A2019

1. On applique le principe fondamental de la dynamique à la masse assimilée à un point matériel dans le référentiel du laboratoire supposé galiléen :

$$m\ddot{\vec{r}} = m\vec{g} + \vec{R} - k(x - l_0)\vec{e_x}$$

Sur $\vec{e_y}$ : en supposant que la masse ne se déplace pas, $\vec{R} = -m\vec{g}$

Sur $\vec{e_x}$ : $\ddot{x} + \omega_0^2 x = l_0\omega_0^2$ avec $\omega_0 = \sqrt{\frac{k}{m}}$

2. La seule force qui travaille

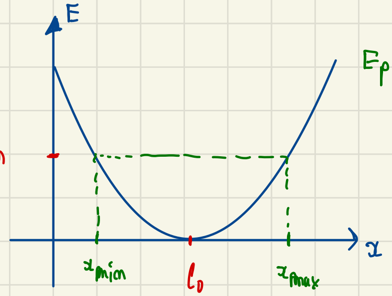
$$\vec{F}_{res} = -k(x - l_0)\vec{e_x}$$

$$= -\vec{\nabla}\left(\frac{1}{2}k(x - l_0)^2\right) = -\vec{\nabla}E_p(x)$$

Donc $E_p(x) = \frac{1}{2}k(x - l_0)^2$. La quantité $E_p(x)$ ne dépend pas du chemin suivi et vérifie la propriété remarquable

$E_p(x) + E_c = E_m =$ constante.

3.




$$E_m = \frac{1}{2}m\dot{x}^2 + E_p(x)$$

donc $E_p(x) \leq E_m$

$$\frac{1}{2}k(x_0 - l_0)^2 = \frac{1}{2}k\left(x_\pm - l_0\right)^2$$

$x_+ = x_0$

$x = 2l_0 - x_0$

4. $\ddot{x} + \omega_0^2 x = \omega_0^2 \ell_0$

solut° homogène : $x_H(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$

$x_p = \ell_0$

Donc $x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) + \ell_0$

$\begin{cases} \dot{x}(0) = 0 \\ x(0) = x_0 \end{cases} \Rightarrow \begin{cases} A + \ell_0 = x_0 \Rightarrow \boxed{A = x_0 - \ell_0} \\ B\omega_0 = 0 \text{ donc } B = 0 \end{cases}$

Finalement : $\boxed{x(t) = (x_0 - \ell_0)\cos(\omega_0 t) + \ell_0}$ $\quad x_0 > \ell_0$

et donc $\begin{cases} x_{max} = x_0 \\ x_{min} = 2\ell_0 - x_0 \end{cases}$

5. en $x = x_e$, $\Sigma\vec{F} = \vec{0}$ donc $\vec{\nabla}E(x) = 0$ soit en 1D :

$\boxed{\left.\frac{dE(x)}{dx}\right)_{x=x_e} = 0}$ caractérise la position d'équilibre

$m\ddot{x} + \frac{dE}{dx}(x) = 0$ DL2 de $E$ autour de $x_e$ :

$E(x) = E(x_e) + \left.\frac{dE}{dx}\right)_{x=x_e} \cdot (x - x_e) + \frac{1}{2}\left.\frac{d^2E}{dx^2}\right)_{x=x_e} (x-x_e)^2$

et donc $\ddot{x} + \frac{1}{2m}\left.\frac{d^2E}{dx^2}\right|_{x=x_e}(x-x_e)^2 = 0$ en posant $\varepsilon = x - x_e$

$\ddot{\varepsilon} + \frac{1}{m}\frac{d^2E}{dx^2}\varepsilon = 0$ donc $\left.\frac{d^2E}{dx^2}\right|_{x=x_e} > 0 \Rightarrow$ oscillation stable autour de l'éq donc équilibre stable

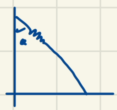
$\left.\frac{d^2E}{dx^2}\right|_{x=x_e} < 0 \Rightarrow$ éq instable

Si $\left.\frac{d^2E}{dx^2}\right|_{x=x_e} = 0$, il faut regarder les ordres supérieurs

6. $E_p(x) = \frac{1}{2}k(x-\ell_0)^2$ donc $\frac{dE_p}{dx}(x_e) = 0$

$\Rightarrow \boxed{x_e = \ell_0}$

7. $E_p(x) = \frac{1}{2}k(r-\ell_0)^2$



et $r = \sqrt{x^2+a^2}$

Finalement $E_p(x) = \frac{1}{2}k\left(\sqrt{x^2+a^2}-\ell_0\right)^2$

$= \frac{1}{2}k\ell_0^2\left(\sqrt{\alpha^2+\left(\frac{x}{\ell_0}\right)^2}-1\right)^2$

8. $\frac{dE_p}{dx} = 0 \iff k\ell_0^2\cdot\left(\sqrt{\alpha^2+\left(\frac{x}{\ell_0}\right)^2}-1\right)\cdot\frac{2\frac{x}{\ell_0^2}}{2\sqrt{\alpha^2+\left(\frac{x}{\ell_0}\right)^2}} = 0$

$x = 0$

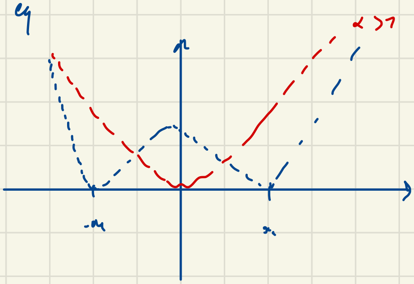
$\iff 1 - \frac{1}{\sqrt{\alpha^2+\left(\frac{x^2}{\ell_0^2}\right)^2}} = 0$

$1 = \alpha^2 + \left(\frac{x^2}{\ell_0^2}\right)^2$ donc si $\underline{\alpha > 1}$ : pas de sol°

si $\alpha < 1$ : $\frac{x_e^2}{\ell_0^2} = \sqrt{1-\alpha^2}$

donc $\boxed{x_e = \pm \ell_0 \sqrt{1-\alpha^2}}$ $\underline{\alpha < 1}$

$x_e = 0$ vj eq




$\alpha < 1$ : $x_e = 0$ instable, $x_e$ et $-x_e$ stable

$\alpha > 1$ : $x_e = 0$ stable

10. $\frac{dE_p}{dx} = kx\left(1 - \frac{1}{\sqrt{\alpha^2 + \left(\frac{x}{\ell_0}\right)^2}}\right)$

$\frac{d^2E_p}{dx^2} = kx \cdot \frac{1}{2} \cdot \frac{1}{\left(\alpha^2 + \left(\frac{x}{\ell_0}\right)^2\right)^{3/2}} \cdot \frac{2x}{\ell_0^2} + k\left(1 - \frac{1}{\sqrt{\alpha^2 + \left(\frac{x}{\ell_0}\right)^2}}\right)$

en $x = x_e = 0$ : $\left.\frac{d^2E_p}{dx^2}\right)_{x=0} = k \cdot \left(1 - \frac{1}{\alpha}\right) > 0$

et donc $\omega_0^2 = \frac{1}{m}\frac{d^2E}{dx^2}$

d'où $\omega_0 = \sqrt{\frac{k}{m}} \cdot \sqrt{\frac{\alpha^2 - 1}{\alpha^2}}$ $T_{sup} = \frac{2\pi}{\omega_0}$

$T_{sup} = 2\pi \sqrt{\frac{m\alpha}{k(\alpha - 1)}} = T_0 \sqrt{\frac{\alpha}{\alpha - 1}}$

11. $\left.\frac{d^2E}{dx^2}\right)_{x=x_e} = \frac{k x_e^2}{\ell_0^2} = k(1-\alpha^2)$

donc $\boxed{T_{img} = T_0 \cdot \frac{1}{\sqrt{1-\alpha^2}}}$

12. $\alpha \to 0$, $T_{img} \to T_0$. $\alpha \to 0 \Rightarrow \frac{d}{\ell_1} \to 0$ donc $\ell_1 \gg d$

On revient au cas 1D.

$\alpha \to 1$, $T_{img} \to \infty$
$T_{sup} \to \infty$

13. $S + S_{th}$

$\Delta S + \Delta S_{th} \geq 0$

$\Delta S + \frac{Q_{th}}{T_0} \geq 0$ $\qquad q_{th} = \Delta U_{th} + p_0 \Delta V_{th} = -[\Delta U + p_0 \Delta V]$

donc $\Delta S - \frac{(\Delta U + p_0 \Delta V)}{T_0} \geq 0$

Finalement $\Delta(U + p_0 V - T_0 S) \leq 0$

A l'éq : $dG^* = d(U + p_0 V - T_0 S) = 0$

14. à l'éq $P = P_0$ et $T = T_0$ donc $G = G^*$

$m_g = \frac{\frac{4}{3}\pi r^3}{v_s}$ et donc $G = g_e \cdot \left(m - \frac{4\pi r^3}{3 v_s}\right) + g_g \cdot \frac{4\pi r^3}{3 v_s} = \boxed{g_e m + (g_s - g_e) \cdot \frac{4\pi r^3}{3 v_s} = G}$

15. $G^*$ minimal qd $r \to r_{max} = \left(\frac{3v_s}{4\pi}\right)^{1/3}$

alors $G = G_{min} = m g_s$ donc $\frac{dG}{dr} < 0 \Rightarrow \boxed{g_s - g_e < 0}$

16. $G = m g_e + (g_s - g_e) \cdot \frac{4\pi r^3}{3 v_s} + 4\pi \gamma r^2$


$\frac{dG}{dr} = 0$

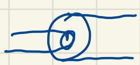
$\Rightarrow \frac{r^2}{v_s} \cdot (g_s - g_e) = -2\gamma r$

donc $\boxed{r = \frac{2\gamma v_s}{g_e - g_s}}$

$r < r_c \Rightarrow$ disp° du germe

18. AN : $r_c \sim 1 mm$

20.



$T r' = T r$ $\quad \theta r' = \ell'$ donc $\frac{\ell'}{\ell} = \frac{r'}{r} < 1$

donc $T' = T \cdot \frac{r}{r'} > T$ $\quad \theta r = \ell$

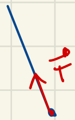
21. Dépend du référentiel

projection radiale de la force d'inertie d'entraînement.

22. $m a_r = mg \cos\theta - T + m r \dot{\theta}^2$

projection radiale du poids

23.




$dr \cdot T = \delta W > 0$ : on fournit de l'énergie

$\delta E_m = \delta W$ : on fournit de l'En

24. $$\frac{W_s}{E} = \frac{mgh - \frac{1}{2} m v_0^2}{mgh} = 1 - \frac{v_0^2}{2gh}$$

AN : $\boxed{\frac{W_s}{E} = 0{,}13}$

26. $m\ell_0^2 \ddot{\theta} = 0$ car les forces sont de moments nuls

donc $m\ell^2\dot{\theta} = cste \Rightarrow$ $m\ell_0 v_c = m(\ell_0 + \Delta\ell) v_B$

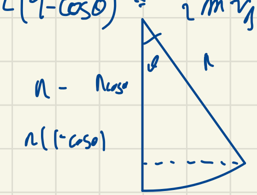
$$v_B = \left(1 - \frac{\Delta\ell}{\ell}\right) v_c$$

27.

$\Delta E_{pe} = -mg\Delta\ell\cos\theta$

$F_{ie} = m\frac{v_B^2}{R}$ donc $T = m\left(g + \frac{v_B^2}{R}\right)$

$mgR(1-\cos\theta) = \frac{1}{2} m v_1^2$




donc $v_B = \sqrt{2gR(1-\cos\theta)}$

Finalement $F_{ie} = m \cdot 2g\,(1-\cos\theta)$

donc $T = m \cdot 2g(1-\cos\theta) + mg$

$$\boxed{T = mg(3-2\cos\theta)}$$

TEM : $\Delta E_{BC} = mg\Delta\ell\,(3-2\cos\theta)$

Em prenant $\theta = \pi/2$ $\quad \dfrac{\Delta E_{BC} + \Delta E_{DE}}{mgr} = \boxed{\dfrac{3\Delta\ell}{\ell} = g}$

$$\Delta E = -mg\Delta\ell\cos\theta_{m+1} + mg\Delta\ell(3-2\cos\theta_{m+1})$$

$$= 3mg\Delta\ell\,(1-\cos\theta_{m+1})$$

De plus $\quad E_{m+1} = E_m + \Delta E$

$\Rightarrow \quad mg\ell_0(1-\cos\theta_{m+1}) = mg\ell_0(1-\cos\theta_m) + 3mg\Delta\ell\,(1-\cos\theta_{m+1})$

$$\cos\theta_{m+1} + 3\frac{\Delta\ell}{\ell}(1-\cos\theta_{m+1}) = \cos\theta_m$$

$\Rightarrow \quad \cos\theta_{m+1}\left(1-\dfrac{3\Delta\ell}{\ell}\right) = -\dfrac{3\Delta\ell}{\ell} + \cos\theta_m$

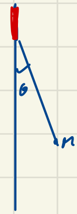
$$\cos\theta_{m+1} = -\frac{3\Delta\ell}{\ell}\left(1+3\frac{\Delta\ell}{\ell}\right) + \cos\theta_m\left(1+3\frac{\Delta\ell}{\ell}\right)$$

$$= -\frac{3\Delta\ell}{\ell} + \cos\theta_m\left(1+3\frac{\Delta\ell}{\ell}\right)$$

$$\cos\theta_{m+1} = \cos\theta_m + \frac{3\Delta\ell}{\ell}(\cos\theta_m - 1) = \cos\theta_m - \frac{3\Delta\ell}{\ell}(1-\cos\theta_m)$$

30.

$$mL^2\ddot{\theta} = -mgL\sin\theta + \mathcal{M}_{ie}$$



$$\vec{F}_{ie} = -m\vec{a}_e = +m\,a\,\omega_g^2\cos(\omega_g t)\,\vec{e}_y$$

$$\mathcal{M}_{ie} = -mLa\omega_g^2\cos(\omega_g t)\sin\theta$$

$$\ddot{\theta} = -\left[\frac{g}{L} + \frac{a\omega_g^2}{L}\cos(\omega_g t)\right]\sin\theta$$

$$\ddot{\theta} + \omega_0^2\left[1 + \frac{a\omega_g^2}{g}\cos(\omega_g t)\right]\sin\theta = 0$$

donc $\boxed{h = \frac{a\omega_g^2}{g}, \quad \omega_0 = \sqrt{\frac{g}{\ell}}}$

31. $\theta \ll 1$ donc $\quad \ddot{\theta} + \omega_0^2\left(1 + \frac{a\omega_g^2}{g}\cos(\omega_g t)\right)\theta = 0$

32. $\mu$ vérifie $\quad \mu^4 + 2\mu^2(1+u^2) + (u^2-1)^2 - \frac{h^2}{4} = 0$

$$\Delta = 4(1+u^2)^2 - 4(u^2-1)^2 + h^2$$

$$= 4(2u^2\cdot 2) - h^2$$

$$= 16u^2 + h^2 > 0 \qquad a_m = \frac{-2(1+u^2) + \sqrt{16u^2+h^2}}{2} > 0$$

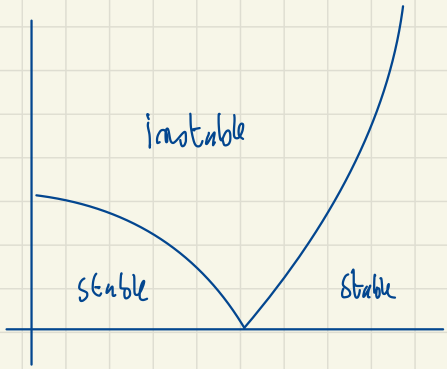
$$\sqrt{16u^2+h^2} > 2(1+u^2)$$

$$\Rightarrow h^2 > 4(1+u^2)^2 - 16u^2 = 4 + 8u^2 + 4u^4 - 16u^2 = 4 - 8u^2 + 4u^2$$

$= 4(\alpha^2-1)$

Donc $\mu^2 \geq 0$ cad $\mu \geq 0$ soit pour $\boxed{h > h_c = 4(\alpha^2-1)}$

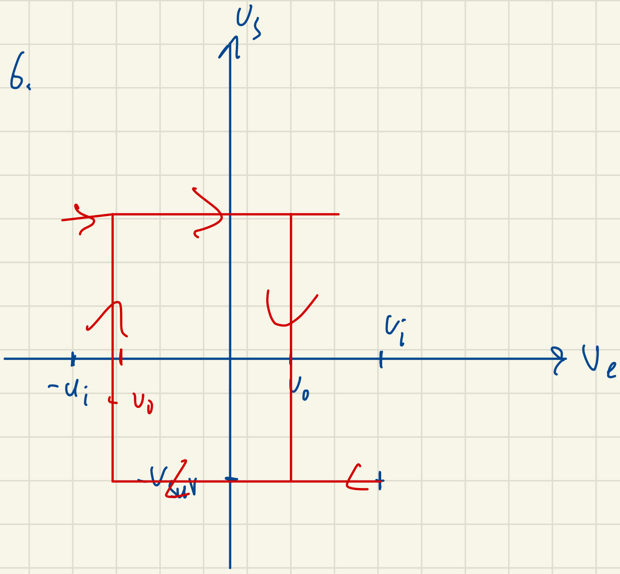
33.

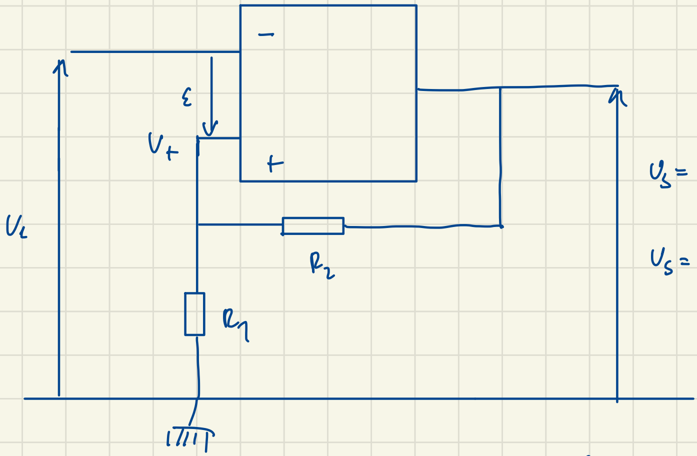



34. $v = \frac{\omega_s}{2\omega_0}$ , $\omega_s$ grand ⇒ peut d'instabilité

$\omega_s \to 2\omega_0$ , $v \to 1$ et $h \to 0$

36.



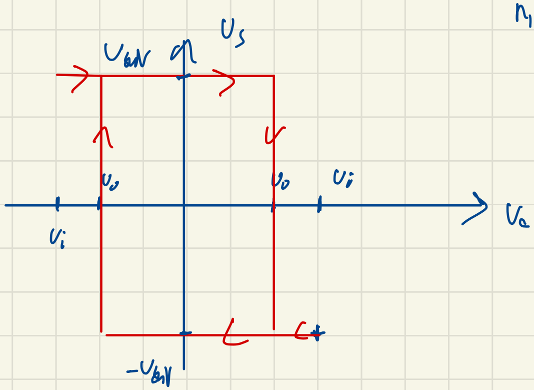



$$U_s = +U_{sat} \quad \varepsilon > 0$$

$$U_s = -U_{sat} \quad \varepsilon < 0$$

$$V_+ = U_S \cdot \frac{R_1}{R_1 + R_2}$$

$$U_0 = \frac{R_1}{R_1 + R_2} V_{sat}$$




$$\varepsilon = -V_{sat} \frac{R_1}{R_1 + R_2} = -U_0 - v_i < 0$$