Laboratoire de l'Informatique du Parallélisme

École Normale Supérieure de Lyon
Unité Mixte de Recherche CNRS-INRIA-ENS LYON n^o 5668

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A new bound on the 2-dimension of partially ordered sets

Thierry Eric

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École Normale Supérieure de Lyon
46 Allée d'Italie, 69364 Lyon Cedex 07, France
Téléphone : +33(0)4.72.72.80.37
Télécopieur : +33(0)4.72.72.80.80
Adresse électronique : `lip@ens-lyon.fr`

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Abstract

This paper provides a new upper bound on the 2-dimension of partially ordered sets. The 2-dimension of an ordered set P is the smallest cardinality of a set S such that there exists an order-embedding of P into the boolean lattice 2^S (all the subsets of S ordered by inclusion). The proof is non-contructive and uses a probabilistic argument. We link the result and the proof with two known theorems of the theory of ordered sets.

Keywords: partially ordered set, 2-dimension, order-embedding, boolean lattice.

Résumé

Ce papier présente une nouvelle borne supérieure sur la 2-dimension des ensembles ordonnés. La 2-dimension d'un ordre P est le cardinal minimum d'un ensemble S tel qu'il existe un plongement d'ordre de P dans le treillis booléen 2^S (composé de tous les sous-ensembles de S ordonnés par l'inclusion). La preuve est non-contructive et utilise un argument probabiliste. Nous rapprochons ce résultat et sa preuve de deux théorèmes connus en théorie des ensembles ordonnés.

Mots-clés: ensemble ordonnées, 2-dimension, plongements d'ordre, treillis booléen.

A new bound on the 2-dimension of partially ordered sets

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Abstract

This paper provides a new upper bound on the 2-dimension of partially ordered sets. The 2-dimension of an ordered set P is the smallest cardinality of a set S such that there exists an order-embedding of P into the boolean lattice 2^S (all the subsets of S ordered by inclusion). The proof is non-contructive and uses a probabilistic argument. We link the result and the proof with two known theorems of the theory of ordered sets.

1 Definitions and notations

Let $P = (X, \leq_P)$ be a partial order (or *order*) on the ground set X. We only consider finite orders and we also denote by $|P|$ the cardinal of X. The same order relation $\leq_P$ restricted to a subset Y of X is called a *suborder* of P and also referred as the *order induced* by P on Y. Let $x, y \in X$, $x \neq y$, we say that x and y are *comparable* in P if either $x \leq_P y$ or $y \leq_P x$. Otherwise we say that x and y are *incomparable.* an order where every pair of elements is comparable is called a *chain.* By extension, for the order $P = (X, \leq_P)$, a nonempty subset Y of X is called a *chain* of P if every pair of elements of Y is comparable in P. The maximum cardinality of a chain of P minus 1 is called the *height* of P and is denoted by $h(P)$. An element $x \in X$ is called the *maximum* (resp. *minimum*) of P if for all $y \in X$, $y \leq_P x$ (resp. $x \leq_P y$).

The *strict order relation* for $P = (X, \leq_P)$ is denoted by $<_P$ and defined for all $x, y \in X$ as $x <_P y$ if $x \leq_P y$ and $x \neq y$. For each $x \in X$, we will consider the set of *predecessors* (resp. *successors*) of x in P defined by $Pred_P(x) = \{y \in X | y <_P x\}$ (resp. $Succ_P(x) = \{y \in X | x <_P y\}$). For each $x \in X$, we will also consider its *ideal* $\downarrow_P x = Pred_P(x) \cup \{x\}$ and its *filter* $\uparrow_P x = Succ_P(x) \cup \{x\}$.

The *dual* of P, denoted by P^d, is the order $P^d = (X, \leq_{P^d})$ where $x \leq_{P^d} y$ if and only if $y \leq_P x$.

Let $x, y \in X$, the couple (x, y) is called an *arrow pair* if $x \not\leq_P y$ and for all $a \in X$, $a <_P x \Rightarrow a \leq_P y$ and for all $a \in X$, $y <_P a \Rightarrow x \leq_P a$. It is denoted by $x \swarrow\!\!\nearrow_P y$. It is known that for all $x, y \in X$, $x \not\leq_P y$, there exist $u, v \in X$ such that $u \swarrow\!\!\nearrow_P v$, $u \leq_P x$ and $y \leq_P v$ [9].

A lattice $L = (X, \leq_L)$ is an order such that for all $x, y \in X$, the pair $\{x, y\}$ has an infimum $x \wedge_L y$ and a supremum $x \vee_L y$. For instance, the set of all the subsets of a fixed set S ordered by inclusion is a lattice. It is denoted by 2^S for short and called a *boolean lattice of dimension* $|S|$. All boolean lattices of dimension k are isomorphic.

Let $P = (X, \leq_P)$ and $Q = (Y, \leq_Q)$ be two orders. An *embedding* from P into Q is an mapping ϕ from X into Y such that for all $x, y \in X$, $x \leq_P y$ if and only if $\phi(x) \leq_Q \phi(y)$. By requiring that Q should belong to a particular class of orders, different interesting classes of embeddings can be defined.

This note provides a new result about *bit-vector encodings* of orders which are embeddings into boolean lattices and are associated with the parameter called *2-dimension.* Let $P = (X, \leq_P)$ be an order, a *bit-vector encoding* of P is an mapping ϕ from X into 2^S (the set of all the subsets of a set S, ordered by inclusion) such that for all $x, y \in X$, $x \leq_P y$ if and only if $\phi(x) \subseteq \phi(y)$. We will only consider finite orders and the *size* of the encoding ϕ is the cardinal of S. There always exists a *canonical* bit-vector encoding embedding P into 2^X and defined for all $x \in X$ by $\phi(x) = \{y \in X | y \leq_P x\}$. Given an order P, the smallest size of a bit-vector encoding of P is called the *2-dimension* of P and denoted by $\mathbf{Dim_2}(P)$.

Bit-vectors encodings and the 2-dimension were originally studied from a mathematical point of view and later from an algorithmic point of view. They have found applications in the fields of databases, knowledge representation or object oriented programming. The interested reader is referred to the syntheses of Nourine and Habib [3], Habib et al [4] or Thierry [6].

2 The upper bound

Some interesting bounds on the 2-dimension have been provided in the litterature for particular classes of orders: trees, crowns, ideals of booleans lattices, distributive and extremal lattices (see [6]). However in the general case, we only had the trivial classical bounds given in Proposition 1.

Proposition 1 *Let $P = (X, \leq_P)$ be an order, then*

$$h(P) \leq \mathbf{Dim_2}(P) \leq |P|.$$

These lower and upper bounds are reached for some orders. Trotter even gave a characterization of the orders such that $\mathbf{Dim_2}(P) = |P|$ [7] and Habib and al proved that it is $\mathcal{NP}$-complete to recognize the orders such that $\mathbf{Dim_2}(P) = h(P)$ [4] even though they are numerous (for instance chains reach the lower bound). However there was a hope that we could bound more precisely the 2-dimension of an order using some other parameters of the order. This was suggested by the results previously obtained in the theory of ordered sets and it lies within a more general framework: finding tight bounds for parameters hard to compute (e.g. $\mathcal{NP}$-complete) in terms of parameters easy to compute.

Theorem 1 *Let $P=(X, \leq_P)$ be an order, $D(P)=\max\{|\downarrow_P x|, x\in X\}$ (resp. $U(P)=\max\{|\uparrow_P x|, x\in X\}$). Then*

$$\mathbf{Dim_2}(P) \leq \lceil 2e(D(P)+1)\ln|X|\rceil \quad (\text{resp. } \mathbf{Dim_2}(P) \leq \lceil 2e(U(P)+1)\ln|X|\rceil\ ).$$

To prove this result, we resort to a probabilistic argument. Given $P = (X, \leq_P)$, take a set S of a fixed cardinal t, then consider the space $\mathcal{E}$ of all applications from X into 2^S. The idea is to find an appropriate cardinal t and an appropriate probability distribution on $\mathcal{E}$ such that we can prove that the probability that an application of $\mathcal{E}$ is an embedding, is strictly positive.

One simple way to define a probability distribution on $\mathcal{E}$ is to associate with each element i of S a real number p_i, $0 \leq p_i \leq 1$, and say that for the applications $\phi \in \mathcal{E}$, $Prob(i \in \phi(x)) = p_i$ for all $i \in S$ and $x \in X$, and where all these events are independent.

However, for a given t and a probability distribution on $\mathcal{E}$, proving directly by counting/probabilistic arguments that the probability that there exists one embedding of P in $\mathcal{E}$ is strictly positive, is rather difficult since the constraints involved by embeddings are highly codependent and can not be easily turned into tight inequalities.

One way to bypass this difficulty is to relax the properties of the searched object (namely the embedding) by looking for a variation object (here a "quasi-embedding") which enables to reconstruct a searched object and which is more frequent (useful for counting and proving that there exists one such varaition object).

Here is the definition of the variation object we will consider in our case. Given the order $P = (X, \leq_P)$, let S be a set of cardinal t and $\mathcal{E}$ the set of all applications from X into 2^S. An application ϕ of $\mathcal{E}$ is a *quasi-embedding* if the application $\tilde{\phi}$ defined for all $x \in X$ by $\tilde{\phi}(x) = \bigcup_{y \leq_P x} \phi(y)$ is an embedding of P (note that any embedding is a quasi-embedding). Figure 1 presents an order with *(i)* a bit-vector encoding (of optimal size), *(ii)* a quasi-embedding ϕ and *(iii)* the associated embedding $\tilde{\phi}$.

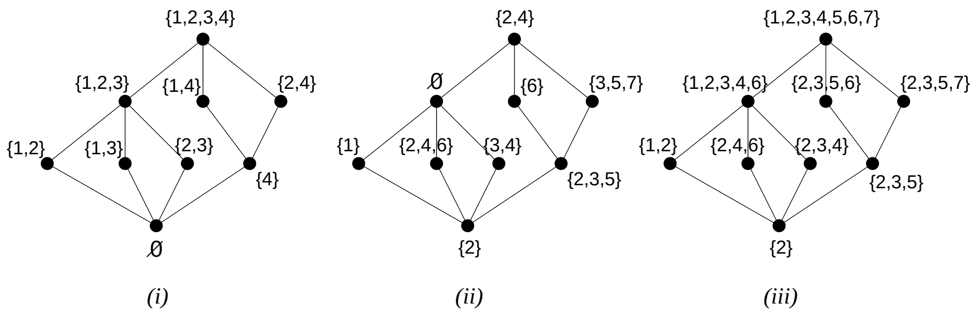

Figure 1: (i) an embedding, (ii) a quasi-embedding with (iii) its associated embedding.

The next proposition provides a characterization of quasi-embeddings.

Proposition 2 *Let $P = (X, \leq_P)$ be an order and ϕ an application from X into 2^S. Then the following statements are equivalent:*

(1) ϕ is a quasi-embedding.

(2) for all $x, y \in X$, $x \not\leq_P y$, there exists $i \in S$ such that $\exists u \leq_P x$, $i \in \phi(u)$ and $\forall v \leq_P y$, $i \notin \phi(v)$.

(3) for all $x, y \in X$, $x \swarrow\!\!\!\nearrow_P y$, there exists $i \in S$ such that $i \in \phi(x)$ and $\forall v \leq_P y$, $i \notin \phi(v)$.

Proof
(1) $\implies$ (2) since $\tilde{\phi}$ as defined before is an embedding and $x \not\leq_P y$ implies $\tilde{\phi}(x) \not\subseteq \tilde{\phi}(y)$. Let $i \in \tilde{\phi}(x) \backslash \tilde{\phi}(y)$, it clearly satifies the condition (2) due to the definition of $\tilde{\phi}$.

(2) $\implies$ (1) since the definition of $\tilde{\phi}$ implies that for all $x, y \in X$, $x \leq_P y$ implies $\tilde{\phi}(x) \subseteq \tilde{\phi}(y)$. Now let $x, y \in X$, $x \not\leq_P y$ and $i \in S$ satisfying the condition (2), then by definition of $\tilde{\phi}$, we have $i \in \tilde{\phi}(x) \backslash \tilde{\phi}(y)$ and thus $\tilde{\phi}(x) \not\subseteq \tilde{\phi}(y)$. Consequently, $\tilde{\phi}$ is an embedding.

(2) $\implies$ (3) since for all $x, y \in X$, $x \swarrow\!\!\!\nearrow_P y$, we have $x \not\leq_P y$. Thus by condition (2), there exists $i \in S$ such that $\exists u \leq_P x$, $i \in \phi(u)$ and $\forall v \leq_P y$, $i \notin \phi(v)$. If $u <_P x$, due to the definition of $\swarrow\!\!\!\nearrow_P$, $u \leq_P y$ and thus $i \notin \phi(u)$. We necessarily have $u = x$ and i satisfies the condition (3).

(3) $\implies$ (2) since for all $x, y \in X$, $x \not\leq_P y$, we know that there exist $u, v \in X$ such that $u \swarrow\!\!\!\nearrow_P v$, $u \leq_P x$ and $y \leq_P v$. The condition (3) implies that there exists $i \in S$ such that $i \in \phi(u)$ and $\forall w \leq_P v$, $i \notin \phi(w)$. Thus u satisfies condition (2) and $\forall w \leq_P y$, we have $w \leq_P v$ and $i \notin \phi(w)$. $\square$

Proof of Theorem 1

Given $P = (X, \leq_P)$ a fixed order, take a set S of cardinal t, namely $S = \{1, \ldots, t\}$ and consider the space $\mathcal{E}$ of all applications from X into 2^S. Define a probability distribution on $\mathcal{E}$ as mentioned before: associate with each element i of S a real number p_i, $0 \leq p_i \leq 1$, and say that for the applications $\phi \in \mathcal{E}$, $Prob(i \in \phi(x)) = p_i$ for all $i \in S$ and $x \in X$, and assume that all these events are independent. This correctly defines a probability distribution on $\mathcal{E}$. Now let ϕ be a random application of $\mathcal{E}$ with respect to our distribution, we can evaluate the probability that ϕ is a quasi-embedding.

Due to Proposition 2, ϕ is a quasi-embedding if and only if for all $x, y \in X$, $x \swarrow\!\!\!\nearrow_P y$, we have $\mathcal{A}_{x,y,i}$ where $\mathcal{A}_{x,y,i}$ is the event "there exists $i \in S$ such that $i \in \phi(x)$ and $\forall v \leq_P y$, $i \notin \phi(v)$". For fixed parameters x,y,i, this event occures with probability $Prob(\mathcal{A}_{x,y,i}) = p_i q_i^{D(y)}$ where $q_i = 1 - p_i$ and $D(y) = |\downarrow_P y|$. Thus $Prob(\mathcal{A}_{x,y,i}) \leq p_i q_i^{D(P)}$.

Let us bound the probability that ϕ is not a quasi-embedding:

$$Prob(\phi \text{ is not a quasi-embedding}) = Prob(\bigcup_{x \swarrow\!\!\!\nearrow_P y} \bigcap_{1 \leq i \leq t} \overline{\mathcal{A}_{x,y,i}}) \leq \sum_{x \swarrow\!\!\!\nearrow_P y} \prod_{1 \leq i \leq t} (1 - p_i q_i^{D(P)}).$$

A rough upper bound of the number of couples (x, y) such that $x \swarrow\!\!\!\nearrow_P y$ is n^2 where $n = |X|$. Thus

$$Prob(\phi \text{ is not a quasi-embedding}) \leq n^2 \prod_{1 \leq i \leq t} (1 - p_i q_i^{D(P)}).$$

Now we look for values of t and the p_i's such that $n^2 \prod_{1 \leq i \leq t}(1 - p_i q_i^{D(P)}) < 1$. It will ensures that $Prob(\phi$ is not a quasi-embedding$) < 1$ and thus that there exists at least one quasi-embedding in $\mathcal{E}$, which is equivalent to say that there exists at least one embedding from P into 2^S or to say that $dim_2(P) \leq t$.

$$n^2 \prod_{1 \leq i \leq t} (1 - p_i q_i^{D(P)}) < 1 \text{ if and only if } 2\ln(n) + \sum_{1 \leq i \leq t} \ln(1 - p_i q_i^{D(P)}) < 0.$$

This is true if $2\ln(n) < \sum_{1 \leq i \leq t} p_i q_i^{D(P)}$ (since $\ln(1 - x) \leq -x$ for all real $x < 1$). We are free to choose the p_i's, so we take the same value p for all of them: $p_1 = \ldots = p_t = p$ (and we denote $q = 1 - p$). The preceding inequality is now equivalent to $2\ln(n) < tpq^{D(P)}$ or (if $p \neq 0$ and $p \neq 1$)

$$2\ln(n) \frac{1}{(1 - q) q^{D(P)}} < t$$

We are interested in the smallest values of t such that this inequality holds (since we wish to give a small upper bound to $dim_2(P)$). A careful analysis (using derivation) of the function f defined on $]0,1[$ by $f(q) = \frac{1}{(1-q)q^{D(P)}}$ shows that it decreases from $+\infty$ down to a minimum at $q_{\min} = D(P)/(D(P)+1)$ (which belongs to $]0,1[$) and then it increases to $+\infty$. Consequently, we choose to take $q = D(P)/(D(P)+1)$ for the probability distribution, then our sufficient inequality ensuring the existence of quasi-embeddings (and thus embeddings into 2^S) is

$$2\ln(n)(D(P)+1)\big(1+\frac{1}{D(P)}\big)^{D(P)} < t$$

We always have $(1+1/D(P))^{D(P)} < e$, thus to conclude, if $2e(D(P)+1)\ln(n) < t$, then for the choosen probability distribution $Prob(\phi$ is a quasi-embedding$) > 0$, which implies that there exists at least one embedding from P into 2^S (in other words, $dim_2(P) \leq t$). □

3 Remarks and open questions

Links to previous results

As mentionned previously, the form of the upper bound of Theorem 1 is close to known results in the theory of ordered sets: similar theorems exist for the 2-dimension of particular orders and for the dimension of general orders. Here is the first theorem which bounds the 2-dimension of some ideals of boolean lattices. Its proof by Kierstead can be found in [5] (Theorem 2.4, page 224).

Theorem 2 (Kierstead [5]) *Let $n \geq k \geq 1$ and $\mathcal{B}(1,k;n)$ the set of all 1-subsets of $\{1,\ldots,n\}$ and all k-subsets of $\{1,\ldots,n\}$ (ordered by inclusion). Let $dim_2(1,k;n)$ be the 2-dimension of this order, then:*

$$dim_2(1,k;n) \leq e(k+1)^2\ln(n).$$

A careful look at the proof shows that it is exactly the same reasoning as the one of Theorem 1 in this very particular case (it can be seen that the characterization given in their proposition 0.8 corresponds to the definition of quasi-embeddings for this particular case). This proof contains the fact that we can use parameters in the definition of the probability distribution (so that we can adjust them later) instead of basically using a uniform distribution. However it lacks the definition of quasi-embeddings for an extension to the general case.

The other close theorem by Füredi and Kahn [1] deals with the dimension of orders, a proof can be found in [8] (Theorem (4.1), page 166).

Theorem 3 (Füredi, Kahn [1]) *Let $P = (X, \leq_P)$ be an order and $D(P) = \max\{|\downarrow_P x|, x \in X\}$, then*

$$dim(P) < 2(D(P)+1)\ln(n)$$

The probabilistic proof uses the notion of quasi-realizers (instead of the usual realizers for the dimension). A careful look at the definition of quasi-realizers shows that it exactly corresponds to quasi-embeddings in the case of the dimension. In fact, a general definition involving these two notions can be given in the framework of embeddings into products of chains. For $k \geq 1$, we denote by $[1,k]$ the usual order on the integers $\{1,\ldots,k\}$.

Definition 1 *Let $\Pi = [1,k_1] \times \cdots \times [1,k_d]$ be the product of d chains (ordered with the usual product order). Let $P = (X, \leq_P)$ be an order. An application $\phi = (\phi_1,\ldots,\phi_d)$ from X into $[1,k_1] \times \cdots \times [1,k_d]$ is called a Π-quasi-embedding if the application $\tilde{\phi}$ defined for all $x \in X$ by $\tilde{\phi}(x) = (\max_{y \leq_P x} \phi_1(y), \ldots, max_{y \leq_P x} \phi_d(y))$, is an embedding of P into Π.*

Our quasi-embeddings clearly correspond to Π-quasi-embedding when $\Pi = [1,2] \times \cdots \times [1,2]$, and quasi-realizers correspond to Π-quasi-embedding when $\Pi = [1,n] \times \cdots \times [1,n]$ (where $n = |X|$).

Links between the 2-dimension and the dimension

These similar bounds in the study of the dimension and the 2-dimension asks the question of the existence of links between these two notions. For now, such links are not perfectly clear. Their definitions are quite similar (embeddings into cartesian products of chains) and thus they share some good properties. However they differ on many points:

- *qualitatively*: e.g. the dimension is a comparability invariant, the 2-dimension is not,
- *quantitatively*: the dimension is often much more smaller than the 2-dimension, even if the only inequality which is always true is $\mathbf{Dim}(P) \leq \mathbf{Dim_2}(P)$,
- *computationally*: both parameters are associated with $\mathcal{NP}$-complete problems, but the 2-dimension seems more difficult to compute (non-approximability results and see the case of trees).

A thorough study of the bounds we can obtain for the 2-dimension and the comparison with the dimension would certainly lead to some new insights in the study of these interesting but complex parameters of orders.

Questions

At first, it would be interesting to know whether the bound of Theorem 1 is tight. For any fixed integer $d \geq 1$, does there exist an infinite class of orders P such that $D(P) = d$ and such that their 2-dimension reach our upper bound (up to a constant factor) ?

Our proof is non-constructive due to the use of some probabilistic arguments. Is it possible to generate efficiently bit-vector encodings that achieve this upper bound ?

Of course, it may be possible to improve our bound or to find other bounds on the 2-dimension depending on some other parameters of orders, either in the general case or for specific classes of orders. For instance there is a conjecture still open concerning the 2-dimension of lattices, which is presented in [2, 6].

In the perspective of practical applications of bit-vector encodings, the knowledge of tight bounds on the 2-dimension in function of easily computed parameters (such as $D(P)$) would enable to check if there is a chance that the encodings provide a storage of the order with a good space compression.

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