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Ordering the order of a distributive lattice by itself

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Abstract

We order the ordering relation of an arbitrary poset **P** component-wise by itself, obtaining a poset $\Phi(\mathbf{P})$ extending **P**. In particular, the effects of Φ on $\mathbf{L} \in \mathsf{DLAT01}$, the category of all bounded distributive lattices, are studied, mainly with the aid of Priestley duality. We characterize those $\mathbf{L} \in \mathsf{DLAT01}$ which occur as $\Phi(\mathbf{K})$ for some $\mathbf{K} \in \mathsf{DLAT01}$, decide this situation in polynomial time for finite **L**, characterize fixpoints of Φ within $\mathsf{DLAT01}$ and relate them to free objects in $\mathsf{DLAT01}$.
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1. Introduction

In January 2005, at the end of a long seminar day devoted to ordered sets and lattices, van der Zypen wondered what happens if one orders the order relation of a poset component-wise – that is, by itself. Since no interesting fixed points of this operation came up, the topic was dropped quickly. Soon after Krebs fired up the AlgebraWorkBench (AWB) (see [9,8]) – a software package developed by Sprenger for the handling of finite (universal) algebras in an e-learning environment – fed it the two-element chain and performed the above operation on the latter repeatedly. To general surprise, AWB returned the first few freely generated distributive 0–1-lattices in succession, and eventually led to Krebs' Ph.D. thesis [6], in whose title Φ stands – for lack of a better idea – for the operation sketched above. This note collects some results, partially contained in [6], on the effects of "ordering the order by itself" within bounded distributive lattices.

The paper is organized as follows: Section 2 collects, mostly without proofs, the basic properties of Φ we need; for worked-out proofs and details one might consult [7,6]. Section 3 contains the germinating example which sparked our interest in Φ, with a brute force direct proof (tailored to the finite case) showing that $\Phi^n(\mathbf{2})$ is nothing else than $\mathbf{FD}_{01}(n)$. In Section 4, the effects of Φ on a bounded distributive lattices **L** are studied in topological terms, based on Priestley duality, the main result being that $\Phi(\mathbf{L})$ is the free product of **L** with the three-element chain. Returning to the finite case, we show in Section 5 how to decide in polynomial time whether a given finite distributive lattice occurs

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as the Φ-image of another one. Section 6 characterizes those $\mathbf{L}$ which stay fixed under Φ and relates them to free bounded distributive lattices on infinitely many generators. The final Section 7 considers the special case of Boolean lattices and lists some open questions. The authors' sincere thanks go to referee who with his constructive and detailed criticism substantially improved the presentation of this paper.

2. Set-up

A partially ordered set – for short: a *poset* – is a pair $\mathbf{P} = (P, R_P)$ consisting of a carrier set P and a binary relation R_P or just R on P which is reflexive, antisymmetric and transitive, that is, an *order* (*relation*). We will freely confuse $\mathbf{P}$ and P if the order is clear from the context; also, instead of R_P we will mostly use the standard infix notation $\leq_P$ or just $\leq$. Write POSET for the category of all posets together with order-preserving maps.

Let $\mathbf{P} \times \mathbf{P}$ be the cartesian square of $\mathbf{P}$, having carrier $P \times P$ and the product order $\leq_{P\times P}$ given by $(a, a') \leq_{P\times P} (b, b')$ iff $a \leq b$ and $a' \leq b'$, for all $(a, a'), (b, b') \in P \times P$. Now R_P as a binary relation is actually a subset of $P \times P$, consisting of all pairs $(a, a') \in P \times P$ satisfying $a \leq a'$, and thus carries the canonical order induced by $\leq_{P\times P}$. For some of the properties of $\Phi(\mathbf{P}) = (R_P, \leq_{P\times P}\restriction_{R_P})$ for an arbitrary poset $\mathbf{P}$, the reader may wish to consult [6].

This paper is about $\Phi(\mathbf{P})$ in the special case when $\mathbf{P}$ is a bounded distributive lattice. Such lattices will be written $\mathbf{L} = (L; \wedge, \vee, 0, 1)$, and DLAT01 stands for the category of all bounded distributive lattices satisfying $0 \neq 1$, together with all 0–1-preserving lattice homomorphisms. Here's the formal definition of Φ for this case:

Definition 2.1. For $\mathbf{L} \in$ DLAT01 with order $\leq_L \subseteq L \times L$, let $\Phi(\mathbf{L})$ be the 0–1-sublattice of $\mathbf{L} \times \mathbf{L}$ with carrier $\leq_L$.

Given $f : \mathbf{K} \longrightarrow \mathbf{L}$ in DLAT01, define $\Phi(f) : \Phi(\mathbf{K}) \longrightarrow \Phi(\mathbf{L})$ by $\Phi(f)(a, a') := (f(a), f(a'))$ for all $(a, a') \in \Phi(K)$, that is, $\Phi(f) = f \times f$.

Fact 2.2. Φ is a functor from DLAT01 to DLAT01.

Proof. This follows easily since, for $\mathbf{L} \in$ DLAT01, the sup of $(a, a'), (b, b') \in \Phi(\mathbf{L})$ is given by $(a \vee b, a' \vee b')$, and their inf by $(a \wedge b, a' \wedge b')$. □

Define $\Delta_L : \mathbf{L} \longrightarrow \Phi(\mathbf{L})$ by $\Delta_L(a) := (a, a)$ for $a \in L$. The subscript L will be omitted if clear from the context.

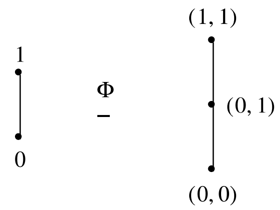
Fact 2.3. $\Delta(\mathbf{L})$ is a 0–1-sublattice of $\Phi(\mathbf{L})$ isomorphic to $\mathbf{L}$ for any $\mathbf{L} \in$ DLAT01, thus $\mathbf{L} \cong \Delta(\mathbf{L}) \subseteq \Phi(\mathbf{L}) \subseteq \mathbf{L} \times \mathbf{L}$ (as 0–1-sublattices).

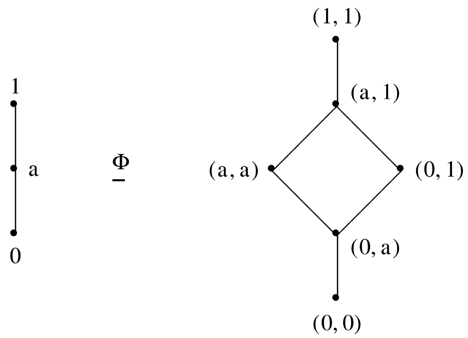
Proof. Routine. □

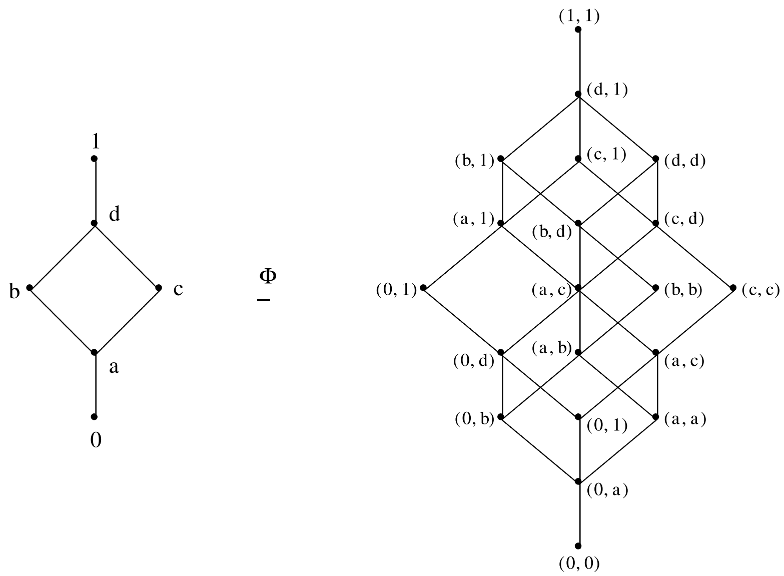
One may check that Φ preserves, e.g., injectivity of morphisms (if $\mathbf{K}$ is a 0–1-sublattice of $\mathbf{L}$, then $\Phi(\mathbf{K})$ a 0–1-sublattice of $\Phi(\mathbf{L})$), arbitrary direct products ($\Phi(\prod_{i\in I} \mathbf{L}_i) \cong \prod_{\in I} \Phi(\mathbf{L}_i)$ for any index set I and lattices $\mathbf{L}_i \in$ DLAT01), and even completeness (if $\mathbf{L}$ is a complete lattice, so is $\Phi(\mathbf{L})$), but we do not need these facts for the purposes of this paper.

3. Unleashing Φ

So far, we have not seen Φ doing any real work. The simplest nontrivial candidate for testing Φ will be the two-element chain $\mathbf{2}$ with carrier $\{0, 1\}$ and order $\{(0, 0), (0, 1), (1, 1)\}$. The following figures show $\Phi^0(\mathbf{2}) := \mathbf{2}$, $\Phi^1(\mathbf{2}) = \Phi(\mathbf{2})$, $\Phi^2(\mathbf{2})$ and $\Phi^3(\mathbf{2})$.







Especially the last diagram will immediately ring a bell for anyone having been exposed to an introductory "Order and Lattices" course: It depicts the free algebra in DLAT01 on three generators, and invites us to recognize the other three as the diagrams of the free algebras on 0, 1 and 2 generators. To formulate the expected theorem, we write $\mathbf{FD}_{01}(n)$ for the lattice freely generated in DLAT01 by n generators.

Theorem 3.1. $\mathbf{FD}_{01}(n) \cong \Phi^n(\mathbf{2})$ *for all* $n \in \omega$.

Proof. Put $\mathbf{L}_n := \mathbf{FD}_{01}(n)$. We show $\Phi(\mathbf{L}_n) \cong \mathbf{L}_{n+1}$ for all $n \in \omega$ by brute force, transforming an arbitrary n-element set freely generating $\mathbf{L}_n$ into a $(n+1)$-element set freely generating $\Phi(\mathbf{L}_n)$. The procedure is based on a peculiar property of free generating sets in DLAT01: G is such for $\mathbf{L} \in$ DLAT01 iff G generates $\mathbf{L}$ and $\bigwedge H \leq \bigvee K$ implies $H \cap K \neq \emptyset$, for any nonempty subsets $H, K \subseteq G$ (see, e.g., [2, section V.3]).

So assume $G = \{g_1, \ldots, g_n\}$ freely generates $\mathbf{L}_n$ (if $n = 0$, G is empty), and put $G' := \Delta[G] \cup \{(0, 1)\}$. Consider an arbitrary element $(a, b) \in \Phi(\mathbf{L}_n)$, thus $a, b \in \mathbf{L}_n$ and $a \leq b$. We see that $(a, b) = ((0, 1) \wedge \Delta(b)) \vee \Delta(a)$ since meets and joins are computed component-wise in $\Phi(\mathbf{L}_n)$. Since $a \in \mathbf{L}_n$, we may write $a = p(g_1, \ldots, g_n)$ for some (lattice) polynomial p, and thus $\Delta(a) = p(\Delta(g_1), \ldots, \Delta(g_n))$. Analogously, $\Delta(b) = q(\Delta(g_1), \ldots, \Delta(g_n))$ for a suitable polynomial, hence, (a, b) is in the 0–1-sublattice of $\Phi(\mathbf{L}_n)$ generated by G'.

It remains to show that G' *freely* generates $\Phi(\mathbf{L}_n)$. Assume that, for some $H', K' \subseteq G'$, we have $\bigwedge H' \leq \bigvee K'$. Put $H := \Delta^{-1}(H')$ and $K := \Delta^{-1}(K')$, thus $H, K \subseteq G$. If $(0, 1) \in H' \cap K'$, we are done, so we may assume that $(0, 1)$ occurs in at most one of H' resp. K'.

Suppose $(0, 1) \in H'$ but $(0, 1) \notin K'$. Hence $K \neq \emptyset$ and $\bigvee K' = (\bigvee K, \bigvee K)$. It follows that $\bigwedge H' \neq (0, 1)$ since $(0, 1) \leq (\bigvee K, \bigvee K)$ implies $1 \leq \bigvee K$ which is not possible for any nonempty $K \subseteq G$, G being a free generating set for $\mathbf{L}_n$. But $\bigwedge H' \neq (0, 1)$ is only possible if $H \neq \emptyset$, in which case $\bigwedge H' = (0, \bigwedge H)$. We conclude that $(0, \bigwedge H) = \bigwedge H' \leq \bigvee K' = (\bigvee K, \bigvee K)$ and thus $\bigwedge H \leq \bigvee K$. The last comparability is in $\mathbf{L}_n$ and we infer that $H \cap K \neq \emptyset$, G being a free generating set. Pick $g_i \in H \cap K$, then $(g_i, g_i) \in H' \cap K'$ and we are done.

The dual argument works for $(0, 1) \notin H'$ but $(0, 1) \in K'$. Finally, if $(0, 1) \notin H' \cup K'$, the desired conclusion $H \cap K \neq \emptyset$ is obtained directly. □

So the orbit of Φ on $\mathbf{2}$ gives the complete list of all finitely generated free algebras in DLAT01. The obvious question at this point is what other – if any – lattices $\mathbf{K} \in$ DLAT01 are of the form $\Phi(\mathbf{L})$ for some $\mathbf{L} \in$ DLAT01. As it turns out, the best way to characterize these lattices is to use Priestley's topological duality theory for DLAT01.

4. Priestley duality at work

A *Priestley space* $\mathbf{X} = (X, \leq, \tau)$ is a (partially) ordered topological space (that is, $(X, \leq)$ is a poset and (X, τ) is a topological space) where the topology τ is compact and related to the order $\leq$ in the following way: If $x, y \in X$ and $x \not\leq y$, there exists a clopen down-set $C \subseteq X$ containing y but missing x (a subset $C \subseteq X$ is a *down-set* iff $c \in C, x \in X$ and $x \leq c$ jointly imply that $x \in C$; *up-sets* are defined dually). This property is called *total order-disconnectedness*; it is easily seen to imply that τ is Hausdorff. Let TOD be the category of all Priestley spaces together with all continuous order-preserving maps.

Priestley duality for bounded distributive lattices essentially says that DLAT01 and TOD are dually equivalent categories: Spaces in TOD correspond bijectively to lattices in DLAT01; in the direction we are interested in one assigns to $\mathbf{X} \in$ TOD the lattice $\mathbf{E}(\mathbf{X})$ of all clopen down-sets in $\mathbf{X}$ (with set intersection and union as lattice operations). We do not specify how to construct the space $\mathbf{D}(\mathbf{L}) \in$ TOD corresponding to $\mathbf{L} \in$ DLAT01 (since we will not need it) but note that $\mathbf{E}(\mathbf{D}(\mathbf{L}))$ and $\mathbf{L}$ are naturally isomorphic as lattices, and similarly $\mathbf{D}(\mathbf{E}(\mathbf{X}))$ and $\mathbf{X}$ are naturally order-homeomorphic as Priestley spaces. We will also use $\cong$ to denote order-homeomorphism between Priestley spaces as there is no danger of confusion.

The functors $\mathbf{D}$ and $\mathbf{E}$ defining the duality are contravariant. In particular, (i) surjective maps $p : \mathbf{X}_1 \longrightarrow \mathbf{X}_2$ in TOD correspond bijectively to lattice embeddings $\mathbf{E}(p) : \mathbf{E}(\mathbf{X}_2) \longrightarrow \mathbf{E}(\mathbf{X}_1)$ with $\mathbf{E}(p)$ being given by $\mathbf{E}(p) = p^{-1}$, and (ii) $\mathbf{D}$ takes (direct) products in DLAT01 to coproducts in TOD. We refer the reader interested in more to [5] for details. A broader picture of when, why and how dualities of the type considered here actually work may be found in [3].

Lemma 4.1. *If* $\mathbf{L} \in$ DLAT01, *then* $\mathbf{D}(\Phi(\mathbf{L})) \cong \mathbf{D}(\mathbf{L}) \times \mathbf{2}$.

Proof. As $\Phi(\mathbf{L})$ is a 0–1-sublattice of $\mathbf{L} \times \mathbf{L}$, we start from the Priestley space corresponding to the latter. The space $\mathbf{D}(\mathbf{L} \times \mathbf{L})$ is order-homeomorphic to the coproduct $\mathbf{D}(\mathbf{L}) \coprod \mathbf{D}(\mathbf{L})$ which coincides with the topological and order-theoretic sum $\mathbf{D}(\mathbf{L}) \dot\cup \mathbf{D}(\mathbf{L})$ of two disjoint copies of $\mathbf{D}(\mathbf{L})$.

Write $\overline{\mathbf{2}}$ for the two-element antichain with carrier $\{0, 1\}$. Equipped with the discrete topology $\overline{\mathbf{2}}$ is turned into a Priestley space. The topological and order-theoretic product $\mathbf{D}(\mathbf{L}) \times \overline{\mathbf{2}}$ is then also Priestley, and the map $f : \mathbf{D}(\mathbf{L}) \dot\cup \mathbf{D}(\mathbf{L}) \longrightarrow \mathbf{D}(\mathbf{L}) \times \overline{\mathbf{2}}$, given by $f(x) = (x, 0)$ if x comes from the first copy of $\mathbf{D}(\mathbf{L})$, and by $f(x) = (x, 1)$ if it comes from the second, obviously defines an order-homeomorphism $\mathbf{D}(\mathbf{L}) \coprod \mathbf{D}(\mathbf{L}) \cong \mathbf{D}(\mathbf{L}) \times \overline{\mathbf{2}}$.

Analogously, we get a Priestley space by equipping the two-element chain $\mathbf{2}$ with the discrete topology, and the topological and order-theoretic product $\mathbf{D}(\mathbf{L}) \times \mathbf{2}$ is also Priestley. Define a map $p : \mathbf{D}(\mathbf{L}) \times \overline{\mathbf{2}} \longrightarrow \mathbf{D}(\mathbf{L}) \times \mathbf{2}$ by

$p(x, 0) := (x, 1)$ and $p(x, 1) := (x, 0)$, for all $x \in \mathbf{D}(\mathbf{L})$. It is clear that p is surjective (in fact, even bijective), continuous and order-preserving (since there is no nontrivial order to be preserved on $\overline{\mathbf{2}}$).

The clopen down-sets of $\mathbf{D}(\mathbf{L}) \times \mathbf{2}$ are finite unions of sets of type $C \times \{0\}$ or $C' \times \{0, 1\}$ for clopen down-sets $C, C' \in \mathbf{D}(\mathbf{L})$. Since $\times$ distributes over $\cup$, such unions may be simplified to sets of the form $(B \times \{0\}) \cup (B' \times \{1\})$ with $B, B' \in \mathbf{E}(\mathbf{D}(\mathbf{L}))$ satisfying $B' \subseteq B$. Conversely, any set of the latter type may be written as $(B \times \{0\}) \cup (B' \times \{0, 1\})$ since $B' \times \{0\} \subseteq B \times \{0\}$, and is thus in $\mathbf{E}(\mathbf{D}(\mathbf{L}) \times \mathbf{2})$. Applying $\mathbf{E}(p) = p^{-1}$ to $(B \times \{0\}) \cup (B' \times \{1\})$ we obtain $(B' \times \{0\}) \cup (B \times \{1\}) \in \mathbf{E}(\mathbf{D}(\mathbf{L}) \times \overline{\mathbf{2}}) \cong \mathbf{E}(\mathbf{D}(\mathbf{L}) \dot{\cup} \mathbf{D}(\mathbf{L}))$. $\mathbf{E}(p)$ is injective as p is surjective, so $\mathbf{E}(p)$ embeds $\mathbf{E}(\mathbf{D}(\mathbf{L}) \times \mathbf{2}))$ into $\mathbf{E}(\mathbf{D}(\mathbf{L}) \dot{\cup} \mathbf{D}(\mathbf{L})) \cong \mathbf{E}(\mathbf{D}(\mathbf{L})) \times \mathbf{E}(\mathbf{D}(\mathbf{L}))$ as the sublattice $\mathbf{S}$ consisting of all pairs (B', B) satisfying $B' \subseteq B$, which is nothing else than $\Phi(\mathbf{E}(\mathbf{D}(\mathbf{L})) \cong \Phi(\mathbf{L})$. We conclude that $\mathbf{D}(\mathbf{L}) \times \mathbf{2}$ is the Priestley space of $\Phi(\mathbf{L})$. □

So the question raised at the end of the preceding section may be answeres as follows:

Theorem 4.2. *If* $\mathbf{K} \in \mathsf{DLAT01}$*, then* $\mathbf{K} \cong \Phi(\mathbf{L})$ *for some* $\mathbf{L} \in \mathsf{DLAT01}$ *iff* $\mathbf{D}(\mathbf{K}) \cong \mathbf{Y} \times \mathbf{2}$ *for some* $\mathbf{Y} \in \mathsf{TOD}$.

A different, less categorical, proof of Lemma 4.1 based on an explicit construction of the space $\mathbf{D}(\mathbf{L})$ may be found in [7]. We may stretch the duality a little bit further. $\mathbf{E}$ being contravariant, it takes products in TOD to coproducts in $\mathsf{DLAT01}$ – better known as *free products* in categories of (universal) algebras like $\mathsf{DLAT01}$. In the case we are interested in, we have $\Phi(\mathbf{L}) \cong \mathbf{E}(\mathbf{D}(\Phi(\mathbf{L}))) \cong \mathbf{E}(\mathbf{D}(\mathbf{L}) \times \mathbf{2}) \cong \mathbf{E}(\mathbf{D}(\mathbf{L})) \coprod \mathbf{E}(\mathbf{2}) \cong \mathbf{L} \coprod \mathbf{E}(\mathbf{2})$. Now $\mathbf{E}(\mathbf{2})$ is the lattice of clopen down-sets of the discrete ordered space $\mathbf{2}$ which obviously is the three-element chain $\mathbf{3}$ realized as $\emptyset \subset \{0\} \subset \{0, 1\}$. So we obtain an algebraic variant of Theorem 4.1.

Corollary 4.3. *For* $\mathbf{L} \in \mathsf{DLAT01}$ *we have* $\Phi(\mathbf{L}) \cong \mathbf{L} \coprod \mathbf{3}$.

This gives an alternative proof for Theorem 3.1: For $n \in \omega$, the free distributive 0–1-lattice on n free generators $\mathbf{FD}_{01}(n)$ is in fact the free product of n copies of $\mathbf{FD}_{01}(1)$ (for $n = 0$, there is no free generator and thus $\mathbf{FD}_{01}(0) \cong \mathbf{2}$). So Theorem 3.1 may be proved inductively by using $\mathbf{FD}_{01}(0) \cong \mathbf{2} \cong \Phi^0(\mathbf{2})$ to start the induction and applying Corollary 4.3 for the induction step.

Finally, we would like to mention that Lemma 4.1 may deduced from the main result of [4] by observing that for any poset $\mathbf{P} \in \mathsf{POSET}$, one may view $\Phi(\mathbf{P})$ as the set of all order-preserving maps from $\mathbf{2}$ to $\mathbf{P}$, ordered point-wise. The authors are indebted to Brian Davey for pointing out this fact at the occasion of a workshop at the University of Bern. See [6] for details.

5. Deciding Φ

While Theorem 4.2 provides a clean description – in terms of Priestley duality – of all $\mathbf{K} \in \mathsf{DLAT01}$ occurring as $\Phi(\mathbf{L})$ for some $\mathbf{L} \in \mathsf{DLAT01}$, it is not necessarily of much help when it comes to actually decide whether a given $\mathbf{K}$ is of this type. The purpose of this section is to present an *effective* solution of this decision problem for finite $\mathbf{K} \in \mathsf{DLAT01}$.

Priestley duality takes a much simpler form for finite $\mathbf{L}$: The spaces $\mathbf{D}(\mathbf{L})$ are finite and thus discrete, being Hausdorff, so we may discard the topology. Moreover, the carrier poset of $\mathbf{D}(\mathbf{L})$ may be identified with the poset $\mathbf{J}(\mathbf{L})$ of all non-zero join-irreducible elements of $\mathbf{L}$ in the order inherited from $\mathbf{L}$. Conversely, given any finite poset $\mathbf{P}$, the lattice $\mathbf{E}(\mathbf{P})$ may be identified with the set $\mathbf{O}(\mathbf{P})$ of all down-sets (alias *order ideals*, to justify the notation) of $\mathbf{P}$, equipped with set union and intersection as operations. In fact, $\mathbf{J}$ and $\mathbf{O}$ are functors providing a full-blown duality between finite distributive lattices (and 0–1-preserving lattice homomorphisms) and finite posets (with order-preserving maps), but we do not need their morphism part. This duality is known as *finite Birkhoff duality*.

Consequently, Lemma 4.1 and Theorem 4.2 take a rather simple form in the finite setting:

Corollary 5.1. *For finite* $\mathbf{L} \in \mathsf{DLAT01}$*, the poset dual to* $\Phi(\mathbf{L})$ *is* $\mathbf{J}(\mathbf{L}) \times \mathbf{2}$*. If* $\mathbf{K} \in \mathsf{DLAT01}$*, then* $\mathbf{K} \cong \Phi(\mathbf{L})$ *for some finite* $\mathbf{L} \in \mathsf{DLAT01}$ *iff* $\mathbf{J}(\mathbf{K}) = \mathbf{P} \times \mathbf{2}$ *for some finite poset* $\mathbf{P}$.

The obvious question at this point is: Given a finite poset $\mathbf{P}$, (i) does there exist a poset $\mathbf{Q}$ such that $\mathbf{P} \cong \mathbf{Q} \times \mathbf{2}$ and (ii) if yes, how can we construct it? At first glance, this seems to require checking all subsets of P which are half the size of P – a procedure of exponential complexity. We will show how to accomplish (i) and (ii) in polynomial time (in the size of P), for any poset $\mathbf{P}$.

For $\mathbf{P} \in \mathsf{POSET}$ and $a, a' \in P$, we say that a is *covered* by a' (resp. that a' *covers* a) iff $a \leq x \leq a'$ implies $x \in \{a, a'\}$ for all $x \in P$, and denote this situation by $a \prec a'$; also, a is then said to be a *lower cover* of a' resp. a' a *upper cover* of a. A *covering partition* $\mathcal{C}$ of $\mathbf{P}$ is a bijection $' : P_0 \longrightarrow P_1$ where $\{P_0, P_1\}$ is a partition of P and $a \prec a'$ for all $a \in P_0$. Finally, $x \parallel y$ stands for $x \nleq y$ *and* $y \nleq x$.

Fact 5.2. Suppose $\mathbf{P} = \mathbf{Q} \times \mathbf{2}$. Let $Q_i := Q \times \{i\} \subseteq P$ for $i = 0, 1$. Then $' : Q_0 \longrightarrow Q_1$ given by $(z, 0)' := (z, 1)$ for all $z \in Q$ is a covering partition of $\mathbf{P}$. Moreover, the following are true:

(i) $(a, 0) \prec (u, 0)$ implies $(u, 1) = \sup\{(u, 0), (a, 1)\}$.

(ii) $(l, 1) \prec (b, 1)$ implies $(l, 0) = \inf\{(b, 0), (l, 1)\}$.

(iii) $x \parallel y$ or $x < y$ for any $x \in Q_0$ and $y \in Q_1$.

Proof. It is clear by definition that $' : Q_0 \longrightarrow Q_1$ is a covering partition of $\mathbf{P}$. For (i), let $(a, 0) \prec (u, 0)$, then $a \leq u$. Now $(w, i) \geq (u, 0), (a, 1)$ iff $i = 1$ and $w \geq u, a$, that is, iff $i = 1$ and $w \geq u$, hence $(u, 1) = \sup\{(u, 0), (a, 1)\}$. (ii) is dealt with analogously. For (iii), $x \geq y$ is clearly not possible for $x \in Q_0$ and $y \in Q_1$ by the definition of Q_k. □

Actually, conditions (i)–(iii) in Fact 5.2 characterize *finite* posets of type $\mathbf{P} \cong \mathbf{Q} \times \mathbf{2}$:

Lemma 5.3. *Let* $\mathbf{P}$ *be a finite poset and* $' : P_0 \longrightarrow P_1$ *a covering partition of* $\mathbf{P}$*. Assume the following three conditions are satisfied:*

(i) *Let* $a, u \in P_0$ *and* $a \prec u$*. Then* $u' = \sup\{u, a'\}$.

(ii) *Let* $l', b' \in P_1$ *and* $l' \prec b'$*. Then* $l = \inf\{b, l'\}$.

(iii) *For all* $x \in P_0$ *and* $y \in P_1$*, either* $x \parallel y$ *or* $x < y$.

Then $\mathbf{P}_0 \cong \mathbf{P}_1$ *and* $\mathbf{P} \cong \mathbf{P}_0 \times \mathbf{2} \cong \mathbf{P}_1 \times \mathbf{2}$*, where* $\mathbf{P}_i = (P_1, \leq P{\restriction}_{P_i})$.

Proof. Write $\prec_i$ for the covering relations of $\mathbf{P}_i$ $(i = 0, 1)$. We start by showing that $\prec_i = \prec{\restriction}_{P_i}$. Indeed, let $a, b \in P_0$, $a \prec_0 b$. Suppose there is $y \in P$ such that $a < y < b$. Then $y \notin P_0$ since $a \prec_0 b$, thus $y \in P_1$. By condition (iii) either $b \parallel y$ or $b < y$; since b and y are comparable we obtain $b < y < b$, a contradiction. So y as assumed will not exist and $a \prec b$ as desired. The same argument works for $\prec_1$.

Since P is finite, it follows that in order to show that $'$ is an order-isomorphism, we only need to prove that $a \prec b$ in $\mathbf{P}_0$ iff $a' \prec b'$ in $\mathbf{P}_1$. So let $a, u \in P_0$ and $a \prec u$. Hence $u' = \sup\{u, a'\}$ by condition (i). Suppose $l' \in P_1$ and $a' \leq l' \prec u'$. Hence $l = \inf\{u, l'\}$ by condition (ii). On the other hand, $a = \inf\{u, a'\}$ since $u \neq a'$, $a \prec u$ and $a \prec a'$. We infer that $a \leq u, l'$ since $a' \leq l'$. Thus $a \leq \inf\{u, l'\} = l \leq u$. From $a \prec u$ we get that either $l = a$ or $l = u$. The latter is ruled out for then $l' = u'$. Hence $l = a$ and $l' = a'$ which gives $a' \prec u'$ as desired. The same argument – with the rôles of (i) and (ii) interchanged – will show that $l' \prec b'$ in P_1 implies $l \prec b$ in P_0, and $'$ is seen to be an order isomorphism.

It remains to establish w.l.o.g. that $\mathbf{P}_0 \times \mathbf{2} \cong \mathbf{P}$. Define $\psi : \mathbf{P}_0 \times \mathbf{2} \to \mathbf{P}$ by

$$\psi(p, i) = \begin{cases} p & : \quad i = 0 \\ p' & : \quad i = 1 \end{cases}$$

for $p \in \mathbf{P}_0$. It is clear that ψ is bijective and order-preserving, so it only remains to show that ψ^{-1} is also order-preserving. But this exactly what condition (iii) ensures. □

Lemma 5.3 provides the key for an algorithm which, fed a finite connected poset $\mathbf{P}$, computes a covering partition $' : P_0 \longrightarrow P_1$ satisfying the three conditions of 5.3 – provided such $'$ exists – and outputs the corresponding subposet $\mathbf{P}_0$, or says NO if no such $'$ exists. We shall use a so-called queue structure:

A queue is simply a string to which new arrivals are added at the back and departures take place from the front. The addition of an entry at the back of a queue is called an *enqueue* operation, and a departure from the front of the queue is called a *dequeue* operation. Thus the notation *enqueue(v,Q)* denotes adding v to the end of a queue called

Q; *dequeue(Q)* denotes removal of the entry currently at the front of Q. We will also use a function *front(Q)*, which returns the value of the entry currently at the front of Q.

Algorithm 5.4. Given a connected poset **P**, the following algorithm outputs a subposet $\mathbf{P_0}$ of **P** or reports "NO".

```
procedure DirectProduct (P: connected poset)
{checks if P ≅ P′ × 2 for some P′}

var
  Q: queue of elements;
  N: set of upper neighbours of m;
  P0: set of covered elements;
  P1: set of covering elements;
  m: minimal element;
  product: boolean;

begin
  if |P| even then
  begin
    initialize Q, M, P0, P1 to be empty;
    m := minimal element of P;
    enqueue (m, Q);
    P0 ← P0 ∪ {m};
    rename m to (m, 0);
    mark (m, 0) visited;
    N := set of upper neighbours of m;                    {arbitrarily ordered}
    product := false;
    while N ≠ ∅ and product = false do
    begin
      product := true
      choose n ∈ N;                                       {e.g. the first in the order}
      N ← N \ {n};
      P1 ← P1 ∪ {n};
      rename n to (m, 1);
      mark (m, 1) visited;
      while Q is not empty and product = true do
      begin
        for each upper neighbour x of (front(Q),0) do
        begin
          if x not visited then
          begin
            enqueue(x, Q);
            P0 ← P0 ∪ {x};
            rename x to (x, 0);
            mark (x, 0) visited;
            if sup((x, 0), (front(Q), 1)) exists then
            begin
              P1 ← P1 ∪ {sup((x, 0), (front(Q), 1))};
              rename sup((x, 0), (front(Q), 1)) to (x, 1);
              mark (x, 1) visited;
            end;
            else product = false;
```

```
            end;                                                    {if}
          end;                                                      {for}
          for each lower neighbour y of (front(Q),1) do
          begin
            if y not visited then
            begin
              enqueue(y, Q);
              P1 ← P1 ∪ {y};
              rename y to (y, 1);
              mark (y, 1) visited;
              if inf((y, 1), (front(Q), 0)) exists then
              begin
                P0 ← P0 ∪ {inf((y, 1), (front(Q), 0))};
                rename inf((y, 1), (front(Q), 0)) to (y, 0);
                mark (y, 0) visited;
              end;
              else product = false;
            end;                                                    {if}
          end;                                                      {for}
          dequeue(Q);
        end;                                                        {while}
        if not all elements visited then product = false;
        if (∃x ∈ P0∃y ∈ P1 : y < x) then product = false;
      end;                                                          {while}
    end;                                                            {if even then}
    else product = false;
    if product = true then write("P is a direct product of 2 and P0")
    else write("P is not a direct product with one of the factors equal to 2");
end;                                                                {procedure}
```

Lemma 5.5. *Algorithm 5.4 outputs a subposet* $\mathbf{P_0}$ *of* $\mathbf{P}$ *if and only if* $\mathbf{P} \cong \mathbf{P_0} \times \mathbf{2}$.

Proof. We show that Algorithm 5.4 outputs such $\mathbf{P}_0$ iff $\mathbf{P}$ satisfies the conditions of Lemma 5.3. Start by selecting a minimal element $m \in P$ and label it $(m, 0)$ (as $\mathbf{P}$ is connected, it does not matter which minimal element we choose). Next, we try to find a covering partition of $\mathbf{P}$. Any upper neighbour of $(m, 0)$ in $\mathbf{P}$ is a candidate for the desired covering element $(m, 1)$ of $(m, 0)$. So pick an arbitrary upper cover of $(m, 0)$ to be $(m, 1)$ and check whether this choice does not force – eventually – a violation of one of the three conditions stated in 5.3. We start by "moving along squares", i.e. we test conditions (i) and (ii). Since we store all upper neighbours of $(m, 0)$ and all lower neighbours of $(m, 1)$ in queue, we are able to move from one pair to another, and may repeat the test with the most recently found pairs (this amounts more or less to a breadth-first search of the graph of the covering relation of $\mathbf{P}$). As soon as we discover a missing infimum or supremum, we give up $(m, 1)$ as covering element of $(m, 0)$, pick a new one and repeat the procedure. We do the same if we can not visit all elements, that is, if no covering partition of $\mathbf{P}$ results (connectedness of $\mathbf{P}$ ensures that non-visitation results from finding a missing infimum or supremum). Finally, we check condition (iii) (that is, whether no element of $\mathbf{P_1}$ is smaller than any element of $\mathbf{P_0}$). If this happens, we again discard $(m, 1)$ and look for a new partner for $(m, 0)$.

Consequently, Algorithm 5.4 effectively constructs a covering partition satisfying all conditions of Lemma 5.3 whenever this is possible. Since these conditions are, by Fact 5.2, also necessary for $\mathbf{P}$ to be of the required type, we get an output $\mathbf{P}_0$ if and only if $\mathbf{P} \cong \mathbf{P_0} \times \mathbf{2}$. □

Proposition 5.6. *Let* $\mathbf{P}$ *be a connected poset with n elements. Then the decision problem "Is* $\mathbf{P}$ *of the form* $\mathbf{P}_0 \times \mathbf{2}$ *for some subposet* $\mathbf{P}_0 \subseteq \mathbf{P}$*" is solvable with time complexity* $\mathrm{O}(n^7)$.

Proof. We assume w.l.o.g. that we are given both the cover table and the comparability table of $\mathbf{P}$ since either one can be converted into the other in time at most $O(n^3)$. From these data, a list of all suprema and infima (of two elements) existing in $\mathbf{P}$ may be compiled in $O(n^4)$ time, concluding the preprocessing stage.

For the at most $\binom{n}{2}$ possible covering pairs, 5.4 checks at most $\binom{n}{2}$ possible meets and $\binom{n}{2}$ possible joins. In the worst case, we have to repeat the process for at most n possible upper neighbours of m, resulting in an overall time complexity $n \cdot \binom{n}{2}^3 = O(n^7)$. $\square$

Suppose that $\mathbf{P}$ is a disjoint sum $\mathbf{P} = \mathbf{P}_1 \dot{\cup} \mathbf{P}_2$. Then $\mathbf{P} \cong \mathbf{Q} \times \mathbf{2}$ for some poset $\mathbf{Q}$ iff there are posets $\mathbf{Q}_i$ $(i = 1, 2)$ such that $\mathbf{Q} = \mathbf{Q}_1 \dot{\cup} \mathbf{Q}_2$ and $\mathbf{P}_i \cong \mathbf{Q}_i \times \mathbf{2}$.

Corollary 5.7. *For an arbitrary poset* $\mathbf{P}$ *of size n, the decision problem 5.6 is solvable in time* $O(n^8)$.

Proof. The connected components of $\mathbf{P}$ may be found in $O(n + \binom{n}{2}) = O(n^2)$ time using a standard algorithm for undirected graphs applied to the cover table of $\mathbf{P}$. Perform 5.4 on each component. $\square$

Theorem 5.8. *Let* $\mathbf{L} \in \mathsf{DLAT01}$ *be finite with n elements. The decision problem "Does* $\mathbf{L}$ *belong to the range of* Φ*?" is solvable in time* $O(n^8)$.

Proof. Assume we are given the cover table, the comparability table and the operation tables of L since all of these may be obtained from a given one in in time at most $O(n^3)$. To obtain $\mathbf{P} = \mathbf{J}(\mathbf{L})$ we have to find the elements of $\mathbf{L}$ with exactly one lower neighbor, which can be done in $O(n)$ time. Perform 5.4 on $\mathbf{P}$. $\square$

6. Fixpoints of Φ

A fixpoint of Φ (in $\mathsf{DLAT01}$) is any $\mathbf{L} \in \mathsf{DLAT01}$ satisfying $\Phi(\mathbf{L}) \cong \mathbf{L}$. It is easy to see that there no *finite* fixpoints in $\mathsf{DLAT01}$: Since $0 \neq 1$, we have $(0, 1) \in \Phi(\mathbf{L})$ for any $\mathbf{L} \in \mathsf{DLAT01}$, so $|\Phi(\mathbf{L})| > |\Delta(\mathbf{L})| = |\mathbf{L}|$ for any finite $\mathbf{L}$. We will show that there are many *infinite* fixpoints of Φ in $\mathsf{DLAT01}$, using Lemma 4.1.

Theorem 6.1. *Every free algebra* $\mathbf{FD}_{01}(\kappa) \in \mathsf{DLAT01}$*, where* κ *is an infinite cardinal, is a fixpoint of* Φ.

Proof. Let $\mathbf{L} = \mathbf{FD}_{01}(\kappa)$ and recall that $\mathbf{D}(\mathbf{L}) \cong \mathbf{2}^\kappa$. Since κ is infinite, there is a bijection from κ to $\kappa + 1$ and therefore an isomorphism from $\mathbf{2}^\kappa$ to $\mathbf{2}^{\kappa+1}$. Then we have

$$\mathbf{D}(\Phi(\mathbf{L})) \cong \mathbf{D}(\mathbf{L}) \times \mathbf{2} \cong \mathbf{2}^\kappa \times \mathbf{2} = \mathbf{2}^{\kappa+1} \cong \mathbf{2}^\kappa \cong \mathbf{D}(\mathbf{L})$$

and consequently $\Phi(\mathbf{L}) \cong \mathbf{L}$. $\square$

So the actions of Φ in $\mathsf{DLAT01}$ and free algebras in $\mathsf{DLAT01}$ are related in two opposite ways: In the finite case, Φ produces them sequentially, while it fixes them in the infinite case. In the rest of section we will establish another connection: If $\mathbf{L}$ is any fixpoint of Φ, then $\mathbf{L}$ must contain a copy of $\mathbf{FD}_{01}(\omega)$ as a 0–1-sublattice.

So suppose $\Phi(\mathbf{L}) \cong \mathbf{L}$. By Lemma 4.1, we obtain $\mathbf{D}(\Phi(\mathbf{L})) \cong \mathbf{D}(\mathbf{L}) \times \mathbf{2} \cong \mathbf{D}(\mathbf{L})$. Put $\mathbf{X} := \mathbf{D}(\mathbf{L})$ and fix an order-homeomorphism $\alpha : \mathbf{X} \times \mathbf{2} \longrightarrow \mathbf{X}$. Obviously, $\mathbf{X}$ is order-homeomorphic with $\mathbf{X} \times \mathbf{2}^n$ for any $n \in \omega$; based on α, we will construct a family of compatible order-homeomorphisms $\iota_n : \mathbf{X} \times \mathbf{2}^n \longrightarrow \mathbf{X}$ for $n \in \omega$. We write the elements $s \in \mathbf{2}^n$ as finite 0–1-sequences $s_0, \ldots, s_{n-1}$ of *length* $|s| = n$.

Let $n > 1$ and for any $x \in \mathbf{X}$, $s \in 2^n$ define homeomorphisms $\epsilon_n : \mathbf{X} \times \mathbf{2}^n \longrightarrow \mathbf{X} \times \mathbf{2}^{n-1}$ by $\epsilon_n(x, s_0, \ldots, s_{n-1}) := (\alpha(x, s_{n-1}), s_0, \ldots, s_{n-2})$. Let $\iota_n : \mathbf{X} \times \mathbf{2}^n \longrightarrow \mathbf{X} := \epsilon_1 \circ \epsilon_2 \circ \ldots \circ \epsilon_n$. The ι_n are also order-homeomorphisms; and $\iota_1 = \epsilon_1 = \alpha$.

For $n \geq 1$ and $s \in \mathbf{2}^n$, $\mathbf{Y}_s := \mathbf{X} \times \{(s_0, \ldots, s_{n-1})\}$ defines a clopen subspace of $\mathbf{X} \times \mathbf{2}^n$ which is a order-homeomorphic copy of $\mathbf{X}$. With s ranging over $\mathbf{2}^n$, these subspaces form a partition of $\mathbf{X} \times \mathbf{2}^n$. Note that $\mathbf{Y}_s$ is the disjoint union of $\mathbf{Y}_{t'}$ and $\mathbf{Y}_{t''}$ where $t', t'' \in \mathbf{2}^{n+1}$ and $t' = s \cup \{(n, 0)\}$ resp. $t'' = s \cup \{(n, 1)\}$.

Define subspaces $\mathbf{X}_s$ of $\mathbf{X}$ by $\mathbf{X}_s := \iota_n[\mathbf{Y}_s]$. In this way, we obtain a partition $\mathcal{P}_n = \{\mathbf{X}_s;\, s \in \mathbf{2}^n\}$ of $\mathbf{X}$ into clopen subspaces, all of them order-homeomorphic with $\mathbf{X}$. We let $\mathcal{P}_0 := \{\mathbf{X}\}$ for completeness and end up with a sequence of partitions $(\mathcal{P}_n)_{n\in\omega}$ with $\mathcal{P}_{n+1}$ refining $\mathcal{P}_n$ and $|\mathcal{P}_n| = 2^n$.

For $x \in \mathbf{X}$ define $\hat{x} \in \mathbf{2}^\omega$ as follows: Put, for each $n \in \omega$, $\hat{x}(n) := s(n)$ where $\mathbf{X}_s$ is the unique subspace in $\mathcal{P}_{n+1}$ such that $x \in \mathbf{X}_s$ (e.g., if $x \in \mathbf{X_{101}}$, we have $\hat{x}(0) = 1$, $\hat{x}(1) = 0$ and $\hat{x}(2) = 1$ and so on). Finally, define $\sigma : \mathbf{X} \longrightarrow \mathbf{2}^\omega$ by $\sigma x := \hat{x}$, for all $x \in \mathbf{X}$.

Lemma 6.2. *σ is surjective, continuous and order-preserving.*

Proof. We start with surjectivity. Let $t \in \mathbf{2}^\omega$ and consider the set of all $\mathbf{X}_s \in \mathcal{P}_n$ where $n \geq 1$ and s is the initial segment of t with $|s| = n$. This produces a descending chain of nonempty closed sets in a compact space, and thus $\bigcap \mathbf{X}_s \neq \emptyset$. Evidently, $\sigma x = t$ for any $x \in \bigcap \mathbf{X}_s$.

For continuity, recall that a subbasic open set in $\mathbf{2}^\omega$ is of the form $S_{n,i} = \{t \in \mathbf{2}^\omega;\ t(n) = i\}$ for some $n \in \omega$ and $i \in \{0, 1\}$. But then $\sigma^{-1}(S_{n,i})$ equals the set union of all $\mathbf{X}_s \in \mathcal{P}_n$ satisfying $s(n-1) = i$; this is open as a finite union of clopen sets.

It remains to show that σ preserves order. Consider $x, y \in \mathbf{X}$, $x \leq y$. By the definition of σ, we have $\sigma x \leq \sigma y$ iff $s'(n) \leq s''(n)$ for all $n \in \omega$, where $\mathbf{X}_{s'}$ resp. $\mathbf{X}_{s''}$ are the unique subspaces in $\mathcal{P}_{n+1}$ containing x resp. y. Let $(x^*, s'_0, \ldots, s'_n) \in \mathbf{Y}_{s'}$ resp. $(y^*, s''_0, \ldots, s''_n) \in \mathbf{Y}_{s''}$ be the the preimages of x resp. y under ι_{n+1}. Since ι_{n+1} is an order-isomorphism, we have $x \leq y$ in $\mathbf{X}$ iff $(x^*, s'_0, \ldots, s'_n) \leq (y^*, s''_0, \ldots, s''_n)$ in $\mathbf{X} \times \mathbf{2}^{n+1}$; in particular, it follows that $s'_n \leq s''_n$, as desired. □

Turning on Priestley duality, we immediately obtain that $\mathbf{E}(\mathbf{2}^\kappa)$ embeds into $\mathbf{E}(\mathbf{X})$ and with that

Theorem 6.3. *If $\mathbf{L} \in \mathsf{DLAT01}$ is a fixpoint of Φ, then $\mathbf{L}$ contains a copy of $\mathbf{FD}_{01}(\omega)$, the distributive 0–1-lattice on ω free generators, as a 0–1-sublattice.*

Note that the converse is not true: *Not* every lattice having a copy of $\mathbf{FD}_{01}(\omega)$ as a sublattice is a fixpoint of Φ. Let $\mathbf{B}$ the minimal Boolean extension of $\mathbf{FD}_{01}(\omega)$ (with Priestley space $\overline{\mathbf{2}}^\omega$). We cannot have $\Phi(\mathbf{B}) \cong \mathbf{B}$ since $\Phi(\mathbf{B}) \subseteq \mathbf{B} \times \mathbf{B}$ will contain $(0, 1)$ but not $(1, 0)$ and thus fails to be complemented.

7. Remarks, and loose ends

7.1. The Boolean case

As we have seen in the remark following Theorem 6.3, Φ does not send Boolean algebras to Boolean algebras, so we are tempted to ask whether a reasonable analogue of Φ for Boolean algebras exists. An answer may be obtained by looking at the duality, this time Stone duality. Let BA the category of Boolean algebras and Boolean homomorphisms, and TD that of Stone spaces and continuous maps, where a Stone space is a totally disconnected compact Hausdorff space (so Stone duality may be viewed as the "unordered case" of Priestley duality). Again, BA and TD are dually equivalent categories, and we also write $\mathbf{D}$ and $\mathbf{E}$ for the responsible functors.

Mimicking Lemma 4.1, we define, for any $\mathbf{B} \in \mathsf{BA}$ with Stone space $\mathbf{D}(\mathbf{B}) = \mathbf{X}$, a Boolean algebra $\Phi'(\mathbf{B})$ as the algebra with Stone space $\mathbf{X} \times \overline{\mathbf{2}}$. Starting from the Boolean algebra $\mathbf{2}$ (with complementation as a basic operation) having $\overline{\mathbf{2}}$ as its Stone space, successive applications of Φ' produce the discrete spaces $\overline{\mathbf{2}}^n$ for $n \geq 1$, which are the dual spaces of the Boolean algebras $\mathbf{2}^{2^n}$ – i.e., the finite free Boolean algebras, in complete analogy with Theorem 3.1. For any infinite cardinal κ, one has $\overline{\mathbf{2}}^\kappa \cong \overline{\mathbf{2}}^{\kappa+1}$ which implies, in complete analogy with Theorem 6.1, that $\mathbf{FBA}(\kappa)$, the Boolean algebra on κ free generators, is a fixpoint of Φ'. Also the proof of Lemma 6.2 carries over and produces the fact, corresponding to Theorem 6.3 that if a Boolean algebra is a fixpoint of Φ', then it contains a copy of $\mathbf{FBA}(\omega)$ as a subalgebra.

But what is Φ' in algebraic terms? Consider $\mathbf{B} \in \mathsf{BA}$ with Stone space $\mathbf{X} \in \mathsf{TD}$. The Stone space of $\Phi'(\mathbf{B})$ is then, by definition, given as $\mathbf{X} \times \overline{\mathbf{2}}$ which is the disjoint sum (or coproduct) of two copies of $\mathbf{X}$. But this means that $\Phi'(\mathbf{B})$ is nothing else than $\mathbf{B} \times \mathbf{B}$. So we may conclude that(i) $\mathbf{FBA}(\kappa) \times \mathbf{FBA}(\kappa) \cong \mathbf{FBA}(\kappa)$ for any infinite cardinal κ and

(ii) that any Boolean algebra **B** satisfying $\mathbf{B} \times \mathbf{B} \cong \mathbf{B}$ must contain a copy of $\mathbf{FBA}(\omega)$ as a subalgebra. These feels like known facts, but we haven't been able to locate a specific reference in the literature.

7.2. Beyond distributivity

The focus of this note is on the category DLAT01 which is a variety in the sense of universal algebra, that is, closed under the formation of subalgebras, direct products and homomorphic images. That it is also closed under Φ thus follows directly from the definition of Φ as a subalgebra of a direct product. Consequently, every class of lattices closed under $\mathbb{S}$ and $\mathbb{P}$ is also closed under Φ, in particular, every quasivariety generated by a single finite lattice. So a natural question is that for fixpoints in such classes.

Moving into the opposite direction, we may look at distributive lattices with additional structure. While $\Phi(\mathbf{L})$ for $\mathbf{L} \in$ DLAT01 will fail to be complemented (see the remark following Theorem 6.3), it will be pseudocomplemented whenever **L** is, the pseudocomplement of $(a, b) \in \Phi(\mathbf{L})$ being (b^*, b^*) where * denotes pseudocomplementation in **L**. Moreover, one has $\Delta(a^*) = (a^*, a^*)$, so the embedding $\Delta : \mathbf{L} \longrightarrow \Phi(\mathbf{L})$ preserves pseudocomplements. Let PALG be the category of *p-algebras*, that is, of distributive pseudocomplemented lattices with pseudocomplementation as a basic operation. It follows that Φ induces a functor from PALG to PALG. The proper subvarieties of PALG form an ω-chain $\mathsf{L}_0, \mathsf{L}_1, \ldots, \mathsf{L}_n, \ldots$ with $\mathsf{L}_0 =$ BA, the so-called *Lee classes*; it would be interesting to know how Φ interacts with these, and where possible fixpoints are located in this hierarchy.

To define pseudocomplements, all one needs is a zero and a meet-operation. So the questions raised in the preceding paragraph may as well be applied to PSEMI, the category of all pseudocomplemented semilattices. Here BA is the only proper subvariety. As shown recently, pseudocomplemented semilattices can simulate any category of (universal) algebras; indeed, PSEMI is *universal* (in the category-theoretic sense) relative to BA, see [1]. So knowing what Φ does to PSEMI might be worth the effort.

7.3. Any relation instead of order

Playing with posets and lattices just reflects the authors' preferences. In fact, any relation may be exposed to an analogous treatment: Let R be a k-ary relation on a carrier set A, that is, $R \subseteq A^k$. Define a k-ary relation ΦR on R by putting k-tuples $\mathbf{a}_i = (a_{i1}, \ldots, a_{ik}) \in R$ $(1 \leq i \leq k)$ into ΦR iff $(a_{1j}, \ldots, a_{kj}) \in R$ for all $1 \leq j \leq k$. Specifically, we may subject $graph(f) \subseteq A^{k+1}$ to this procedure for any k-ary operation f on A. However, in order to obtain a k-ary operation Φf on $graph(f)$, f has to satisfy a straightforward compatibility condition which, for $k = 2$, will hold – in particular – provided f is associative and commutative. So certainly semilattice operations will fall into this category.

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