-SUPAERO, Toulouse)

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# The problem

We consider optimization problems arising in the training of artificial neural networks:

$$\min_p \mathcal{L}(p, z) \qquad z \in \mathcal{T}$$

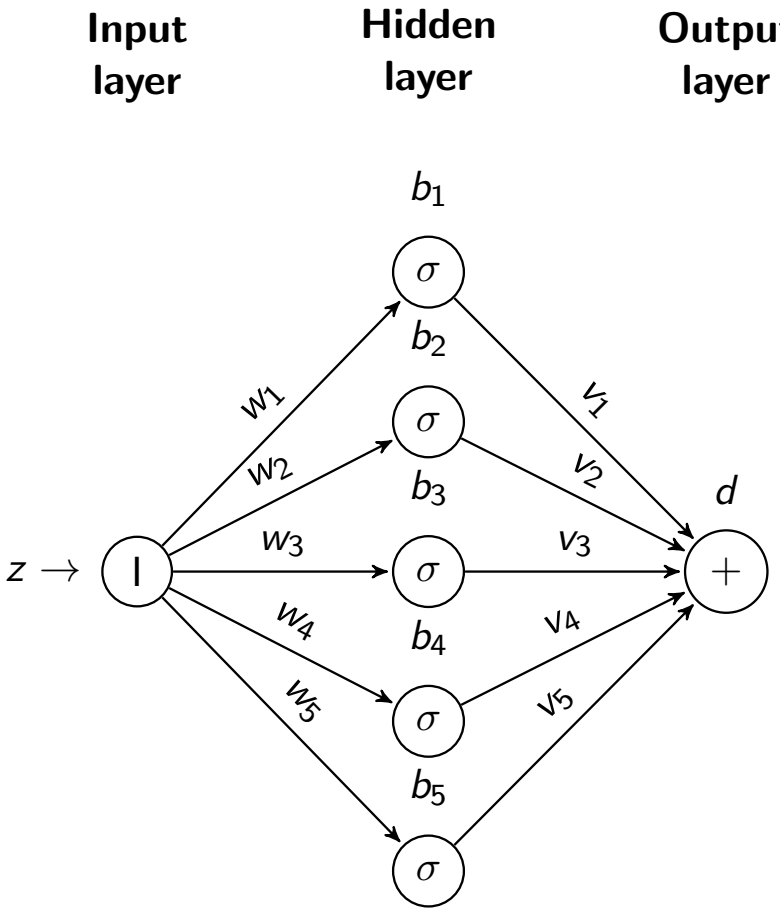
where $\mathcal{L}$ is the loss function, $p$ is the vector of weights and biases of the network, $z$ is the problem's variable and $\mathcal{T}$ is the training set.

### Example: approximate $y = g(z)$

Given a training set $\{(z_1, y_1), \ldots, (z_t, y_t)\}$ and denoted with $\hat{g}$ the output of the network, we define

- $L_1$ loss: $\mathcal{L}(p, z) = \frac{1}{t} \sum_{i=1}^{t} |y_i - \hat{g}(z_i, p)|$,
- $L_2$ loss: $\mathcal{L}(p, z) = \frac{1}{t} \sum_{i=1}^{t} (y_i - \hat{g}(z_i, p))^2$,
- Logistic loss: $\mathcal{L}(p, z) = \frac{1}{t} \sum_{i=1}^{t} \frac{1}{1+e^{y_i - \hat{g}(z_i, p)}}$.

# Network architecture




$p = [v, w, b, d]^T$

$\rightarrow \hat{g}(p, z) = \sum_{i=1}^{r} v_i \sigma(w_i z + b_i) + d$

Activation funct. $\sigma$,
sigmoid: $\sigma(z) = \frac{e^z - 1}{e^z + 1}$,
tanh: $\sigma(z) = \frac{e^{2z} - 1}{e^{2z} + 1}$,
logit: $\sigma(z) = \frac{e^z}{e^z + 1}$,
softplus: $\sigma(z) = \log(e^z + 1)$.

# Large-scale problems

The optimization problem may be a large-scale problem, for example if $g$ is an oscillatory function. Many nodes may be necessary to have a network able to accurately approximate it.

We look for an efficient scalable optimization method to solve the training problem.

$$\Downarrow$$

Can we exploit the structure of the network?

# Idea

We have to solve a large-scale problem

$$\min_p \mathcal{L}(p, z) = \mathcal{F}(\hat{g}(p, z) - y), \qquad z \in \mathcal{T}.$$

Can we exploit the structure of the network to build a hierarchy of problems approximating the original one?

**Hierarchy of problems**

$\{\mathcal{F}_l(\hat{g}_l(p_l, z) - y)\}$, $p_l \in \mathcal{D}_l$ such that $|\mathcal{D}_l| < |\mathcal{D}_{l+1}|$ and $\mathcal{F}_l$ is cheaper to optimize compared to $\mathcal{F}_{l+1}$.

This is the idea on which classical multigrid methods are based

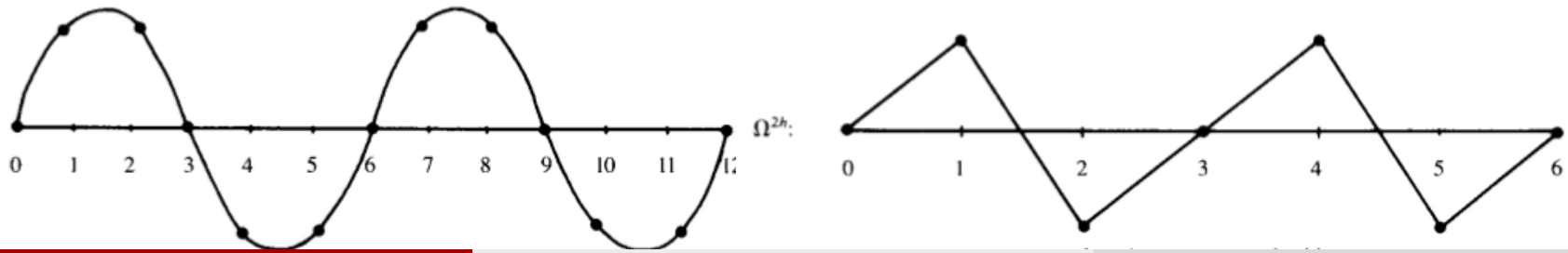
# Classical multigrid methods

- Consider a linear elliptic PDE: $D(z, u(z)) = f(z)$ $z \in \Omega$ + b.c.
- Discretize on grid $h$.
- Get a large-scale linear system $A_h x_h = b_h$.

## Multigrid methods

Consider the discretization of the same PDE problem on a coarser grid: $A_H x_H = b_H$, $H > h$.

- Relaxation methods fails to eliminate smooth components of the error efficiently.
- Smooth components projected on a coarser grid appear more oscillatory.



# Coarse problem construction

Define transfer grid operators: $P$ prolongation and $R$ restriction to project vectors from a grid to another: $x_H = Rx_h$, $x_h = Px_H$, such that $R = \alpha P^T$.

## Geometry exploitation

The geometrical structure of the problem is exploited to build $R$ and $P$.

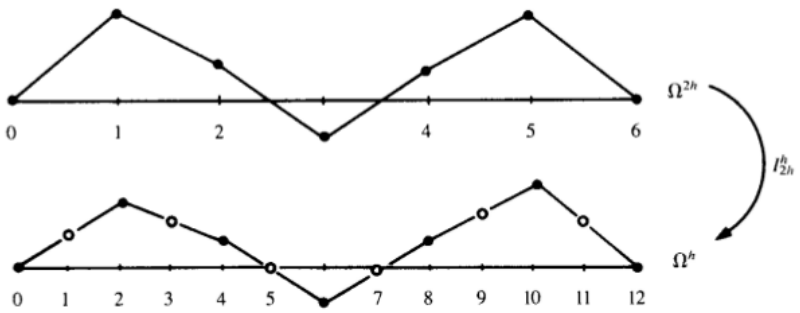



Figure 3.2: *Interpolation of a vector on coarse grid $\Omega^{2h}$ to fine grid $\Omega^h$.*

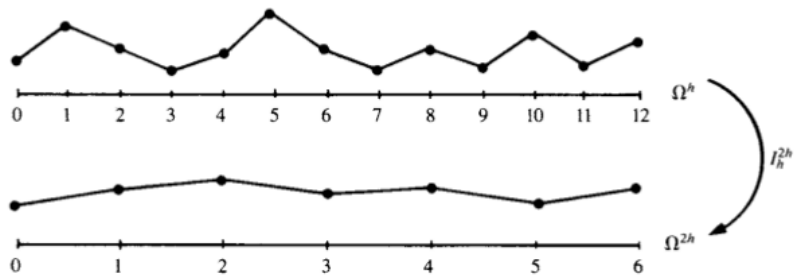



Figure 3.4: *Restriction by full weighting of a fine-grid vector to the coarse grid.*

## Remark

This strategy is also available in the nonlinear case (Full Approximation Scheme (FAS) algorithm).

# Optimization methods

We have a nonlinear problem to solve

$$\min_x f(x)$$

Classical iterative optimization methods:

$$f(x_k + s) \simeq T_q(x_k, s) = f(x_k) + s^T \nabla f(x_k) + \frac{1}{2} s^T B_k s$$

with $T_q(x_k, s)$ Taylor model of order $q = 1, 2$, $B_k$ approximation to Hessian matrix. At each iteration we compute a step $s_k$ to update the iterate:

$$\min_s m_k(x_k, s) = T_q(x_k, s) + \frac{\lambda_k}{q+1} \|s\|^{q+1}, \qquad \lambda_k > 0$$

# Higher-order models

Classical choices:

- Least-squares: Levenberg-Marquardt (LM), $q = 1$, $B_k = J(x_k)^T J(x_k)$.
- Adaptive Cubic Regularization method (ARC), $q = 2$, $B_k = \nabla^2 f(x_k)$.

## Extension to higher-order methods

[Birgin, Gardenghi, Martnez, Santos, and Toint, 2017] extension to order $q > 2$.
Unifying framework for global convergence is presented.

# Basic iterative optimization algorithm

Until convergence

- Define the local model $m_k$ of $f$ around $x_k$, depending on $\lambda_k$
- Compute a trial point $x_k + s_k$ that decreases this model
- Compute the predicted reduction $m_k(x_k) - m_k(x_k + s_k)$
- Evaluate change in the objective function $f(x_k) - f(x_k + s_k)$
- If achieved change $\sim$ predicted reduction then
  - Accept trial point as new iterate $x_{k+1} = x_k + s_k$

  else
  - Reject the trial point $x_{k+1} = x_k$
  - Increase $\lambda_k$

# Subproblem solution

Solving

$$\min_s T_q(x_k, s) + \frac{\lambda_k}{q+1}\|s\|^{q+1}$$

represents greatest cost per iteration, which depends on the size of the problem.

$$\Downarrow$$

Recursive multilevel trust region method [Gratton, Sartenaer, Toint, 2008]

## Assumption

1. Assume to have at disposal a sequence of approximations $\{f_l\}$ to the objective function $f$ such that $f_l$ is cheaper to optimize than $f_{l+1}$.
2. Assume to have linear full-rank operators $R_l$ and $P_l$ to move from a level to another, such that $R_l = P_l^T$ (up to a scalar).

# Multigrid setting

- At each level $l$, $x \in \mathbb{R}^{n_l}$. $l_{\max}$ finest level, 0 coarsest level.

| | | | | |
|---|---|---|---|---|
| level $l_{\max}$ | $\mathbb{R}^n$ | $x^{l_{\max}}$ | $f^{l_{\max}} = f$ | $\mu^{l_{\max}} = f$ |
| $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ |
| level $l+1$ | $\mathbb{R}^{n^{l+1}}$ | $x^{l+1}$ | $f^{l+1}$ | $\mu^{l+1}$ |
| | $R^{l+1} \Downarrow \quad \Uparrow P^{l+1}$ | | | |
| level $l$ | $\mathbb{R}^{n^l}$ | $x^l$ | $f^l$ | $\mu^l$ |
| $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ |
| level 0 | $\mathbb{R}^{n^0}$ | $x^0$ | $f^0$ | $\mu^0$ |

- $f^l$ represent $f$ on the coarse spaces (it is e.g. the discretization of $f$ on a coarse space)
- The functions $\mu^l$ are modifications of the $f^l$'s to ensure inter-level coherence.

# Coherence between levels

Lower level model:

- Let $x_0^{l-1} = Rx_k^l$. Model with first order correction:

$$\mu^{l-1} = f^{l-1}(x_0^{l-1} + s^{l-1}) + (R^l \nabla f^l(x_k^l) - \nabla f^{l-1}(x_k^{l-1}))^T s^{l-1}$$

This ensures that

$$\nabla \mu^{l-1}(x_0^{l-1}) = R^l \nabla f^l(x_k^l)$$

$\rightarrow$ first-order behaviours of $f^l$ and $\mu^{l-1}$ are coherent in a neighbourhood of the current approximation. If $s^l = P^l s^{l-1}$

$$\nabla f^l(x_k^l)^T s^l = \nabla f^l(x_k^l)^T P^l s^{l-1} = \frac{1}{\alpha} \nabla \mu^{l-1}(x_0^{l-1})^T s^{l-1}.$$

# Multilevel strategy

At level $l$, let $x_k^l$ be the current approximation. We look for a correction $s_k^l$ to define the new approximation $x_{k+1}^l = x_k^l + s_k^l$. Two choices:

1. minimize regularized Taylor model, get $s_k^l$,
2. choose lower level model $\mu^{l-1}$:

$$
\begin{array}{ccc}
x_k^l & & x_{k+1}^l = x_k^l + s_k^l \\
\Big\downarrow R^l & & \Big\uparrow s_k^l = P^l(x_*^{l-1} - x_0^{l-1}) \\
R^l x_k^l := x_0^{l-1} & \xrightarrow{\quad\mu^{l-1}\quad} & x_*^{l-1}
\end{array}
$$

# Our contribution

In [Gratton, Sartenaer, Toint, 2008] second-order models are considered ($q = 2$).

- We combine ideas from [Gratton, Sartenaer, Toint, 2008] and [Birgin, Gardenghi, Martnez, Santos, and Toint, 2017] and we propose a family of scalable, multilevel optimization methods of order $q \geq 1$, which are proved to be globally convergent.
- We propose a suitable mechanism to construct a hierarchy of problems for the specific case of neural network training.
- We specialize the training method to least-squares problems (recursive multilevel Levenberg-Marquardt method).

On scalable multilevel optimization strategies for large-scale problems arising in the training of artificial neural networks

# Recursive multi-scale $q$-order methods

Until convergence

- Choose $q \geq 1$. Choose either a Taylor or a (useful) recursive model.
  - Taylor model: compute a Taylor step satisfying a sufficient decrease property
  - Recursive: apply the algorithm recursively
- Evaluate change in the objective function
- If achieved change $\sim$ predicted reduction then
  - Accept trial point as new iterate

  else
  - Reject the trial point
  - Increase $\lambda$

The algorithm is proved globally convergent to first order critical points

# Exploit multi-scale method for training of ANNs

How to build the coarse problem?

## Remark

The variables to be optimized are the network's weights:

$$\min_{p} \mathcal{L}(p, z) \qquad z \in \mathcal{T}$$

NO evident geometrical structure to exploit!

## Algebraic multigrid

We can take inspiration from algebraic multigrid techniques.
When solving linear systems $Ax = b$, the structure is discovered through the matrix $A$. $R$ and $P$ are built just looking at the entries of the matrix.

# Ruge and Stueben AMG

To build the coarse problem, the variables are divided into two sets, set $C$ of coarse variables and set $F$ of fine variables.

## Ruge and Stueben $C/F$ splitting

- Two variables $i, j$ are said to be *coupled* if $a_{i,j} \neq 0$.
- We say that a variable $i$ is strongly coupled to another variable $j$, if

$$-a_{i,j} \geq \epsilon \max_{a_{i,k}<0} |a_{i,k}|$$

  for a fixed $0 < \epsilon < 1$, usually $\epsilon = 0.25$.
- Each $F$ variable is required to have a minimum number of its strong couplings be represented in $C$. The $C/F$ splitting is usually made choosing some first variable $i$ to become a coarse variable. Then, all variables strongly coupled to it become $F$ variables. The process is repeated until all variables have been split.

# Which matrix should we use?

Assume to use a second-order model. At each iteration we have to solve a linear system of the form:

$$(B_k + \tilde{\lambda}_k I)s = -\nabla f(x_k)$$

for $\tilde{\lambda}_k > 0$. As in AMG for linear systems, we use information contained in matrix $B_k$.

**Remark**

Variables are coupled! $\mathcal{L}(p, z) = \mathcal{F}(\hat{g}(p, z) - y)$ and $\hat{g}(p, z) = \sum_{i=1}^{r} v_i \sigma(w_i z + b_i) \rightarrow p = \{(v_i, w_i, b_i)\}$.

We do not use the full matrix $B_k$ and we define $A$ as:

$$B_k = \begin{bmatrix} A_{v,v} & .. & .. \\ .. & A_{w,w} & .. \\ .. & .. & A_{b,b} \end{bmatrix} \rightarrow A = \frac{A_{v,v}}{\|A_{v,v}\|_\infty} + \frac{A_{w,w}}{\|A_{w,w}\|_\infty} + \frac{A_{b,b}}{\|A_{b,b}\|_\infty}$$

We define the coarse/fine splitting based on the auxiliary matrix $A$.

# Preliminary results: solution of PDEs

Approximate the solution $u$ of a PDE:

$$D(z, u(z)) = g(z), \ z \in (a, b);$$
$$u(a) = A, \ u(b) = B.$$

We approximate $u \sim \hat{u}(p, z)$ for $p \in \mathbb{R}^n$ and we define

$$\mathcal{L}(p, z) = \frac{1}{2t}(\|D(z, u(z)) - g(z)\|^2 + \lambda_p(\|u(a) - A\|^2 + \|u(b) - B\|^2))$$

for $z \in \mathcal{T}$ training set.

Least-squares problem $\rightarrow$ multi-scale Levenberg-Marquardt method

# Choice of the true solution

$$D(z, u(z)) = g(z), \ z \in (a, b);$$

- We choose $g$ to have true solution $u_T(z, \nu)$ depending on $\nu$

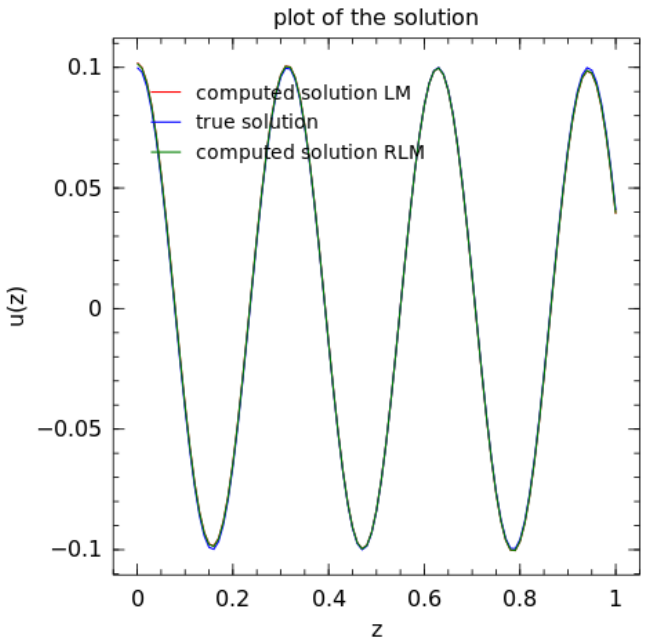
**Remark**

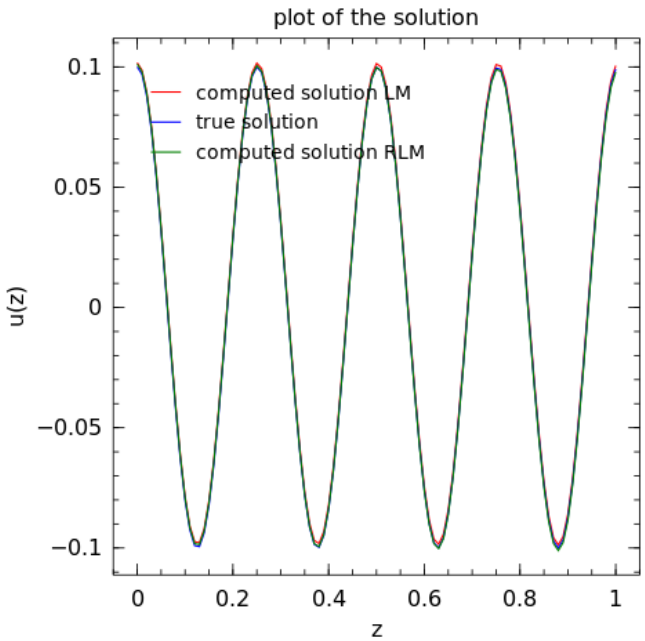
- As $\nu$ increases the function becomes more oscillatory and it is harder to approximate.
- The size of the problem increases with the number of nodes.
- $\mathcal{T}$: equispaced points in $(0, 1)$ with $h = \frac{1}{3\nu}$ (Shannon's criterion).

# Poisson's equation, $u_T(z,\nu) = cos(\nu z)$, 5 runs

| Problem | $\nu = 20$ | | $r = 2^9$ | $\nu = 25$ | | $r = 2^{10}$ |
|---|---|---|---|---|---|---|
| Solver | iter | RMSE | save | iter | RMSE | save |
| LM | 282 | 1.e-3 | | 632 | 1.e-2-1.e-3 | |
| RLM | 193 | 1.e-3 | 1.2-1.75 | 347 | 1.e-2-1.e-3 | 1.2-3.15 |

save=ratio between total number of flops required for matrix-vector products



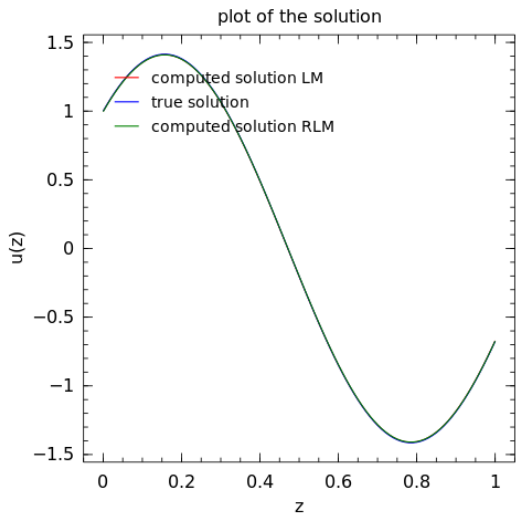



# Helmholtz's equation, 5 runs

Equation: $\Delta u(z) + \nu^2 u(z) = 0$ , $u_T(z, \nu) = sin(\nu z) + cos(\nu z)$

| Problem | | $\nu = 5$ | $r = 2^{10}$ |
|---|---|---|---|
| Solver | iter | RMSE | save |
| LM | 1243 | 1.e-2-1.e-3 | |
| RLM | 1229 | 1.e-2-1.e-3 | 1.2-3.1 |

save=ratio between total number of flops required for matrix-vector products



# Conclusions and future work

- We have presented a class of high-order methods for optimization and proved their global convergence.
- We have proposed a AMG strategy to build coarse representations of the problem to use these methods for the training of artificial neural networks.
- Preliminary tests show encouraging results. In future work we will consider further applications.

**Thank you for your attention!**