Regularizing trust-region and Levenberg-Marquardt approaches for ill-posed nonlinear least squares problems

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Outline

- Introduction to iterative regularization methods.
- Zero residual problems:
  - Regularizing Levenberg-Marquardt (LM) methods.
  - Regularizing trust-region (TR) approaches.
  - Numerical tests: LM versus TR
- Small residual problems: Elliptical trust-region methods.
- Open issues and future developments.

Ill-posed problems

Let us consider the following inverse problem: given $F : \mathbb{R}^n \to \mathbb{R}^m$ with $m \geq n$, nonlinear, continuously differentiable and $y \in \mathbb{R}^m$, solve

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} \| F(x) - y \|^2.$$

We consider ill-posed problems:

- no finite bounds on the norm of the inverse of $J(x)$ can be used in the analysis;
- the solution does not depend continuously on the data.

Noisy case

In realistic situation only noisy data y^δ are given:

$$\|y - y^\delta\| \leq \delta,$$

where δ is the noise level.

- Applications: Data assimilation, geophysics, seismic inversion, fitting of exponentials, discretization of problems with compact operator
- Classical methods used for well-posed problems are not suitable in this contest.

Need for regularization

Iterative regularization methods

Let $x^\dagger$ be a solution of $\min \frac{1}{2}\|F(x) - y\|^2$.

Iterative regularization methods generate a sequence $\{x_k^\delta\}$. If the process is stopped at iteration $k^*(\delta)$ the method is supposed to guarantee the following properties:

- $x_{k^*(\delta)}^\delta$ is an approximation of $x^\dagger$;
- $\{x_{k^*(\delta)}^\delta\}$ tends to $x^\dagger$ if δ tends to zero;
- local convergence to $x^\dagger$ in the noise-free case.

Regularizing methods for zero residual problems

- Landweber (gradient-type method)[Hanke, Neubauer, Scherzer, 1995,Kaltenbacher, Neubauer, Scherzer, 2008]
- Truncated Newton - Conjugate Gradients [Hanke,1997, Rieder, 2005]
- Iterative Regularizing Gauss-Newton [Bakushinsky, 1992, Blaschke, Neubauer, Scherzer, 1997]
- Levenberg-Marquardt [Hanke,1997,2010,Vogel 1990, Kaltenbacher, Neubauer, Scherzer, 2008]
- Trust region methods [Wang, Yuan 2002,B., Morini, Riccietti 2016]

Most of these methods are analyzed only under local assumptions.

Levenberg-Marquardt method

- Given $x_k^\delta \in \mathbb{R}^n$ and $\lambda_k > 0$, we denote with $J \in \mathbb{R}^{m\times n}$ the Jacobian matrix of F. The step $p_k \in \mathbb{R}^n$ is the minimizer of

$$m_k^{LM}(p) = \frac{1}{2}\|F(x_k^\delta) - y^\delta + J(x_k^\delta)p\|^2 + \frac{1}{2}\lambda_k\|p\|^2;$$

- $p_k = p(\lambda_k)$ is the solution of

$$(B_k + \lambda_k I)p_k = -g_k$$

  with $B_k = J(x_k^\delta)^T J(x_k^\delta)$, $g_k = J(x_k^\delta)^T(F(x_k^\delta) - y^\delta)$;
- The step is then used to compute the new iterate

$$x_{k+1}^\delta = x_k^\delta + p_k.$$

Regularizing LM method for zero residual problems

- The parameter $\lambda_k > 0$ satisfies:

$$\|F(x_k^\delta) - y^\delta + J(x_k^\delta)p(\lambda_k)\| = q\|F(x_k^\delta) - y^\delta\|$$

  with $q \in (0,1)$;

- With noisy data the process is stopped at iteration $k^*(\delta)$ such that $x^\delta_{k^*(\delta)}$ satisfies the discrepancy principle:

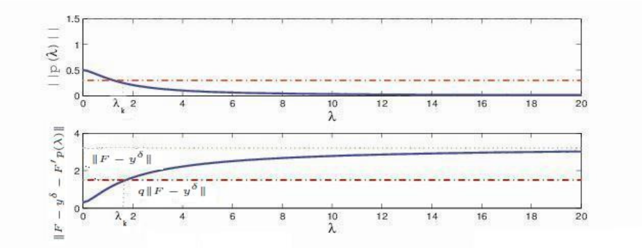
$$\|F(x^\delta_{k^*(\delta)}) - y^\delta\| \leq \tau\delta < \|F(x_k^\delta) - y^\delta\|$$

  for $0 \leq k < k^*(\delta)$ and $\tau > 1$ suitable parameter.

[Hanke, 1997,2010]

The role of the q-condition

- A sufficiently small step is needed in order to prevent to approach the solution of the noisy problem and to leave the region around $x^\dagger$
- The q-condition prevents to take too long steps



$\|p(\lambda)\|$ and $\|F - y^\delta + F'(p(\lambda))\|$ varying λ.

Local analysis

Hypothesis for the local analysis:
Given the starting guess x_0, it exist positive ρ and c such that

- the system $F(x) = y$ is solvable in $B_\rho(x_0)$;
- for $x, \tilde{x} \in B_{2\rho}(x_0)$

$$\|F(x) - F(\tilde{x}) - J(x)(x - \tilde{x})\| \leq c\|x - \tilde{x}\|\|F(x) - F(\tilde{x})\|.$$

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well posed problems: $\|F(x) - F(\tilde{x}) - J(x)(x - \tilde{x})\| \leq c\|x - \tilde{x}\|^2.$

- Due to the ill-posedness of the problem it is not possible to assume that a finite bound on the inverse of the Jacobian matrix exists.
- The Jacobian may be singular at the solution.

Regularizing properties of the LM method

Choosing λ_k solution of the q-condition

$$\|F(x_k^\delta) - y^\delta + J(x_k^\delta)p(\lambda_k)\| = q\|F(x_k^\delta) - y^\delta\|$$

and stopping the process when the discrepancy principle

$$\|F(x_{k^*(\delta)}^\delta) - y^\delta\| \leq \tau\delta < \|F(x_k^\delta) - y^\delta\|$$

is satisfied, Hanke proves that:

- With exact data ($\delta = 0$): local convergence to $x^\dagger$,
- With noisy data ($\delta > 0$): Choosing x_0 close to $x^\dagger$ the discrepancy principle is satisfied after a finite number of iterations $k^*(\delta)$ and $\{x_{k^*(\delta)}^\delta\}$ converges to a solution of $F(x) = y$ if δ tends to zero.

Regularizing method.