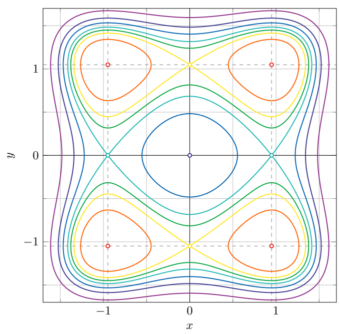
A computer-assisted proof for $\mathcal{H}(4) \geqslant 24$ in Hilbert's sixteenth problem

Florent Bréhard Nicolas Brisebarre Mioara Joldeș Warwick Tucker

October 18, 2019

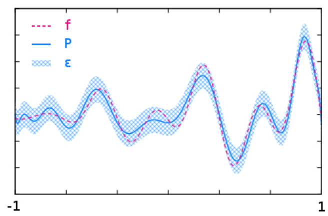
We provide a computer-assisted proof for the existence of a degree 4 polynomial vector field with at least 24 limit cycles, thus giving a lower on $\mathcal{H}(4)$ in the Hilbert 16th problem.



Level curves of potential function

Hilbert's 16th problem is part of Hilbert's famous list of 23 problems presented in 1900 at the International Congress of Mathematicians in Paris. The second part of this problem asks whether there exists, for each natural integer n, a finite upper bound $\mathcal{H}(n)$ on the number of limit cycles a polynomial planar vector field of degree n can have [1]. For now, a proof of this conjecture seems out of reach, even for $n = 2$. A simpler version of this problem restricts the investigation to perturbed Hamiltonian systems [2]. The upper bound $\mathcal{Z}(n)$ for such systems of degree n was proved to be finite for all n. A key ingredient is the Poincaré-Pontryagin theorem that relates the number of limit cycles with the number of zeros of so-called Abelian integrals along the level curves of the potential function associated to the Hamiltonian system.

The quartic system we investigate is not Hamiltonian but still integrable, so that the Poincaré-Pontryagin theorem still applies. Our work consists in approximating the Abelian integral associated to each monomial of the perturbation (as function of the energy level of the potential function), adjusting the coefficients to maximize the number of zeros of the resulting linear combination, and finally computing rigorous enclosures of the integral at various points to certify the number of sign changes. For that purpose, we use Rigorous Polynomial Approximations [4, 3] via our free C library available here[1], which allows us to perform rigorous and efficient evaluations of the Abelian integral.



A Rigorous Polynomial Approximation is a polynomial with an error bound, defining a tube around the exact function.

[1] The TchebyApprox experimental library : `https://gforge.inria.fr/projects/tchebyapprox/`

In the second part of the talk, we will discuss the possibility to use rigorous numerics for the *Wronskian* of the system of Abelian integrals. This, indeed, gives an upper bound for the number of limit cycles one can hope to obtain thanks to a well-chosen perturbation of the integrable system. Computer algebra and symbolic-numeric methods play an essential role here: obtaining linear differential equations satisfied by the Abelian integrals, using Laplace transform and formal asymptotics to deduce initial conditions, computing a rigorous polynomial approximation for the Wronskian.

On a last note, we are currently working towards a formalization of this result using certified Rigorous Polynomial Approximations in the Coq proof assistant [5].

References

[1] Y. Ilyashenko, *Centennial history of Hilbert's 16th problem*, Bull. Amer. Math. Soc. (N.S.), 39 (2002), pp. 301–354, `https://doi.org/10.1090/S0273-0979-02-00946-1`.

[2] V. I. Arnol'd, *Ten problems*, vol. 1 of Adv. Soviet Math., Amer. Math. Soc., Providence, RI, 1990.

[3] F. Bréhard, N. Brisebarre, and M. Joldes, *Validated and numerically efficient Chebyshev spectral methods for linear ordinary differential equations*, ACM Trans. Math. Softw., 44 (2018), pp. 44:1–44:42, `https://doi.org/10.1145/3208103`.

[4] N. Brisebarre and M. Joldes, *Chebyshev interpolation polynomial-based tools for rigorous computing*, in Proceedings of the 2010 International Symposium on Symbolic and Algebraic Computation, ACM, 2010, pp. 147–154.

[5] F. Bréhard, A. Mahboubi and D. Pous, *A Certificate-Based Approach to Formally Verified Approximations*, in Proceedings of the 10th International Conference on Interactive Theorem Proving (ITP 2019), 2019, pp. 8:1–8:19.