Statistical mechanics and kinetic theory of the 2D Euler and stochastic Navier-Stokes equations

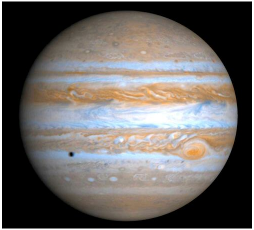
F. BOUCHET – ENS-Lyon and CNRS

"Kinetic Theory and Fluid Mechanics" – March 2012 – Lyon

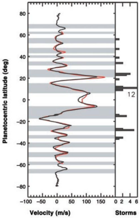
Outline

Kinetic Theory of Earth and Jupiter's Zonal Jets

These jets does not seem to be maximum entropy states



Jupiter atmosphere



Jupiter Zonal wind (Voyager and Cassini, from Porco et al 2003)

How far are we to reproduce such phenomena in numerical simulations ? How to theoretically predict such velocity profile ?

The 2D Stochastic-Navier-Stokes (SNS) Equations

- The simplest model for two dimensional turbulence
- Navier Stokes equation with random forces

$$\frac{\partial \omega}{\partial t} + \mathbf{u}.\nabla\omega = \nu\Delta\omega - \alpha\omega + \sqrt{\sigma} f_s$$

where $\omega = (\nabla \wedge \mathbf{u}).\mathbf{e}_z$ is the vorticity, f_s is a random force, α is the Rayleigh friction coefficient.

- An academic model with experimental realizations (Sommeria and Tabeling experiments, rotating tanks, magnetic flows, and so on). Analogies with geophysical flows (Quasi Geostrophic and Shallow Water layer models)

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Kinetic theory of the 2D Euler and Navier-Stokes equations

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Irreversibility in Fluid Mechanics and in Turbulence

Do we need viscosity to explain irreversible behavior of turbulent flows ?

- In many fluid mechanics or turbulence textbooks, it is stated, for example, that "Viscosity, whatever small, is necessary to explain the irreversible behavior of turbulent flows".
- Based on "D'Alembert's Paradox" (Euler and Lagrange theorems) (about potential flows) and Prandtl boundary layer analysis.
- The reversibility paradox of very small Reynolds number flows.

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- Irreversibility of turbulent flows should be explained independently of microscopic irreversible phenomena.
- Today the case of 2D turbulent flows and the irreversible behavior of the 2D Euler equations

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The 2D Euler Eq.: a Hamiltonian Reversible Dynamical System

- 2D Euler equations

$$\frac{\partial \Omega}{\partial t} + \mathbf{V}.\nabla\Omega = 0$$

- Vorticity $\Omega = (\nabla \wedge \mathbf{V}).\mathbf{e}_z$. $\Omega = \Delta\psi$
- The 2D Euler Eq. are symmetric under time reversal symmetry:

$$\Omega(\mathbf{r},t) \to \Omega(\mathbf{r},t) \text{ and } \mathbf{V}(\mathbf{r},t) \to -\mathbf{V}(\mathbf{r},-t)$$

- The 2D Euler Eq. has a irreversible macroscopic behavior: relaxation of the largest scales towards equilibrium

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Nonlinear Landau Damping
Clément Mouhot, and Cédric Villani, 2010

- Vlasov equation (dynamics of electrons in a plasma). μ-space density $f(x,p,t)$:

$$\frac{\partial f}{\partial t}+p\frac{\partial f}{\partial x}-\frac{dV}{dx}\frac{\partial f}{\partial p}=0.$$

- Hamiltonian and time reversible. A transport equation by a non-divergent flow, like the 2D Euler equations.
- Base state: a steady state $f=f_0(p)$. Understanding of the linearized equation by Landau (1946)
- Proof of the irreversible convergence, for large times, of f (weak topology) and ρ (strong topology) towards homogeneous densities (Mouhot, and Villani, 2010)

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The 2D Linearized Euler Eq.

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$$\frac{\partial \Omega}{\partial t} + \mathbf{V}.\nabla\Omega = 0$$

- Base state : a stable steady state $\mathbf{v}_0 = U(y)\mathbf{e}_x$, with vorticity ω_0: $\mathbf{v}_0.\nabla\omega_0 = 0$
- The 2D Euler equation, linearized close to $\mathbf{v}_0$, $\Omega = \omega_0(y) + \omega$ and $\mathbf{V} = \mathbf{v} + U(y)\mathbf{e}_x$

$$\frac{\partial \omega}{\partial t} - U''(y)\,v_y + U(y)\frac{\partial \omega}{\partial x} = 0$$

The Case of a Constant Shear in a Channel

Easily solvable (trivial) – For pedagogical purpose

$$\frac{\partial \omega}{\partial t} - U''(y)\,v_y + U(y)\frac{\partial \omega}{\partial x} = 0$$

- $U(y) = sy$. $-l \le y \le l$. Then $\omega_0' = -U''(y) = 0$. A drastic simplification.

$$\frac{\partial \omega}{\partial t} + sy\frac{\partial \omega}{\partial x} = 0$$

- Fourier series for the spatial variable $\omega(x,y,t) = \omega(y,t)\exp(ikx)$:

$$\frac{\partial \omega}{\partial t} + iksy\omega = 0 \text{ then } \omega(y,t) = \omega(y,0)\exp(-iksyt)$$

The solution for the vorticity is trivial in that case

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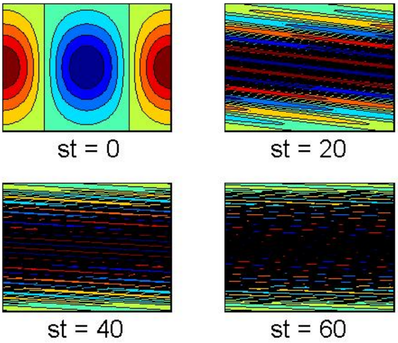
Vorticity Evolution in the Case of Constant Shear

Deterministic evolution - The Orr mechanism - Base flow $U(y) = sy$



Evolution of the perturbation vorticity $\omega(t)$, advected by a constant shear s

Velocity Asymptotics in the Case of Constant Shear

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$$\omega(y,t) = \omega(y,0)\exp(-iksyt)$$

- We look at the solution for the velocity $\mathbf{v}(y,t)$:

$$\mathbf{v}(y,t) = \int dy\, \mathbf{G}_k(y,y')\omega(y',0)\exp(-iksy't)$$

- We have an oscillating integral. For large times:

$$v_x(y,t) \underset{t\to\infty}{\sim} \frac{v_{x,\infty}(y)}{t}\exp(-iksyt) \text{ and } v_y(y,t) \underset{t\to\infty}{\sim} \frac{v_{y,\infty}(y)}{t^2}\exp(-iksyt)$$

The velocity decreases algebraically

Orr mechanism-Case (1969)

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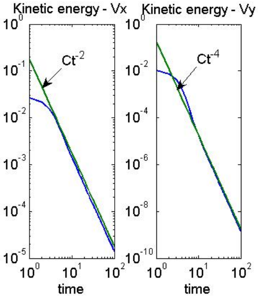

Evolution of the perturbation kinetic energy for the transverse and longitudinal components of the velocity $\mathbf{v}(t)$

- The shear acts as an effective dissipation (Phase mixing)

The Linearized Euler Eq. close to Shear Flows

- Base flow : $\mathbf{v}_0(\mathbf{r}) = U(y)\mathbf{e}_x$. The linearized Euler equation:

$$\frac{\partial \omega}{\partial t} + ikU(y)\omega - ik\psi U''(y) = 0 \quad (1)$$

with $\omega(x,y,t) = \omega(y,t)\exp(ikx)$ and $\omega = \frac{d^2\psi}{dy^2} - k^2\psi$

- Laplace transform: $\phi(y,c,\varepsilon) = \int_0^\infty dt\, \Psi(y,t)\exp(ik(c+i\varepsilon)t)$

$$\left(\frac{d^2}{dy^2} - k^2\right)\phi - \frac{U''(y)}{U(y)-c-i\varepsilon}\phi = \frac{\omega(y,0)}{ik(U(y)-c-i\varepsilon)} \quad (2)$$

- This is the celebrated Rayleigh equation. A one century old classical problem in fluid mechanics, applied mathematics and mathematics. Rayleigh (1842-1919)

- Large time asymptotic is related to the limit $\varepsilon \to 0$
- Singularity of the equation : critical layer $U(y_c) = c$

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Asymptotic Behavior of the Linearized Euler Eq.

Historical results

1) Base flows without stationary points: for any y, $U'(y) \neq 0$ (monotonic profile)

- Rayleigh (1880) Mode equation
- Case (Phys. Fluid. 1960) Algebraic laws for the velocity field in the case of constant shear (wrong)
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- Brown,Stewartson (JFM 1980), Lundgren (Phys. Fluid. 1982) Ansatz for the asymptotic expansion, for large time
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2) Base flows with one or several stationary streamlines: $U'(y_0) = 0$

No results!

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Base Flows with Stationary Streamlines

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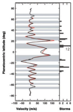

Most of geophysical jets have stationary streamlines (they do not verify the Rayleigh or Rayleigh-Kuo stability criteria)

- Stationary streamlines: $U'(y_0) = 0$. Velocity extrema - No shear - No Orr mechanism
- The Case velocity asymptotics (Brown and Stewartson asymptotic expansion) is not self-consistent
- $\mathbf{v}(y,t) = \int dy\, \mathbf{G}_k(y,y')\omega(y',0)\exp(-ikU(y)t)$. Stationary phase approximation: $v \underset{t\to\infty}{\propto} C/\sqrt{t}$
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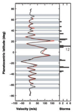

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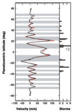

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Asymptotic Behavior of the Linearized Euler Eq.

Base flow with stationary streamlines: $U'(y_0) = 0$

- Mathematical methods : Laplace transform and detailed analysis of the singularities due to the critical layers and stationary streamlines
- By contrast with what was previously believed, we can deal with the difficulty related to the stationary streamlines

Theory : a) Asymptotic oscillatory vorticity field

$$\omega(y,t) \underset{t\to\infty}{\sim} \omega_\infty(y)\exp(ikU(y)t) + \mathcal{O}\left(\frac{1}{t^\alpha}\right)$$

b) DEPLETION OF THE VORTICITY PERTURBATION:
For any stationary streamline of the flow (y_0 such that $U'(y_0) = 0$)

$$\omega_\infty(y_0) = 0$$

- + Prediction of the asymptotic vorticity $\omega_\infty(y)$

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1. Inviscid relaxation of the 2D Euler equations
   - Irreversibility in turbulence
   - Large Time Asymptotics of the linearized 2D Euler Eq.
   - The Kolmogorov flow
2. Kinetic theory of systems with long range interactions
   - Beyond the Vlasov equation
   - Plasma and self-gravitating systems with stochastic forces
   - The Lenard-Balescu equation and aging correlations
3. Kinetic theory of the stochastic Navier-Stokes equations
   - The 2D Stochastic Navier–Stokes Equations
   - The kinetic approach
   - The 2D linearized Euler Eq. with random forces

An Example: the Kolmogorov Flow

- $U(y) = \cos(y)$ in the doubly periodic domain $(0, 2\pi/\delta)x(0, 2\pi)$; δ is the aspect ratio
- Two stationary streamlines $U'(y_0) = 0$, for $y_0 = 0$ or $y_0 = \pi$
- Usual criteria for stability (Rayleigh, Arnold) do not apply
- The Kolmogorov flow is stable for $\delta > 1$ (Lyapunov stability), spectrally and linearly stable (easily proved)
- This flow has no neutral modes

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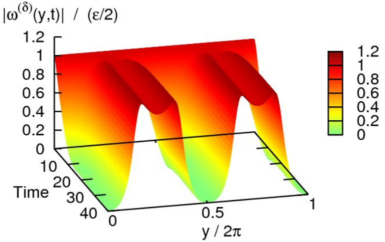
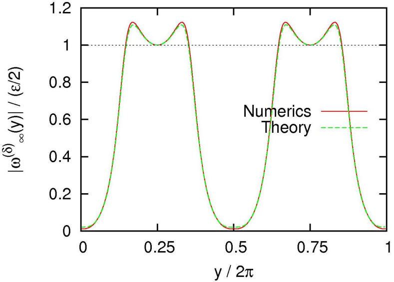
Base flow with stationary streamlines: depletion of the vorticity perturbation at the stationary streamlines





Evolution of the perturbation vorticity $\omega(t)$, advected by a shear flow $U(y) = \cos(y)$ with stationary points in $y = 0$ and $y = \pi$

Asymptotic Behavior of the Linearized Euler Eq.

Base flow with stationary streamlines : the velocity field

Theorem: algebraically decaying asymptotic velocity field

$$v_x(y,t) \underset{t\to\infty}{\sim} \frac{v_{x,\infty}(y)}{t} \exp\left(-ikU(y)t\right) \tag{3}$$

$$v_y(y,t) \underset{t\to\infty}{\sim} \frac{v_{y,\infty}(y)}{t^2} \exp\left(-ikU(y)t\right) \tag{4}$$

- What about stationary streamlines? They should give contributions of order $1/t^{1/2}$!
- No contribution from the stationary streamlines thanks to the depletion of the vorticity perturbation at stationary streamlines

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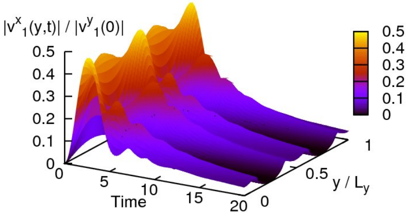

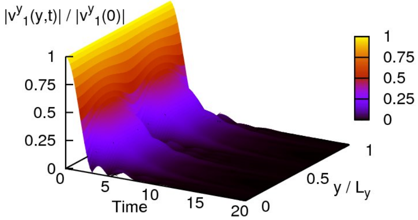

Evolution of the perturbation velocity, components $v_x(t)$ and $v_y(t)$, advected by a constant shear flow $U(y)$ with stationary streamlines

Asymptotic Behavior of the Linearized Euler Eq.

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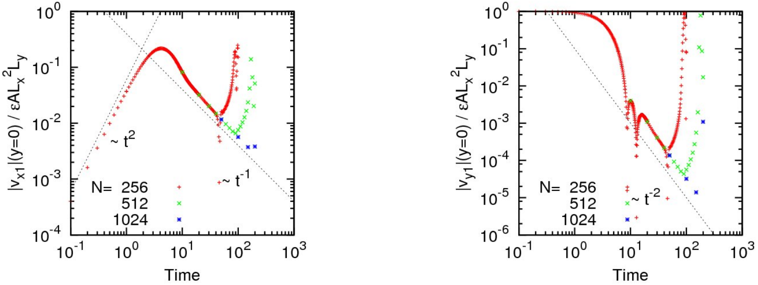

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The velocity perturbation converges to zero (asymptotic stability) even without dissipation

Asymptotic Behavior of the Linearized Euler Eq.: Conclusions

- Asymptotically oscillating vorticity fields
- Algebraic decay of the velocity field with $1/t$ or $1/t^2$ laws, whatever the cases (except at stationary streamlines).
- All cases of base flow with any type of shear have been treated
- Depletion of the vorticity perturbation at the stationary streamlines
- Axisymmetric vortices should behave the same way
- The perturbation converges (weak topology) towards a Young measure

Outline

Systems with Long Range Interactions

- Long range interactions = non-integrable potential

$$H=\frac{1}{2}\sum_{k=1}^{N}p_k^2+\frac{1}{2N}\sum_{k,l=1}^{N}V(q_k-q_l)$$

- In the algebraic case $V(r)\propto_{r\to\infty}1/r^{\alpha}$; non integrable when $\alpha<d$
- 2D Euler $G(r)\propto_{r\to\infty}\log(r)$:

$$E=\frac{1}{2}\int_{\mathscr{D}}d^2xd^2y\,G(\mathbf{x}-\mathbf{y})\omega(\mathbf{x})\omega(\mathbf{y})\simeq\frac{1}{2N^2}\sum_{i,j,k,l=1}^{N}G_{ij,kl}\omega_{ij}\omega_{kl}$$

- Examples: self-gravitating stars, 2D and geophysical flows, plasma, cold atoms, etc ?

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$$E = \frac{1}{2}\int_{\mathscr{D}} d^2x d^2y\, G(\mathbf{x}-\mathbf{y})\omega(\mathbf{x})\,\omega(\mathbf{y}) \simeq \frac{1}{2N^2}\sum_{i,j,k,l=1}^{N} G_{ij,kl}\omega_{ij}\omega_{kl}$$

- Examples: self-gravitating stars, 2D and geophysical flows, plasma, cold atoms, etc ?

Kinetic Theory for Systems with Long Range Interactions

A classical and fascinating framework

$$H=\frac{1}{2}\sum_{k=1}^{N}p_k^2+\frac{1}{2N}\sum_{k,l=1}^{N}V(q_k-q_l)$$

- We will assume V smooth and $x \in \mathscr{D}$ with $\mathscr{D}$ bounded. For instance $\mathscr{D}$ a d-dimensional torus.
- A common framework for many systems: plasma physics, self gravitating systems, point vortex model
- Classical approach: BBGKY hierarchy, small parameter, chaotic hypothesis, derivation of kinetic equations
- Vlasov equation (40's), Landau equation, Lenard–Balescu equation (60's,70's)

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The Mean Field Limit and the Vlasov Equation

$$H = \frac{1}{2}\sum_{k=1}^{N} p_k^2 + \frac{1}{2N}\sum_{k,l=1}^{N} V(q_k - q_l)$$

- Dynamics of the empirical measure $f_e(q,p,t) = \frac{1}{N}\sum_k \delta(q - q_k(t), p - p_k(t))$
- The mean field limit: $N \to \infty$ for times $t \ll C\log(N)$. f_e remain close to the solution of the Vlasov equation :

$$\frac{\partial f}{\partial t} + p\frac{\partial f}{\partial q} - \frac{\partial \Phi[f]}{\partial q}\frac{\partial f}{\partial p} = 0$$

- With ϕ the mean field potential

$$\Phi[f](q) \equiv \int \mathrm{d}q_1 \mathrm{d}p_1\, V(q - q_1) f(q_1, p_1, t)$$

Beyond the Mean Field Limit

$$H = \frac{1}{2}\sum_{k=1}^{N} p_k^2 + \frac{1}{2N}\sum_{k,l=1}^{N} V(q_k - q_l)$$

- Dynamics of the empirical measure

$$\frac{\partial f_e}{\partial t} + p\frac{\partial f_e}{\partial q} - \frac{\partial \Phi[f_e]}{\partial q}\frac{\partial f_e}{\partial p} = 0$$

- We see the Vlasov equation as a consequence of the law of large numbers $\lim_{N\to\infty} f_e = f$
- Beyond the law of large numbers: Gaussian fluctuations (of order $1/\sqrt{N}$). $f_e = f + \delta f/\sqrt{N}$
- What is the dynamics of those fluctuations ?

Rough Ideas About Corrections to the Vlasov EquationsFluctuation Dynamics

$$\frac{\partial f_e}{\partial t}+p\frac{\partial f_e}{\partial q}-\frac{\partial \Phi[f_e]}{\partial q}\frac{\partial f_e}{\partial p}=0$$

- Formal asymptotic expansion $f_e=f+\delta f/\sqrt{N}$

$$\frac{\partial f}{\partial t}+p\frac{\partial f}{\partial q}-\frac{\partial \Phi[f]}{\partial q}\frac{\partial f}{\partial p}=\frac{1}{N}\left\langle\frac{\partial \Phi[\delta f]}{\partial q}\frac{\partial \delta f}{\partial p}\right\rangle$$

$$\frac{\partial \delta f}{\partial t}+L_f\left[\delta f\right]=0$$

- At leading order the fluctuations are transported by the linearized Vlasov equation

Rough Ideas About Fluctuation Dynamics

$$\frac{\partial \delta f}{\partial t} + L_f\left[\delta f\right] = 0$$

- Second order correlation function $g(q_1,p_1,q_2,p_2,t) = \langle \delta f(q_1,p_1,t)\delta f(q_2,p_2,t)\rangle$

$$\frac{\partial g}{\partial t} + L_f^1\left[g\right] + L_f^2\left[g\right] = S$$

- Fluctuations dynamics is governed by a Lyapunov equation (the equation for the two-points correlation of an Ornstein-Uhlenbeck process). This is a general rule. See for instance fluctuating hydrodynamics.

The Lenard-Balescu Equation

- After solving the Lyapunov equations for the second order correlation function we obtain the Lenard-Balescu equation:

$$\frac{\partial f}{\partial t}+\text{Vlasov}\,[f]=\frac{1}{N}\left\langle \frac{\partial \Phi[\delta f]}{\partial q}\frac{\partial \delta f}{\partial p}\right\rangle=\frac{1}{N}\mathscr{L}\mathscr{B}\,[f]$$

- The Vlasov equation is an approximation of the Lenard-Balescu equation. In plasma, the Vlasov operator is a good approximation of the Lenard-Balescu one for scales much larger than the Debye length.

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Mean Field Hamiltonian and Stochastic Forces

A simpler framework from a mathematical point of view

$$H = \frac{1}{2}\sum_{k=1}^{N} p_k^2 + \frac{1}{2N}\sum_{k,l=1}^{N} V(q_k - q_l)$$

- Hamiltonian dynamics plus stochastic forces

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \text{and} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} - \alpha p_i + \sqrt{\alpha}\, F(q_i, t)$$

- α: friction constant. $F(q,t)$ is a homogeneous Gaussian process with zero mean and variance

$$\langle F(q,t) F(q',t') \rangle = C(|q-q'|)\delta(t-t')$$

- Force spectrum: $c_k \equiv \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}q \; C(q) e^{-ikq} > 0$
- We expect fluctuations of order $\sqrt{\alpha}$ due to the stochastic force and $1/\sqrt{N}$ due to the potential

Energy Balance

$$H=\frac{1}{2}\sum_{k=1}^{N}p_k^2+\frac{1}{2N}\sum_{k,l=1}^{N}V(q_k-q_l)$$

$$\dot{q}_i=\frac{\partial H}{\partial p_i}, \qquad \text{and} \qquad \dot{p}_i=-\frac{\partial H}{\partial q_i}-\alpha p_i+\sqrt{\alpha}\,F(q_i,t)$$

- A close equation for the average specific energy $e=H/N$

$$\left\langle \frac{de}{dt}\right\rangle=-2\alpha\left\langle \kappa\right\rangle+\frac{\alpha}{2}C(0)$$

- $\kappa=\sum_{i=1}^{N}p_i^2/(2N)$ is the kinetic energy per particle
- Average kinetic energy, for the stationary state

$$\left\langle \kappa\right\rangle_{ss}=C(0)/4$$

N-particle Fokker-Planck Equation

- Evolution of the N-particle distribution function $f_N(q_1,...,q_N,p_1,...,p_N,t)$ (after averaging over the noise realization)

$$\frac{\partial f_N}{\partial t}+\sum_{i=1}^{N}p_i\frac{\partial f_N}{\partial q_i}+\frac{1}{2N}\sum_{i,j=1}^{N}V'(q_i-q_j)\left[\frac{\partial}{\partial p_i}-\frac{\partial}{\partial p_j}\right]f_N=...$$

$$...=\alpha\sum_{i=1}^{N}\frac{\partial(p_if_N)}{\partial p_i}+\frac{\alpha}{2}\sum_{i,j=1}^{N}C(q_i-q_j)\frac{\partial^2 f_N}{\partial p_i\partial p_j}$$

- With $\alpha=0$, we get the Liouville equation for the Hamiltonian dynamics

The BBGKY Hierarchy

- The n particle distribution function

$$f_n(q_1,p_1,...,q_n,p_n,t) = \int \prod_{i=n+1}^{N} \mathrm{d}q_i \mathrm{d}p_i \, f_N(q_1,p_1,...,q_N,p_N,t)$$

- We denote $f(q,p,t) = f_1(q,p,t)$
- We anticipate that at leading order we expect loss of correlation (Stosszahl ansatz)

$$f_n(q_1,p_1,...,q_n,p_n,t) = \prod_{i=1}^{N} f(q_i,p_i,t) + \alpha g_n(q_1,p_1,...,q_n,p_n,t)$$

- Each g_n is governed by an equation involving $(f, g_2, ..., g_{n+1})$. This is called the BBGKY hierarchy
- g_2 is denoted g

The Two First Equation of the BBGKY Hierarchy

- For pedagogical reasons, we assume an homogenous state: $f(q,p) = f(p)$ and $g(q_1,q_2,p_1,p_2) = g(q_1-q_2,p_1,p_2)$

$$\frac{\partial f}{\partial t} + \text{Vlasov}[f] = \alpha\left[\frac{\partial}{\partial p}(pf) + \frac{C(0)}{2}\frac{\partial^2 f}{\partial p^2} + \frac{\partial}{\partial p}\int \mathrm{d}q_2\mathrm{d}p_2\, V'(q_2)g(q_2,p,p_2,t)\right]$$

$$\frac{\partial g}{\partial t} + \left[p_1\frac{\partial g}{\partial q_1} - \frac{\partial f}{\partial p}(p_1)\int \mathrm{d}q_3\mathrm{d}p_3\, V'(q_1-q_3)g(q_3-q_2,p_3,p_2,t)\right] + \{1\leftrightarrow 2\} = \ldots$$

$$\ldots = C(q_1-q_2)\frac{\partial f}{\partial p}(p_1)\frac{\partial f}{\partial p}(p_2) + \frac{\alpha}{N}\mathcal{N}_2(f,g) + \alpha\mathcal{N}_3(f,g,g_3)$$

- The hierarchy appears as an ordered expansion both in powers of α and $1/N$
- We discuss for instance the mean-field stochastic regime $1/N \ll \alpha \ll 1$ (the limit $\lim_{\alpha\to 0}\lim_{N\to\infty}$)

Truncation at First Non Trivial Order

$$\frac{\partial f}{\partial t}+\text{Vlasov}\left[f\right]=\alpha\left[\frac{\partial}{\partial p}\left(pf\right)+\frac{C(0)}{2}\frac{\partial^2 f}{\partial p^2}+\frac{\partial}{\partial p}\int \mathrm{d}q_2\mathrm{d}p_2\,V'(q_2)g(q_2,p,p_2,t)\right]$$

$$\frac{\partial g}{\partial t}+L_f^1\left[g\right]+L_f^2\left[g\right]=C(q_1-q_2)\frac{\partial f}{\partial p}(p_1)\frac{\partial f}{\partial p}(p_2)$$

- We assume that $f(p)$ is a stable steady solution of the Vlasov equation, in order for f to evolve over a slow time scale (for instance $1/\alpha$)
- We assume, that for fixed f, the Lyapunov equation converge for large times to $g_\infty[f]$
- Bogolyubov hypothesis: the kinetic equation is

$$\frac{\partial f}{\partial t}+\text{Vlasov}\left[f\right]=\alpha\left[\frac{\partial}{\partial p}\left(pf\right)+\frac{C(0)}{2}\frac{\partial^2 f}{\partial p^2}+\frac{\partial}{\partial p}\int \mathrm{d}q_2\mathrm{d}p_2\,V'(q_2)g_\infty[f](q_2,p,p_2)\right]$$

Kinetic Equation

- All computations can be performed explicitly. Then

$$\frac{\partial f}{\partial t}+\text{Vlasov}\,[f]=\alpha\left[\frac{\partial(pf)}{\partial p}+\frac{\partial}{\partial p}\left(D[f]\frac{\partial f}{\partial p}\right)\right]$$

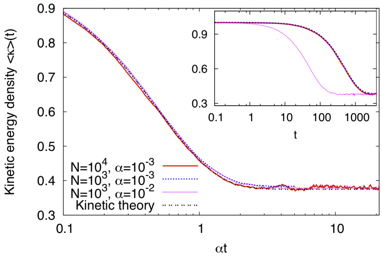
$$D[f](p)=\frac{1}{2}C(0)+2\pi\sum_{k=1}^{\infty}V_k c_k\int^{*}\mathrm{d}p_1\left[\frac{1}{|\varepsilon(k,kp)|^2}+\frac{1}{|\varepsilon(k,kp_1)|^2}\right]\frac{1}{p_1-p}\left.\frac{\partial f}{\partial p}\right|_{p_1}$$

$\int^*$ indicates the Cauchy principal value of the integral, and the dielectric function ε is

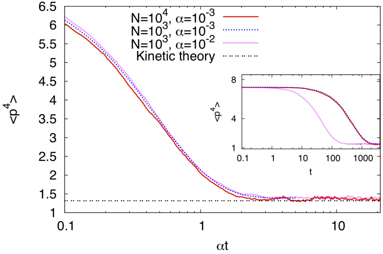
$$\varepsilon(k,\omega)=\lim_{\eta\to0^+}\left[1-2\pi i v_k k\int\mathrm{d}p\,\frac{1}{-i(\omega+i\eta)+ikp}\frac{\partial f}{\partial p}\right]$$

- The equation correctly predicts the energy balance

Kinetic Evolution from Numerical Simulation



Kinetic energy $\langle \kappa \rangle$ as a function of αt

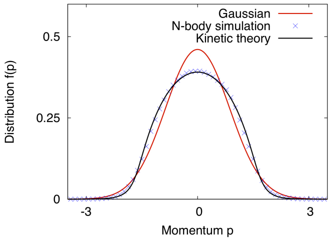


$\langle p^4 \rangle$ as a function of αt

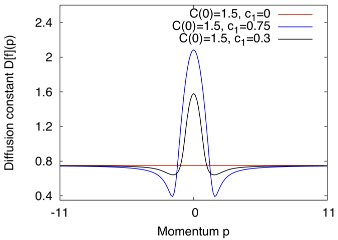
$(C(0) = 1.5$, and $c_1 = 0.75)$

- This is an evidence of kinetic evolution over a time scale $1/\alpha$

Kinetic Equation Predicts the Stationary Distribution



Stationary distribution ($\alpha = 0.01$, $C(0) = 1.5$, and $c_1 = 0.75$)



Stationary diffusion coefficient

- Very good agreement between kinetic theory and N-particle numerical simulations

Mathematical Study of the Kinetic Equation ?

- All computations can be performed explicitly. Then

$$\frac{\partial f}{\partial t} + \text{Vlasov}\,[f] = \alpha \left[\frac{\partial (pf)}{\partial p} + \frac{\partial}{\partial p} \left(D[f] \frac{\partial f}{\partial p} \right) \right]$$

$$D[f](p) = \frac{1}{2} C(0) + 2\pi \sum_{k=1}^{\infty} V_k c_k \int^{*} \mathrm{d}p_1 \left[\frac{1}{|\varepsilon(k,kp)|^2} + \frac{1}{|\varepsilon(k,kp_1)|^2} \right] \frac{1}{p_1 - p} \frac{\partial f}{\partial p}(p_1)$$

$\int^{*}$ indicates the Cauchy principal value of the integral, and the dielectric function ε is

$$\varepsilon(k,\omega) = \lim_{\eta \to 0^+} \left[1 - 2\pi i v_k k \int \mathrm{d}p \, \frac{1}{-i(\omega + i\eta) + ikp} \frac{\partial f}{\partial p} \right]$$

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A classical and fascinating framework

$$H = \frac{1}{2}\sum_{k=1}^{N} p_k^2 + \frac{1}{N}\sum_{k,l=1}^{N} V(x_k - x_l)$$

- For instance a plasma in the weak coupling limit.
- A common framework for many systems: plasma physics, self gravitating systems, point vortex model
- Classical approach: BBGKY hierarchy, small parameter, chaotic hypothesis, derivation of kinetic equations
- Vlasov equation (40's), Landau equation, Lenard Balescu equation (60's,70's)

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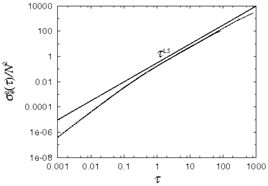
Kinetic Theory: Diluted Gases Versus Long Range Systems

Two opposite limits but strong analogies for the kinetic theories

	Short-ranged (gases)	Long-range
Small parameter	$a/l = 1/\left(\pi a^2 n\right)$	$1/N$
Initial evolution Late relaxation	Collisionless Boltzmann Boltzmann equation	Vlasov equation Lenard-Balescu
Vanishing correlations Boltzmann entropy Stosszahl Ansatz	Yes Yes Yes	Yes Yes Yes
Steady states of the initial evolution	Local thermal equilibrium	Quasistationary states
Relaxation time scale	$\propto l/\bar{v}$ or larger	$\propto N$ or larger
Long temporal correlations and algebraic decays	Yes Yes	? ?
Anomalous diffusion		?

Algebraic Decays and Anomalous Diffusion

Long range system : the relaxation of a test particle in a bath



- Using kinetic theory, we derive a Fokker-Planck equation for a particle in a bath. Rapidly decaying diffusion coefficient.
- The Fokker Planck equation has a continuous spectrum
- Algebraic decay for large times of correlation functions
- We compute analytically the exponent using matched asymptotic expansions

F. Bouchet, T. Dauxois (2004) Phys. Rev. E,
Y. Yamaguchi, F. Bouchet, and T. Dauxois (2007) J. Stat. Mech.

The Lenard–Balescu Equation

- After solving the Lyapunov equations for the second order correlation function we obtain the Lenard-Balescu equation:

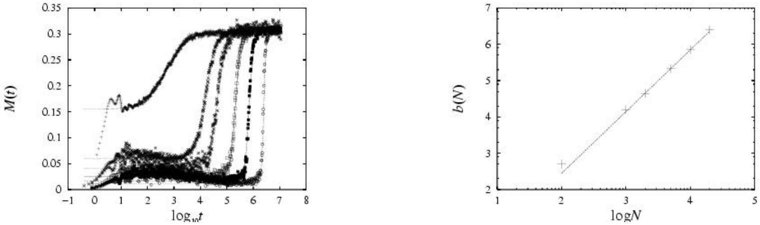
$$\frac{\partial f}{\partial t} + \text{Vlasov}\,[f] = \frac{1}{N}\left\langle \frac{\partial \Phi[\delta f]}{\partial q}\frac{\partial \delta f}{\partial p}\right\rangle = \frac{1}{N}\mathscr{L}\mathscr{B}\,[f]$$

- Valid when f is close to a stable steady state of the Vlasov equation
- What are the limits of the Vlasov and Lenard–Balescu equations ?

What are the Limits of Kinetic Theory ?

How long are the Vlasov equation valid, or the Lenard-Balescu equation valid ?

- The $C\log(N)$ upper bound for the validity of the Vlasov equation is optimal K. Jain, F. Bouchet, and D. Mukamel (2007), J.Stat. Mech.



$\tau \propto N^{1.7}$. This is not consistent with simple use of kinetic theory

Y.Y. Yamaguchi, J. Barré, F. Bouchet, T. Dauxois and S. Ruffo, (2004) Physica A, F. Bouchet, T. Dauxois (2004) Phys. Rev. E

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The 2D Stochastic Navier–Stokes (SNS) Equations

- Navier Stokes equation with a random force

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla \omega = \nu \Delta \omega - \alpha \omega + \sqrt{\sigma} f_s$$

  where $\omega = (\nabla \wedge \mathbf{u}).\mathbf{e}_z$ is the vorticity, α is the Rayleigh friction coefficient
- f_s is a random Gaussian field

$$\left\langle f_S(\mathbf{x},t) f_S(\mathbf{x}',t') \right\rangle = C(\mathbf{x},\mathbf{x}')\delta(t-t')$$

- An academic model with experimental realizations (Sommeria, Tabeling, Ecke experiments, rotating tanks, magnetic flows, soap films, and so on). Analogies with geophysical flows

The 2D Stochastic Navier-Stokes Equations

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla \omega = \nu \Delta \omega + \sqrt{\nu} f_s$$

- Some recent mathematical results: Kuksin, Sinai, Shirikyan, Bricmont, Kupiainen, Hairer, etc;
    - Existence of a stationary measure μ_ν. Existence of $lim_{\nu \to 0}\ \mu_\nu$,
    - In this limit, almost all trajectories are solutions of the Euler equation.
- We would like to obtain more physical results:
    - What is the link of this limit $\nu \to 0$ with the RSM theory?
    - Will we stay close to some steady solutions of the Euler equation?
    - Can we describe these statistically stationary states and their properties?

Balance Relations (Energy Conservation)

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla\omega = \nu\Delta\omega - \alpha\omega + \sqrt{\sigma} f_s$$

Energy conservation

$$\frac{d\langle E\rangle}{dt} = -2\alpha\langle E\rangle - \nu\langle\Omega_2\rangle + \sigma$$

In a statistically stationary regime:

$$\langle E\rangle_S = \frac{\sigma}{2\alpha} - \frac{\nu}{2\alpha}\langle\Omega_2\rangle_S$$

Time unit change, in order to fix an energy of order one (the turnover time will be of order one):
$t' = \sqrt{\sigma/2\alpha}\, t$; $\omega' = \sqrt{2\alpha/\sigma}\,\omega$; $\alpha' = (2\alpha)^{3/2}/\left(2\sigma^{1/2}\right)$ and $\nu' = \nu(2\alpha/\sigma)^{1/2}$

Balance Relations (Energy Conservation)

$$\frac{\partial \omega}{\partial t}+\mathbf{v}.\nabla\omega=\nu\Delta\omega-\alpha\omega+\sqrt{\sigma}f_s$$

Energy conservation

$$\frac{d\left\langle E\right\rangle}{dt}=-2\alpha\left\langle E\right\rangle-\nu\left\langle \Omega_2\right\rangle+\sigma$$

In a statistically stationary regime:

$$\left\langle E\right\rangle_S=\frac{\sigma}{2\alpha}-\frac{\nu}{2\alpha}\left\langle \Omega_2\right\rangle_S$$

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The 2D Stochastic Navier-Stokes Equations

- The 2D Stochastic Navier Stokes equations:

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla\omega = \nu\Delta\omega - \alpha\omega + \sqrt{2\alpha} f_s,$$

  where f_s is a random force (white in time, smooth in space). We study the limit $\nu \to 0$.
- The time unit has been chosen such that $E = 1 - \mathscr{O}(\nu/\alpha)$.
- $1/\alpha = U/\alpha L$ is a Reynolds' number based on the large scale friction.
- Two main regimes

1. $\alpha \gg 1$: the flow is not much affected by the box. The energy is dissipated before reaching the box size. Cascade regime (possibly)
2. $\alpha \ll 1$: most of the energy will stand at the box scale. Regime of large scale coherent flows

The 2D Stochastic Navier-Stokes Equations

- The 2D Stochastic Navier Stokes equations:

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla\omega = \nu\Delta\omega - \alpha\omega + \sqrt{2\alpha} f_s$$

- We use very small Rayleigh friction, to observe large scale energy condensation (this is not the inverse cascade regime).
- We study the limit: $\lim_{\alpha\to 0}\lim_{\nu\to 0}$ $(\nu \ll \alpha)$ $(Re \gg R_\alpha \gg 1)$ (Weak forces and dissipation).
- We have time scale separations:

turnover time $= 1 \ll 1/\alpha =$ forcing or dissipation time.

The 2D Stochastic Navier-Stokes Equations

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Large Scale Structures and Euler Eq. Steady States

$$\frac{\partial \omega}{\partial t}+\mathbf{v}.\nabla\omega=\nu\Delta\omega-\alpha\omega+\sqrt{2\alpha}f_s \tag{5}$$

- Time scale separation: magenta terms are small.
- At first order, the dynamics is nearly a 2D Euler dynamics. The flow self organizes and converges towards steady solutions of the Euler Eq.:

$$\mathbf{v}.\nabla\omega=0 \text{ or equivalently } \omega=f(\psi)$$

where the Stream Function ψ is given by: $\mathbf{v}=\mathbf{e}_z\times\nabla\psi$.

- Steady states of the 2D Euler equations will play a crucial role. Degeneracy : what does select f ?

Large Scale Structures and Euler Eq. Steady States

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla\omega = \nu\Delta\omega - \alpha\omega + \sqrt{2\alpha} f_s \qquad (5)$$

- Time scale separation: magenta terms are small.
- At first order, the dynamics is nearly a 2D Euler dynamics. The flow self organizes and converges towards steady solutions of the Euler Eq.:

$$\mathbf{v}.\nabla\omega = 0 \text{ or equivalently } \omega = f(\psi)$$

where the Stream Function ψ is given by: $\mathbf{v} = \mathbf{e}_z \times \nabla\psi$.

- Steady states of the 2D Euler equations will play a crucial role. Degeneracy : what does select f ?

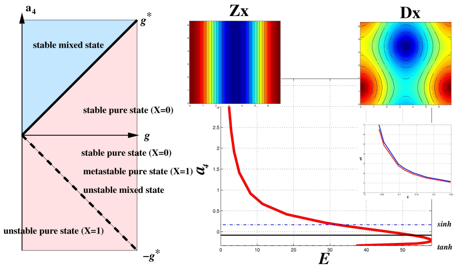
Steady States of Euler Eq. as Maxima of Variational Problems

Energy-Casimir Variational Problems

$$S(E) = \max_{\omega} \left\{ \int_{\mathscr{D}} d\mathbf{r}\, s(\omega) \,\middle|\, \frac{1}{2}\int_{\mathscr{D}} d\mathbf{r}\, \frac{\mathbf{v}^2}{2} = E \right\}.$$

- Numerical results : Z. Yin, D. C. Montgomery, and H. J. H. Clercx, Phys. Fluids (2003).
- Maxima: $\omega = \Delta\psi = \left(s^{'}\right)^{-1}(\beta\psi)$ (stable steady states of the Euler Eq.).
- In the following, normal form analysis with $s(\omega) = -\frac{\omega^2}{2} + a_4\frac{\omega^4}{4} + \ldots$
- Geometry parameter $g = E(\lambda_1 - \lambda_2) \propto (L_x - L_y)$.

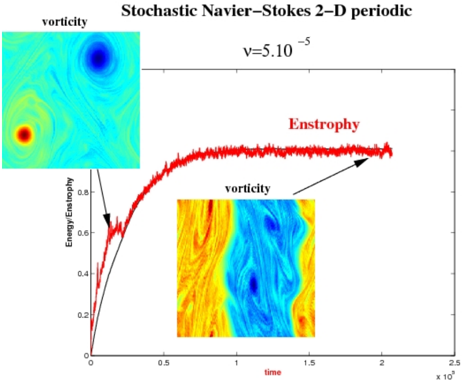
Steady States for the 2D-Euler Eq. (doubly periodic)



Bifurcation analysis : degeneracy removal, either by the domain geometry (g) or by the nonlinearity of the vorticity-stream function relation (f, parameter a_4).

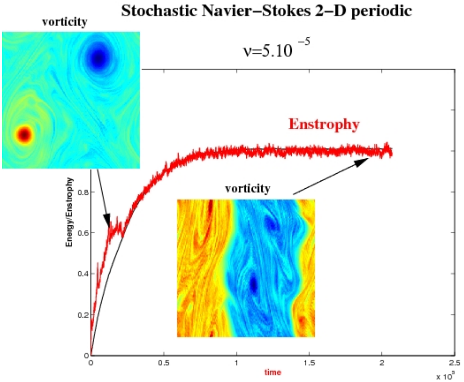
Derivation: normal form for an Energy-Casimir variational problem.
A general degeneracy removal mechanism.

Numerical Simulation of the 2D Stochastic NS Eq.



Very long relaxation times. 10^5 turnover times.

Numerical Simulation of the 2D Stochastic NS Eq.



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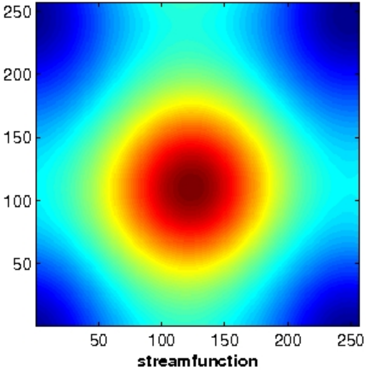
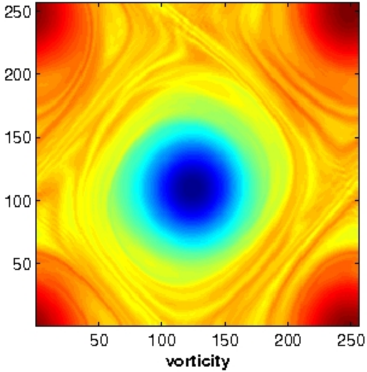
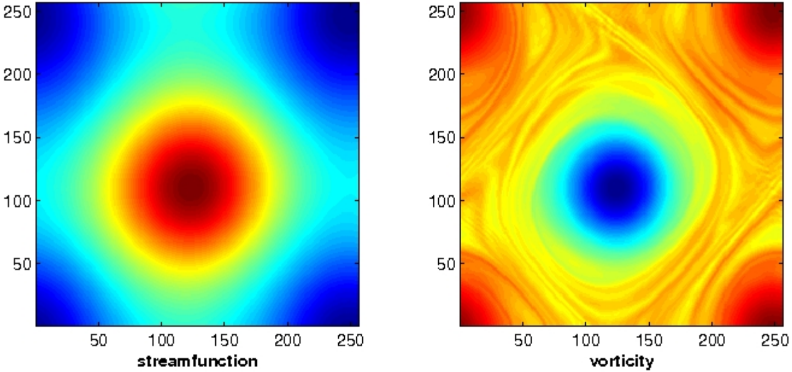
Out of Equilibrium Stationary States: Dipoles





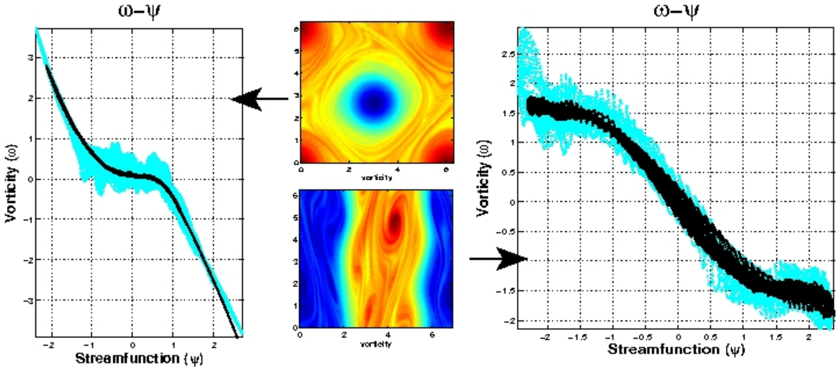
Are we close to some steady states of the Euler Eq.?

Out of Equilibrium Stationary States: Dipoles



Are we close to some steady states of the Euler Eq.?

Vorticity-Streamfunction Relation



Conclusion: we are close to steady states of the Euler Eq.

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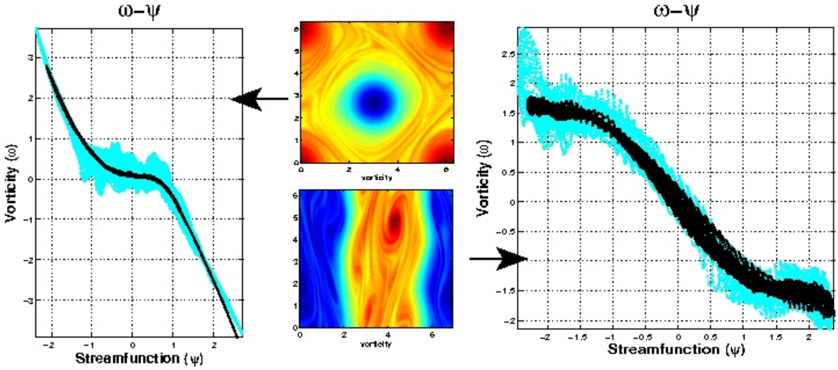

Conclusion: we are close to steady states of the Euler Eq.

Outline

Leading Order: 2D Euler Steady States

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla\omega = \nu\Delta\omega - \alpha\omega + \sqrt{2\alpha} f_s$$

- At leading order we expect a 2D Euler steady state $\mathbf{V}_0$ with vorticity Ω_0: $\mathbf{V}_0.\nabla\Omega_0 = 0$
- We assume for simplicity a stable parallel base flow $\Omega_0(y)$ and $\mathbf{V}_0 = U(y)\mathbf{e}_x$
- The linearized 2D Euler equations

$$\partial_t \delta\omega + L_{\Omega_0}[\delta\omega] = 0 \text{ with } L[\delta\omega] = \mathbf{V}_0.\nabla\delta\omega + \delta\mathbf{v}\cdot\nabla\Omega_0$$

Projection over Slow and Fast Variables

$$\frac{\partial \omega}{\partial t}+\mathbf{v}.\nabla\omega=-\alpha\omega+\sqrt{2\alpha}f_s$$

- We decompose the field into slow (parallel flows) and fast variables

$$\Omega_0(y)=P[\omega]=\frac{1}{2\pi}\int_{\mathscr{D}}\mathrm{d}x\,\omega \text{ and } \sqrt{\alpha}\delta\omega=\omega-\Omega_0$$

- Projected equations (exact)

$$\partial_t\Omega_0=-\alpha\Omega_0-\alpha P[\delta\mathbf{v}\cdot\nabla\delta\omega]+\sqrt{2\alpha}P[f_S(\mathbf{x},t)]$$

$$\partial_t\delta\omega+L_{\Omega_0}[\delta\omega]=-\sqrt{\alpha}\delta\omega+\sqrt{2}(1-P)[f_S(t,\mathbf{x})]-\sqrt{\alpha}(1-P)[\delta\mathbf{v}\cdot\nabla\delta\omega]$$

- Kinetic (or quasilinear approach) $\sqrt{\alpha}(1-P)[\delta\mathbf{v}\cdot\nabla\delta\omega]$ is neglected

The Quasilinear Approximation

$$\partial_t \Omega_0 = -\alpha \Omega_0 - \alpha P\left[\delta \mathbf{v} \cdot \nabla \delta \omega\right] + \sqrt{2\alpha} P\left[f_S(\mathbf{x},t)\right]$$

$$\partial_t \delta \omega + L_{\Omega_0}\left[\delta \omega\right] = -\sqrt{\alpha}\delta \omega + \sqrt{2}(1-P)\left[f_S(\mathbf{x},t)\right]$$

- The unforced quasilinear approximation conserves energy and enstrophy
- For fixed Ω_0 the second equation is an Ornstein Uhlenbeck process that can be studied precisely
- We will study the quasilinear approximation and discuss if this approximation is self-consistent in the limit $\alpha \to 0$
- In the limit $\alpha \to 0$, we have a two time-scale problem, that we treat adiabatically

The Lyapunov Equation

$$\partial_t \delta\omega + L_{\Omega_0}[\delta\omega] = -\sqrt{\alpha}\delta\omega + \sqrt{2}(1-P)[f_S(\mathbf{x},t)]$$

- The statistics of this Gaussian process is characterized by the two-points correlation function $g(\mathbf{x}_1,\mathbf{x}_2,t) \equiv \langle \delta\omega(\mathbf{x}_1,t)\delta\omega(\mathbf{x}_2,t)\rangle$

$$\frac{\partial g}{\partial t} + L^1_{\Omega_0}[g] + L^2_{\Omega_0}[g] = -\sqrt{\alpha}g + 2C_P(\mathbf{x}_1,\mathbf{x}_2)$$

- We will prove that this equation has a limit

$$g(\mathbf{x}_1,\mathbf{x}_2,t) \underset{t\to\infty}{\to} g_\infty[\Omega_0](\mathbf{x}_1,\mathbf{x}_2)$$

- Any two point correlation function has a limit. For instance

$$P\langle \delta\mathbf{v}(\mathbf{x},t)\cdot\nabla\delta\omega(\mathbf{x},t)\rangle \underset{t\to\infty}{\to} \mathcal{N}\mathcal{L}[\Omega_0](\mathbf{x})$$

The Kinetic Equation

$$\partial_t \Omega_0 = -\alpha \Omega_0 - \alpha P\left[\delta \mathbf{v} \cdot \nabla \delta \omega\right] + \sqrt{2\alpha} P\left[f_S(\mathbf{x}, t)\right]$$

- For instance, we look at the evolution of the average

$$\partial_t \Omega_0 = -\alpha \left(\Omega_0 + \mathscr{N}\mathscr{L}\left[\Omega_0\right]\right)$$

Outline

The 2D Stochastic Linearized Euler Equations

General considerations

$$\partial_t \omega + \mathbf{v}.\nabla\Omega_0 + \mathbf{V}_0.\nabla\omega = \sqrt{\sigma} f_S(\mathbf{x},t) \text{ with } \left\langle f_S(\mathbf{x},t) f_S(\mathbf{x}',t') \right\rangle = C(\mathbf{x},\mathbf{x}')\delta(t-t')$$

- Linear: Gaussian process (two point correlations, Lyapunov equation)
- Theoretical difficulty : the deterministic linearized operator is non normal (no mode decomposition)
- Infinite dimensional linear operator : non trivial non-exponential behaviors
- Landau damping or Orr mechanism

Linear SDE: Brownian Motion and Ornstein Uhlenbeck Process

Some trivial remarks

- First case: without dissipation. Ex : Brownian motion

$$dx = \sqrt{\sigma} dW_t \text{ with } x(0) = 0$$

$$\left\langle x^2 \right\rangle (t) = \sigma t$$

- Second case: with dissipation. A simple 1-d Ornstein Uhlenbeck process (with dissipation)

$$dx = -\alpha x dt + \sqrt{\sigma} dW_t \text{ with } x(0) = 0$$

$$\left\langle x^2 \right\rangle_S = \frac{\sigma}{2\alpha}$$

A linear stochastic differential equation with dissipation leads to a statistically stationary process

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The 2D Linearized Stochastic Euler Equation

Linearized close to a stable steady state

$$\partial_t \omega + \mathbf{v}.\nabla\Omega_0 + \mathbf{V}_0.\nabla\omega = \sqrt{\sigma} f_S(\mathbf{x},t) \text{ and } \omega(t=0)=0$$

- Lyapunov equation for $g(\mathbf{x}_1,\mathbf{x}_2,t) \equiv \langle \omega(\mathbf{x}_1,t)\omega(\mathbf{x}_2,t)\rangle$

$$\frac{\partial g}{\partial t} + L^1_{\Omega_0}[g] + L^2_{\Omega_0}[g] = \sigma C(\mathbf{x}_1,\mathbf{x}_2)$$

- In which case are we? A linear variance growth or the stabilization to a stationary process?
- There is no dissipation (reversible conservative dynamics), we should be in the first case. We expect a linear growth
- The result will depend on which functional space is g.

The 2D Linearized Stochastic Euler Equation

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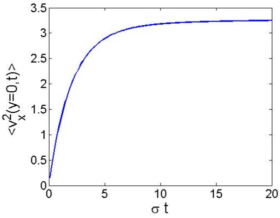
The 2D Linearized Stochastic Euler Eq.

Stochastic Orr mechanism

- $\langle \mathbf{v}(r,t+\tau)\mathbf{v}(r',t)\rangle \to \langle \mathbf{v}(r,\tau)\mathbf{v}(r',0)\rangle_S \quad \langle \mathbf{v}(r,t)\mathbf{v}(r',t)\rangle = \mathscr{O}(\sigma)$.
  (The velocity autocorrelation function has a finite limit)



The velocity v_x variance (base flow $v_0(y) = \sigma y$)

- Even without dissipation, the velocity process converges towards a stationary process (Stochastic Orr mechanism)

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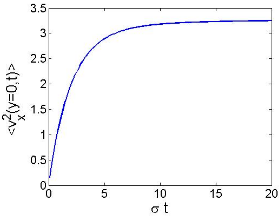

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A Simple Explanation

$$\frac{\partial g}{\partial t} + L^1_{\Omega_0}[g] + L^2_{\Omega_0}[g] = \sigma \sum_k C_k(\mathbf{x}_1) C_k(\mathbf{x}_2)$$

- Solution in terms of the deterministic solutions of the linearized equation

$$g(\mathbf{x}_1,\mathbf{x}_2,t) = \sigma \sum_k \int_0^t \mathrm{d}u \exp\left(L_{\Omega_0} u\right)[C_k](\mathbf{x}_1) \exp\left(L_{\Omega_0} u\right)[C_k](\mathbf{x}_2)$$

- For the 2D linearized Euler Eq., for velocity variables, even without dissipation $[\exp(Lu)] \underset{u\to\infty}{\rightarrow} 0$ and the integral converges
- Thanks to both the Orr mechanism and the depletion of the perturbation vorticity, for the velocity, L converges to zero even without dissipation or friction

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Asymptotic Behavior of the Linearized Euler Eq.

Base flow with stationary streamlines : the velocity field

Theorem: algebraically decaying asymptotic velocity field

$$v_x(y,t) \underset{t\to\infty}{\sim} \frac{v_{x,\infty}(y)}{t}\exp\left(-ikU(y)t\right)$$

$$v_y(y,t) \underset{t\to\infty}{\sim} \frac{v_{y,\infty}(y)}{t^2}\exp\left(-ikU(y)t\right)$$

- What about stationary streamlines? They should give contributions of order $1/t^{1/2}$!
- No contribution from the stationary streamlines thanks to the depletion of the vorticity perturbation at stationary streamlines

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The 2D Linearized Stochastic Euler Eq.: Conclusions

The results for shear flows should also hold for axisymmetric vortex

- $\langle \mathbf{v}(y,t+\tau)\mathbf{v}(y',t)\rangle \to \langle \mathbf{v}(y,\tau)\mathbf{v}(y',0)\rangle_S \quad \langle \mathbf{v}(y,t)\mathbf{v}(y',t)\rangle = \mathscr{O}(\sigma)$.
(The velocity autocorrelation function has a finite limit)
- R.m.s. value of the vorticity and vorticity gradient:

$$\sqrt{\langle \nabla \omega^2 \rangle} \underset{t\to\infty}{\sim} \sigma^{1/2} C(\mathbf{r})\, t^{3/2} \text{ and } \sqrt{\langle \omega^2 \rangle} \underset{t\to\infty}{\sim} \sigma^{1/2} C(\mathbf{r})\, t^{1/2}$$

- $\langle \omega(y,t)\omega(y',t)\rangle = \underset{t\to\infty}{\sim} \sigma S(y)\delta(y-y')$ (Resonance over streamlines for the vorticity autocorrelation function)

The Lyapunov Equation

$$\partial_t \delta\omega + L_{\Omega_0}[\delta\omega] = -\sqrt{\alpha}\delta\omega + \sqrt{2}(1-P)[f_S(\mathbf{x},t)]$$

- Gaussian : the statistics is characterized by the two-points correlation function $g(\mathbf{x}_1,\mathbf{x}_2,t) \equiv \langle \delta\omega(\mathbf{x}_1,t)\delta\omega(\mathbf{x}_2,t)\rangle$

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The Kinetic Equation

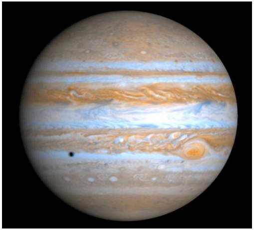
- The kinetic equation for the 2D Stochastic Navier-Stokes equations is a well defined object

$$\partial_t \Omega_0 = -\alpha \left(\Omega_0 + \mathscr{N}\mathscr{L}\left[\Omega_0\right] \right)$$

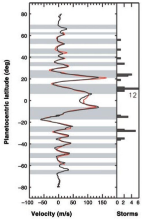
- Ongoing work : study of this nonlinear Fokker-Planck equation, at a mathematical and physical level
- Comparison with experiments and direct numerical simulations

Kinetic Theory of Earth and Jupiter's Zonal Jets

These jets does not seem to be maximum entropy states



Jupiter atmosphere



Jupiter Zonal wind (Voyager and Cassini, from Porco et al 2003)

How far are we to reproduce such phenomena in numerical simulations ? How to theoretically predict such velocity profile ?

Summary

Messages :

- For the 2D Euler equations, the equivalent of the Landau damping is the Orr mechanism
- For perturbations of any stable shear flows, the decay of the velocity perturbation is asymptotically algebraic
- We propose a kinetic theory for stochastic mean field Hamiltonians. The kinetic equation describes the relaxation towards the stationary momentum distribution
- A similar kinetic approach is proposed for the 2D stochastic Navier-Stokes equations
- We have thoroughly studied the Ornstein-Uhlenbeck process for the 2D linearized Euler equations and its asymptotics

F. Bouchet, and A. Venaille, Physics Reports, 2011, Statistical mechanics of

Publications

1. F. Bouchet, Physica D, 2008 Simplified variational problems for the statistical equilibria of 2D flows.
2. F. Bouchet and E. Simonnet, PRL (March 2009), Random changes of flow topology in 2D and geophysical turbulence.
3. A. Venaille and F. Bouchet, PRL (March 2009), Phase transitions, ensemble inequivalence and Fofonoff flows.
4. F. Bouchet and H. Morita, Physica D (April 2010), Asymptotic stability of the 2D Euler and of the 2D linearized Euler equations.
5. A. Venaille and F. Bouchet, to be submitted to J. Phys. Oceanography. Are strong mid-basin eastward jets (Gulf Stream, Kuroshio) statistical equilibria?
6. F. Bouchet and A. Venaille, to be submitted to Physics Reports: Statistical mechanics of two dimensional and geophysical flows.
7. F. Bouchet and M. Corvellec, submitted to J. Stat. Mech. Invariant measures of the 2D Euler and Vlasov equations.