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NOTE

DIRECTED CARTESIAN-PRODUCT GRAPHS HAVE UNIQUE FACTORIZATIONS THAT CAN BE COMPUTED IN POLYNOMIAL TIME

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The *cartesian product* of directed, simple graphs $D_1=(V_1,A_1)$ and $D_2=(V_2,A_2)$ is a digraph D with $V(D)=V_1\times V_2$ and $A(D)=\{(v_1,v_2)\to(w_1,w_2)$: $v_1=w_1$ and $v_2\to w_2\in A_2$ or $v_2=w_2$ and $v_1\to w_1\in A_1\}$. In this paper, we prove that directed graphs have unique prime factorizations under cartesian multiplication and that we can find the prime factorizations of weakly connected digraphs in polynomial time. This work extends recent work by Feigenbaum, Hershberger, Schäffer, and Winkler on cartesian factoring of undirected graphs.

The *cartesian product* of undirected, simple graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ is a graph G with $V(G)=V_1\times V_2$ and $E(G)=\{(v_1,v_2)-(w_1,w_2)$: $v_1=w_1$ and $v_2-w_2\in E_2$ or $v_2=w_2$ and $v_1-w_1\in E_1\}$. More concretely, for each node in G_1, insert a copy of G_2 and draw in the trivial isomorphism for each pair of copies corresponding to adjacent nodes. Sabidussi showed that every connected, undirected graph G has a prime factorization $G_1\times\cdots\times G_m$ that is unique up to isomorphism [2]. Feigenbaum, Hershberger, Schäffer, and Winkler have recently shown how to compute this factorization in polynomial time [1, 4]. The purpose of this note is to prove the analogous results for directed graphs, which we will call *digraphs*.

The cartesian product D of two digraphs $D_1=(V_1,A_1)$ and $D_2=(V_2,A_2)$ is analogous to that of undirected graphs: $V(D)=V(D_1)\times V(D_2)$ and $A(D)=\{(v_1,v_2)\to(w_1,w_2)$: $v_1=w_1$ and $v_2\to w_2\in A_2$ or $v_2=w_2$ and $v_1\to w_1\in A_1\}$. We use the terminology and notation of [1] and extend it when appropriate for digraphs. In particular, if G can be written as a product $G_1\times G_2$, where both $|V(G_1)|$ and $|V(G_2)|$ are greater than one, then G is *cartesian-factorable* and $G_1\times G_2$ is a *proper factorization*; if no such factorization exists, then G is *prime*. If $G=G_1\times G_2$ is a proper factorization, then $V(G)$ can be partitioned into $|V(G_1)|$ copies of G_2, and

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we denote by G_2^x the unique copy containing $x \in V(G)$; because cartesian multiplication is commutative (that is, $G_1 \times G_2 \cong G_2 \times G_1$), we can also partition $V(G)$ into $|V(G_2)|$ copies of G_1 and identify G_1^x for each x. Two copies G_i^x and G_i^y that correspond to adjacent nodes in the other factor are *adjacent copies.*

Let D be a digraph. Form a graph G with node set $V(D)$ and edge set $E(G) = \{x\text{-}y : x \to y \in A(D) \text{ or } y \to x \in A(D)\}$. Then G is called the *underlying graph* of D and is denoted $U(D)$. We say that D is *weakly connected* if $U(D)$ is connected. The factors D_1 and D_2 are weakly connected if and only if $D_1 \times D_2$ is as well. For both digraphs and undirected graphs that are disconnected, the cartesian factoring problem is at least as hard as the graph isomorphism problem; in [1, 4] the authors restrict attention to connected graphs in order to get a polynomial time algorithm, and here we consider only weakly connected digraphs.

Suppose that $G = G_1 \times \cdots \times G_m$ is the prime factorization of G, and fix this order so that we may refer to the ith prime factor of G. Suppose further that the edges E_i of $E(G)$ that belong to the ith factor are labeled i; so $E(G) = \bigcup_{i=1}^m E_i$. With this labeling, G_i^x can be computed, for a specific x, using depth-first search [1]. If A_i is the set of arcs in $A(D)$ whose directions we removed to form G, then we say that *E_i lifts to A_i*.

If G is cartesian-factorable, then for any prime factor G_i, there is a proper factorization $G \cong H \times G_i$, where H is the cartesian product of all of the prime factors except G_i, and a partition of the nodes of G into sets whose induced subgraphs are the copies G_i^x. The edges of H present isomorphisms between each pair of adjacent copies G_i^x, G_i^y. There is a corresponding partition of D into induced digraphs D_i^x; together the edges of the copies G_i^x make up the set E_i, and similarly the arcs of the D_i^x's make up A_i. We use our terminology and notation liberally and say that a subgraph D' of D has underlying subgraph G', that G' lifts to D', and that a proper factorization $B \times C$ of D has underlying factorization $U(B) \times U(C)$ of $G = U(D)$. The natural question is whether D has m prime factors, $D_1, \ldots, D_m$, with $D_i = U(G_i)$. This questions reduces to that of whether the edges between copies G_i^x lift to arcs that give directed isomorphisms between copies D_i^x.

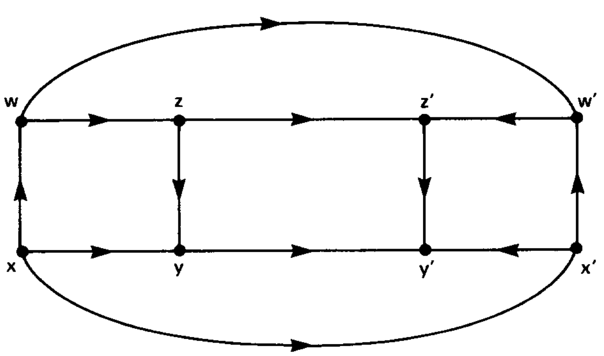

Fig. 1a. Unfactored directed graph D.

Fig. 1a shows why the answer to this question is no. The underlying graph G is just a cube, the cartesian product of three copies of the two-node graph K_2. The copies G_i^a, $a \in V(G)$, $i \in \{1, 2, 3\}$ are labeled in Fig. 1b; for example $G_1^x = (\{x, w\}, \{x - w\})$. If the corresponding subgraphs D_i^a were copies of true factors of D, then the arcs $w \to z$ and $x \to y$ of A_2 would give an isomorphism between the adjacent copies D_1^x and D_1^y of D_1. As shown in Fig. 1c, these arcs do not give an isomor-

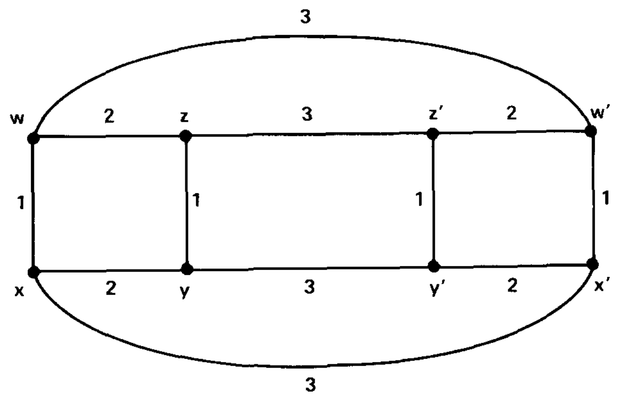

Fig. 1b. The underlying undirected graph $G = U(D) \cong K_2 \times K_2 \times K_2$.

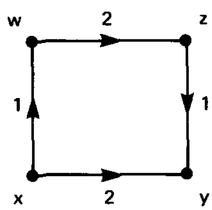

Fig. 1c. Four-node subgraph showing conflict between arc classes 1 and 2.

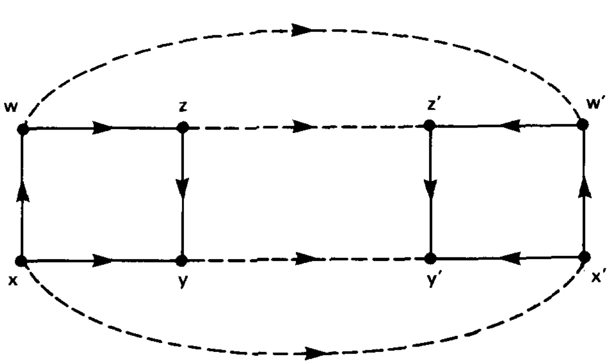

Fig. 1d. Factored directed graph. Arcs of the left factor are dashed and those of the right factor solid.

phism, because they do not respect direction. They take the tail of the arc $x \to w$ to the head of the arc $z \to y$.

If we merge E_1 and E_2 and then lift the merged class, as in Fig. 1d, we see that D does have a non-trivial factorization $D_1 \times D_2$, with $D_1^x = (\{x, y, z, w\}, \{x \to w, w \to z, z \to y, x \to y\})$, $D_2^x = (\{x, x'\}, \{x \to x'\})$. This process of merging edge classes for which the corresponding arcs go in conflicting directions is the main step in constructing the prime factors of D from those of G.

Every proper factorization $B \times C$ of D corresponds to a proper factorization $H \times K$ of G, where $H = U(C)$ and $K = U(D)$. Because G factors uniquely into the prime graphs G_i, the factorization $H \times K$ is just a way of partitioning these prime factors into two sets. So if there is a prime factorization $D_1 \times \cdots \times D_p$ of D, the underlying graphs of the factors D_j are given by a partition of the prime factors of G. Conversely, any partition of the E_i's lifts to a partition of the A_i's that presents a legal factorization of D *unless there is a conflict in the directions of the arcs between adjacent copies of one of the factors.* Hence the only question is whether there is a unique way to partition the factors G_i so that, in each set of the partition, the product of the G_i's in that set lifts to a legal, irreducible factor of D.

Let E_i and E_j, $i \neq j$, be the edge classes corresponding to two prime factors of G. We want to decide whether the corresponding arc classes A_i and A_j are in conflict and hence must belong to the same prime factor of D. Suppose u-v-v'-u'-u is a chordless four-cycle in G such that $\{u\text{-}v, u'\text{-}v'\} \subseteq E_i$ and $\{u\text{-}u', v\text{-}v'\} \subseteq E_j$ (Fig. 2a). This means that in G, v is in some copy G_i^u, u'-v' is the corresponding edge in adjacent copy $G_i^{u'}$, and the j-edges u-u' and v-v' are part of an isomorphism drawn between G_i^u and $G_i^{u'}$ to signify that they replaced adjacent nodes in G_j. (By commutativity, we can also regard the j-edges u-u' and v-v' as corresponding edges in adjacent copies G_j^u and G_j^v and the i-edges u-v and u'-v' as part of an isomorphism.) This four-cycle lifts to a four-node subgraph C of D, one pair of whose opposite arcs belongs to the class A_i and the other to class A_j. The subgraph C demonstrates a conflict between A_i and A_j if one edge, say u-v, lifts to two arcs $u \to v$, $v \to u$ while the opposite edge lifts to a single arc (Fig. 2b-1), or if opposite edges lift to single arcs that go in opposite directions (Fig. 2b-2). The possibilities for C that do *not* demonstrate a conflict are those shown in Fig. 2c, or any digraph isomorphic to one of them.

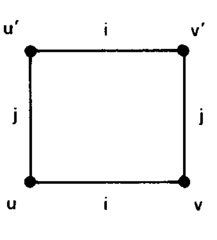

Fig. 2a. Chordless four-cycle containing edge classes i and j in $U(D)$.

We can check in polynomial time whether arc classes A_i and A_j conflict: For each pair of adjacent copies G_i^x, G_i^y, for each edge u–v in G_i^x, there is at most one j-edge u–u' such that u' is in G_i^y; it takes constant time to check whether the subgraph of D to which the four-cycle u–u'–v'–v–u lifts is isomorphic to one of the digraphs in Fig. 2c.

Let $R=\{(A_{i_1}, A_{j_1}), \ldots, (A_{i_t}, A_{j_t})\}$ be a complete list of pairs of arc sets for which we can find four-node subgraphs that demonstrate conflict. Consider R as a binary symmetric relation on $\{A_1, \ldots, A_m\}$ and form its reflexive, transitive closure R^*. There are no conflicts between arc sets A_i and A_j that lie in different equivalence classes in R^*. Thus the partition of the A_i's given by R^* represents a legal factorization of D; furthermore, it is the finest such partition, and hence each factor that it gives is irreducible. So this is the prime factorization of D.

We summarize the factoring algorithm as follows:

Factor (D) [D is a weakly connected digraph.]
  $G:=U(D)$;
  Find the prime factorization $G_1 \times \cdots \times G_m$ of G;
  Let $A_1, \ldots, A_m$ be the arc classes corresponding to edge classes $E_1, \ldots E_m$;
  Initialize R to the empty relation;

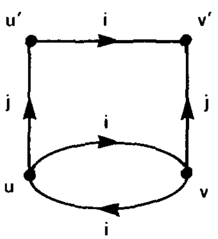

Fig. 2b-1. Arc classes i and j conflict because, between adjacent copies D_j^u and D_j^v, the *number* of i-arcs is inconsistent.

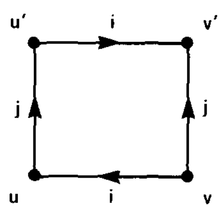

Fig. 2b-2. Arc classes i and j conflict because, between adjacent copies D_j^u and D_j^v, the *direction* of the i-arcs is inconsistent.

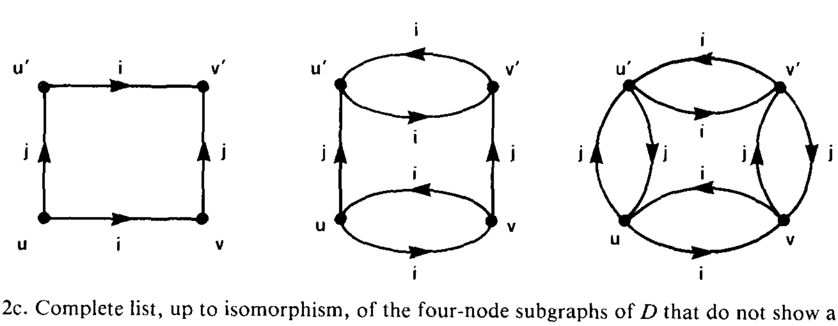

Fig. 2c. Complete list, up to isomorphism, of the four-node subgraphs of D that do not show a conflict between arc classes i and j.

L1: For each pair of arc classes A_i, A_j
L2: For each pair of adjacent copies G_i^x, G_i^y
For each pair of edges u–$v \in E_i$, u–$u' \in E_j$, $u \in G_i^x$, $u' \in G_i^y$
If the four-cycle u–u'–v'–v–u lifts to a
subgraph of D that shows a conflict between A_i and A_j
Then $R := R \cup \{(A_i, A_j)\}$;
$R^* :=$ the reflexive, transitive closure of R;
$p :=$ the number of equivalence classes in R^*;
For $i := 1$ **to** p
Begin
Let $\{A_{i_1}, \ldots, A_{i_{l_i}}\}$ be the ith class in R^*;
Let $\{E_{i_1}, \ldots, E_{i_{l_i}}\}$ be the corresponding edge classes in G;
$H := G_{i_1}^x \cup \cdots \cup G_{i_{l_i}}^x$ for some $x \in V(G)$;
Let D_i be the subgraph of D to which H lifts;
The ith prime factor of D is isomorphic to D_i;
End

The prime factors $G_1, \ldots, G_m$ can be computed in time $O(n^{4.5})$, where $n = |V(D)|$, using the algorithm in [1], or in time $O(n^4)$ using the one in [4]. There are at most $\log n$ prime factors of G and hence at most $\log n$ classes A_i and $\binom{\log n}{2} = O(\log^2 n)$ pairs A_i, A_j to be considered in the For loop **L1**. For a given pair, if there are k copies G_i^x, then there are $O(k^2)$ pairs of adjacent copies and $O((n/k)^2)$ four cycles to be considered in each one, giving a bound of $O(n^2)$ for the For loop **L2**. So the whole nested loop **L1** runs in time $O(n^2 \log^2 n)$. The number of pairs in R is bounded above by $\log^2 n$, and thus R^* can be computed straightforwardly by min-plus matrix multiplication in time $O(\log^6 n)$.

In conclusion, we have the following

Theorem. *Weakly connected digraphs have unique prime factorizations that can be found in polynomial time. The most time consuming task in the given algorithm is to find the prime factors of the underlying undirected graph.*

Coincident with the preparation of this manuscript, Walker showed that any two cartesian factorizations of a connected graph have a strict common refinement and used this fact to derive an independent proof that cartesian digraph factorizations are unique and can be found in polynomial time [3].

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