On the iterative solution of systems of the form $A^\top Ax = A^\top b + c$

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30th July, 2019

Context

Given $A \in \mathbb{R}^{m \times n}$, $m \geq n$ with $\operatorname{rank}(A) = n$, $b \in \mathbb{R}^m$ and $x, c \in \mathbb{R}^n$, solve

$$A^\top A x = A^\top b + \mathbf{c} \tag{SYS}$$

or

$$\min_x \|Ax - b\|^2 - x^\top \mathbf{c}$$

Remarks

- This is a generalization of the normal equations for least-squares problems (case $c = 0$)

Motivating applications (I)

- Multilevel Levenberg-Marquardt method

Calandra, H., Gratton, S., Riccietti, E., Vasseur, X., *On the approximation of the solution of partial differential equations by artificial neural networks trained by a multilevel Levenberg-Marquardt method*, arXiv e-print, 2019

$$\min_{x\in\mathbb{R}^n} f(x) = \frac{1}{2}\|F(x)\|^2.$$

We have at disposal an approximation to the objective function:

$$f^H(x^H) = \frac{1}{2}\|F^H(x^H)\|^2, \quad x^H \in \mathbb{R}^{n_H},\ n_H < n$$

Coarse model:

$$m_k^H(x_k^H, s^H) = \frac{1}{2}\|F^H(x_k^H) + J^H(x_k^H)s^H\|^2 + \frac{\lambda_k}{2}\|s^H\|^2 + (R\nabla f(x_k) - \nabla f^H(x_0^H))^\top s^H,$$

with $J^H(x_k^H)$ the Jacobian matrix of F^H at x_k^H, R a full-rank linear restriction operator and $x_0^H = Rx_k$.

Motivating applications (II)

- Penalty function method

Fletcher, R., *A class of methods for nonlinear programming: III. Rates of convergence*, Numerical Methods for Nonlinear Optimization, 1973

Estrin, R. and Orban, D. and Saunders, M. A., *LNLQ: An iterative method for least-norm problems with an error minimization property*, technical report, 2018

$$\min_x f(x)$$
$$\text{s.t. } g(x) = 0,$$

Penalty function :

$$\Phi_\sigma(x) = f(x) - g(x)^\top y_\sigma(x),$$

where $y_\sigma(x) \in \mathbb{R}^m$ is defined as the solution of the following minimization problem:

$$\min_y \|A(x)^\top y - \nabla f(x)\|^2 + \sigma g(x)^\top y,$$

with $A(x)$ the Jacobian matrix of $g(x)$ at x and $\sigma > 0$, a given real-valued penalty parameter.

Interesting questions

- What is the conditioning of $A^\top Ax = A^\top b + c$?
  - Standard theory for linear systems do no take into account structured perturbations and gives underwhelming results
  - Structured conditioning analysis is necessary. Presence of c results in a different mapping from data to solution
- What is the backward error?
  - Different set of admissible perturbations on the matrix
- How to numerically solve it by an iterative method?
  - Methods for normal equations such as CGLS cannot be used.

THEORETICAL RESULTS

Conditioning, case $c = 0$

Let $\delta x = x - \hat{x}$, $\hat{x}$ a perturbed solution.

Forward error bound

From standard theory on linear systems:

$$\frac{\|\delta x\|}{\|x\|} \leq \kappa(A)^2 u$$

For least squares problems:

$$\frac{\|\delta x\|}{\|x\|} \leq \gamma_m \kappa_{LS} u, \quad \kappa_{LS} = \kappa(A)\left(1 + \frac{\|A^\dagger\|\|r\|}{\|x\|}\right), \quad r = b - Ax$$

Underwhelming result!

The conditioning of the problem depends on $\kappa(A)^2$ only if $\|r\|$ is large!

Conditioning

Definition

If F is a continuously differentiable function

$$\begin{aligned} F : \mathcal{X} &\to \mathcal{Y} \\ x &\longmapsto F(x), \end{aligned}$$

the absolute condition number of F at x is the scalar $\|F'(x)\|_{\mathrm{op}}$. The relative condition number of F at x is

$$\frac{\|F'(x)\|_{\mathrm{op}} \, \|x\|_{\mathcal{X}}}{\|F(x)\|_{\mathcal{Y}}}.$$

J . R . Rice, *A theory of condition*, SIAM J . Numer . Anal ., 1966

Conditioning, case $c = 0$

Definition of F

We consider F as the function that maps A, b to the solution x of a least squares problem:

$$F : \mathbb{R}^{m \times n} \times \mathbb{R}^m \to \mathbb{R}^n$$
$$(A, b) \longmapsto F(A, b) = A^\dagger b.$$

Explicit formula for the conditioning

The absolute condition number of a least-squares problem, with Euclidean norm on the solution and Frobenius norm on the data[a], is given by

$$\kappa_{NE} = \|A^\dagger\| \sqrt{1 + \|x\|^2 + \|A^\dagger\|^2 \|r\|^2}$$

Gratton, S., *On the condition number of linear least squares problems in a weighted Frobenius norm*, BIT Numerical Mathematics, 1996

[a] $\|[A, b]\|_F^2 := \|A\|_F^2 + \|b\|^2$

A formula for the condition number, $c \neq 0$

Lemma

The absolute condition number of the problem SYS is given by

$$\|F'(A,b,c)\|_{\mathrm{op}} = \|[(r^\top \otimes (A^T A)^{-1})L_T + x^\top \otimes A^\dagger, A^\dagger, (A^T A)^{-1}]\|,$$

where L_T is the linear operator such that $\mathrm{vec}(A^\top) = L_T \mathrm{vec}(A)$ and $r = b - Ax$.

Case $c = 0$

$$\|F'(A,b,c)\|_{\mathrm{op}} = \|[(r^\top \otimes (A^T A)^{-1})L_T + x^\top \otimes A^\dagger, A^\dagger]\|.$$

An explicit formula for the condition number, $c \neq 0$

We consider F as the function that maps A, b, c to the solution x of SYS

$$F : \mathbb{R}^{m\times n} \times \mathbb{R}^{m} \times \mathbb{R}^{n} \to \mathbb{R}^{n}$$
$$(A, b, c) \longmapsto F(A, b, c) = A^{\dagger} b + A^{\dagger}(A^{\dagger})^{\top} c.$$

Theorem

The absolute condition number of problem SYS, with Euclidean norm on the solution and Frobenius norm on the data[a], is $\sqrt{\|\bar{M}\|}$, with $\bar{M} \in \mathbb{R}^{n\times n}$ given by

$$\bar{M} = (1 + \|r\|^2)(A^{\top}A)^{-2} + (1 + \|x\|^2)(A^{\top}A)^{-1} - 2\,\mathrm{sym}(B),$$

with $B = A^{\dagger} r x^{\top}(A^{T}A)^{-1}$, $\mathrm{sym}(B) = \frac{1}{2}(B + B^{\top})$ and x the exact solution of SYS.

[a] $\|[A, b, c]\|_F^2 := \|A\|_F^2 + \|b\|^2 + \|c\|^2$

Upper bound for the condition number

$$\sqrt{\|\bar{M}\|} \leq (1 + \|r\| + 2\sqrt{\|c\|\|x\|})\|A^{\dagger}\|^2 + (1 + \|x\|)\|A^{\dagger}\|.$$

Backward error analysis

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ and $\tilde{x}$ a perturbed solution to SYS. Find the smallest perturbation E of A such that the vector $\tilde{x}$ exactly solves

$$(A+E)^\top (A+E)x = (A+E)^\top b + c,$$

i.e. given

$$\mathcal{G} := \{E \in \mathbb{R}^{m \times n} : (A+E)^\top (A+E)\tilde{x} = (A+E)^\top b + c\},$$

we want to compute the quantity:

$$\eta(\tilde{x}) = \min_{E \in \mathcal{G}} \|E\|_F.$$

Set of admissible perturbations on the matrix

Theorem

Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c, \tilde{x} \in \mathbb{R}^n$ and assume that $\tilde{x} \neq 0$. Let $\tilde{r} = b - A\tilde{x}$ and define two sets $\mathcal{E}, \mathcal{M}$ by

$$\begin{aligned}\mathcal{E} =& \{E \in \mathbb{R}^{m\times n} : \ (A+E)^\top (b - (A+E)\tilde{x}) = -c\},\\ \mathcal{M} =& \{v\left(\alpha c^\top - v^\dagger A\right) + (I_m - vv^\dagger)(\tilde{r}\tilde{x}^\dagger + Z(I_n - \tilde{x}\tilde{x}^\dagger)) \ :\\ & v \in \mathbb{R}^m,\ Z \in \mathbb{R}^{m\times n}, \alpha \in \mathbb{R},\ s.t.\ \alpha\|v\|^2(v^\dagger b - \alpha c^\top \tilde{x}) = -1\}.\end{aligned}$$

Then $\mathcal{E} = \mathcal{M}$.

Case $c = 0$

$$\begin{aligned}\mathcal{E} =& \{E \in \mathbb{R}^{m\times n} : \ (A+E)^\top (b - (A+E)\tilde{x}) = 0\},\\ \mathcal{M} =& \{-vv^\dagger A + (I_m - vv^\dagger)(\tilde{r}\tilde{x}^\dagger + Z(I_n - \tilde{x}\tilde{x}^\dagger)) \ : v \in \mathbb{R}^m,\ Z \in \mathbb{R}^{m\times n}\}.\end{aligned}$$

Lower bound on the backward error

Lemma

The set of admissible perturbations $\mathcal{E}$ defined in Theorem is such that $\mathcal{E} \subseteq \mathcal{M}_2$, with

$$\mathcal{M}_2 = \{v\left(\alpha c^\top - v^\dagger A\right) + (I_m - vv^\dagger)(\tilde{r}\tilde{x}^\dagger + Z(I_n - \tilde{x}\tilde{x}^\dagger)) \;:\; v \in \mathbb{R}^m,\, Z \in \mathbb{R}^{m\times n},\, \alpha \in \mathbb{R}\}.$$

Then,

$$\min_{\mathcal{E}} \|E\|_F^2 \geq \min_{\mathcal{M}_2} \|E\|_F^2 = \frac{\|\tilde{r}\|^2}{\|\tilde{x}\|^2} + \min\{\lambda_*, 0\},$$

for $\lambda_* = \lambda_{\min}\left(A(I_n - cc^\top)A^\top - \dfrac{\tilde{r}\tilde{r}^\top}{\|\tilde{x}\|^2}\right)$, with $\lambda_{\min}(M)$ denoting the smallest eigenvalue of the matrix M.

Case $c = 0$

$$\min_{\mathcal{E}} \|E\|_F^2 = \frac{\|\tilde{r}\|^2}{\|\tilde{x}\|^2} + \min\{\lambda_*, 0\}, \quad \lambda_* = \lambda_{\min}\left(AA^\top - \frac{\tilde{r}\tilde{r}^\top}{\|\tilde{x}\|^2}\right).$$

NUMERICAL SOLUTION OF THE SYSTEM

CG vs CGLS for normal equations

Same method in exact arithmetic, different performance in finite precision for some problems:

- in CGLS $d_k = b - Ax_k$ is recurred and $r_k = A^\top d_k$.

Algorithm 1 CG for $A^\top Ax = A^\top b$

Input: A, b, x_0.
Define $r_0 = A^\top(b - Ax_0)$, $p_1 = r_0$.
for $k = 1, 2, \ldots$ **do**

$$\alpha_k = \frac{r_{k-1}^\top r_{k-1}}{\|Ap_k\|^2},$$
$$x_k = x_{k-1} + \alpha_k p_k,$$
$$r_k = r_{k-1} - \alpha_k A^\top(Ap_k),$$
$$\beta_k = \frac{r_k^\top r_k}{r_{k-1}^\top r_{k-1}},$$
$$p_{k+1} = r_k + \beta_k p_k.$$

end for

Algorithm 2 CGLS for $A^\top Ax = A^\top b$

Input: A, b, x_0.
Define $d_0 = b - Ax_0$, $r_0 = A^\top d_0$, $p_1 = r_0$.
for $k = 1, 2, \ldots$ **do**

$$t_k = Ap_k,$$
$$\alpha_k = \frac{r_{k-1}^\top r_{k-1}}{\|t_k\|^2},$$
$$x_k = x_{k-1} + \alpha_k p_k,$$
$$d_k = d_{k-1} - \alpha_k t_k,$$
$$r_k = A^\top d_k,$$
$$\beta_k = \frac{r_k^\top r_k}{r_{k-1}^\top r_{k-1}},$$
$$p_{k+1} = r_k + \beta_k p_k.$$

end for

Paige, C. C. and Saunders, M. A., *LSQR: An Algorithm for Sparse Linear Equations and Sparse Least Squares,* ACM Trans. Math. Softw., 1982

Björck, A. and Elfving, T. and Strakos, Z. , *Stability of conjugate gradient and Lanczos methods for linear least squares problems,* SIMAX, 1998

CG for $A^T A x = A^T b + c$

Initial rounding error due to the product $r_0 = A^T b + c - A^T A x_0$:

$$\|\delta x\| \le \kappa(A)^2 u \left(\frac{\|b\|}{\|A\|} + \frac{\|c\|}{\|A\|^2} \right).$$

This initial error cannot be canceled, and the best error bound we can hope for will include the term given above.
Optimal bound:

$$\|\delta x\| \le \sqrt{\|\bar{M}\|}\,\|[A, b, c]\|_F\, u$$

If

$$\|b\|\|A\| + \|c\| >> \left[1 + \|r\| + 2\sqrt{\|c\|\|x\|} + \frac{1 + \|x\|}{\|A^\dagger\|} \right] \sqrt{\|A\|_F^2 + \|b\|^2 + \|c\|^2}$$

CG can be expected to produce less than optimal accuracy.

IDEA to design a stable method

- Extend the successful algorithmic procedures to the case $c \neq 0$
- Need to factorize matrix A in both the left and right hand sides

$$A^T(A^Tx - b)$$

Two solution methods

We propose two iterative methods based on two different reformulations of the problem

Proposed methods (I) CGLSϵ

Given $\epsilon > 0$, let us then define

$$A_\epsilon = \begin{bmatrix} A \\ \epsilon c^\top \end{bmatrix}, \quad b_\epsilon = \begin{bmatrix} b \\ 1/\epsilon \end{bmatrix}.$$

We then consider the following linear least squares problem:

$$\min_x \|A_\epsilon x - b_\epsilon\|^2,$$

with normal equations

$$(A^\top A + \epsilon^2 cc^\top)x = A^\top b + c. \qquad (\mathrm{SYS}\epsilon)$$

CGLSϵ solves SYSϵ with CGLS method

Lemma

Let x_ϵ be the solution of SYSϵ and x be the solution of SYS. Then, $\lim_{\epsilon\to 0} x_\epsilon = x$ and the relative norm of the error satisfies

$$\frac{\|x_\epsilon - x\|}{\|x\|} \leq \epsilon^2 \frac{\|c\|\|w\|}{1 + \epsilon^2 c^\top w}, \quad w = (A^\top A)^{-1}c.$$

Remarks (I)

- Will a really small ϵ may cause large errors in finite arithmetic?
- A perturbed solution $\tilde{x}_\epsilon = x_\epsilon + \delta x_\epsilon$ will be such that:

$$(A_\epsilon^\top A_\epsilon)(\delta x_\epsilon) = \delta(A_\epsilon^\top b_\epsilon). \quad |\delta(A_\epsilon^\top b_\epsilon)| \leq \gamma_{m+1}|A_\epsilon^\top||b_\epsilon|$$

This overestimates the error!

$$\mathrm{fl}(A_\epsilon^\top b_\epsilon) = \mathrm{fl}(A^\top b) + \mathrm{fl}\left(\epsilon c \frac{1}{\epsilon}\right) + \delta_s,$$

with δ_s error due to the summation.

- If $\epsilon = 2^i$ for $i \in \mathbb{Z}$, then $\mathrm{fl}\left(\epsilon\ c\ \frac{1}{\epsilon}\right) = c$. Then,

$$\mathrm{fl}\left(A_\epsilon^\top b_\epsilon\right) = A^\top b + c + \delta_p + \delta_s, \quad \text{with} \quad |\delta_s| \leq u|\mathrm{fl}(A^\top b) + c|,\ |\delta_p| \leq \gamma_m|A||b|,$$

and the bound does not depend on ϵ.

Remarks (II)

- What about the conditioning of the problem?
- Due to the presence of small ϵ in the right-hand side the residual will generally be really large.
- Standard conditioning analysis of least squares problems is not well-suited in this case
- We can show that the conditioning does not depend on $\|b_\epsilon - A_\epsilon x_\epsilon\|$, that will be really large, but rather on $\|r_\epsilon\| = \|b - Ax_\epsilon\|$, that will be indeed much smaller

Conditioning

- Let F_ϵ be the function that maps A, b, c to the solution x_ϵ of SYSϵ

$$F_\epsilon : \mathbb{R}^{m\times n} \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$$
$$(A, b, c) \longmapsto F_\epsilon(A, b, c) = (A_\epsilon^\top A_\epsilon)^{-1}(A^\top b + c),$$

and let $r_\epsilon = b - Ax_\epsilon$.

- The absolute condition number of problem SYSϵ, with Euclidean norm on the solution and Frobenius norm on the data, is then given by:

$$\|F_\epsilon'(A, b, c)\|_{\mathrm{op}} = \|[(r_\epsilon^\top \otimes (A_\epsilon^\top A_\epsilon)^{-1})L_T + x_\epsilon^\top \otimes (A_\epsilon^\top A_\epsilon)^{-1}A^\top,$$
$$(A_\epsilon^\top A_\epsilon)^{-1}A^\top, (1 - 2\epsilon c^\top x_\epsilon)(A_\epsilon^\top A_\epsilon)^{-1}]\|.$$

- Computable formula: $\sqrt{\|\bar{M}_\epsilon\|}$, with

$$\bar{M}_\epsilon = ((1 - 2\epsilon c^\top x_\epsilon)^2 + \|r_\epsilon\|^2)(A_\epsilon^\top A_\epsilon)^{-2}$$
$$+ (1 + \|x_\epsilon\|^2)(A_\epsilon^\top A_\epsilon)^{-1}A^\top A(A_\epsilon^\top A_\epsilon)^{-1} - 2\,\mathrm{sym}(B_\epsilon)$$

with $B_\epsilon = (A_\epsilon^\top A_\epsilon)^{-1}A^\top r_\epsilon x_\epsilon^\top (A_\epsilon^\top A_\epsilon)^{-1}$ and $\mathrm{sym}(B_\epsilon) = \frac{1}{2}(B_\epsilon + B_\epsilon^\top)$.

Proposed method (II) CGLSI

Given $\hat{I} \in \mathbb{R}^{(m+1)\times(m+1)}$, we define $\hat{A} \in \mathbb{R}^{(m+1)\times n}$ and $\hat{b} \in \mathbb{R}^{m+1}$ as:

$$\hat{A} = \begin{bmatrix} A \\ c^\top \end{bmatrix}, \qquad \hat{I} = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}, \qquad \hat{b} = \begin{bmatrix} b \\ 1 \end{bmatrix}.$$

We then reformulate SYS as:

$$\hat{A}^\top \hat{I} \hat{A} x = \hat{A}^\top \hat{b}$$

- Possible to factorize $\hat{A}^\top$ in both the right and the left-hand sides:
  - no need of recurring the residual $r = \hat{A}^\top(\hat{I}\hat{A}x - \hat{b})$ (simply update $\hat{d} = \hat{I}\hat{A}x - \hat{b}$ along the iterations and form r by multiplication with $\hat{A}^\top$)
  - computation of $p_k^\top A^\top A p_k$ as $\|\hat{I}\hat{A}p_k\|^2$

We can therefore expect the same benefits of CGLS as compared to CG.

Algorithm

Algorithm 3 CGLSI for $A^\top Ax = A^\top b + c$

Input: $\hat{A}$, $\hat{b}$, x_0

Define $\hat{d}_0 = \hat{b} - \hat{A}x_0,\ r_0 = \hat{A}^\top(\hat{b} - \hat{A}x_0),\ p_1 = r_0.$

for $k = 1, 2, \ldots$ **do**

$\hat{t}_k = \hat{I}\hat{A}p_k,$

$\alpha_k = \dfrac{r_{k-1}^\top r_{k-1}}{\hat{t}_k^\top \hat{t}_k},$

$x_k = x_{k-1} + \alpha_k p_k,$

$\hat{d}_k = \hat{d}_{k-1} - \alpha_k \hat{t}_k,$

$r_k = \hat{A}^\top \hat{d}_k,$

$\beta_k = \dfrac{r_k^\top r_k}{r_{k-1}^\top r_{k-1}},$

$p_{k+1} = r_k + \beta_k p_k.$

end for

First order approximation for the forward error

First order approximation for the forward error can be obtained as

$$\frac{\|x - \hat{x}\|}{\|x\|} \sim \frac{\kappa_{SYS} \|[A, b, c]\|_F}{\|x\|} u, \quad u \text{ machine precision}$$

We define the following error estimates:

$$\hat{E}_{CGLSI} := \frac{\sqrt{\|\bar{M}\|}\,\|[A, b, c]\|_F}{\|x\|} u,$$

$$\hat{E}_{CGLS_\epsilon} := \epsilon^2 \frac{\|c\|\,\|w\|}{1 + \epsilon^2 c^\top w} + \frac{\sqrt{\|\bar{M}_\epsilon\|}\,\|[A, b, c]\|_F}{\|x\|} u \left\| I_n - \frac{\epsilon^2 w c^\top}{1 + \epsilon^2 c^\top w} \right\|,$$

u being the machine precision.

- CGLSϵ: the error on the computed solution $\hat{x}_\epsilon$ depends on two terms:

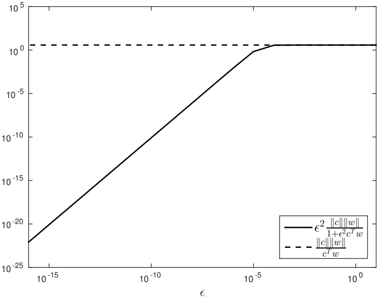
$$\frac{\|x - \hat{x}_\epsilon\|}{\|x\|} \leq \frac{\|x - x_\epsilon\|}{\|x\|} + \frac{\|x_\epsilon - \hat{x}_\epsilon\|}{\|x\|} = \frac{\|x - x_\epsilon\|}{\|x\|} + \frac{\|x_\epsilon - \hat{x}_\epsilon\|}{\|x_\epsilon\|} \frac{\|x_\epsilon\|}{\|x\|}.$$

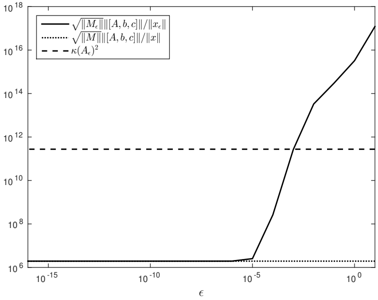
NUMERICAL TESTS

Numerical tests: setting

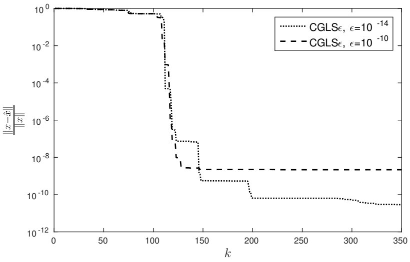
- All the numerical methods have been implemented in Matlab
- $A \in \mathbb{R}^{m \times n}$, $A = U\Sigma V^\top$, where U and V from *gallery('orthog',m/n,j)*, $j = 1, \dots, 6$.
- C1 : $\Sigma_{ii} = a^{-i}$, for $a > 0$,
  C2 : $\Sigma_{ii} = u_i$, $u = \mathrm{linspace}(dw, up, n)$, with $\mathrm{dw}, \mathrm{up} > 0$,
  for $i = 1, \dots, n$.
- Matrix dimensions: $m = 40$ and $n = 20$ for the tests and $m = 100$, $n = 50$ for performance profiles
- Performance profiles: 40 matrices, with condition number between 1 and 10^{10}. The optimality measure is $\frac{\|x - \hat{x}\|}{\|x\|}$, with x the exact solution ($x = (n - 1 : -1 : 0)$). A simulation is considered unsuccessful if the relative solution accuracy is larger than 10^{-2}.

How to choose ϵ?





$$\frac{\|x-\hat{x}_\epsilon\|}{\|x\|} \leq \frac{\|x-x_\epsilon\|}{\|x\|} + \frac{\|x_\epsilon-\hat{x}_\epsilon\|}{\|x\|} = \frac{\|x-x_\epsilon\|}{\|x\|} + \frac{\|x_\epsilon-\hat{x}_\epsilon\|}{\|x_\epsilon\|}\frac{\|x_\epsilon\|}{\|x\|}.$$



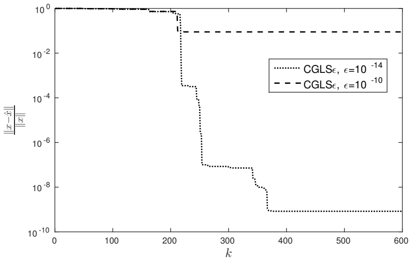
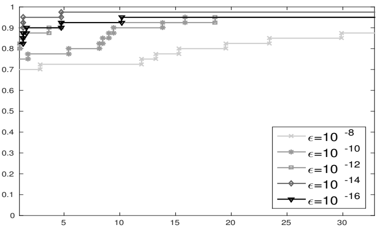


Figure: Left: right hand side of small norm, Right: right hand side of large norm



Comparison with CG

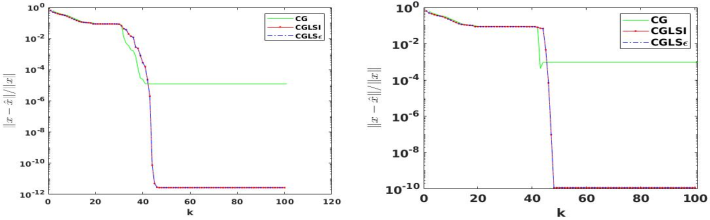
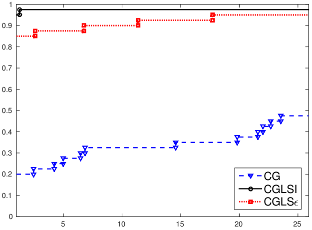


Figure: Left: $\kappa(A) = 10^5$, $\kappa(\hat{A}) = 10^5$. Right: $\kappa(A) = 10^7$, $\kappa(\hat{A}) = 10^{10}$.



Performance of CGLSI and CGLSϵ is comparable but

- CGLSI is parameter free
- CGLSI is less sensible to the right hand side

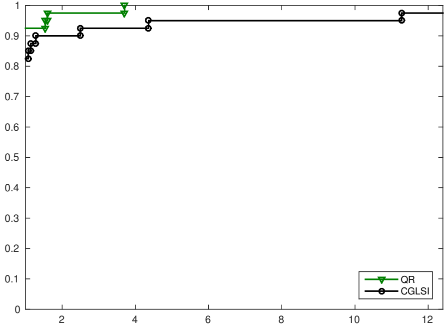
Much better performance than CG

Validation of error bounds

Problem	$\kappa(A)^2 u$	E_{CGLSI}	$\hat{E}_{CGLSI}$	$E_{CGLS\epsilon}$	$\hat{E}_{CGLS\epsilon}$
$a = 2.0$	10^{-10}	10^{-11}	10^{-8}	10^{-9}	10^{-9}
$a = 2.5$	10^{-2}	10^{-9}	10^{-6}	10^{-9}	10^{-6}
$a = 1.5$	10^{-10}	10^{-13}	10^{-11}	10^{-13}	10^{-11}
$a = 1.3$	10^{-12}	10^{-13}	10^{-13}	10^{-14}	10^{-13}
$a = 1.1$	10^{-14}	10^{-14}	10^{-13}	10^{-14}	10^{-14}
$a = 0.7$	10^{-10}	10^{-12}	10^{-12}	10^{-12}	10^{-12}
$\mathrm{up} = 1$	10^{-2}	10^{-9}	10^{-7}	10^{-8}	10^{-8}
$\mathrm{up} = 1$	10^{-2}	10^{-8}	10^{-4}	10^{-4}	10^{-6}
$a = 1.5$	10^{-10}	10^{-13}	10^{-12}	10^{-10}	10^{-10}

- Better performance that standard CG, both in terms of accuracy and of rate of convergence.
- The error bounds much better predict forward errors than classical bounds.

Comparison with QR method



- Propose method can compare with direct methods in terms of solution accuracy

THANK YOU FOR YOUR ATTENTION

Calandra, H., Gratton, S., Riccietti, E., Vasseur, X., **On the solution of systems of the form $A^T Ax = A^T b + c$, In preparation**

Effect of large right-hand sides

Let us assume to apply CG to SYS and CGLS to SYSϵ. We would respectively compute:

$$\alpha_1 = \frac{\|r_0\|^2}{p_1^\top A^\top A p_1} = \frac{\|A^\top b + c\|^2}{\|A(A^\top b + c)\|^2}, \qquad x_1 = \alpha_1 (A^\top b + c) = \alpha_1 p_1,$$

and

$$\alpha_1(\epsilon) = \frac{\|A^\top b + c\|^2}{\|A(A^\top b + c)\|^2 + \epsilon \|c^\top (A^\top b + c)\|}, \qquad x_1(\epsilon) = \alpha_1(\epsilon) p_1(\epsilon) = \alpha_1(\epsilon) p_1.$$

Notice that if ϵ tends to zero, so does the term $\epsilon\|c^\top(A^\top b + c)\|$ in the denominator of $\alpha_1(\epsilon)$. Consequently $\alpha_1(\epsilon)$ tends toward α_1 and $x_1(\epsilon)$ tends toward x_1. If ϵ has to be fixed, its value should be small enough to let $\epsilon\|c^\top(A^\top b + c)\|$ be small compared to $\|A(A^\top b + c)\|^2$, otherwise the found approximation will be close to a solution of SYSϵ rather than to one of SYS. This choice is then particularly difficult when $\|A^\top b + c\|$ is large.