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LETTER

Kinetic theory for non-equilibrium stationary states in long-range interacting systems

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Abstract. We study long-range interacting systems perturbed by external stochastic forces. Unlike the case of short-range systems, where stochastic forces usually act locally on each particle, here we consider perturbations by external stochastic fields. The system reaches stationary states where the external forces balance the dissipation on average. These states do not respect detailed balance and support non-vanishing fluxes of conserved quantities. We generalize the kinetic theory of isolated long-range systems to describe the dynamics of this non-equilibrium problem. The kinetic equation that we obtain applies to plasmas, self-gravitating systems, and to a broad class of other systems. Our theoretical results hold for homogeneous states, but may also be generalized to apply to inhomogeneous states. We obtain an excellent agreement between our theoretical predictions and numerical simulations. We discuss possible applications to describe non-equilibrium phase transitions.

Keywords: exact results, stochastic particle dynamics (theory), stochastic processes (theory), stationary states

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Contents

1. Introduction

Most physical systems are out of equilibrium either because of coupling to thermal baths at different temperatures or because of external forces that break detailed balance. The study of non-equilibrium stationary states is an active area of research in modern statistical mechanics. It is indeed a lasting challenge to achieve for non-equilibrium systems a level of theoretical understanding similar to the one established for equilibrium systems [1]–[3].

In this letter, we consider systems of particles interacting through two-body non-integrable potentials, also called long-range interactions. Examples include plasmas and self-gravitating systems (globular clusters, galaxies), where particles interact through repulsive or attractive Coulomb and attractive Newton potentials, respectively. Our work also applies to a large class of models with non-integrable interactions, such as spins, vortices in two dimensions, and others, which have been studied extensively in recent years [4]–[8].

Systems with non-integrable potentials are often forced through external stochastic fields. For example, globular clusters are affected by the gravitational potential of their galaxy, thereby producing a force that fluctuates along their physical trajectories. In addition, galaxies themselves feel the random potential of other surrounding galaxies, and their halos are subjected to transient and periodic perturbations due, for example, to the passing of dwarfs or to orbital decay [9]. Plasmas may also be subjected to fluctuating interactions imposed by environmental electric or magnetic fields [10]. These physical situations often lead to a stationary state where the power injected by the external random fields balances on average the dissipation. To the best of our knowledge, such non-equilibrium stationary states in systems with non-integrable potentials have not been studied before, and this work provides a first step in this direction.

Unlike systems with short-range interactions, stochastic perturbations in long-range interacting systems often act coherently on all particles and not independently on each particle. Moreover, unlike short-range systems, it is not natural to consider long-range systems as being coupled to thermal baths at the boundaries. Thus, the non-equilibrium

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stationary states that we study are rather different from the ones in systems with short-range interactions. These states do not verify detailed balance and support non-zero fluxes of conserved quantities, which are basic ingredients of non-equilibrium stationary states.

Theoretical results on isolated systems with long-range interactions include the kinetic theory description of relaxation toward equilibrium. In plasma physics, this approach leads to the Lenard–Balescu equation or to the approximate Landau equation [11, 12]. These equations, or some of their approximations, are grouped as the collisional Boltzmann equation in the astrophysical context. The main theoretical result of this letter is a generalization of the kinetic theory to describe non-equilibrium stationary states, valid for small external perturbations and spatially homogeneous stationary states.

The non-equilibrium kinetic equation that we obtain describes the temporal evolution of the one-particle distribution function. When the system is not far from equilibrium, it is natural to expect that the system settles into a stationary state. We find that in such a state, the one-particle momentum distribution is non-Gaussian. The kinetic equation describes the evolution of the kinetic energy, and its prediction of the stationary state compares very well with N-body numerical simulations.

2. Stochastically forced long-range interacting systems

Consider a system of N particles interacting through a long-ranged pair potential. The Hamiltonian of the system is

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^{N} v(q_i - q_j), \tag{1}$$

where q_i and p_i are, respectively, the coordinate and the momentum of the ith particle, while $v(q)$ is the two-body interaction potential. The particles are taken to be of unit mass. In this paper, for simplicity, we consider the q_is to be scalar periodic variables of period 2π; generalization to $q_i \in \mathbb{R}^n$, with $n = 1, 2$ or 3, is straightforward.

In self-gravitating systems, since the dynamics is dominated by collective effects, it is natural and usual to rescale the time in such a way that the parameter $1/N$ multiplies the interaction potential [13]. In plasma physics, the typical number of particles with which one particle interacts is given by the coupling parameter $\Gamma = n\lambda_{\mathrm{D}}^3$, where n is the number density and λ_{D} is the Debye length. It is then usual to rescale the time such that the inverse of a power of Γ multiplies the interaction term [11]. These reasons justify the rescaling of the potential energy by $1/N$ in equation (1), known as the Kac scaling in systems with long-range interactions [14].

We perturb the system (1) by the action of the stochastic field $F(q_i, t)$. The resulting equations of motion are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \text{and} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} - \alpha p_i + \sqrt{\alpha}\, F(q_i, t), \tag{2}$$

where α is the friction constant, and $F(q, t)$ is a statistically homogeneous Gaussian process with zero mean and variance given by

$$\langle F(q, t) F(q', t') \rangle = C(|q - q'|)\delta(t - t'). \tag{3}$$

The hypothesis that the Gaussian fields are statistically homogeneous, i.e., the correlation function depends solely on $|q-q'|$, holds for any perturbation which does not break space homogeneity. Such a hypothesis will also be essential for the following discussions where we consider homogeneous stationary states. Now, $C(q)$ represents correlation, and is therefore a positive-definite function [15]. Its Fourier components are thus positive:

$$c_k \equiv \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}q\, C(q)\mathrm{e}^{-\mathrm{i}kq} > 0, \qquad C(q) = c_0 + 2\sum_{k=1}^{\infty} c_k \cos(kq). \tag{4}$$

It will be convenient to use the following equivalent Fourier representation of the Gaussian field $F(q,t)$:

$$F(q,t) = \sqrt{c_0}\; X_0 + \sum_{k=1}^{\infty} \sqrt{2c_k}[\cos(kq)\, X_k + \sin(kq)\, Y_k], \tag{5}$$

where X_k and Y_k are independent scalar Gaussian white noises satisfying $\langle X_k(t)\, X_{k'}(t')\rangle = \delta_{k,k'}\delta(t-t')$, $\langle Y_k(t)\, Y_{k'}(t')\rangle = \delta_{k,k'}\delta(t-t')$, and $\langle X_k(t)\, Y_{k'}(t')\rangle = 0$.

Using the Itō formula [16] to compute the time derivative of the energy density $e = H/N$ and averaging over noise realizations give

$$\left\langle \frac{\mathrm{d}e}{\mathrm{d}t} \right\rangle + \langle 2\alpha\kappa\rangle = \frac{\alpha}{2}C(0), \tag{6}$$

where $\kappa = \sum_{i=1}^{N} p_i^2/(2N)$ is the kinetic energy density. The average kinetic energy density in the stationary state is thus $\langle\kappa\rangle_{\mathrm{ss}} = C(0)/4$.

In the dynamics (2), the fluctuations of the intensive observables due to stochastic forcing are of order $\sqrt{\alpha}$, while those due to finite-size effects are of order $1/\sqrt{N}$. Moreover, the typical timescale associated with the effect of stochastic forces is $1/\alpha$ (see, e.g., equation (6)), while the one associated with relaxation to equilibrium due to finite-size effects is of order N, see [4, 5].

In the following, we analyze the dynamics (2) in the joint limit $N \to \infty$ and $\alpha \to 0$. While the first limit is physically motivated on the grounds that most long-range systems indeed contain a large number of particles, the second one allows us to study non-equilibrium stationary states for small external forcing. Moreover, for small α, we will be able to develop a complete kinetic theory for the dynamics.

For simplicity, we discuss in this letter the continuum limit $N\alpha \gg 1$, when stochastic effects are predominant with respect to finite-size effects. Generalization to other cases ($N\alpha$ of order one, or $N\alpha \ll 1$) is straightforward, as discussed in the conclusion.

3. Kinetic theory

A natural framework to study the dynamics (2) is the kinetic theory. We now describe the theoretical approach to derive this theory, while some of the technical results will be explicitly obtained in a longer paper [17]. The central result is the kinetic equation (10) below, which describes the time evolution of the single-particle distribution function.

We consider the Fokker–Planck equation associated with the equations of motion (2). It describes the evolution of the N-particle distribution function $f_N(q_1,\ldots,q_N,p_1,\ldots,p_N,t)$ (after averaging over the noise realization, f_N is the probability

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density to observe the system with coordinates and momenta around the values $\{q_i, p_i\}_{1\le i\le N}$ at time t). This equation can be derived by standard methods [16]. We get

$$\frac{\partial f_N}{\partial t} = \sum_{i=1}^{N}\left[-p_i\frac{\partial f_N}{\partial q_i} + \frac{\partial(\alpha p_i f_N)}{\partial p_i}\right] + \frac{1}{2N}\sum_{i,j=1}^{N}\frac{\partial v(q_i-q_j)}{\partial q_i}\left[\frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_j}\right] f_N + \frac{\alpha}{2}\sum_{i,j=1}^{N} C(q_i-q_j)\frac{\partial^2 f_N}{\partial p_i \partial p_j}. \tag{7}$$

We have proved by analyzing the so-called potential conditions [18] for this Fokker–Planck equation that a sufficient condition for the stochastic process (2) to verify detailed balance is that the Gaussian noise is white in space, that is, $c_k = c$ for all k. This condition is not satisfied for a generic correlation function C. Steady states are then true non-equilibrium ones, with non-vanishing currents and a balance between external forces and dissipation.

Similarly to the Liouville equation for Hamiltonian systems, the N-particle Fokker–Planck equation is a very detailed description of the system. Using kinetic theory, we want to describe the evolution of the one-particle distribution function $f(z_1,t) = \int\prod_{i=2}^{N} \mathrm{d}z_i\, f_N(z_1,\ldots,z_N,t)$ (we use the notation $z_i = (q_i,p_i)$ whenever convenient). Note that the normalization is $\int \mathrm{d}z\, f(z,t) = 1$.

In plasmas and self-gravitating systems, due to the long-range nature of the interactions, the one-particle distribution function is not affected by the two-particle distribution function at leading order in $1/N$, and, therefore, its evolution is described at leading order by the Vlasov equation. Finite-size effects, however, induce weak correlations whose effects on the long-time evolution of the one-particle distribution can be computed self-consistently in the framework of the kinetic theory by using perturbation theory. A complete treatment of the problem leads to the Lenard–Balescu equation [11, 12]. In a similar way, for our problem, the evolution will be described at leading order by the Vlasov equations due to the long-range nature of the interactions. Weak stochastic forces lead to weak correlations that affect the long-time evolution. This case can be treated by following a generalized kinetic approach, as we now describe.

Substituting in the N-particle Fokker–Planck equation (7) the reduced distribution function $f_s(z_1,\ldots,z_s,t) = \int\prod_{i=s+1}^{N}\mathrm{d}z_i\, f_N(z_1,\ldots,z_N,t)$, and using standard techniques [19], we get a hierarchy of equations, similar to the BBGKY hierarchy. We split the reduced distribution functions into connected and non-connected parts, e.g., $f_2(z_1,z_2,t) = f(z_1,t)f(z_2,t) + \alpha g(z_1,z_2,t)$, and then neglect the effect of the connected part of the three-particle correlation on the evolution of the two-particle correlation function. This scheme is consistent at leading order in the small parameter α, and is the simplest closure scheme for the hierarchy. For simplicity, we moreover assume that the system is homogeneous: f depends on p, and g depends on $|q_1-q_2|$, p_1 and p_2, only. The first two equations of the hierarchy are then

$$\frac{\partial f}{\partial t} - \alpha\frac{\partial}{\partial p}[pf] - \frac{\alpha}{2}C(0)\frac{\partial^2 f}{\partial p^2} = \alpha\frac{\partial}{\partial p}\int \mathrm{d}q\,\mathrm{d}p_2\, v'(q)g(q,p,p_2,t), \tag{8}$$

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and

$$\frac{\partial g}{\partial t} + \left[p_1 \frac{\partial g}{\partial q_1} - \left.\frac{\partial f}{\partial p}\right|_{p_1} \int \mathrm{d}q_3\, \mathrm{d}p_3\, v'(q_1 - q_3) g(q_3 - q_2, p_3, p_2, t) \right] + \{1 \leftrightarrow 2\}$$
$$= C(q_1 - q_2) \left.\frac{\partial f}{\partial p}\right|_{p_1} \left.\frac{\partial f}{\partial p}\right|_{p_2}, \tag{9}$$

where the symbol $\{1 \leftrightarrow 2\}$ means an expression obtained from the bracketed one by exchanging 1 and 2, while the prime denotes differentiation.

To obtain from these equations a single kinetic equation for the distribution function f, we have to solve equation (9) for g as a function of f and plug the result into the right-hand side of equation (8). From these two equations, we readily see that the two-particle correlation g evolves over a timescale of order one, whereas the one-particle distribution function $f(p,t)$ evolves over a timescale of order $1/\alpha$. We use this timescale separation, and compute the long-time limit of g from equation (9) by assuming f to be constant. This procedure is equivalent to making the Bogoliubov hypothesis for deriving the kinetic theory of isolated long-range systems. For the timescale separation to be valid, it is also required that the one-particle distribution function $f(p,t)$ is a stable solution of the Vlasov equation at all times.

The solution of equations of the type (9) is quite technical (see the long appendix in Nicholson's book [11]). Equation (9) differs from the corresponding equation for an isolated long-range system in that the term on the right-hand side is different in the two cases, and cannot be solved by methods known in the literature. The main technical achievement that aided this work was being able to solve equation (9). In a future paper [17], we will give the details of the solving procedure. In brief, the method relies on making a parallel between the Lyapunov equations for infinite-dimensional Ornstein–Uhlenbeck processes and their general solutions, and equation (9). Using this method, we get the desired kinetic equation:

$$\frac{\partial f}{\partial t} - \alpha \frac{\partial (pf)}{\partial p} - \alpha \frac{\partial}{\partial p}\left[D[f] \frac{\partial f}{\partial p} \right] = 0, \tag{10}$$

where

$$D[f](p) = \frac{1}{2} C(0) + 2\pi \sum_{k=1}^{\infty} v_k c_k \int^* \mathrm{d}p_1 \left[\frac{1}{|\epsilon(k, kp)|^2} + \frac{1}{|\epsilon(k, kp_1)|^2} \right] \frac{1}{p_1 - p} \left.\frac{\partial f}{\partial p}\right|_{p_1}. \tag{11}$$

Here, v_k is the kth Fourier coefficient of the pair potential $v(q)$, the quantity c_k is defined in equation (4), while $\int^*$ indicates the Cauchy integral, and the dielectric function ϵ is

$$\epsilon(k, \omega) = \lim_{\eta \to 0^+} \left[1 - 2\pi \mathrm{i} v_k k \int \mathrm{d}p \frac{1}{-\mathrm{i}(\omega + \mathrm{i}\eta) + \mathrm{i}kp} \frac{\partial f}{\partial p} \right]. \tag{12}$$

The kinetic equation (10) is the central result of this letter.

This kinetic equation has the form of a non-linear Fokker–Planck equation, since the diffusion coefficient $D[f](p)$ itself is a function of the unknown distribution function f. As equation (11) shows, this coefficient has two parts, namely, (i) a linear part, $C(0)/2$,

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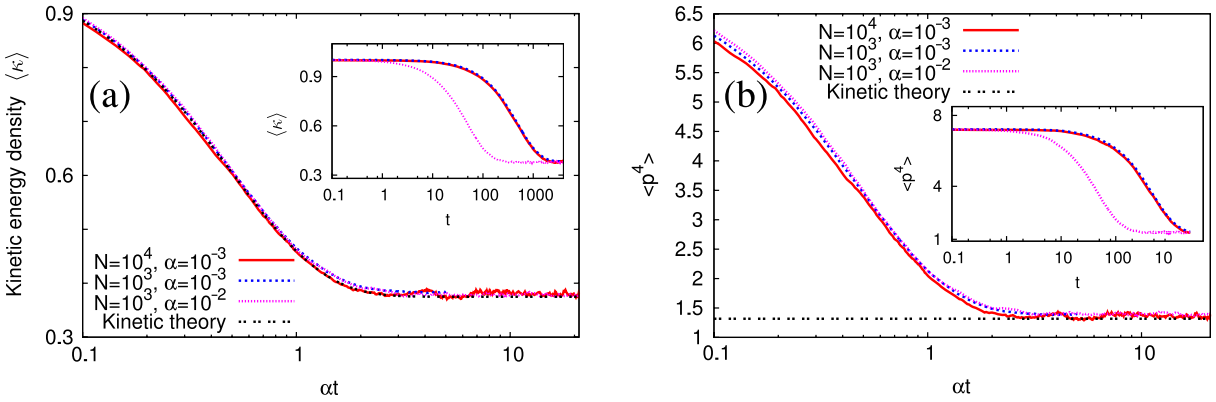

Figure 1. (a) Kinetic energy density $\langle\kappa\rangle$ and (b) $\langle p^4\rangle$ as a function of αt, for the values $C(0) = 1.5$ and $c_1 = 0.75$. The data for different N and α values are obtained from numerical simulations of the stochastically forced HMF model, and involve averaging over 50 histories for $N = 10^4$ and 10^3 histories for $N = 10^3$. The data collapse implies that α is the timescale of relaxation to the stationary state. The inset shows the data without time rescaling by α.

which is due to the mean-field effect of the stochastic forces, and (ii) a non-linear part due to correlations induced in the system by the stochastic forces. The contributions of different modes of the stochastic force are independent of each other, that is, contributions proportional to c_k do not couple with $v_{k'}$ with $k \neq k'$.

For consistency, the prediction of the evolution of the kinetic energy from the kinetic equation has to agree with equation (6). We have checked this by using equations (10) and (11), and proving that the integrals in the non-linear part of the diffusion coefficient give no contribution to the kinetic energy.

As already mentioned, and as is evident from equation (10), the timescale for the kinetic evolution is $1/\alpha$. This has been checked by performing direct numerical simulations, see figure 1 and section 4. Thus, α can be eliminated from the kinetic equation (10) by a redefinition of time. Therefore, even for vanishingly small values of α, if a stationary distribution exists, it will be at distance of order one from a Gaussian momentum distribution.

While a linear Fokker–Planck equation with non-degenerate diffusion coefficient can be proven to converge to a unique stationary distribution [18], this is not true in general for non-linear Fokker–Planck equations such as equation (10). We expect that if the system is not too far from equilibrium, the kinetic equation will have a unique stationary state. Far from equilibrium, the kinetic equation could lead to very interesting dynamical phenomena, such as bistability, limit cycle or more complex behaviors. The main issue is then the analysis of the evolution of the kinetic equation. Although some methods to study this type of equation exist [20], in order to provide some preliminary answers, we have devised a numerical iterative scheme to compute some of the stationary states of the kinetic equation (10). We now describe the scheme.

A linear Fokker–Planck equation whose diffusion coefficient $D(p)$ is strictly positive admits a unique stationary state

$$f_{\rm ss}(p) = A \exp\left[-\int_0^p \mathrm{d}p' \, \frac{p'}{D(p')}\right]. \tag{13}$$

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For a given distribution $f_n(p)$, we compute the diffusion coefficient $D_n(p)$ from equation (11), and then f_{n+1} using D_n and equation (13). This procedure defines an iterative scheme. Whenever convergent, this scheme leads to a stationary state of equation (10). Each iteration involves integrations, so we expect the method to be robust enough when starting not too far from an actual stationary state. However, we have no detailed mathematical analysis yet.

In section 4, we discuss numerical results on N-particle simulations, and the computation of stationary states from the iterative method mentioned above.

4. The stochastically forced HMF model

Until now, we have presented our theoretical analysis for a general two-particle interaction $v(q)$. In order to perform simple numerical simulations, we now consider the case of the stochastically forced attractive Hamiltonian mean-field (HMF) model, which corresponds to the choice $v(q) = 1 - \cos q$.

The HMF model serves as a paradigm to study long-range interacting systems, and describes particles moving on a circle under deterministic Hamiltonian dynamics [21, 22]. This model has been studied a lot in recent times. It displays many features of generic long-range interacting systems, such as the existence of quasistationary states [4, 22]. In equilibrium, the system displays a second-order phase transition from a high-energy homogeneous phase to a low-energy inhomogeneous phase at the energy density $e_c = 3/4$.

Since the Fourier transform of the HMF interparticle potential is, for $k \neq 0$, $v_k = -[\delta_{k,1} + \delta_{k,-1}]/2$, where $\delta_{k,i}$ is the Kronecker delta, we see from the kinetic equation (10) that only the stochastic force with wavenumber $k = 1$ contributes to the non-linear part of the diffusion coefficient; all the other stochastic forces give only a mean-field contribution through the term $C(0)$. Thus, the two parameters that dictate the evolution of the stochastically forced HMF model are $C(0)$ and c_1. From (6), we know that $C(0) = 4\langle\kappa\rangle_{\rm ss}$ is proportional to the kinetic energy in the final stationary state. Moreover, equation (4) implies that $c_1 \leq C(0)/2$.

If $c_1 = 0$, the kinetic equation reduces to a linear Fokker–Planck equation with diffusion coefficient $C(0)/2$. This equation also describes the HMF model coupled to a Langevin thermostat, studied in [23, 24]. As the kinetic equations are the same, the dynamics coincide at leading order in α. However, we know that at higher orders, detailed balance is broken in our case, whereas it holds for the Langevin dynamics.

In the case $c_1 = 0$, the homogeneous stationary states of the kinetic equation have Gaussian momentum distribution $f(p)$. As has been studied thoroughly in the context of canonical equilibrium of the HMF model, these states are stable for kinetic energies greater than $1/4$, i.e., for $C(0) > 1$.

For values of $C(0)$ and c_1 such that $C(0) > 1$ and $c_1 \ll C(0)$, we then expect the stationary states to be close to homogeneous states with Gaussian momentum, so that the numerical iterative scheme to locate stationary states of the kinetic equation is expected to converge for well-chosen initial conditions. We have checked the convergence for the set of values of c_1 used in the simulations reported in this letter.

To check the theory, we have performed numerical simulations of the stochastically forced HMF model. In figure 1, we show the evolution of the kinetic energy and $\langle p^4\rangle = (1/N)\sum_{i=1}^{N} p_i^4$, and compare them with theoretical predictions. In the latter case,

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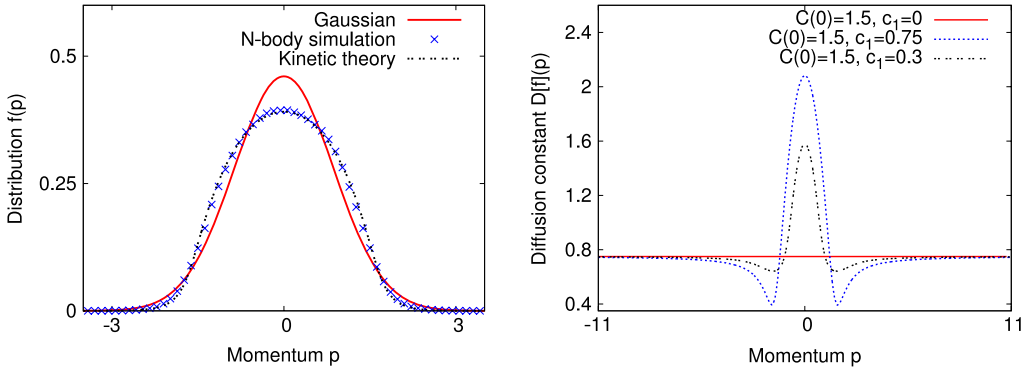

Figure 2. The panel on the left shows the stationary momentum distribution $f(p)$ for $\alpha = 0.01$, $C(0) = 1.5$, and $c_1 = 0.75$. The data denoted by crosses are results of N-body simulations of the stochastically forced HMF model with $N = 10\,000$, while the black broken line refers to the theoretical prediction from the kinetic theory. For comparison, the red continuous line shows the Gaussian distribution with the same kinetic energy (stationary state at $C(0) = 1.5$, $c_1 = 0$). The panel on the right shows the diffusion coefficient $D[f](p)$ for the stationary momentum distribution $f(p)$ for different values of $C(0)$ and c_1.

we have compared the long-time asymptotic value with the kinetic theory prediction for the stationary state, computed using the iterative scheme. In both cases, we observe a very good agreement between the theory and simulations.

For a more accurate comparison, we have obtained the stationary momentum distribution from both N-body simulations and the numerical iterative scheme. The comparison between the two is shown in figure 2, left panel, where we also show the Gaussian distribution with the same kinetic energy. The agreement between theory and simulations is excellent.

5. Conclusions

In this work, we studied the effect of external stochastic fields on Hamiltonian long-range interacting systems by generalizing the kinetic theory of isolated long-range systems. Our theoretical results are general, being applicable to any long-range interparticle potential, space dimensions and boundary conditions. In this letter, we demonstrated an excellent agreement between the theory and numerical simulations for one representative case.

Here, we discussed the kinetic theory in the limit $N\alpha \gg 1$. The extension to general values of $N\alpha$ is straightforward: because of the linearity of the equations of the BBGKY hierarchy, the finite-N and stochastic effects give independent contributions. The kinetic equation at leading order of both stochastic and finite-size effects is

$$\frac{\partial f}{\partial t} = L_\alpha[f] + L_N[f], \tag{14}$$

where L_α is the operator described in equation (10) and L_N (of order $1/N$) is the Lenard–Balescu operator [11]. For instance, in the case $N\alpha \ll 1$ and in dimensions greater than one, the operator L_N is responsible for the relaxation to Boltzmann equilibrium after a

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timescale of order N, whereas the smaller effect of L_α selects the actual temperature after a longer timescale of order $1/\alpha$.

We note that an equivalent approach to derive the kinetic theory is to write an evolution equation for the noise-averaged empirical density $\rho(p,q,t) = (1/N)\sum_{i=1}^{N}\langle\delta(q_i(t)-q)\delta(p_i(t)-p)\rangle$, by analogy with the Klimontovich approach for isolated systems. The noise appears in the resulting equation as a multiplicative term. This equation can be treated perturbatively, and may be shown to lead to the kinetic equation (10).

Let us mention some open issues. For technical simplicity, we assumed a homogeneous state in our approach. Recently, Heyvaerts [25] has generalized the Lenard–Balescu equation to some non-homogeneous cases; his approach could be used to generalize the theory developed here to inhomogeneous states. There is no difficulty in principle, although actual computation could be more involved.

An interesting follow up to this work is to study the dynamics of the kinetic equation (10), both analytically and numerically. This may unveil very interesting behaviors, such as bistability or limit cycles. Bistability was observed in two-dimensional turbulence with stochastic forcing [26], in a framework which has deep connection with the one studied in this letter. One of the motivations for this work was to make a first step in formulating a kinetic theory for the point vortex model and the Euler equations in two-dimensional turbulence [8]. This subject will be the topic of further investigations.

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