ENS de Lyon - M2 **Tuesday, December 18, 2018**

Stochastic calculus exam - Answers

EXERCISE I : A nonnegative semimartingale with a positive bias.

We consider B a standard Brownian motion, $\lambda > 0$ some fixed parameter, and X a continuous semimartingale. For $x \geq 0$, we denote by

$$T_x := \inf\{t \geq 0, X_t = x\}$$

the hitting time of x by the semimartingale X. We further suppose that X satisfies

$$X_t = 1 + B_{t\wedge T_0} + \int_0^{t\wedge T_0} \frac{\lambda}{X_s} \mathrm{d}s.$$

In particular, X solves the SDE

$$\begin{cases} X_0 &= 1 \\ \mathrm{d}X_t &= \mathbb{1}_{X_t>0}\mathrm{d}B_t + \mathbb{1}_{X_t>0}\frac{\lambda}{X_t}\mathrm{d}t. \end{cases}$$

1. Show that for $b > 1$, we have $T_0 \wedge T_b$ is almost surely finite.
   Writing $T_b(B) := \inf\{t \geq 0, B_t = b\}$, we know $T_b(B)$ is a.s. finite (1-dimensional brownian motion is recurrent). Observing the elementary property $X_t \geq B_t \mathbb{1}_{t\leq T_0}$, we deduce $T_0 \wedge T_b \leq T_b(B)$. So $T_0 \wedge T_b$ is a.s. finite.
2. For $\alpha \in \mathbb{R}$ and $a \in (0, 1)$, we consider $F_{\alpha,a}$ a C^2 function that coincides with $x \mapsto x^\alpha$ on $[a, +\infty)$. Write down Itō formula for the semimartingale $F_{\alpha,a}(X_t)$.
   Itō formula gives
   $$F_{\alpha,a}(X_t) = 1 + \int_0^t F'_{\alpha,a}(X_s)\mathrm{d}B_s + \int_0^t \left(\frac{F''_{\alpha,a}(X_s)}{2} + \frac{\lambda F'_{\alpha,a}(X_s)}{X_s}\right)\mathrm{d}s.$$
3. We suppose $\lambda \neq 1/2$. For a parameter $\alpha = \alpha(\lambda)$ that you will determine, deduce that the process $(X^\alpha_{t\wedge T_a})_{t\geq 0}$ is a local martingale, for any $a \in (0, 1)$.
   We deduce from last question
   $$F_{\alpha,a}(X_{t\wedge T_a}) = 1 + \int_0^{t\wedge T_a} F'_{\alpha,a}(X_s)\mathrm{d}B_s + \int_0^{t\wedge T_a} \left(\frac{F''_{\alpha,a}(X_s)}{2} + \frac{\lambda F'_{\alpha,a}(X_s)}{X_s}\right)\mathrm{d}s.$$

But on $[a,+\infty)$, the function $F_{\alpha,a}$ and its derivatives coincide with those of $x \mapsto x^\alpha$, so we get, if $\alpha \notin \{0,1\}$,

$$X^\alpha_{t\wedge T_a} = 1 + \int_0^{t\wedge T_a} \alpha X_s^{\alpha-1} \mathrm{d}B_s + \int_0^{t\wedge T_a} \left(\frac{\alpha(\alpha-1)X_s^{\alpha-2}}{2} + \lambda\alpha X_s^{\alpha-2} \right) \mathrm{d}s.$$

Choosing now $\alpha = \alpha(\lambda) = 1-2\lambda \notin \{0,1\}$, the second integral giving the process of finite variation part of the semimartingale is 0, and so $(X^\alpha_{t\wedge T_a})_{t\geq 0}$ is a local martingale.

4. For $a \in (0,1)$ and $b > 1$, compute $\mathbb{P}(T_a < T_b)$.
   From question 1, we know that $T_a \wedge T_b$ is a.s. finite. Now, $(X^\alpha_{t\wedge T_a \wedge T_b})_{t\geq 0}$ is a bounded martingale closed by $X^\alpha_{T_a\wedge T_b}$, whence

$$1 = \mathbb{E}[X^\alpha_{T_a\wedge T_b}] = a^\alpha \mathbb{P}(T_a < T_b) + b^\alpha \mathbb{P}(T_b < T_a).$$

   This together with $\mathbb{P}(T_a < T_b) + \mathbb{P}(T_b < T_a) = 1$ easily gives

$$\mathbb{P}(T_a < T_b) = \frac{b^\alpha - 1}{b^\alpha - a^\alpha}.$$

5. In this question only, we suppose $\lambda > 1/2$. Show X_t tends to $+\infty$ a.s. as $t \to +\infty$. (One may observe that $(X^\alpha_{t\wedge T_a})_{t\geq 0}$ is a bounded martingale).
   In that case we have $\alpha = 1-2\lambda < 0$, and we deduce from last question

$$\mathbb{P}(\exists b > 1, T_a < T_b) = \lim_{b\to\infty} \mathbb{P}(T_a < T_b) = a^{-\alpha}.$$

   But T_b tends to $+\infty$ as $b \to +\infty$ so we get $\mathbb{P}(T_a < +\infty) = a^{-\alpha}$. On the event $T_a = +\infty$, we have a.s., for any $b > 1$, $T_b = T_a \wedge T_b < +\infty$, and thus $\limsup X_t = +\infty$. But the process $(X^\alpha_{t\wedge T_a})_{t\geq 0}$ is a bounded martingale, which therefore converges a.s. We deduce that X_t tends to $+\infty$ a.s. on the event $T_a = +\infty$. Finally, X_t tends to $+\infty$ a.s. on the event

$$\cup_{a\in(0,1)}\{T_a = +\infty\},$$

   which has probability $\lim_{a\to 0}(1 - a^{-\alpha}) = 1$.

6. In this question only, we suppose $\lambda < 1/2$. Show that T_0 is a.s. finite.
   In that case we have $\alpha = 1-2\lambda > 0$. On the event $T_0 = +\infty$, we have $T_b = T_0 \wedge T_b < +\infty$ a.s., and there exists $a \in (0,1)$ such that $T_b < T_a$. (This is statisfied for a smaller than $\inf\{X_t, t \leq T_b\} > 0$). So

$$\mathbb{P}(T_0 = +\infty) \leq \mathbb{P}(\exists a \in (0,1), T_b < T_a) = \lim_{a\to 0} \mathbb{P}(T_b < T_a) = \lim_{a\to 0} \frac{1-a^\alpha}{b^\alpha - a^\alpha} = b^{-\alpha}.$$

   As $b > 1$ was arbitrary, we deduce $\mathbb{P}(T_0 = +\infty) = 0$.

7. Finally, we suppose $\lambda = 1/2$. Proceed similarly as in the previous questions to show that a.s., the hittings times T_x are finite for all $x \in (0, +\infty)$. In other words, the process X_t is recurrent on $(0, +\infty)$.

   In the case $\lambda = 1/2$, we obtain similarly, for $a \in (0,1)$, that $\log(X_{t \wedge T_a})$ is a local martingale, and we deduce

$$\mathbb{P}(T_a < T_b) = \frac{\log b}{\log(b/a)}.$$

   But now we deduce, by letting $b \to \infty$,

$$\mathbb{P}(T_a < \infty) = \mathbb{P}(\exists b > 1, T_a < T_b) = 1,$$

   and by letting $a \to 0$,

$$\mathbb{P}(T_b = \infty) = \mathbb{P}(T_0 < T_b) \leq \mathbb{P}(\forall a \in (0,1), T_a < T_b) = 0.$$

   Finally, we deduce that a.s., for $n \in \mathbb{N}$, the hitting times of n and of $1/n$ are finite, and so the process is recurrent.

EXERCISE II : Hitting time of the sphere for the biased planar Brownian motion.

We consider $B_t = (B_t^{(1)}, B_t^{(2)})_{t \geq 0}$ a planar Brownian motion started from 0, and T the hitting time of the sphere $\mathbb{S} = \{(x,y), x^2 + y^2 = 1\}$.

1. Explain briefly why $(B_T^{(1)}, B_T^{(2)})$ is independent from T and uniform on the sphere.

   This follows from the invariance of the law of planar Brownian motion (started from 0) by an isometry.

2. Introducing an appropriate martingale, deduce that the Laplace transform of T is given for $\lambda \geq 0$ by

$$\mathbb{E}[e^{-\lambda T}] = \frac{1}{\phi\left(\sqrt{2\lambda}\right)},$$

   where ϕ is the function defined for $x \geq 0$ by

$$\phi(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{x \cos t} \mathrm{d}t.$$

   Note you are not asked to compute this integral.

   For $\lambda \geq 0$, we consider the exponential martingale $M_t = \exp(\sqrt{2\lambda} B_t^{(1)} - \lambda t)$, and observe that $(M_{t \wedge T})_{t \geq 0}$ is a bounded martingale (bounded by $\exp(\sqrt{2\lambda})$), and closed by M_T. We thus have

$$\begin{aligned} 1 &= \mathbb{E}[M_0] = \mathbb{E}[M_T] = \mathbb{E}[T e^{\sqrt{2\lambda} B_T^{(1)}}] \\ &= \mathbb{E}[T]\mathbb{E}[e^{\sqrt{2\lambda}\cos(\theta_T)}] = \mathbb{E}[T]\phi(\sqrt{2\lambda}), \end{aligned}$$

   where θ_T, defined uniquely in $[0, 2\pi)$ by $(B_T^{(1)}, B_T^{(2)}) = (\cos\theta_T, \sin\theta_T)$, is uniform on $[0, 2\pi)$.

3. For a parameter $c \in \mathbb{R}$, we introduce now $C_t = (C_t^{(1)}, C_t^{(2)})_{t\geq 0}$ the Brownian motion with drift ct, defined for $t \geq 0$ by $C_t^{(1)} = B_t^{(1)} + ct$ and $C_t^{(2)} = B_t^{(2)}$. We also let T_C be the hitting time of the sphere for the process C. Show that, for any F measurable and bounded functional on the Wiener space $C(\mathbb{R}_+)$ of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$, we have

$$\mathbb{E}\left[F((C_{t\wedge T_C})_{t\geq 0})\right] = \mathbb{E}\left[F((B_{t\wedge T})_{t\geq 0})e^{cB_T - \frac{c^2}{2}T}\right].$$

Define the processes L and D by $L_t := cB_{t\wedge T}^{(1)}$ and $D_t := \mathcal{E}(L)_t = \exp(cB_{t\wedge T}^{(1)} - \frac{c^2(t\wedge T)}{2})$. The local martingale D is actually a true bounded martingale, closed by

$$D_\infty = \exp(cB_T^{(1)} - \frac{c^2 T}{2}).$$

Moreover, we have $\langle B^{(1)}, L\rangle_t = c(t\wedge T)$. We consider the probability measure $\mathbb{Q}$ which is absolutely continuous with respect to $\mathbb{P}$, and with Radon-Nikodym derivative D_∞. It follows from Girsanov theorem that the process $(B_t^{(1)} - c(t\wedge T), B_t^{(2)})_{t\geq 0}$ is a planar Brownian motion under $\mathbb{Q}$ (its coordinates are local martingales, with bracket those expected from a Brownian motion). Thus the law of the process $(C_{t\wedge T})_{t\geq 0}$ (under $\mathbb{P}$) is equal to the law of $(B_{t\wedge T})_{t\geq 0}$ under $\mathbb{Q}$. Thus, for F measurable and bounded functional,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[F((C_{t\wedge T_C})_{t\geq 0})\right] &= \mathbb{E}_{\mathbb{Q}}\left[F((B_{t\wedge T})_{t\geq 0})\right] \\ &= \mathbb{E}_{\mathbb{P}}\left[F((B_{t\wedge T})_{t\geq 0})e^{cB_T - \frac{c^2}{2}T}\right].\end{aligned}$$

4. Deduce the Laplace transform of T_C is given by

$$\mathbb{E}[e^{-\lambda T_C}] = \frac{\phi(c)}{\phi\left(\sqrt{2\lambda + c^2}\right)}.$$

From last question, we have

$$\begin{aligned}\mathbb{E}[e^{-\lambda T_C}] &= \mathbb{E}[e^{cB_T^{(1)} - \frac{c^2 T}{2}} e^{-\lambda T}] \\ &= \mathbb{E}[e^{cB_T^{(1)}}]\mathbb{E}[e^{-(\lambda + \frac{c^2}{2})T}] = \frac{\phi(c)}{\phi(\sqrt{2\lambda + c^2})},\end{aligned}$$

using the independence of T and $B_T^{(1)}$, and last questions.

PROBLEM : Local martingales with product of finite variation

This problem starts with two preliminary questions, and then studies the local martingales whose product is a process of finite variation.

1. In this question, we consider Y a stochastic process with continuous paths started from $Y_0 = 1$, and staying null after its first hitting time of 0. Thus, writing $T_0 := \inf\{t \geq 0, Y_t = 0\}$, we have $Y_t = 0$ for every $t \geq T_0$. We further suppose that there exists a nondecreasing sequence of stopping times $(S_n)_{n\geq 0}$ converging to T_0 and such that for every $n \geq 0$, the stochastic process $(Y_{t\wedge S_n})_{t\geq 0}$ is a local martingale. Prove Y is a local martingale.

   Hint : You may first suppose that the process Y is bounded.

   As suggested by the hint, we first suppose that the process Y is bounded, in which case the processes Y^{S_n} defined by $Y_t^{(S_n)} = Y_{t\wedge S_n}$ are true martingales. Thus, for $0 \leq s \leq t$, we have, for every n,

$$\mathbb{E}[Y_t^{S_n}|\mathcal{F}_s] = Y_s^{S_n}.$$

   Moreover, the sequence $(Y_s^{S_n})_{n\geq 0}$ converges a.s. to Y_t and is bounded, thus we can use the conditional dominated convergence theorem and get

$$\mathbb{E}[Y_t|\mathcal{F}_s] = Y_s.$$

   In the general case, we define $T_k = \inf\{t \geq 0, Y_t \geq k\}$ for $k \geq 2$ integer. Then $Y_{t\wedge T_k}$ is bounded and the processes $Y_{t\wedge T_k \wedge S_n}$ are of course still martingales. We thus deduce that $(Y_{t\wedge T_k})_{t\geq 0}$ is a martingale. Finally, Y is a local martingale reduced by the sequence of stopping times $(T_k)_{k\geq 2}$.

2. Suppose M is a local martingale started from 0 such that $\langle M\rangle_\infty$ is a.s. finite. Show
   (a) The probability of the event $\{\langle M\rangle_\infty \leq u, M_\infty + \frac{1}{2}\langle M\rangle_\infty \geq v\}$ goes to 0 when $v \to \infty$, uniformly on the chosen local martingale M.
   (b) The probability of the event $\{M_\infty + \frac{1}{2}\langle M\rangle_\infty \leq u, \langle M\rangle_\infty \geq v\}$ goes to 0 when $v \to \infty$, uniformly on the chosen local martingale M.

   We use the generalized version of Dubins-Schwarz theorem to write

$$M_t = B_{\langle M\rangle_t},$$

   for B a brownian motion started from 0. On the event $E_{u,v} := \{\langle M\rangle_\infty \leq u, M_\infty + \frac{1}{2}\langle M\rangle_\infty \geq v\}$, there exists $t \leq u$ such that $B_t + \frac{t}{2} \geq v$. Thus, writing $B_u^\star := \sup_{t\leq u} B_t$, we can write

$$\mathbb{P}(E_{u,v}) \leq \mathbb{P}(B_u^\star \geq v - \frac{u}{2})$$

   This gives an upper bound independent of the chosen local martingale M, and which converges to 0 when $v \to +\infty$.

   On the event $\widetilde{E}_{u,v} = \{M_\infty + \frac{1}{2}\langle M\rangle_\infty \leq u, \langle M\rangle_\infty \geq v\}$, there exists $t \geq v$ such that $\frac{B_t}{t} \leq \frac{u}{t} - \frac{1}{2}$. Thus, for $v \geq 4u$,

$$\mathbb{P}(\widetilde{E}_{u,v}) \leq \mathbb{P}(\inf_{t\geq v} \frac{B_t}{t} \leq \frac{1}{4}).$$

   Again, the upper bound does not depend on the choice of the local martingale. To see it converges to 0, it suffices to use the property that B_t/t tends to 0 a.s. as $t \to +\infty$.

3. We now consider (X, Y) a pair of local martingales and suppose their product $Z_t = X_t Y_t$ is a process of finite variation.
   (a) Show the local martingale N defined by

$$N_t = \int_0^t X_s \mathrm{d}Y_s + \int_0^t Y_s \mathrm{d}X_s$$

   is indistinguishable from 0.

   The integration by parts for local martingales gives

$$Z_t = Z_0 + N_t + \langle X, Y \rangle_t.$$

   But Z is by hypothesis a process of finite variation. It follows that N is both a process of finite variation and a local martingale, therefore it is indistinguishable from 0.

   (b) Compute $\langle N \rangle$ and deduce that the process

$$t \mapsto \int_0^t Z_s \mathrm{d}\langle X, Y \rangle_s$$

   is nonincreasing.

   Since N is indistinguishable from 0, its quadratic variation is 0, from which it follows, for $t \geq 0$,

$$\int_0^t X_s^2 \mathrm{d}\langle Y \rangle_s + \int_0^t Y_s^2 \mathrm{d}\langle X \rangle_s + 2 \int_0^t Z_s \mathrm{d}\langle X, Y \rangle_s = 0.$$

   Rewriting this as

$$\int_0^t Z_s \mathrm{d}\langle X, Y \rangle_s = -\frac{1}{2} \left(\int_0^t X_s^2 \mathrm{d}\langle Y \rangle_s + \int_0^t Y_s^2 \mathrm{d}\langle X \rangle_s \right),$$

   it is now clear that this process is nonincreasing.

   (c) Deduce that Z_t^2 is nonincreasing.

   Using again the integration by parts formula (or Itō formula) for the process of finite variation Z, we get

$$Z_t^2 = Z_0^2 + 2 \int_0^t Z_s \mathrm{d}Z_s = Z_0^2 + 2 \int_0^t Z_s \mathrm{d}\langle X, Y \rangle_s,$$

   and thus Z^2 is nonincreasing.

   As a consequence, if $X_0 = 0$, then for every $t \geq 0$, we have $Z_0 = 0$ and thus either $X_t = 0$ or $Y_t = 0$. In the following, we suppose $X_0 = Y_0 = 1$ and the processes X and Y stay still after their first hitting time of 0.

4. In this question only, we further suppose that X and Y stay (strictly) positive. Show there is a local martingale L starting from 0 such that $X_t = \mathcal{E}(L)_t$ and $Y_t = \mathcal{E}(-L)_t$. Reciprocally, for every local martingale L starting from 0, show the product of the local martingales $\mathcal{E}(L)$ and $\mathcal{E}(-L)$ is indeed a process of finite variation.

Under the hypotheses that the processes X and Y start from 0 and stay positive, the unique way to write $X = \mathcal{E}(L)$ and $Y = \mathcal{E}(\widetilde{L})$ is by taking

$$L_t := \int_0^t \frac{1}{X_s} \mathrm{d}X_s, \widetilde{L}_t := \int_0^t \frac{1}{Y_s} \mathrm{d}Y_s.$$

Using the definition of N and the fact it is indistinguishable from 0, we deduce

$$0 = \int_0^t \frac{1}{X_s Y_s} \mathrm{d}N_s = L_t + \widetilde{L}_t.$$

Thus $\widetilde{L} = -L$. Reciprocally, if L is a local martingale starting from 0, then the product of the local martingales $\mathcal{E}(L)\mathcal{E}(-L)$ is equal to the process of finite variation $(e^{-\langle L \rangle_t})_{t \geq 0}$.

5. For $\varepsilon \in (0, 1)$, we let $T_\varepsilon := \inf\{t \geq 0, X_t = \varepsilon \text{ or } Y_t = \varepsilon\}$ and define

$$L_t^\varepsilon := \int_0^{t \wedge T_\varepsilon} \frac{1}{X_s} \mathrm{d}X_s.$$

(a) Show we have $X^{T_\varepsilon} = \mathcal{E}(L^\varepsilon)$ and $Y^{T_\varepsilon} = \mathcal{E}(-L^\varepsilon)$.
We just use last question with the processes X^{T_ε} and Y^{T_ε}.

(b) We also let $T_0 := \inf\{t \geq 0, X_t = 0 \text{ or } Y_t = 0\}$ and $T_0(X) := \inf\{t \geq 0, X_t = 0\}$. On the event $T_0 = T_0(X) < +\infty$, show that $\langle L^\varepsilon \rangle_{+\infty}$ tends to $+\infty$ a.s. when ε decreases to 0.
Observe that $\langle L^\varepsilon \rangle_\infty = \int_0^{T_\varepsilon} \frac{1}{X_s^2} \mathrm{d}\langle X \rangle_s$ is monotone in ε, therefore if it does not go to $+\infty$ then it is bounded.
Moreover, on the event $T_0 = T_0(X) < +\infty$, we have $X_{T_\varepsilon} \to 0$, whence

$$L_\infty^\varepsilon - \frac{1}{2} \langle L^\varepsilon \rangle_\infty \underset{\varepsilon \to 0}{\to} -\infty.$$

Finally, it follows from question 2.(a) that the probability of the event

$$\{\langle L^\varepsilon \rangle_\infty \textit{ is bounded}, -L_\infty^\varepsilon + \frac{1}{2} \langle L^\varepsilon \rangle_\infty \underset{\varepsilon \to 0}{\to} +\infty\}$$

is 0, and the result follows.

(c) Deduce that, on the event $T_0 < +\infty$, we have $X_{T_0} = Y_{T_0} = 0$ a.s.
It suffices to prove that, on the event $T_0 = T_0(X) < +\infty$, we have $Y_{T_0} = 0$ almost surely. To get this, observe first that we have on this event $\langle L^\varepsilon \rangle_\infty \to +\infty$ almost surely. Now, by question 2.(b), the probability of the event that $\langle L^\varepsilon \rangle_\infty$ tends to $+\infty$ but $L_\infty^\varepsilon + \frac{1}{2} \langle L^\varepsilon \rangle_\infty$ stays bounded, is 0. We deduce that, on the event $T_0 = T_0(X) < +\infty$, we have a.s. $\liminf Y_{T_\varepsilon} = 0$, whence $Y_{T_0} = 0$.

6. In this question, suppose X is any local martingale starting from 1 and staying null after its first hitting time of 0. Deduce from previous questions that there is a unique local martingale Y starting from 1 and staying null after its first hitting time of 0 such that XY is a process of finite variation, and propose a description of it.

   By previous questions and in particular 5.(a) and 5.(c), if Y is a process satisfying the conditions, then we must have $Y_t = \frac{1}{X_t} e^{-\int_0^t \frac{1}{X_s^2} \mathrm{d}\langle X \rangle_s}$ for every $t \leq T_\varepsilon$, and the process Y has to hit 0 at time $T_0(X)$. These properties actually define unambiguously the process Y (up to indistinguishability). We thus now define the process Y by

$$Y_t := \begin{cases} \frac{1}{X_t} e^{-\int_0^t \frac{1}{X_s^2} \mathrm{d}\langle X \rangle_s} & \text{if } t < T_0(X) \\ 0 & \text{if } t \geq T_0(X) \end{cases}$$

   By question 4, the product XY, stopped at time T_ε, is nonincreasing. We prove, just like[1] in 5.(c), that $Y_{T_0(X)} = 0$, and hence Y is continuous, as well as the product XY, which is thus nonincreasing on the whole $\mathbb{R}_+$ (and stays null after hitting 0). Finally, Y^{T_ε} is a local martingale for any $\varepsilon > 0$, and by question 1, we deduce that Y is a local martingale.

1. The only reason why we cannot use directly 5.(c), is that in 5.(c) we already know that Y is a local martingale