Asynchronous Behavior of Double-quiescent Elementary Cellular Automata

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Abstract

In this paper we propose a probabilistic analysis of the relaxation time of elementary finite cellular automata (i.e., $\{0, 1\}$ states, radius 1 and unidimensional) for which both states are quiescent (i.e., $(0, 0, 0) \mapsto 0$ and $(1, 1, 1) \mapsto 1$), under α-asynchronous dynamics (i.e., each cell is updated at each time step independently with probability $0 < \alpha \leqslant 1$). It has been experimentally shown in previous work that introducing asynchronism in the global function of a cellular automaton may perturb its behavior, but as far as we know, only few theoretical work exist on the subject. This work generalizes previous work in [3], in the sense that we study here a continuous range of asynchronism that goes from full asynchronism to full synchronism. We were able to characterize formally the relaxation times for 52 of the 64 considered automata, and to provide a complete study of their sensitivity to asynchronism. Our work relies on the design of several probabilistic tools that enable to predict the global behaviour by counting local configuration patterns. These tools may be of independent interest since they provide a convenient framework to deal *exhaustively* with the tedious case analysis inherent to this kind of study. The remaining 12 automata (only 5 after symmetries) exhibit interesting complex phenomena (such as polynomial/exponential/infinite phase transitions on their relaxation time), for which we provide a comprehensive set of hints that yield a set of conjectures.

1 Introduction

The aim of this article is to analyze the asynchronous behavior of unbounded finite cellular automata. Cellular automata are widely used to model systems involving a huge number of interacting elements such as agents in economy, particles in physics, proteins in biology, distributed systems, etc. In most of these applications, in particular in many real system models, agents are not synchronous. One typical example consists of a network where each cell have two states, e.g., "I have a token" and "I don't have a token", and where transitions from one state to the other depends on the states of the neighbours, e.g., "I get a token if both of my neighbors have one" or "I have a token if and only if my right neighbor has one", etc. One natural question for such systems, ask for the *relaxation time*, i.e. the time needed to reach a stable configuration (e.g., "everyone has a token" or "no one has a token"). Depending on the transition rules, the behaviour of the system may vary widely when asynchronism perturbs the dynamics. More generally one can ask how much does asynchronous in real system perturbs computation. Interestingly enough, in spite of this lack of synchronism, real living systems are very resilient over time. One might then expect the cellular automata used to model these systems to be robust to asynchronism and other kind of failure as well (such as misreading the state of the neighbors). Surprisingly enough, it turns out that the resilience to asynchronism widely varies from one automata to another (e.g., [1, 2]).

Only few theoretical studies exist on the influence of asynchronism. Most of them usually focused on one specific cellular automata (e.g., [6, 5, 10]) and do not address the problem globally. Recently, Gács shows in [7] that it is undecidable to determining if in a given automota, the sequences of states followed by a given cell is independent of the history of the updates. Related work on the existence of stationary distribution on infinite configurations for probabilistic automata can be found in [9]. One can see cellular automata has physical systems where cell states changes according to local constraints (the transition rules). As opposed to classic work in asynchronous distributed computing, where one tries to *design* efficient transitions rules that guarantees fast convergence to a stable configuration (e.g., [4]), we study here how asynchrony affects the global evolution of the system given an *arbitrary* set of local constraints, and in particular how does asynchronicity affects its relaxation time.

In [3], the authors carried out a complete analysis of the class of one-dimensional double quiescent elementary automata (DQECA), where each cell has two states `0` and `1` which are quiescent (i.e., a cell such that every cell in its neighbourhood are in the same state, remains in the same state) and where each cell updates according to its state and the states of its two immediate neighbours. They study the behaviour of these automata under fully asynchronous dynamics, where only one random cell is updated at each time step. They show that one can classify the 64 DQECAs in six categories according to their relaxation times (constant, logarithmic, linear, quadratic, exponential or infinite) and furthermore that the relaxation time characterizes their behaviour, i.e., that all automata with equivalent relaxation times present the same kind of space-time diagrams.

The present paper extends this study to a continuous range of asynchyronism from fully asynchronous dynamics to fully synchronous dynamics: the α-*asynchronous dynamics*, where $0 < \alpha \leqslant 1$. In this setting, each cell is updated independently with probability α at each time step. When α varies from 1 to 0, the α-asynchronous dynamics evolves from the fully synchronous regime to a more and more asynchronous regime. As α approaches 0, the probability that updates take place on a single cell, tends to 1, and the dynamics gets closer and closer to a kind of fully asynchronous dynamics up to a time rescaling by a factor $1/\alpha$. Abusing of the notation, we thus refer the fully asynchronous dynamics as the 0-asynchronous regime.

Figure 1 page 6 presents the space-time diagrams of the 24 representatives of the DQECAs as α increases (by steps of 0.25) starting from the same random configuration of length $n = 100$. The last column plots the density of black cells at time step $t = 1000/\alpha$ from one single random configuration. This class exhibits a rich variety of behaviours. Thirteen representatives of the DQECAs (ECAs 204 to 128, 198, and 142 on Fig. 1) appear to be marginally sensitive to asynchronism. Six of them (ECAs 242 to 170, 194, and 138 on Fig. 1) present a brutal transition from the synchronous to asynchronous dynamics: they converge in polynomial time to an all-zero or all-one configuration as soon as (even a small amount of) asynchronism is introduced, while diverge under synchronous dynamics. One can observed that their space-time diagrams exhibit random walks like behaviour. The most interesting behaviour are observed for the remaining five representants. The relaxation time of ECAs 210 and 214 are respectively exponential and infinite under fully asynchronous dynamics, and both infinite under synchronous dynamics, but appears to be polynomial under α-asynchronous dynamics. The relaxation time as well as the time-space diagrams of EACs 178 and 146 evolve continuously as α increases, but seem to present an interesting phase transition at some α_c and α'_c, respectively, such that the relaxation time appears to be polynomial for $\alpha < \alpha_c^{(\prime)}$, and exponential for $\alpha > \alpha_c^{(\prime)}$. Finally, the relaxation time of ECA 150 appears to be exponential when $0 < \alpha < 1$, but is infinite otherwise.

Section 2 introduce the main definitions and presents our main result. Section 3 presents the key phenomena that differentiate the different dynamics: fully synchronous, α-asynchronous, and fully asynchronous. These observation will guide the design of probabilistic tools that are presented in Section 4 and used in Section 5 to boudn the relaxation time. The last section 6 sum up the intuitions, hints and conjectures on the behaviours of the remaining automata that could not be treated theoretically here, leaving the determination of their relaxation time open.

2 Definitions, Notations and Main Results

In this paper, we consider two-state cellular automata on finite size configurations with periodic boundary conditions.

Definition 1 *An* Elementary Cellular Automata (ECA) *is given by its* transition function $\delta : \{0,1\}^3 \to \{0,1\}$. *We denote by* $Q = \{0,1\}$ *the set of* states. *A state* q *is* quiescent *if* $\delta(q,q,q) = q$. *An ECA is* double-quiescent (DQECA) *if both states* 0 *and* 1 *are quiescent.*

We denote by $U = \mathbb{Z}/n\mathbb{Z}$ *the set of* cells. *A* finite configuration *with periodic boundary conditions* $x \in Q^U$ *is a word indexed by* U *with letters in* Q.

Definition 2 *For a given pattern* $w \in Q^U$, *we denote by* $|x|_w = \#\{i \in U : x_{i+1} \dots x_{i+|w|} = w\}$ *the number of occurrences of* w *in configuration* x.

We will use the following labeling introduced in [3] which will simplify the analysis of the probabilistic evolution of the ECAs.

Notation 1 We say that a transition is *active* if it changes the state of the cell where it is applied. Each ECA is fully determined by its active transitions. We label each active transition by a letter as follow:

label	A	B	C	D	E	F	G	H
$x\ y\ z$	000	001	100	101	010	011	110	111
$\delta(x,y,z)$	1	1	1	1	0	0	0	0

We label each ECA by the set of its active transitions.

Note that with these notations, the DQECAs are exactly the ECAs having a label containing neither A nor H. We consider three kinds of dynamics for ECAs: the synchronous dynamics, the α-asynchronous and the fully asynchronous dynamics. The synchronous dynamics is the classic dynamics of cellular automata, where the transition function is applied at each (discrete) time step on each cell simultaneously.

Definition 3 (Synchronous Dynamics) *The* synchronous dynamics $S_\delta : Q^U \to Q^U$ *of an ECA* δ, *associates deterministically to each configuration* x *the configuration* y, *such that for all* $i \in U$, $y_i = \delta(x_{i-1}, x_i, x_{i+1})$.

Definition 4 (Asynchronous Dynamics) *An* asynchronous dynamics AS_δ *of an ECA* δ *associates to each configuration* x *a random configuration* y*, such that* $y_i = x_i$ *for* $i \notin S$*, and* $y_i = \delta(x_{i-1}, x_i, x_{i+1})$ *for* $i \in S$*, where* S *is a random subset of* U *chosen by a daemon. We consider two types of asynchronous dynamics:*

- *in the* α-asynchronous dynamics*, the daemon puts each cell* i *in* S *independently with probability* α *where* $0 < \alpha \leqslant 1$*. The random function which associates the random configuration* y *to* x *according to this dynamics is denoted* AS_δ^α*.*
- *in the* fully asynchronous dynamics*, the daemon chooses a cell* i *uniformly at random and sets* $S = \{i\}$*. The random function which associates the random configuration* y *to* x *according to this dynamics is denoted* AS_δ^F*.*

For a given ECA δ*, we denote by* x^t *the random variable for the configuration obtained by* t *applications of the asynchronous dynamics function* AS_δ *on configuration* x*, i.e.,* $x^t = (AS_\delta)^t(x)$*. Note that* $(x^t)_{t\in\mathbb{N}}$ *is an homogeneous Markov chain on* Q^U*.* AS_δ *could equivalently be seen as a function with two arguments, the configuration* x *and the random subset* $S \subseteq U$ *chosen according to the processes listed above.*

Definition 5 (Fixed point) *We say that a configuration* x *is a* fixed point *for* δ *under asynchronous dynamics if* $AS_\delta(x) = x$ *whatever the choice of* S *(the cells to be updated) is.* $\mathfrak{F}_\delta$ *denotes the set of fixed points for* δ*.*

The set of fixed points of the asynchronous dynamics is clearly identical to $\{x : S_\delta(x) = x\}$ the set of fixed points of the synchronous dynamics. The set of fixed points of an automaton can be easily deduce from its labeling as shown in [3]. Every DQECA admits two *trivial fixed points*, $\mathtt{0}^n$ and $\mathtt{1}^n$.

Definition 6 (Relaxation Time) *Given an ECA* δ *and a configuration* x*, we denote by* $T_\delta(x)$ *the random variable for the time to reach a fixed point from configuration* x *under asynchronous dynamics, i.e.,* $T_\delta(x) = \min\{t : x^t \in \mathfrak{F}_\delta\}$*. The* relaxation time *of ECA* δ *is* $\max_{x\in Q^U} \mathbb{E}[T_\delta(x)]$*.*

The process $(x^t)_{t\in\mathbb{N}}$ always converges to a stationary distribution, but we will abusively say that an ECA *diverges from an initial configuration* x if the probability to reach a fixed point from x is 0. We can now state our main theorem.

Theorem 1 (Main result) *Under* α*-asynchronous dynamics, among the sixty-four DQECAs, we can determine the behaviour of 52 of them:*

- *forty-eight converge almost surely to a random fixed point from any initial configuration, and the relaxation times of these forty-eight convergent DQECAs are* 0*,* $\Theta(\frac{\ln n}{\ln(1-\alpha)})$*,* $\Theta(\frac{n}{\alpha})$*,* $\Theta(\frac{n}{\alpha} + \frac{1}{\alpha(1-\alpha)})$*,* $O(\frac{n}{\alpha(1-\alpha)})$*,* $O(\frac{n}{\alpha^2(1-\alpha)})$*,* $\Theta(\frac{n^2}{\alpha(1-\alpha)})$
- *two diverge from any initial configuration that is neither* $\mathtt{0}^n$*, nor* $\mathtt{1}^n$*, nor* $(\mathtt{01})^{n/2}$ *when* n *is even.*
- *two converge with a small probability from few initial configurations when* n *is even and diverge otherwise.*

The twelves others have different behaviours that we cannot prove for the moment. Some seem to exhibit a phase transition but their mathematical analysis remains a challenging problem. All the results and the conjectures (with question marks) are summed up in table 1.

Table 1: Behavior of DQECAs under asynchronous and synchronous dynamics (see Section 2 for explanations).

ECA (#)	Rule	01	10	010	101	Shift	Spawn	Fork	Annihil.	Fully Asynchr.	α-Asynchr.	Synchr.
204 (1)	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	0	0	0
200 (2)	E	$\cdot$	$\cdot$	+	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\Theta(n \ln n)$	$\Theta(\frac{\ln n}{\ln(1-\alpha)})$	1
232 (1)	DE	$\cdot$	$\cdot$	+	+	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\Theta(n \ln n)$	$O(\frac{n}{\alpha} + \frac{1}{\alpha(1-\alpha)})$	∞
206 (4)	B	$\leftarrow$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\Theta(n^2)$	$\Theta(\frac{n}{\alpha})$	$\Theta(n)$
222 (2)	BC	$\leftarrow$	$\rightarrow$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	+	$\Theta(n^2)$	$\Theta(\frac{n}{\alpha})$	$\Theta(n)$
192 (4)	EF	$\rightarrow$	$\cdot$	+	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\Theta(n^2)$	$\Theta(\frac{n}{\alpha})$	$\Theta(n)$
128 (2)	EFG	$\rightarrow$	$\leftarrow$	+	$\cdot$	$\cdot$	$\cdot$	$\cdot$	+	$\Theta(n^2)$	$\Theta(\frac{n}{\alpha})$	$\Theta(n)$
234 (4)	BDE	$\leftarrow$	$\cdot$	+	+	+	$\cdot$	$\cdot$	$\cdot$	$\Theta(n^2)$	$\Theta(\frac{n}{\alpha} + \frac{1}{\alpha(1-\alpha)})$	∞
202 (4)	BE	$\leftarrow$	$\cdot$	+	$\cdot$	+	$\cdot$	$\cdot$	$\cdot$	$\Theta(n^2)$	$\Theta(\frac{n}{\alpha} + \frac{1}{\alpha(1-\alpha)})$	∞
250 (2)	BCDE	$\leftarrow$	$\rightarrow$	+	+	+	$\cdot$	+	+	$\Theta(n^2)$	$O(\frac{n}{\alpha} + \frac{1}{\alpha(1-\alpha)})$	∞
218 (2)	BCE	$\leftarrow$	$\rightarrow$	+	$\cdot$	+	$\cdot$	+	+	$\Theta(n^2)$	$O(\frac{n}{\alpha} + \frac{1}{\alpha(1-\alpha)})$	∞
242 (4)	BCDEF	$\leftrightsquigarrow$	$\rightarrow$	+	+	+	+	+	+	$\Theta(n^2)$	$\Theta(\frac{n}{\alpha(1-\alpha)})$	∞
130 (4)	BEFG	$\leftrightsquigarrow$	$\leftarrow$	+	$\cdot$	+	+	$\cdot$	+	$\Theta(n^2)$	$\Theta(\frac{n}{\alpha(1-\alpha)})$	∞
170 (2)	BDEG	$\leftarrow$	$\leftarrow$	+	+	+	$\cdot$	$\cdot$	$\cdot$	$\Theta(n^3)$	$\Theta(\frac{n^2}{\alpha(1-\alpha)})$	∞
138 (4)	BEG	$\leftarrow$	$\leftarrow$	+	$\cdot$	+	$\cdot$	$\cdot$	$\cdot$	$\Theta(n^3)$	$\Theta(\frac{n^2}{\alpha(1-\alpha)})$	∞
226 (2)	BDEF	$\leftrightsquigarrow$	$\cdot$	+	+	+	+	$\cdot$	$\cdot$	$\Theta(n^3)$	$O(\frac{n^2}{\alpha(1-\alpha)})$	∞
194 (4)	BEF	$\leftrightsquigarrow$	$\cdot$	+	$\cdot$	+	+	$\cdot$	$\cdot$	$\Theta(n^3)$	$O(\frac{n}{\alpha^2(1-\alpha)})$	∞
178 (1)	BCDEFG	$\leftrightsquigarrow$	$\leftrightsquigarrow$	+	+	+	+	+	+	$\Theta(n^3)$	phase transition ? poly. for $\alpha < \alpha_c$? exp. for $\alpha > \alpha_c$?	∞
146 (2)	BCEFG	$\leftrightsquigarrow$	$\leftrightsquigarrow$	+	$\cdot$	+	+	+	+	$\Theta(n^3)$	phase transition ? poly. for $\alpha < \alpha'_c$? exp. for $\alpha > \alpha'_c$?	∞
210 (4)	BCEF	$\leftrightsquigarrow$	$\rightarrow$	+	$\cdot$	+	+	+	+	$\Theta(n2^n)$	polynomial ?	∞
214 (4)	BCF	$\leftrightsquigarrow$	$\rightarrow$	$\cdot$	$\cdot$	$\cdot$	+	$\cdot$	+	∞	polynomial ?	∞
150 (1)	BCFG	$\leftrightsquigarrow$	$\leftrightsquigarrow$	$\cdot$	$\cdot$	$\cdot$	+	$\cdot$	+	∞	exponential ?	∞
198 (2)	BF	$\leftrightsquigarrow$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	+	$\cdot$	$\cdot$	∞	∞	∞
142 (2)	BG	$\leftarrow$	$\leftarrow$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	∞	∞	∞

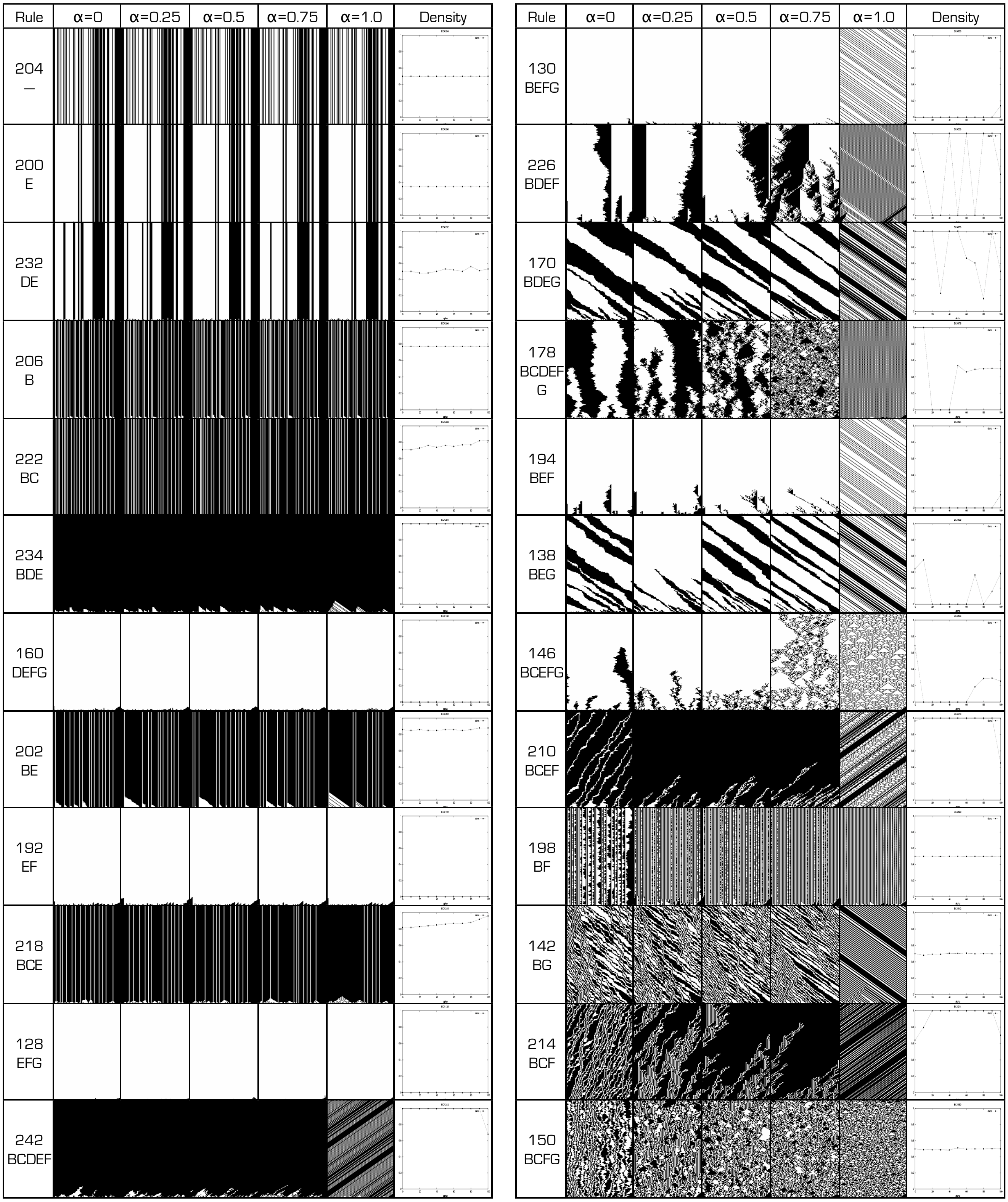


Figure 1: Behaviour of DQECAs as a function of the synchronicity rate α (Zoom in for details).

3 Key Observations

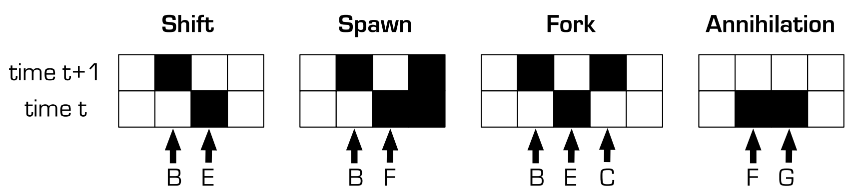
Due to 0/1 and horizontal symmetries of configurations, we shall w.l.o.g. only consider the 24 DQECAs listed in Tab. 1 among the 64 DQECAs. For each of these 24 DQECAs, the number of the equivalent automata under symmetries is written within parentheses after their classic ECA code in the table.

From now on, we only consider the α-asynchronous dynamics; this will be implicit in all the following propositions. Our results rely on the observation of the evolution of the *0-regions* and *1-regions* in the space-time diagram (i.e., of the intervals of consecutive 0s or consecutive 1s in configuration x^t). We will now enumerate the different ways to affect the regions.

First we consider the case where a cell updates and none of its two neighbours update:

- Transitions D and E are thus responsible for decreasing the number of regions in the space-time diagram: D “erases” the isolated 1s and E the isolated 0s.
- Transitions B and F act on patterns 01. Intuitively, transition B moves a pattern 01 to the left, and transition F moves it to the right. In particular, patterns 01 perform a kind of random walk for DQECA with both transitions B and F if no others phenomena occurs. The arrows in Tab. 1 represent the different behavior of the patterns: $\leftarrow$ or $\rightarrow$, for left or right moves of the patterns 01 or 10; $\leftrightsquigarrow$, for random walks of these patterns.
- Similarly, transitions C and G act on patterns 10. Transition C moves a pattern 10 to the right, and transition G moves it to the left.

One important observation made during the study of the fully asynchronous in [3] is that the number of regions can only decrease and each activation of D or E makes the number of regions decrease by one. This statement is not true anymore in the case of the α-asynchronous dynamics, as we will see now. Here are the new phenomena when two or three neighboring cells update at the same time:



- **Shift** phenomenon appears with the activation of rules B and E or C and E or F and D or G and D together: in this case an isolated 0 or an isolated 1 is shifted. Here a transition D or E is activated but no regions is erased.
- **Spawn** phenomenon appears with the activation of rules B and F or C and G together: a pattern 0011 could create a new region. This is an important phenomenon because it makes the number of regions increase by one each time it occurs.
- **Fork** phenomenon appears with the activation of rules B, C and E or F, G and D together: here three neighboring cellules upadte at the same time and an isolated point is duplicated. This phenomenon makes the number of regions increasing by one each time it occurs.
- **Annihilation** phenomenon appears with the activation of rules B and C or F and G together: the activation of two rules could erase a region of length 2. This is a very important phenomenon because it is another way to make the number of region decrease. In particular,

it is the only way to make the number of regions decrease in automaton where neither D or neither E is activated.

The next section is devoted to the tools which will be used to prove our main theorem.

4 Lyapunov functions based on local neighbourhoods

Definition 7 (Mask) *A* mask $\dot{m}$ *is a word on* $\{0, 1, \dot{0}, \dot{1}\}$ *containing exactly one dotted letter in* $\{\dot{0}, \dot{1}\}$*. We say that the cell* i *in configuration* x *matches the mask* $\dot{m} = m_{-k} \dots m_{-1}\dot{m}_0 m_1 \dots m_l$ *if* $x_{i-k} \dots x_i \dots x_{i+l} = m_{-k} \dots m_0 \dots m_l$*. We denote by* m *the undotted word* $m_{-k} \dots m_0 \dots m_l$*.*

Fact 2 *The number of cells matching a given mask* $\dot{m}$ *in a configuration* x *is exactly* $|x|_m$*, the number of occurrences of the undotted word* m*.*

Definition 8 (Masks basis) *A* masks basis $\mathcal{B}$ *is a finite set of masks such that for any configuration* x *and any cell* i*, there exists an unique* $\dot{m} \in \mathcal{B}$ *that matches cell* i*.*

A masks basis $\mathcal{B}$ can be represented by a binary tree where the children of a node are labelled by adding 0 and 1 to the node label, on the right or the left (the children of the root receive $\dot{0}$ and $\dot{1}$), and where the masks of $\mathcal{B}$ are the labels of the leaves. Reciprocally, any binary tree observing these properties defines a masks basis by taking the labels of its leaves. Figure (b) page 10 illustrates the construction of the tree for the masks basis $\mathcal{B} = \{1\dot{1}, 00\dot{1}0, 00\dot{1}1, 010\dot{1}, 110\dot{1}, \dot{0}0, 0\dot{0}10, 0\dot{0}11, 01\dot{0}1, 11\dot{0}1\}$.

Masks basis will be used to define Lyapunov weight functions from local patterns and provide an efficient tool to validate exhaustive case analysis.

Definition 9 (Local weight function) *A* local weight function f *is a function from a masks basis* $\mathcal{B}$ *to* $\mathbb{Z}$*. The local weight of the cell* i *in configuration* x *given by* f *is* $F(x, i) = f(\dot{m})$ *where* $\dot{m}$ *is the unique mask in* $\mathcal{B}$ *matching cell* i*. The weight of a configuration* x *given by* f *is defined as* $F(x) = \sum_i F(x, i)$*.*

Fact 3 *Given a local weight function* $f : \mathcal{B} \to \mathbb{Z}$*, the weight of configuration* x *is equivalently defined as:* $F(x) = \sum_{\dot{m} \in \mathcal{B}} f(\dot{m}) \cdot |x|_m$*.*

Notation 2 *For a given random sequence of configurations* $(x^t)_{t \in \mathbb{N}}$ *and a weight function* F *on the configurations, we denote by* $(\Delta F(x^t))_{t \in \mathbb{N}}$ *the random sequence* $\Delta F(x^t) = F(x^{t+1}) - F(x^t)$*.*

The next lemma provides upper bounds on stopping times for the markovian sequence of configurations $(x^t)_{t \in \mathbb{N}}$ subject to a weight function F decreasing or remaining constant on average (a *Lyapunov function*). Their proofs can be found in [3] and are based on classical results from Lyapunov functions and martingale theory [8]. We assume that the values of F on configurations belongs to $\{0, \dots, m\}$ where m is a non-negative integer, and ϵ is a positive real. Two other similar lemmas has been postponed to the appendix due to space constraints.

Lemma 4 *Assume that if* $F(x^t) > 0$*, then* $\mathbb{E}[\Delta F(x^t) | x^t] \leqslant -\epsilon$*. Let* $T = \min\{t : F(x^t) \leqslant 0\}$ *denote the random variable for the first time* t *where* $F(x^t) \leqslant 0$*. Then with* $c_0 = \mathbb{E}[F(x^0)]$*,* $\mathbb{E}[T] \leqslant \frac{m + c_0}{\epsilon}$*.*

5 Relaxation Times

Due to space constraints, we only present the theorems on the relaxation time for the DQECAs **EF**, **EFG** and **BEF**. The results for Identity, **E**, **DE**, **B**, **BC**, **BDE**, **BE**, **BCDE**, **BCE**, **BCDEF**, **BEFG**, **BDEG**, **BEG**, **BDEF**, **BF**, **BG** are given in Tab. 1. The full statements and proofs are postponed to the appendix.

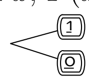
5.1 Automata EF(192) and EFG(128)

The fixed points of these automata are $\mathtt{0}^n$ and $\mathtt{1}^n$. The fixed point $\mathtt{1}^n$ cannot be reached from any other configuration.

Theorem 5 *Under α-asynchronous dynamics, DQECAs **EF** and **EFG** converge a.s. to a fixed point from any initial configuration. The relaxation time is $O\left(\frac{n}{\alpha}\right)$.*

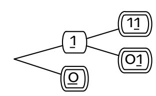
Patterns 01 decrease the number of 1s as they are moved. The difference beetween **EF** and **EFG** is that in the case of **EFG**, the number of 1s decrease also when patterns 01 are moved. But the same proof is true in the two cases. Under totally asynchronous, α-asynchronous or synchronous dynamics, the behaviour of these automata are almost identical.

Proof. We use the masks basis and local weight function f below. We have $F(x) = |x|_{\mathtt{1}}$. Note that for all configuration x, $F(x) \in \{0, \ldots, n\}$ and $F(x) = 0$ if and only if $x = \mathtt{0}^n$.



masks	$\dot{\mathtt{0}}$	$\dot{\mathtt{1}}$
$f(\dot{m})$	0	1

(a) Weight function



(b) Analysis

Lemma 6 $\mathbb{E}[\Delta F(x)] \leqslant -\alpha|x|_{\mathtt{01}}$.

By linearity of expectation, $\mathbb{E}[\Delta F(x)] = \mathbb{E}\left[\sum_{i=0}^{n-1} \Delta F(x,i)\right] = \sum_{i=0}^{n-1} \mathbb{E}[\Delta F(x,i)]$.

We evaluate the variation of $F(x,i)$ using the masks basis (b).

Consider that at step t, cell i matches:

- mask $\dot{\mathtt{0}}$: With probability 1 at the step $t+1$, cell i matches mask $\dot{\mathtt{0}}$. Thus, $\mathbb{E}[\Delta F(x^t,i)] = 0$.
- mask $\mathtt{1}\dot{\mathtt{1}}$: $F(x^t,i) = 1$ thus $F(x^{t+1},i) \leqslant F(x^t,i)$, and, $\mathbb{E}[\Delta F(x^t,i)] \leqslant 0$.
- mask $\mathtt{0}\dot{\mathtt{1}}$:

With probability	α	$1-\alpha$
at the step $t+1$, cell i matches mask	$\dot{\mathtt{0}}$	$\dot{\mathtt{1}}$
and $\Delta F(x^t,i)$	$=-1$	$=0$

Thus, $\mathbb{E}[F(x^t,i)] \leqslant -\alpha$. Finally, $\sum_{i=0}^{n-1} \mathbb{E}[\Delta F(x^t,i)] \leqslant -\alpha|x|_{\mathtt{01}}$. So, as long as x^t is not a fixed point, we have: $\mathbb{E}[\Delta F(x^t)] \leqslant -\alpha|x|_{\mathtt{01}} \leqslant -\alpha$. Using Lemma 4, automata **EF** and **EFG** converge a.s. from any initial configuration (except $\mathtt{1}^n$) to $\mathtt{0}^n$. The relaxation time is $O\left(\frac{n}{\alpha}\right)$. □

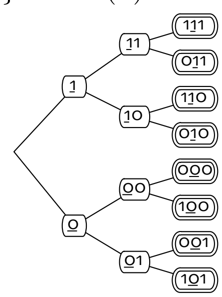
5.2 Automaton BEF(194)

The fixed points of this automaton are $\mathtt{0}^n$ and $\mathtt{1}^n$. Fixed point $\mathtt{1}^n$ cannot be reached from any other configuration.

Theorem 7 *Under α-asynchronous dynamics, DQECA **BEF** converges a.s. to a fixed point from any initial configuration. The relaxation time is $O\left(\frac{n}{\alpha^2(1-\alpha)}\right)$.*

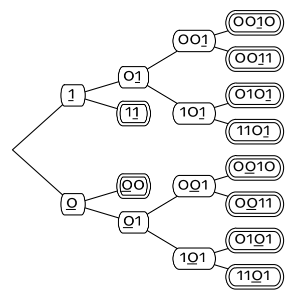
Under fully asynchronous dynamics, the length of any 1-region follows a random walk, and thus converges in $O(n^3)$ in expectation. Here, the Spawn phenomenon(rule **B** and **F** applied together to cells $i-1$ and i) can transform the pattern $\mathtt{000\dot{1}11}$ into the pattern $\mathtt{001011}$ with probability α^2. Even if the number of 0s and 1s are the same in these two patterns, in the pattern $\mathtt{001011}$ two 1s can become 0s at the next step (by applying rules **E** and **F**), while only one 0 can become a 1 at the next step (by applying rule **B**). So the creation of isolated 0s tends to decrease the number of 1s at the next step.

Proof. Let $a = -2c+2, b = -1, c = -\lfloor \frac{3}{\alpha} \rfloor - 1$. We use the masks basis and local weight function f below. We have: $F(x) = a|x^t|_{\mathtt{1}} + b|x^t|_{\mathtt{011}} + c|x^t|_{\mathtt{101}}$. For all configuration x, $F(x) \in \{0, \ldots, 2n(\lfloor \frac{3}{\alpha} \rfloor + 4)\}$ and $F(x) = 0$ if and only if $x = \mathtt{0}^n$.



masks	$\mathtt{1\dot{0}1}$	$\mathtt{0\dot{0}0}$, $\mathtt{0\dot{0}1}$, $\mathtt{1\dot{0}0}$	$\mathtt{0\dot{1}1}$	$\mathtt{1\dot{1}0}$, $\mathtt{1\dot{1}1}$, $\mathtt{0\dot{1}0}$
$f(\dot{m})$	c	0	$a+b$	a

(a) Weight function



(b) Analysis

Lemma 8 $\mathbb{E}[\Delta F(x)] \leqslant -\alpha(1-\alpha)|x|_{\mathtt{01}}$

By linearity of expectation:$\mathbb{E}[\Delta F(x)] = \mathbb{E}\left[\sum_{i=0}^{n-1} \Delta F(x,i)\right] = \sum_{i=0}^{n-1} \mathbb{E}[\Delta F(x,i)]$.

We evaluate the variation of $F(x,i)$ using the masks basis of Figure (b).

Consider that at step t, cell i matches:

- mask $\mathtt{1\dot{1}}$: $F(x^t,i) = a$. With probability 1 at the step $t+1$, cell i matches mask $\mathtt{\dot{1}}$. So $F(x^{t+1},i) \in \{a, a+b\}$. Since $b < 0$, $F(x^{t+1},i) \leqslant F(x^t,i)$. Thus, $\mathbb{E}[(\Delta F(x^t,i)] \leqslant 0$.
- mask $\mathtt{\dot{0}0}$: $F(x^t,i) = 0$. With probability 1 at the step $t+1$, cell i matches mask $\mathtt{\dot{0}}$. So $F(x^{t+1},i) \in \{0, c\}$. Since $c < 0$, $F(x^{t+1},i) \leqslant F(x^t,i)$. Thus, $\mathbb{E}[(\Delta F(x^t,i)] \leqslant 0$.
- mask $\mathtt{00\dot{1}0}$ (and $\mathtt{0\dot{0}10}$ together):

With probability	$\alpha(1-\alpha)$	$\alpha(1-\alpha)$	$(1-\alpha)^2$	α^2
At the step $t+1$, cell i matches mask	$\mathtt{0\dot{0}}$	$\mathtt{1\dot{1}}$	$\mathtt{0\dot{1}}$	$\mathtt{1\dot{0}}$
and $\Delta F(x^t, i-1)$	$=0$	$=a+b$	$=0$	$=a$
and $\Delta F(x^t, i)$	$=-a$	$=0$	$=0$	$=-a$

  Thus, $\mathbb{E}[\Delta F(x^t,i) + \Delta F(x^t,i-1)] = -a\alpha(1-\alpha) + (a+b)\alpha(1-\alpha) = b\alpha(1-\alpha) = -\alpha(1-\alpha)$.

- mask $\mathtt{00\dot{1}1}$ (and $\mathtt{0\dot{0}11}$ together):

With probability	$\alpha(1-\alpha)$	$\alpha(1-\alpha)$	$(1-\alpha)^2$	α^2
at the step $t+1$, cell i matches mask	$\mathtt{0\dot{0}}$	$\mathtt{1\dot{1}}$	$\mathtt{0\dot{1}}$	$\mathtt{1\dot{0}}$
and $\Delta F(x^t, i-1)$	$=0$	$=a+b$	$=0$	$=c-a-b$
and $\Delta F(x^t, i)$	$=-a-b$	$=-b$	$=0$	$=a$

  Thus, $\mathbb{E}[\Delta F(x^t,i) + \Delta F(x^t,i-1)] = (-a-b)\alpha(1-\alpha) + a\alpha(1-\alpha) + (c-b)\alpha^2 \leqslant \alpha(1-\alpha) - 2\alpha \leqslant -\alpha(1-\alpha)$.

- mask $\texttt{110}\dot{\texttt{1}}$ (and $\texttt{11}\dot{\texttt{0}}\texttt{1}$ together):

With probability	α	$(1-\alpha)$
at the step $t+1$, the cell i matches mask	$\texttt{0}\dot{\texttt{0}}$	$\texttt{0}\dot{\texttt{1}}$
and $\Delta F(x, i-1)$	$=-c$	$=0$
and $\Delta F(x, i)$	$=-a-b$	$=0$

Thus, $\mathbb{E}[\Delta F(x^t, i) + \Delta F(x^t, i-1)] = (-a-b-c)\alpha(1-\alpha) \leqslant -\alpha(1-\alpha)$.

- mask $\texttt{010}\dot{\texttt{1}}$ (and $\texttt{01}\dot{\texttt{0}}\texttt{1}$ together):

With probability	α	$(1-\alpha)^2$	$\alpha(1-\alpha)$
at the step $t+1$, the cell i matches mask	$\texttt{0}\dot{\texttt{0}}$	$\texttt{10}\dot{\texttt{1}}$	$\texttt{00}\dot{\texttt{1}}$
and $\Delta F(x^t, i-1)$	$=-c$	$=0$	$=-c$
and $\Delta F(x^t, i)$	$=-a-b$	$=0$	$=0$

Thus, $\mathbb{E}[\Delta F(x^t, i) + \Delta F(x^t, i-1)] = (-a-b-c)\alpha(1-\alpha) - c\alpha(1-\alpha) \leqslant -\alpha(1-\alpha)$.

Finally $\sum_{i=0}^{n-1} \mathbb{E}[\Delta F(x^t, i)] \leqslant -\alpha(1-\alpha)(|x^t|_{\texttt{0010}} + |x^t|_{\texttt{0011}} + |x^t|_{\texttt{1011}} + |x^t|_{\texttt{0101}}) \leqslant -\alpha(1-\alpha)|x^t|_{\texttt{01}}$. So, as long as x^t is not a fixed point, we have $\mathbb{E}[\Delta F(x^t)] \leqslant -\alpha(1-\alpha)|x^t|_{\texttt{01}} \leqslant -\alpha(1-\alpha)$. Using Lemma 4, automaton **BEF** converges a.s. from any intial configuration (except $\texttt{1}^n$) to $\texttt{0}^n$. The relaxation time is $O\left(\frac{n}{\alpha} \times \frac{1}{\alpha(1-\alpha)}\right) = O\left(\frac{n}{\alpha^2(1-\alpha)}\right)$. $\square$

6 Conjectures

This section gathers the remaining DQECAs for which the mathematical analysis is not achieved yet. However by means of simulation and by the study of special patterns, we can give some insights of the phenomena which guide their dynamics and differentiate them from the other DQECAs.

Automaton BCDEFG(178). The fixed points of this automaton are exactly $\texttt{0}^n$ and $\texttt{1}^n$. Simulations show a phase transition concerning the convergence time, which can be also clearly observed on time-space diagrams and seems to appear when $\alpha = \alpha_c \approx 0,5$. If $\alpha < \alpha_c$, the overall behaviour of the automaton does not drastically change when α varies: regions of 0 and 1 merge into larger regions reducing their number, and it seems to converge to $\texttt{0}^n$ or $\texttt{1}^n$ with an $O(n^2/\alpha)$ expected time. While if $\alpha > \alpha_c$, large regions of 0 or 1 crumble quickly at their frontiers and patterns of $\texttt{0101}\cdots\texttt{01}$ fill the space between the regions. The closer α is to 1, the smaller is the probability of formation of large regions. In that case, we conjecture that the relaxation time is exponential in n.
Automaton BCEFG(146). The fixed points of this automaton are exactly $\texttt{0}^n$ and $\texttt{1}^n$. This automaton shows a phase transition which seems to appear when $\alpha = \alpha'_c \approx 0,6$. When $\alpha < \alpha'_c$, regions of 1 quickly disappear and the expected convergence time is conjectured to be polynomial in n. When α is close to 1, like the automaton **BCDEFG**, large regions of 1 do not survive because they tend to crumble very quickly. On the other hand, isolated 1s are easily deleted and seem to multply faster than they disappear. In that case, we conjecture that the relaxation time is exponential in n.
Automaton BCF(214). The fixed points of this automaton are $\texttt{0}^n$, $\texttt{1}^n$ and $(\texttt{01})^{n/2}$ (if n is even). When starting from another configuration, it is impossible to reach one of these fixed points in the fully asynchronous dynamics, since the number of regions remains constant. With the α-asynchronous dynamics, due to the Annihilation phenomenon, any configuration converges a.s. to a fixed point within a finite time. The sizes of large regions of 0 decrease quickly. Only regions with two 0s may disappear, but 10011 patterns may evolve into 11111 or 10101 with the same probability. This could lead to an increase of small regions, tending to slow down the convergence.

However small regions of 0 which are consecutive prevent from this splitting: in a 1001001 pattern, the first 00 region can not split. Thus the number of regions tends to decrease. We conjecture that the relaxation time is polynomial in n and contains an $O(\frac{1}{\alpha^2(1-\alpha)})$ term corresponding to the deletion of 00 regions.

Automaton BCFG(150). The fixed points of this automaton are 0^n, 1^n and $(01)^{n/2}$ (if n is even). In the fully asynchronous dynamics, this automaton does not converge to a fixed point since it is impossible to suppress a region. However in the α-asynchronous dynamics, due to the Annihilation phenomenon, this automaton converges a.s. to a fixed point within a finite time. Simulations suggest that the relaxation time is exponential in n.

Automaton BCEF(210). The fixed points of this automaton are exactly 0^n and 1^n. In the fully asynchrnous dynamics, this automaton converges to 0^n with a exponential expected time. In both fully asynchrounous and α-asynchronous, dynamics, the sizes of regions of 0 tends to decrease quickly. However in the fully asynchronous dynamics, they may only disappear by merging and the last region of 0 will converge to 0 in exponential expected time. The α-asynchronous dynamics introduces the Annihilation phenomenon. On simulations, the convergence to fixed points seems to be polynomial. This case looks like the **BCF** automaton, but the analysis is a bit more complicated since regions of 0 may merge which must be taken into account in the proof of bounds for the convergence time.

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A Lyapunov functions based on local neighbourhoods (Omitted lemmas)

Lemma 9 *Assume that for all t, $\mathbb{E}[\Delta F(x^t)|x^t] = 0$ and if $0 < F(x^t) < m$ then $\Pr\{\Delta F(x^t) \geqslant 1|x^t\} = \Pr\{\Delta F(x^t) \leqslant -1|x^t\} \geqslant \epsilon$. Let $T = \min\{t : F(x^t) \in \{0, m\}\}$. Then with $c_0 = \mathbb{E}[F(x^0)]$,*

$$\mathbb{E}[T] \leqslant \frac{c_0(m - c_0)}{2\epsilon}.$$

Lemma 10 *Assume that for all t, if $F(x^t) < m$ then $\mathbb{E}[\Delta F(x^t)|x^t] = 0$, if $0 < F(x^t) < m$ then $\Pr\{\Delta F(x^t) \geqslant 1|x^t\} = \Pr\{\Delta F(x^t) \leqslant -1|x^t\} \geqslant \epsilon$ and if $F(x^t) = m$ then $\Pr\{F(x^t) \leqslant m-1\} \geqslant \epsilon$. Let $T = \min\{t : F(x^t) = 0\}\}$. Then with $c_0 = \mathbb{E}[F(x^0)]$,*

$$\mathbb{E}[T] \leqslant \frac{c_0(2m + 1 - c_0)}{2\epsilon}.$$

B Relaxation Times (Omitted proofs)

B.1 Automaton E(200)

The fixed points of this automaton are all the configuration without pattern `010`.

Theorem 11 *Under α-asynchronous dynamics, DQECA **E** converges a.s. to a fixed point from any initial configuration. The relaxation time is $O\left(\frac{n}{\alpha}\right)$.*

Proof. At each time step, each isolated `1` disappears with probability α. No other phenomenon occurs for this automaton. This is a classical coupon collector problem.
□

B.2 Automata B(206) and BC(222)

The fixed points of these automata are $\mathtt{0}^n$, $\mathtt{1}^n$ and all configuration without patterns `00`.

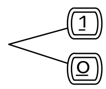
Theorem 12 *Under α-asynchronous dynamics, DQECAs **B** and **BC** converge a.s. to a fixed point from any initial configuration. The relaxation time is $O\left(\frac{n}{\alpha}\right)$.*

The length of the big regions of `1` increase until there are only isolated `0`s.

Proof. We use the masks basis and local weight function f of Figure 2a.
And so:

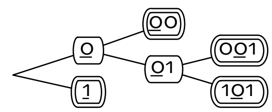
$$F(x) = \sum_{\dot{m} \in \mathcal{B}} f(\dot{m}) \cdot |x|_m = |x|_{\mathtt{0}}$$

For all configuration x, $F(x) \in \{0, \ldots, n\}$ and $F(x) = 0$ if and only if $x = \mathtt{1}^n$.



masks	$\dot{\mathtt{1}}$	$\dot{\mathtt{0}}$
$f(\dot{m})$	1	0

(a) Weight function



(b) Analysis

Figure 2: Masks basis for **B** and **BC**.

Lemma 13 $\mathbb{E}[\Delta F(x)] \leqslant -\alpha |x|_{\mathtt{001}}$.

By linearity of expectation:

$$\mathbb{E}[\Delta F(x)] = \mathbb{E}\left[\sum_{i=0}^{n-1} \Delta F(x,i)\right] = \sum_{i=0}^{n-1} \mathbb{E}[\Delta F(x,i)].$$

We evaluate the variation of $F(x,i)$ using the masks basis of Figure 2b.
Consider that at step t, cell i matches:

- mask $\mathtt{1\dot{0}1}$, $\dot{\mathtt{1}}$,: with probability 1 at the step $t+1$, cell i matches the same mask at the time t. Thus, $\mathbb{E}[\Delta F(x^t,i)] = 0$.
- mask $\mathtt{\dot{0}0}$,: $F(x^t,i) = 1$. Thus , $\mathbb{E}[\Delta F(x^t,i)] \leqslant 0$.
- mask $\mathtt{0\dot{0}1}$: $F(x^t,i) = 1$. With probability α at the step $t+1$, cell i matches mask $\mathtt{1}$ and $\Delta F(x,i) = -1$. Otherwise, it matches the mask $\dot{\mathtt{0}}$ and $\Delta F(x,i) = 0$. Thus, $\mathbb{E}[\Delta F(x^t,i)] = -\alpha$.

Finally:

$$\sum_{i=0}^{n-1} \mathbb{E}[\Delta F(x^t,i)] \leqslant -\alpha |x^t|_{\mathtt{001}}$$

So, as long as x^t is not a fixed point, we have:

$$\begin{aligned} \mathbb{E}[\Delta F(x^t)] &\leqslant -\alpha |x^t|_{\mathtt{001}} \\ &\leqslant -\alpha \end{aligned}$$

Using Lemma 4, the automata **B** and **BC** converge a.s. to a fixed point from any initial configuration (except $\mathtt{0}^n$). The relaxation time is $O\left(\frac{n}{\alpha}\right)$. □

B.3 Automata BDE(234) and BCDE(250)

The fixed points of these automata are $\mathtt{0}^n$ and $\mathtt{1}^n$.

Theorem 14 *Under α-asynchronous dynamics, DQECAs **BDE** and **BCDE** converge a.s. to a fixed point from any initial configuration. The relaxation time is $O\left(\frac{n}{\alpha} + \frac{1}{\alpha(1-\alpha)}\right)$.*

Only rule **E** can erase `1`s in the configuration and this rule can only be applied to isolated `1`s. So, as soon as a configuration has a pattern `11`, it cannot be erased anymore. Rule **B** and rule **D** are sufficient to make this pattern grow and make the automaton converge to $\mathtt{1}^n$. So the proof has two parts. First, we estimate the relaxation time for a configuration without pattern `11` to generate such a pattern (assuming that the automaton has not converged to $\mathtt{0}^n$ meanwhile). Second we compute the relaxation time for a configuration with a pattern `11` to reach the fixed point $\mathtt{1}^n$.

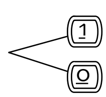
Proof. First, we bound from below the probability that a cell i matching mask $\mathtt{0}\dot{\mathtt{1}}\mathtt{0}$ creates a pattern `11`.

With probability	$(1-\alpha)^2$	$\alpha(1-\alpha)$	$\alpha(1-\alpha)$	α^2
at the step $t+1$, cell i matches the mask	$\mathtt{0}\dot{\mathtt{1}}$	$\mathtt{1}\dot{\mathtt{1}}$	$\mathtt{0}\dot{\mathtt{0}}$	$\mathtt{1}\dot{\mathtt{0}}$

So each pattern `010` creates a pattern `11` with a probability $\alpha(1-\alpha)$. The expected time for a configuration with a pattern `010` to create a pattern `11` assuming that the automaton has not converged to $\mathtt{0}^n$ meanwhile is $O(\frac{1}{\alpha(1-\alpha)})$.

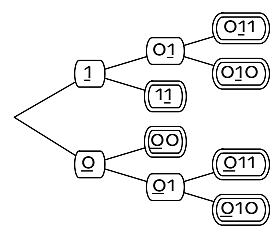
Now we calculate the relaxation time for a configuration x to reach the fixed point $\mathtt{1}^n$ assuming that $|x|_{\mathtt{11}} \geqslant 1$.

We use the masks basis and local weight function f of Figure 3a.



masks	$\dot{\mathtt{0}}$	$\dot{\mathtt{1}}$
$f(\dot{m})$	1	0

(a) Weight function



(b) Analysis

Figure 3: Masks basis for **BDE** and **BCDE**.

And so:

$$F(x) = \sum_{\dot{m} \in \mathcal{B}} f(\dot{m}) \cdot |x|_m = |x|_{\mathtt{0}}$$

For all configuration x, $F(x) \in \{0, \ldots, n-2\}$ and $F(x) = 0$ if and only if $x = \mathtt{1}^n$.

Lemma 15 $\mathbb{E}[\Delta F(x)] \leqslant -\alpha |x|_{\texttt{011}}$.

By linearity of expectation:

$$\mathbb{E}[\Delta F(x)] = \mathbb{E}\left[\sum_{i=0}^{n-1} \Delta F(x,i)\right] = \sum_{i=0}^{n-1} \mathbb{E}[\Delta F(x,i)].$$

We evaluate the variation of $F(x,i)$ using the masks basis of Figure 3b.
Consider that at step t, cell i matches:

- mask $\texttt{1}\dot{\texttt{1}}$, $\texttt{0}\dot{\texttt{1}}\texttt{1}$: $F(x^t,i) = 0$. With probability 1 at the step $t+1$, cell i matches mask $\dot{\texttt{1}}$ and $F(x^{t+1},i) = 0$. Thus, $\mathbb{E}[\Delta F(x^t,i)] = 0$.
- mask $\dot{\texttt{0}}\texttt{0}$: $F(x^t,i) = 1$. Thus, $\mathbb{E}[\Delta F(x^t,i)] \leqslant 0$.
- mask $\dot{\texttt{0}}\texttt{11}$: $F(x^t,i) = 1$. With probability α at the step $t+1$, cell i matches mask $\dot{\texttt{1}}$. Otherwise, it matches mask $\dot{\texttt{0}}$. Thus, $\mathbb{E}[\Delta F(x^t,i)] = -\alpha$.
- mask $\texttt{0}\dot{\texttt{1}}\texttt{0}$ (and $\dot{\texttt{0}}\texttt{10}$ together):

With probability	$(1-\alpha)^2$	$(1-\alpha)\alpha$	$(1-\alpha)\alpha$	α^2
at the step $t+1$, cell i matches mask	$\dot{\texttt{0}}\texttt{1}$	$\dot{\texttt{1}}\texttt{1}$	$\dot{\texttt{0}}\texttt{0}$	$\dot{\texttt{1}}\texttt{0}$
and $\Delta F(x^t, i-1)$	$=0$	$=-1$	$=0$	$=-1$
and $\Delta F(x^t, i)$	$=0$	$=0$	$=1$	$=1$

Thus,

$$\mathbb{E}[\Delta F(x^t,i)] = 0.$$

Finally:

$$\sum_{i=0}^{n-1} \mathbb{E}[\Delta F(x^t,i)] \leqslant -\alpha |x^t|_{\texttt{011}}.$$

So, as long as x^t is not the fixed point $\texttt{1}^n$, we have:

$$\begin{aligned}\mathbb{E}[\Delta F(x^t)] &= \mathbb{E}\left[\sum_{i=0}^{n-1} \Delta F(x^t,i)\right] \\ &\leqslant \sum_{i=0}^{n-1} \mathbb{E}[\Delta F(x^t,i)] \\ &\leqslant -\alpha |x^t|_{\texttt{011}} \\ &\leqslant -\alpha.\end{aligned}$$

Using the Lemma 4, automata **BDE** and **BCDE** converge a.s. from any initial configuration with a pattern 11 to 1^n. The relaxation time is $O\left(\frac{n}{\alpha}\right)$.

Thus, automata **BDE** and **BCDE** converge a.s. from any initial configuration. The relaxation time is $O\left(\frac{n}{\alpha} + \frac{1}{\alpha(1-\alpha)}\right)$.

□

B.4 Automata BE(202) and BCE(218)

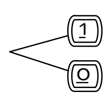
The fixed points of these automata are 0^n, 1^n and all configurations without patterns 00 and 010.

Theorem 16 *Under α-asynchronous dynamics, DQECAs **BE** and **BCE** converge a.s. to a fixed point from any initial configuration. The relaxation time is* $O\left(\frac{n}{\alpha} + \frac{1}{\alpha(1-\alpha)}\right)$.

These cases are almost the same as the previous ones. The first step consists in waiting the spawning of a pattern 11. The second step is slightly different since several fixed points can be reached and patterns 01011 could slow down the expansion of 1-regions.

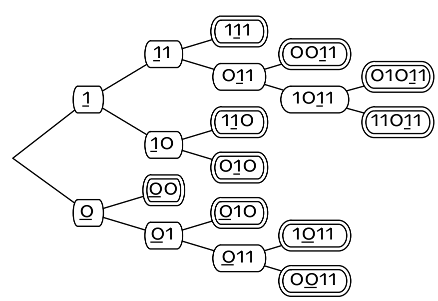
Proof. The first part is similar to the previous automaton. We evaluate the relaxation time for a configuration x to reach a fixed point assuming that $|x|_{11} \geqslant 1$

We use the masks basis and local weight function f of Figure 4a.



masks	$\dot{0}$	$0\dot{1}0$	$010\dot{1}1$	otherwise
$f(\dot{m})$	3	3	1	0

(a) Weight function



(b) Analysis

Figure 4: Masks basis for **BE** and **BCE**.

And so:

$$F(x) = \sum_{\dot{m} \in \mathcal{B}} f(\dot{m}) \cdot |x|_m = 3|x|_{\dot{0}} + 3|x|_{0\dot{1}0} + |x|_{010\dot{1}1}$$

For all configuration x, $F(x) \in \{0, \ldots, 3n-6\}$ and $F(x) = 0$ if and only if $x = 1^n$.

Lemma 17 $\mathbb{E}[\Delta F(x)] \leqslant -\alpha(|x|_{0011} + |x|_{010\dot{1}1})$.

By the linearity of the expectation:

$$\mathbb{E}[\Delta F(x)] = \mathbb{E}\left[\sum_{i=0}^{n-1} \Delta F(x,i)\right] = \sum_{i=0}^{n-1} \mathbb{E}[\Delta F(x,i)].$$

So we evaluate the variation of $F(x,i)$ using the masks basis of Figure 4b.
Consider that at step t, cell i matches:

- mask $\mathtt{1\dot{0}11}$, $\mathtt{1\dot{1}0}$, $\mathtt{1\dot{1}1}$, $\mathtt{110\dot{1}1}$: with probability 1 at the step $t+1$, cell i matches the same mask at the time t. Thus, $\mathbb{E}[\Delta F(x^t,i)] = 0$.
- mask $\mathtt{\dot{0}0}$, $\mathtt{\dot{0}10}$, $\mathtt{0\dot{1}0}$: $F(x^t,i) = 3$. Thus, $\mathbb{E}[\Delta F(x^t,i)] \leqslant 0$.
- mask $\mathtt{0\dot{0}11}$: $F(x^t,i) = 3$. With probability α cell i updates and at the step $t+1$, $F(x^{t+1},i) \leqslant 1$. Otherwise, it matches mask $\mathtt{\dot{0}}$. Thus, $\mathbb{E}[\Delta F(x^t,i)] = -2\alpha$.
- mask $\mathtt{00\dot{1}1}$: $F(x^t,i) = 0$. With probability lower than α, cell $i-2$ updates and cell i matches mask $\mathtt{010\dot{1}1}$ at the step $t+1$ and $\Delta F(x^t,i) = 1$. Otherwise, $\Delta F(x^t,i) = 0$. Thus, $\mathbb{E}[\Delta F(x^t,i)] \leqslant \alpha$.
- mask $\mathtt{010\dot{1}1}$: $F(x^t,i) = 1$. With probability at least α, cell $i-2$ updates and cell i matches mask $\mathtt{00\dot{1}1}$ at the step $t+1$ and $F(x^{t+1},i) = 0$. Otherwise, it matches the mask $\mathtt{010\dot{1}1}$. Thus, $\mathbb{E}[\Delta F(x^t,i)] = -\alpha$.

Finally:

$$\begin{aligned}\sum_{i=0}^{n-1} \mathbb{E}[\Delta F(x^t,i)] &\leqslant -\alpha(2|x^t|_{\mathtt{0011}} - |x^t|_{\mathtt{0011}} + |x^t|_{\mathtt{01011}}) \\ &\leqslant -\alpha(|x^t|_{\mathtt{0011}} + |x^t|_{\mathtt{01011}})\end{aligned}$$

Thus, as long as x^t is not the fixed point $\mathtt{1}^n$, we have:

$$\begin{aligned}\mathbb{E}[\Delta F(x^t)] &= \sum_{i=0}^{n-1} \mathbb{E}[\Delta F(x^t,i)] \\ &\leqslant -\alpha(|x^t|_{\mathtt{0011}} + |x^t|_{\mathtt{01011}}) \\ &\leqslant -\alpha\end{aligned}$$

Using Lemma 4, automata **BE** and **BCE** converge a.s. from any initial configuration with a pattern $\mathtt{11}$. The relaxation time is $O\left(\frac{n}{\alpha}\right)$.

Thus, automata **BE** and **BCE** converge a.s. from any initial configuration. The relaxation time is $O\left(\frac{n}{\alpha} + \frac{1}{\alpha(1-\alpha)}\right)$.

□

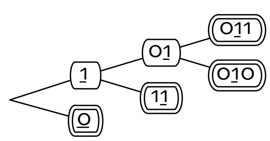
B.5 Automaton BEFG(130)

The fixed points of this automaton are 0^n and 1^n. The fixed point 1^n cannot be reached from any other configuration.

Theorem 18 *Under α-asynchronous dynamics, DQECA **BEFG** converges a.s. to a fixed point from any initial configuration. The relaxation time is $O\left(\frac{n}{\alpha(1-\alpha)}\right)$.*

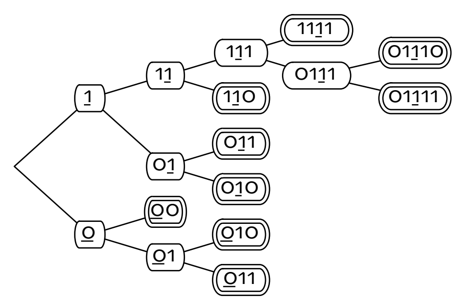
The lengths of the large zones of 1 tend to decrease. Because of the rule **G**, the moves of patterns 110 tend to erase the 1s. But for the borders $\dot{0}11$, the Spawn phenomon (rules **B** on cell i and **F** on cell $i+1$ together) creates new zones. These zones $00\dot{1}011$ can be shifted (ruled **B** on cell $i-1$ and **E** on cell i together). Nevertheless the spawning rate of these zones is not large enough to change the global behaviour of this automaton.

Proof. We use the masks basis and local weight function f of Figure 5a.



masks	$\dot{0}$	$0\dot{1}0$	otherwise ($0\dot{1}1$, $1\dot{1}$)
$f(\dot{m})$	0	6	5

(a) Weight function



(b) Analysis

Figure 5: Masks basis for **BEFG**.

And so:

$$
\begin{aligned}
F(x) &= 5(|x|_{011} + |x|_{11}) + 6|x|_{010} \\
&= 5(|x|_{011} + |x|_{11} + |x|_{010}) + |x|_{010} \\
&= 5|x|_{1} + |x|_{010}.
\end{aligned}
$$

Note that, for all configuration x, $F(x) \in \{0, \ldots, 5n\}$ and $F(x) = 0$ if and only if $x = 0^n$.

Lemma 19 $\mathbb{E}[\Delta F(x)] \leqslant -\alpha(1-\alpha)|x|_{01}$.

By linearity of expectation:

$$
\mathbb{E}[\Delta F(x)] = \mathbb{E}\left[\sum_{i=0}^{n-1} \Delta F(x,i)\right] = \sum_{i=0}^{n-1} \mathbb{E}[\Delta F(x,i)].
$$

So we evaluate the variation of $F(x, i)$ using the masks basis 5b.
Consider that at step t, cell i matches:

- mask $\mathtt{\dot{0}0}$: With probability 1 at the step $t+1$, cell i matches mask $\mathtt{\dot{0}}$. Thus, $\mathbb{E}[\Delta F(x^t, i)] = 0$.
- mask $\mathtt{11\dot{1}1}$ (resp. $\mathtt{01\dot{1}11}$): With probability 1 at the step $t+1$, cell i matches mask $\mathtt{1\dot{1}}$ (resp. $\mathtt{0\dot{1}1}$ or $\mathtt{1\dot{1}}$). Thus, $\mathbb{E}[\Delta F(x^t, i)] = 0$.
- mask $\mathtt{\dot{0}11}$:

With probability	$\geqslant 1-\alpha$	$\leqslant \alpha(1-\alpha)$	$\leqslant \alpha^2$
at the step $t+1$, cell i matches mask	$\mathtt{\dot{0}}$	$\mathtt{\dot{1}1}$	$\mathtt{\dot{1}0}$
and $\Delta F(x^t, i)$	$= 0$	$= 5$	$\leqslant 6$

The two last cases are possible only if $x^t_{i-1} = \mathtt{0}$. Thus,

$$\begin{aligned}\mathbb{E}[\Delta F(x^t, i)] &\leqslant 5\alpha(1-\alpha) + 6\alpha^2 \\ &\leqslant 6\alpha(1-\alpha) + 6\alpha^2 \\ &\leqslant 6\alpha.\end{aligned}$$

- the mask $\mathtt{0\dot{1}0}$ (and $\mathtt{\dot{0}10}$ together):

With probability	$\geqslant \alpha(1-\alpha)$	$\geqslant (1-\alpha)^2$	$\leqslant \alpha(1-\alpha)$	$\leqslant \alpha^2$
at the step $t+1$, cell i matches mask	$\mathtt{0\dot{0}}$	$\mathtt{0\dot{1}}$	$\mathtt{1\dot{1}}$	$\mathtt{1\dot{0}}$
and $\Delta F(x^t, i-1)$	$= 0$	$= 0$	$= 5$	$\leqslant 6$
and $\Delta F(x^t, i)$	$= -6$	$= 0$	$= -1$	$= -6$

The two last cases are possible only if $x^t_{i-2} = \mathtt{0}$. Thus,

$$\mathbb{E}[\Delta F(x^t, i-1) + \Delta F(x^t, i)] \leqslant -2\alpha(1-\alpha).$$

- the mask $\mathtt{0\dot{1}1}$:

With probability	α	$\geqslant (1-\alpha)^2$	$\leqslant \alpha(1-\alpha)$
at the step $t+1$, cell i matches mask	$\mathtt{\dot{0}}$	$\mathtt{\dot{1}1}$	$\mathtt{\dot{1}0}$
and $\Delta F(x^t, i)$	$= -5$	$= 0$	$\leqslant 1$

The last case is possible only if $x^t_{i+2} = \mathtt{0}$. Thus,

$$\mathbb{E}[\Delta F(x^t, i)] \leqslant -5\alpha + \alpha(1-\alpha) \leqslant -4\alpha.$$

- the mask $\texttt{1}\dot{\texttt{1}}\texttt{0}$:

With probability	α	$\geqslant (1-\alpha)^2$	$\leqslant \alpha(1-\alpha)$
at the step $t+1$, cell i matches mask	$\dot{\texttt{0}}$	$\texttt{1}\dot{\texttt{1}}$	$\texttt{0}\dot{\texttt{1}}$
and $\Delta F(x^t, i)$	$=-5$	$=0$	$=1$

The last case is possible only if $x^t_{i-2} = \texttt{0}$. Thus,

$$\mathbb{E}[\Delta F(x^t, i)] \leqslant -4\alpha.$$

- the mask $\texttt{01}\dot{\texttt{1}}\texttt{10}$:

With probability	α^2	$1-\alpha^2$
at the step $t+1$, cell i matches mask	$\texttt{0}\dot{\texttt{1}}\texttt{0}$	other than $\texttt{0}\dot{\texttt{1}}\texttt{0}$
and $\Delta F(x^t, i)$	$=1$	$=0$

Thus,

$$\mathbb{E}[\Delta F(x^t, i)] = \alpha^2 \leqslant \alpha.$$

Finally:

$$\begin{aligned}\sum_{i=0}^{n-1} \mathbb{E}[\Delta F(x^t, i)] \leqslant {} & 6\alpha|x|_{\texttt{011}} - 2\alpha(1-\alpha)|x|_{\texttt{010}} - 4\alpha|x|_{\texttt{011}} \\ & - 4\alpha|x|_{\texttt{110}} + \alpha|x|_{\texttt{01110}}\end{aligned}$$

Since $|x|_{\texttt{011}} = |x|_{\texttt{110}}$ (these patterns count the two borders of any $\texttt{1}$-regions of length at least 2) and since $|x|_{\texttt{01110}} \leqslant |x|_{\texttt{110}}$, we have:

$$\begin{aligned}\sum_{i=0}^{n-1} \mathbb{E}[\Delta F(x^t, i)] & \leqslant -2\alpha(1-\alpha)|x|_{\texttt{010}} - \alpha|x|_{\texttt{011}} \\ & \leqslant -\alpha(1-\alpha)|x|_{\texttt{01}}\end{aligned}$$

So, as long as x^t is not a fixed point, we have:

$$\begin{aligned}\mathbb{E}[\Delta F(x^t)] & \leqslant -\alpha(1-\alpha)|x|_{\texttt{01}} \\ & \leqslant -\alpha(1-\alpha)\end{aligned}$$

Using Lemma 4, automaton **BEFG** converges a.s. from any initial configuration (except $\texttt{1}^n$) to $\texttt{0}^n$. The relaxation time is $O\left(\frac{n}{\alpha(1-\alpha)}\right)$.

□

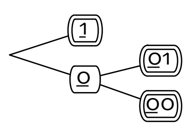
B.6 Automaton BCDEF(242)

The fixed points of this automaton are $\mathtt{0}^n$ and $\mathtt{1}^n$.

Theorem 20 *Under α-asynchronous dynamics, DQECA* ***BCDEF*** *converges a.s. to a fixed point from any initial configuration. The relaxation time is $O\left(\frac{n}{\alpha(1-\alpha)}\right)$.*

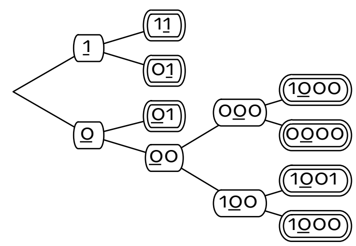
Because of Spawn phenomenon (rules **B** and **F** applied together to cells $i-1$ and i), any pattern $\mathtt{00\dot{1}1}$ can create a new zone. Nevertheless, because of rule **C**, the moves of the patterns $\mathtt{110}$ tend to erase the $\mathtt{0}$s and this is enough to have a fast convergence to a fixed point. the factor $\frac{1}{1-\alpha}$ is necessary for the elimination of the isolated $\mathtt{0}$s.

Proof. We use the masks basis and local weight function f of Figure 6a.



masks	$\dot{\mathtt{0}}\mathtt{0}$	$\dot{\mathtt{0}}\mathtt{1}$	$\dot{\mathtt{1}}$
$F(\dot{m})$	1	2	0

(a) Weight function



(b) Analysis

Figure 6: Masks basis for **BCDEF**.

And so:

$$F(x) = |x^t|_{\mathtt{0}} + |x^t|_{\mathtt{01}}.$$

For all configuration x, $F(x) \in \{0, \dots, n\}$ and $F(x) = 0$ if and only if $x = \mathtt{1}^n$.

Lemma 21 $\mathbb{E}[\Delta F(x)] \leqslant -\alpha(1-\alpha)|x^t|_{\mathtt{01}}$.

By linearity of expectation:

$$\mathbb{E}[\Delta F(x)] = \mathbb{E}\left[\sum_{i=0}^{n-1} \Delta F(x,i)\right] = \sum_{i=0}^{n-1} \mathbb{E}[\Delta F(x,i)].$$

We evaluate the variation of $F(x,i)$ using the following masks basis of Figure 6b.
Consider that cell i matches at the step t:

- mask $\mathtt{0\dot{0}00}$: with probability 1 at the step $t+1$, cell i matches mask $\dot{\mathtt{0}}\mathtt{0}$. Thus, $\mathbb{E}[\Delta F(x,i)] = 0$.
- mask $\mathtt{1\dot{1}}$: with probability 1 at the step $t+1$, cell i matches mask $\dot{\mathtt{1}}$. Thus, $\mathbb{E}[\Delta F(x,i)] = 0$.

- mask $\mathtt{1\dot{0}00}$: with probability α at the step $t+1$, cell i matches mask $\mathtt{\dot{1}}$. Otherwise, it matches mask $\mathtt{\dot{0}0}$. Thus, $\mathbb{E}[\Delta F(x,i)] = -\alpha$.
- mask $\mathtt{0\dot{0}01}$: with probability α at the step $t+1$, cell i matches mask $\mathtt{\dot{0}1}$. Otherwise, it matches mask $\mathtt{\dot{0}0}$. Thus, $\mathbb{E}[\Delta F(x,i)] = \alpha$.
- mask $\mathtt{1\dot{0}01}$:

With probability	α	$(1-\alpha)\alpha$	$(1-\alpha)^2$
at the step $t+1$, cell i matches mask	$\mathtt{\dot{1}}$	$\mathtt{\dot{0}1}$	$\mathtt{\dot{0}0}$
and $\Delta F(x,i)$	$=-1$	$=1$	$=0$

Thus,

$$\mathbb{E}[\Delta F(x,i)] = -\alpha + \alpha(1-\alpha) \leqslant 0.$$

- mask $\mathtt{0\dot{1}}$ (and $\mathtt{\dot{0}1}$ together):

With probability	$(1-\alpha)^2$	$\alpha(1-\alpha)$	$\alpha(1-\alpha)$	α^2
at the step $t+1$, cell i matches mask	$\mathtt{0\dot{1}}$	$\mathtt{1\dot{1}}$	$\mathtt{0\dot{0}}$	$\mathtt{1\dot{0}}$
and $\Delta F(x,i-1)$	$=0$	$=-2$	$=-1$	$=-2$
and $\Delta F(x,i)$	$=0$	$=0$	$\leqslant 2$	$\leqslant 2$

Thus,

$$\mathbb{E}[\Delta F(x,i-1) + \Delta F(x,i)] \leqslant -\alpha(1-\alpha).$$

Finally:

$$\sum_{i=0}^{n-1} \mathbb{E}[\Delta F(x^t,i)] \leqslant -\alpha|x^t|_{\mathtt{1000}} + \alpha|x^t|_{\mathtt{0001}} - \alpha(1-\alpha)|x^t|_{\mathtt{01}}$$

Since $|x^t|_{\mathtt{1000}} = |x^t|_{\mathtt{0001}}$ (these patterns counts the borders of $\mathtt{0}$-regions of length at least 3) , we have:

$$\sum_{i=0}^{n-1} \mathbb{E}[\Delta F(x^t,i)] \leqslant -\alpha(1-\alpha)|x^t|_{\mathtt{01}}.$$

So, as long as x^t is not a fixed point, we have:

$$\begin{aligned} \mathbb{E}[\Delta F(x^t)] &\leqslant -\alpha(1-\alpha)|x^t|_{\mathtt{01}} \\ &\leqslant -\alpha(1-\alpha) \end{aligned}$$

Using Lemma 4, automaton **BCDEF** converges a.s. from any intial configuration. The relaxation time is $O\left(\frac{n}{\alpha(1-\alpha)}\right)$.

□

B.7 Automaton BG(142)

The fixed points of this automaton are $\texttt{0}^n$, $\texttt{1}^n$ and $(\texttt{01})^{n/2}$(only if n is even). These fixed points cannot be reached by any other configuration.

Proof. The number of regions is fixed by the initial configuration. No regions could be destroyed (rules **D** or **E** or Annihilation phenomenon) or created (Spawn or Fork phenomena). So the fixed points cannot be reached by any other configuration. □

Theorem 22 *Under α-asynchronous dynamics, DQECA **BCDEF** diverges a.s. from any initial configuration (except for the fixed points).*

Under totally asynchronous, α-asynchronous or synchronous dynamics, the behaviour of this automaton is identical.

B.8 Automaton BF(198)

The fixed points of this automaton are $\texttt{0}^n$, $\texttt{1}^n$ and $(\texttt{01})^{n/2}$(only if n is even). The fixed points $\texttt{0}^n$ and $\texttt{1}^n$ cannot be reached by any other configuration.

Proof. The number of regions can only increase. No region can be destroyed (rule **D**, rule **E** or Annihilation phenomenon). Spawn phenomenon makes the number of regions increase. So the fixed points $\texttt{0}^n$ and $\texttt{1}^n$ cannot be reached by any other configuration. □

If n is even, the fixed point $(\texttt{01})^{n/2}$ can be reached by a few configurations. A pattern $\texttt{10}^i\texttt{1}^j\texttt{0}$ can create new regions if $i+j>3$. If $i=j=1$, then this pattern is $\texttt{1010}$ and it will never change during the next steps. If $i+j=3$ then the third cell of this pattern will change of state each time it is activated, the pattern oscillates between $\texttt{10010}$ and $\texttt{10110}$. So as soon as $i+j$ is odd, it will lead to the creation of one of these patterns and so the fixed point $(\texttt{01})^{n/2}$ cannot be reached anymore.

So even if all fixed points are unreachable. The automaton will divergence in a set of few configurations. The patterns $\texttt{1010}$, $\texttt{10010}$ and $\texttt{10110}$ are the only ones present in these configurations.

Theorem 23 *Under α-asynchronous dynamics, DQECA **BF** diverges into a set of configurations. The relaxation time for reaching this set is $O\left(\frac{n}{\alpha^2}\right)$.*

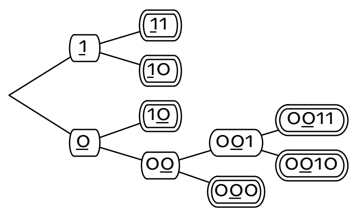
Proof. We use the masks basis and local weight function f of Figure 7a.

And so:

$$F(x) = |x^t|_{\texttt{0011}} + \lceil\frac{3}{\alpha}\rceil|x^t|_{\texttt{10}}.$$

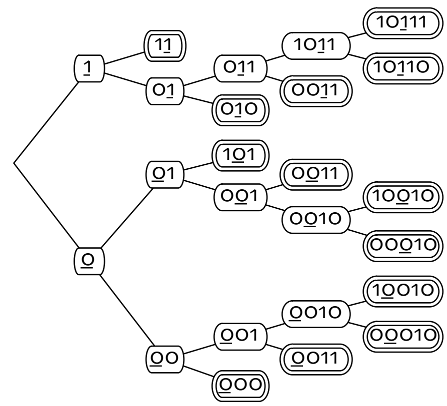
For all configuration x, $F(x) \in \{0, \ldots, \lceil\frac{3}{\alpha}\rceil\frac{n}{2}\}$ and $F(x) = 0$ if and only if $x = \texttt{1}^n$.

Lemma 24 $\mathbb{E}[\Delta F(x)] \geqslant \alpha(|x^t|_{\texttt{10111}} + |x^t|_{\texttt{00010}} + |x^t|_{\texttt{0011}})$



masks	$\dot{1}0$	$0\dot{0}11$	otherwise
$F(\dot{m})$	$\lceil \frac{3}{\alpha} \rceil$	1	0

(a) Weight function



(b) Analysis

Figure 7: Masks basis for **BF**.

By linearity of expectation:

$$\mathbb{E}[\Delta F(x)] = \mathbb{E}\left[\sum_{i=0}^{n-1} \Delta F(x,i)\right] = \sum_{i=0}^{n-1} \mathbb{E}[\Delta F(x,i)].$$

So we evaluate the variation of $F(x,i)$ using the masks basis of Figure 7b.
Consider that cell i matches at the step t:

- mask $1\dot{1}$, $0\dot{1}0$: with probability 1 at the step $t+1$, cell i matches the same mask that the one at time t. Thus, $\mathbb{E}[\Delta F(x^t,i)] = 0$.
- mask $10\dot{1}10$, $10\dot{0}10$, $\dot{0}00$, $1\dot{0}1$, $1\dot{0}010$, $00\dot{0}10$: $F(x^t,i) = 0$. With probability 1 at the step $t+1$, $F(x^{t+1},i) = 0$. Thus, $\mathbb{E}[\Delta F(x^t,i)] = 0$.
- mask $\dot{0}011$, $00\dot{1}1$: $F(x^t,i) = 0$. Thus , $\mathbb{E}[\Delta F(x^t,i)] \geqslant 0$.
- mask $10\dot{1}11$, $0\dot{0}010$: $F(x^t,i) = 0$. With probability α at the step $t+1$, cell i matches mask $0\dot{0}11$. Otherwise, it matches the same mask at the time t. Thus, $\mathbb{E}[\Delta F(x^t,i)] = \alpha$.
- mask $0\dot{0}11$:

With probability	$(1-\alpha)^2$	$(1-\alpha)\alpha$	$(1-\alpha)\alpha$	α^2
at the step $t+1$, cell i matches mask	$\dot{0}1$	$\dot{1}1$	$\dot{0}0$	$\dot{1}0$
and $\Delta F(x^t,i)$	$=0$	$=-1$	$=-1$	$= \lceil \frac{3}{\alpha} \rceil - 1$

Thus,

$$\begin{aligned}
\mathbb{E}[\Delta F(x^t, i)] &= -2(1-\alpha)\alpha - \alpha^2 + \lceil \frac{3}{\alpha} \rceil \alpha^2 \\
&= -\alpha - (1-\alpha)\alpha + \lceil \frac{3}{\alpha} \rceil \alpha^2 \\
&\geqslant -2\alpha + 3\alpha \\
&\geqslant \alpha.
\end{aligned}$$

Finally:

$$\sum_{i=0}^{n-1} \mathbb{E}[\Delta F(x^t, i)] \geqslant \alpha |x^t|_{\texttt{0011}} + \alpha |x^t|_{\texttt{10111}} + \alpha |x^t|_{\texttt{11101}}$$

So, as long as x^t is not a configuration of the final set, we have:

$$\begin{aligned}
\mathbb{E}[\Delta F(x^t)] &\geqslant \alpha |x^t|_{\texttt{0011}} + \alpha |x^t|_{\texttt{10111}} + \alpha |x^t|_{\texttt{11101}} \\
&\geqslant \alpha(1-\alpha)
\end{aligned}$$

Using Lemma 4, automaton **BF** reaches the final set. The relaxation time for reaching this set is $O\left(\frac{n}{\alpha} \times \frac{1}{\alpha}\right) = O\left(\frac{n}{\alpha^2}\right)$.
□

B.9 Automata BEG(138), BDEF(226) and BDEG(170)

The fixed points of these automata are $\texttt{0}^n$ and $\texttt{1}^n$.

Theorem 25 *Under α-asynchronous dynamics, DQECAs **BEG**, **BDEF** and **BDEG** converge a.s. to a fixed point on any initial configuration. The relaxation time is $O\left(\frac{n^2}{\alpha(1-\alpha)}\right)$.*

For these automata, the regions follow a random walk. The expected number of moves before reaching a fixed point is $O(n^2)$, and a move can be done with probability $\alpha(1-\alpha)$.

Proof. Here we directly defined $F(x)$ for a configuration x like this:

$$F(x) = |x|_{\texttt{1}}$$

Each time a transition **B** or **D** is activated the number of $\texttt{1}$ increases by one. Each time a transition **E**, **F** or **G** is activated the number of $\texttt{1}$ decreases by one.

So for **BEG**:

$$\Delta F(x^t) = \alpha(|x^t|_{\texttt{001}} - |x^t|_{\texttt{010}} - |x^t|_{\texttt{110}})$$
$$\leqslant 0$$

For **BDEF**:

$$\Delta F(x^t) = \alpha(|x^t|_{\texttt{001}} + |x^t|_{\texttt{101}} - |x^t|_{\texttt{010}} - |x^t|_{\texttt{011}})$$
$$\leqslant 0$$

For **BDEG**:

$$\Delta F(x^t) = \alpha(|x^t|_{\texttt{001}} + |x^t|_{\texttt{101}} - |x^t|_{\texttt{010}} - |x^t|_{\texttt{110}})$$
$$\leqslant 0$$

Thus for any of these automata $\Delta F(x) \leqslant 0$.

Now we have to calculate the probability of decreasing or increasing $F(x)$ by one or more. We only make the proof for **BDEF**(other ones are almost the same). We look at the evolution of pattern $\dot{\texttt{0}}\texttt{1}$ (and $\texttt{0}\dot{\texttt{1}}$ together):

With probability	$(1-\alpha)^2$	$(1-\alpha)\alpha$	$(1-\alpha)\alpha$	α^2
at the step $t+1$, cell i matches mask	$\dot{\texttt{0}}\texttt{1}$	$\dot{\texttt{1}}\texttt{1}$	$\dot{\texttt{0}}\texttt{0}$	$\dot{\texttt{1}}\texttt{0}$
and $\Delta F(x^t, i) + \Delta F(x^t, i+1)$	$= 0$	$= 1$	$= -1$	$= 0$

So $|\Delta F(x^t, i) + \Delta F(x^t, i)| \geqslant 1$ with probability $2\alpha(1-\alpha)$. But there may be several patterns like this one and the sum of their contributions could be equal to 0. Nevertheless using generating functions of the sum of the contributions of these patterns [8], we can conclude that $|\Delta F(x)| \geqslant 1$ with probability greater than $\alpha(1-\alpha)$.

So using Lemma 9 and 10, DQECA **BEG**, **BDEF** and **BDEG** converge a.s. to a fixed point from any initial configuration. The relaxation time is $O\left(\frac{n^2}{\alpha(1-\alpha)}\right)$.

□