STURM BOUNDS FOR GENERAL CONGRUENCE SUBGROUPS

FRANÇOIS BRUNAULT

Abstract. We prove a Sturm bound for modular forms on general congruence subgroups (improving the statements in the literature when the width of the cusp is greater than 1). We also give some examples.

0.1. **The width of the cusp ∞.** Given a congruence subgroup Γ of $\mathrm{SL}_2(\mathbf{Z})$, the width of the cusp ∞ is the smallest integer $w \geq 1$ such that $\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \in \pm\Gamma$. A local parameter at the cusp $\infty \in X(\Gamma)$ is then given by $q^{1/w}$.

Now let $f \in M_k(\Gamma)$. There are several cases:

- k is even. In this case $f|_k(-I) = f$ and thus $f(z+w) = f(z)$, so that $f = \sum_{n\geq 0} a_n q^{n/w}$.
- k is odd and $-I \in \Gamma$. Then $f = 0$.
- k is odd and $-I \notin \Gamma$. There are two possibilities: either $\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \in \Gamma$ or $\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \in -\Gamma$.
  - If $\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \in \Gamma$, then the cusp ∞ is called regular and $f = \sum_{n\geq 0} a_n q^{n/w}$ as in the case k is even.
  - If $\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \in -\Gamma$, then the cusp ∞ is called irregular and $f = \sum_{n\geq 0} a_n q^{n/(2w)}$. We also have $a_n = 0$ for even n because $f(z+w) = -f(z)$.

Regularity can also be defined in the following way. Let Γ_∞ be the stabiliser of ∞ in Γ. There are three possibilities: $\Gamma_\infty = \left\langle \pm \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \right\rangle$, or $\Gamma_\infty = \left\langle \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \right\rangle$, or $\Gamma_\infty = \left\langle -\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \right\rangle$. In the first two cases ∞ is regular, and in the third case ∞ is irregular.

In any case, every modular form $f \in M_k(\Gamma)$ has a Fourier expansion of the form

$$f = \sum_{n\geq 0} a_n(f) q^{n/w}$$

with n possibly half-integral if k is odd and ∞ is irregular. As in [1, Section 3.2, p. 74], we define the order of vanishing of f at ∞ by

$$\mathrm{ord}_\infty(f) = \min\{n \in \mathbf{Q} : a_n(f) \neq 0\}$$

and $\mathrm{ord}_\infty(0) = +\infty$.

0.2. **Main result.**

Theorem 1. *Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$, et let m be the index of $\pm\Gamma$ in $\mathrm{SL}_2(\mathbf{Z})$. Let $f \in M_k(\Gamma)$ such that*

$$\boxed{\mathrm{ord}_\infty(f) > \frac{km}{12},}$$

in other words $a_n(f) = 0$ for $0 \leq n \leq \lfloor km/12 \rfloor$. Then $f = 0$.

Date: May 28, 2021.

In the case the width of ∞ is greater than 1, or in the case $-I \notin \Gamma$, this theorem improves the classical Sturm bound given in [3] and [2, 9.4.1.2] (see Sturm's Theorem).

Proof. We start with the case $\Gamma = \mathrm{SL}_2(\mathbf{Z})$. Consider the function $g = f^{12}/\Delta^k$, where Δ is the unique newform of weight 12 on $\mathrm{SL}_2(\mathbf{Z})$. The function g has weight 0, in other words it is a modular function on $\mathrm{SL}_2(\mathbf{Z})$. Since Δ doesn't vanish on $\mathcal{H}$, the function g has no pole on $\mathcal{H}$. Moreover

$$\mathrm{ord}_\infty(g) = 12\,\mathrm{ord}_\infty(f) - k\,\mathrm{ord}_\infty(\Delta) = 12\,\mathrm{ord}_\infty(f) - k > 0,$$

so that g has a zero at the cusp ∞. But a modular function which is holomorphic on $X(1) = \mathrm{SL}_2(\mathbf{Z})\backslash(\mathcal{H} \cup \{\infty\})$ must be constant, so here g is constant, and in fact $g = 0$. This implies $f = 0$.

In the general case, let m' be the index of Γ in $\mathrm{SL}_2(\mathbf{Z})$. Consider the function

$$F = \prod_{g \in \Gamma\backslash \mathrm{SL}_2(\mathbf{Z})} f|_k g.$$

It is a modular form of weight km' on $\mathrm{SL}_2(\mathbf{Z})$. Let $a = \mathrm{ord}_\infty(f)$, so that $f = O(q^{a/w})$. Let

$$C = \{\pm\gamma T^\ell : \gamma \in \Gamma,\ \ell \in \mathbf{Z}\}.$$

For every $g \in C$, the Fourier coefficients of $f|_k g$ differ from those of f by multiplying by roots of unity, so that $f|_k g = O(q^{a/w})$. Letting $w' = |\Gamma\backslash C|$, we deduce

$$F = O(q^{aw'/w}).$$

Our assumption on the order of vanishing of f implies

$$\frac{aw'}{w} > \frac{km}{12} \cdot \frac{w'}{w} = \frac{k[\mathrm{SL}_2(\mathbf{Z}) : \pm\Gamma]}{12} \cdot \frac{|\Gamma\backslash C|}{|\pm\Gamma\backslash C|} = \frac{k[\mathrm{SL}_2(\mathbf{Z}) : \pm\Gamma][\pm\Gamma : \Gamma]}{12} = \frac{km'}{12}.$$

The theorem is true in the case $\Gamma = \mathrm{SL}_2(\mathbf{Z})$, so we can apply it to F, and get $F = 0$. Finally, the ring of holomorphic functions on $\mathcal{H}$ is an integral domain, so that $f = 0$. $\square$

0.3. **The case $\Gamma = \Gamma(N)$.** For $N \geq 3$, the Sturm bound for $M_k(\Gamma(N))$ is

$$\frac{k}{24} \cdot \#\,\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z}).$$

This improves by a factor $2N$ the bound found in the literature.

Recall that if f is a modular form on $\Gamma(N)$, then $f(Nz)$ is a modular form on

$$\Gamma_H(N^2) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) : a \equiv d \equiv 1 \bmod N, c \equiv 0 \bmod N^2\right\}.$$

The Sturm bound for $\Gamma_H(N^2)$ is the same as the one for $\Gamma(N)$, because these two congruence subgroups have the same index in $\mathrm{SL}_2(\mathbf{Z})$. (One can show this by hand, but I don't know a conceptual proof.)

0.4. **Another example.** If f is a modular form on $\Gamma(N)$ then for every divisor d of N, the form $f(dz)$ is modular on

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) : a \equiv d \equiv 1 \bmod N, b \equiv 0 \bmod N/d, c \equiv 0 \bmod Nd\right\}.$$

Taken together, all these forms $f(dz)$ are modular on

$$\Gamma := \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) : a \equiv d \equiv 1 \bmod N, b \equiv 0 \bmod N, c \equiv 0 \bmod N^2\right\}.$$

It is clear that $\Gamma \subset \Gamma_H(N^2)$. The map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \overline{b}$ provides a surjective homomorphism $\Gamma_H(N^2) \to \mathbf{Z}/N\mathbf{Z}$ whose kernel is Γ. Therefore $[\mathrm{SL}_2(\mathbf{Z}) : \Gamma] = N \cdot \# \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$.

For example, for $N \geq 3$, the Sturm bound for $M_2(\Gamma)$ is equal to

$$\frac{N \cdot \# \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})}{12}.$$

References

[1] Fred Diamond and Jerry Shurman. *A first course in modular forms*. Graduate Texts in Mathematics, 228, Springer-Verlag, New York, 2005.

[2] William Stein. *Modular forms, a computational approach*. With an appendix by Paul E. Gunnells. Graduate Studies in Mathematics, 79. American Mathematical Society, Providence, RI, 2007.

[3] Jacob Sturm. On the congruence of modular forms. *Number theory* (New York, 1984–1985), 275–280, Lecture Notes in Math., 1240, Springer, Berlin, 1987.

ÉNS Lyon, Unité de mathématiques pures et appliquées, 46 allée d'Italie, 69007 Lyon, France

Email address: `francois.brunault@ens-lyon.fr`

URL: `http://perso.ens-lyon.fr/francois.brunault`