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Prediction of anomalous diffusion and algebraic relaxations for long-range interacting systems, using classical statistical mechanics

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We explain the ubiquity and extremely slow evolution of non-Gaussian out-of-equilibrium distributions for the Hamiltonian mean-field model, by means of traditional kinetic theory. Deriving the Fokker-Planck equation for a test particle, one also unambiguously explains and predicts striking slow algebraic relaxation of the momenta autocorrelation, previously found in numerical simulations. Finally, angular anomalous diffusion are predicted for a large class of initial distributions. Nonextensive statistical mechanics is shown to be unnecessary for the interpretation of these phenomena.

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Modern nonequilibrium statistical mechanics consider the study of kinetic behavior of large ensembles of interacting particles in an attempt to relate macroscopic properties of a system to the microscopic dynamics of its constituent. Recently there has been an increasing interest on long-range interacting systems [1]. The first reason is the broad spectrum of applications: self-gravitating and Coulomb systems, vortices in two-dimensional fluid mechanics [2], wave-particles interaction, and trapped charged particles [3]. Second, unexpected non-Gaussian distributions [4] and nonexponential relaxations for autocorrelations [4,5,7] have recently been observed in such systems. This was a great surprise, indeed, at variance with short-range interacting systems, as all the particles contribute to the local field, mean-field predictions are usually extremely good and often exact for the corresponding equilibrium statistical mechanics. In accordance with this simple picture, usual expectations were that these systems should exhibit gaussian distribution functions and normal exponential relaxations.

The recent discovery of non-Gaussian distributions [4] led to an intense debate on the applicability [8] of usual Boltzmann-Gibbs statistical mechanics to long-range interacting systems (see Tsallis *et al.* in Ref. [1]). Such non-Gaussian distributions have been fitted using Tsallis' distributions [4]. The striking asymptotic algebraic behaviors for momentum autocorrelations have also been fitted using q-exponential functions [4,5], derived from nonextensive statistical mechanics.

In this paper, revisiting this question, we present analytical results explaining these numerical findings, *without* using concepts of nonextensive statistical mechanics. In a recent paper [16], we explained the ubiquitous occurrence of nonequilibrium distributions by linking them with stable Vlasov solutions. Here, we complete this argument by explaining that the time scale for the evolution of such non-Gaussian distributions has to be larger than the number of particles N. In another recent paper, we have derived the Fokker-Planck equation describing a single particle in an equilibrium bath [9]. Here we generalize this result to out of equilibrium bath distributions. More importantly, we also report analytical explicit predictions for the momentum autocorrelations at large time, explaining its algebraic relaxation. Finally, anomalous diffusion phenomena in the asymptotic time limit were found in these systems and also highly debated [6,7]. This concept, raised in a great variety of different fields such as geophysics, chemical engineering, and disordered media, was analytically understood in only very few many-degrees-of-freedom systems, even if the question of Lévy flights is very popular nowadays. Starting from the microscopic dynamics of a long-range interacting system, we present in this paper, analytical predictions showing that there exists a full class of distributions which leads to anomalous diffusion. We give an explicit prediction of the anomalous diffusion exponent in terms of the bath distribution function. We insist that such an exact prediction of this exponent, from the microscopic dynamics of a Hamiltonian N particle system is not common at all.

In this paper, we consider these questions for the Hamiltonian mean-field model [10]

$$H_N = \frac{1}{2}\sum_{j=1}^{N} p_j^2 - \frac{1}{2N}\sum_{i,j=1}^{N} \cos(\theta_i - \theta_j), \qquad (1)$$

since it is nowadays thought to be the simplest model to study dynamical and thermodynamic properties of system with long-range interactions [1]. In addition to its pedagogical properties, it corresponds to a simplification of one-dimensional gravitational interactions and is an excellent first step before the Colson-Bonifacio's model for free-electron lasers [11]. Note that the factor $1/N$ is the appropriate and classical *mean-field scaling*, relevant for long-range interacting systems [12,13]: indeed, the physically interesting limit for such systems amounts to let the number of particles go to infinity at fixed volume, by contrast with the usual thermodynamic limit.

Let us consider the kinetics of the Hamiltonian mean-field (HMF) model using the Klimontovich approach. We have checked that an asymptotical expansion of the Born-Bogoliubov-Green-Kirkwood-Yvon (BBGKY) hierarchy leads to the same results. The state of the N-particles system can thus be described by the *discrete* single particle time-dependent density function $f_d(t,\theta,p)=(1/N)\Sigma_{j=1}^{N}\delta[\theta-\theta_j(t)]\delta[p-p_j(t)]$, where δ is the Dirac function, (θ,p) the Eulerian coordinates of the phase space and (θ_i,p_i) the Lagrangian coordinates of the particles. The dynamics is thus described by the Klimontovich's equation [15]

$$\partial f_d/\partial t + p\partial f_d/\partial\theta - (\mathrm{d}V/\mathrm{d}\theta)(\partial f_d/\partial p) = 0, \qquad (2)$$

where the potential V that affects all particles is $V(t,\theta) \equiv -\int_0^{2\pi}d\alpha\int_{-\infty}^{+\infty}dp\,\cos(\theta-\alpha)f_d(t,\alpha,p)$. This description of the

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Hamiltonian dynamics derived from (1) is *exact*: as the distribution is a sum of Dirac functions it contains the information on the position and velocity of all particles. It is, however, too precise for usual physical quantities of interest but will be a key starting point for the derivation of approximate equations, valid in the limit N large and describing average quantities.

When N is large, it is natural to approximate the discrete density f_d by a continuous one $f(t,\theta,p)$. Considering an ensemble of microscopic initial conditions close to the same initial macroscopic state, one defines the statistical average $\langle f_d\rangle = f_0(\theta,p)$, whereas fluctuations of probabilistic properties are of order $1/\sqrt{N}$. We will assume that f_0 is any stable stationary solution of the Vlasov equation. The discrete time-dependent density function can thus be rewritten as $f_d(t,\theta,p) = f_0(\theta,p) + \delta f(t,\theta,p)/\sqrt{N}$, where the fluctuation δf is of zero average. We define similarly the averaged potential $\langle V\rangle$ and its corresponding fluctuations $\delta V(t,\theta)$ so that $V(t,\theta) = \langle V\rangle + \delta V(t,\theta)/\sqrt{N}$. Inserting both expressions in Klimontovich's Eq. (2) and taking the average, one obtains

$$\frac{\partial f_0}{\partial t} + p\frac{\partial f_0}{\partial \theta} - \frac{d\langle V\rangle}{d\theta}\frac{\partial f_0}{\partial p} = \frac{1}{N}\left\langle \frac{d\delta V}{d\theta}\frac{\partial \delta f}{\partial p}\right\rangle. \qquad (3)$$

The left-hand side (lhs) is the Vlasov equation. The exact kinetic equation (3) suggests that for times much smaller than N, and stationary stable solutions f_0 for the Vlasov equation, the right-hand side (rhs) term, corresponding to the fluctuation of the mean-field potential, can be neglected. This is confirmed, for any finite time, by the Braun and Hepp theorem [13,14]. The innumerable Vlasov stable stationary states, far from equilibrium [16] (see Ref. [17] for a first stability analysis), then explains the generic occurrence of out-of-equilibrium distributions. These quasistationary states do not evolve on time scales much smaller than N, explaining the extremely slow relaxation of the system toward the statistical equilibrium.

Let us now concentrate on stable homogenous distributions $f_0(p)$, which are stationary since $\langle V\rangle = 0$. Subtracting Eq. (3) from Eq. (2) and using $f_d = f_0 + \delta f/\sqrt{N}$, one gets

$$\frac{\partial \delta f}{\partial t} + p\frac{\partial \delta f}{\partial \theta} - \frac{d\delta V}{d\theta}\frac{\partial f_0}{\partial p} = \frac{1}{\sqrt{N}}\left[\frac{d\delta V}{d\theta}\frac{\partial \delta f}{\partial p} - \left\langle \frac{d\delta V}{d\theta}\frac{\partial \delta f}{\partial p}\right\rangle\right].$$

For times much shorter than $\sqrt{N}$, we may drop the rhs encompassing quadratic terms in the fluctuations. The fluctuating part δf are then described, by the linearized Vlasov equation (this is an other result of the Braun and Hepp theorem [13,14]). This suggest to introduce the spatio-temporal Fourier-Laplace transform of δf and δV. This leads to

$$\widetilde{\delta V}(\omega,k) = -\frac{\pi(\delta_{k,1} + \delta_{k,-1})}{\varepsilon(\omega,k)}\int_{-\infty}^{+\infty} dp\frac{\widetilde{\delta f}(0,k,p)}{i(pk-\omega)}, \qquad (4)$$

where

$$\varepsilon(\omega,k) = 1 + \pi k(\delta_{k,1} + \delta_{k,-1})\int_{-\infty}^{+\infty} dp\frac{\frac{\partial f_0}{\partial p}}{(pk-\omega)} \qquad (5)$$

is the dielectric permittivity. The evolution of the potential autocorrelation, can therefore be determined. For homogeneous states, by symetry, $\langle\widetilde{\delta V}(\omega_1,k_1)\widetilde{\delta V}(\omega_2,k_2)\rangle = 0$ except if $k_1 = -k_2 = \pm 1$. One gets, after a transitory exponential decay, the general result

$$\langle \delta V(t_1,\pm 1)\delta V(t_2,\mp 1)\rangle = \frac{\pi}{2}\int_C d\omega e^{-i\omega(t_1-t_2)}\frac{f_0(\omega)}{|\varepsilon(\omega,1)|^2}. \qquad (6)$$

This is an exact result, no approximation has yet been done.

A similar, although longer, calculation shows that the rhs of Eq. (3) identically vanishes at order $1/N$. This proves that Vlasov stable distribution function will not evolve on time scales smaller or equal to N, in agreement with the $N^{1.7}$ scaling law which was numerically reported [16,18]. This is the first result: *generic out of equilibrium distributions evolve on time scales much larger than N.*

Let us now consider relaxation properties of a test particle, indexed by 1, surrounded by a background system of $(N-1)$ particles with a homogeneous distribution. The fluctuation of the potential is thus

$$\delta V(t,\theta) \equiv -\int_0^{2\pi} d\alpha\int_{-\infty}^{+\infty} dp\,\cos(\theta-\alpha)\delta f(t,\alpha,p) - (1/\sqrt{N})\cos(\theta-\theta_1). \qquad (7)$$

Using the equations of motion of the test particle and omitting the index 1 for the sake of simplicity, one obtains $p(t) = p(0) - \int_0^t du\{d\delta V[u,\theta(u)]\}/(d\theta)/\sqrt{N}$. By introducing iteratively the expression of θ in the rhs and expanding the derivative of the potential, one gets the result at order $1/N$. The key point is that this approach does not use the usual ballistic approximation. As a consequence, we obtain an exact result at order $1/N$. This is of paramount importance here to treat accurately the *collective effects*. As the changes in the impulsion are small (of order $1/\sqrt{N}$), the description of the impulsion stochastic process by a Fokker-Planck equation is valid. This last equation is then characterized by the time behavior of the first two moments $\langle[p(t)-p(0)]^n\rangle$. Using the generalization of formula (6) when the effect of the test particle is taken into account, one obtains in the large t limit

$$\langle[p(t)-p(0)]\rangle \overset{t\to+\infty}{\sim} \frac{t}{N}\left(\frac{dD}{dp}(p) + \frac{1}{f_0}\frac{\partial f_0}{\partial p}D(p)\right) \qquad (8)$$

$$\langle[p(t)-p(0)]^2\rangle \overset{t\to+\infty}{\sim} (2t/N)D(p), \qquad (9)$$

where the diffusion coefficient $D(p)$ can be written as

$$D(p) = 2\,\mathrm{Re}\int_0^{+\infty} dt e^{ipt}\langle\delta V(t,1)\delta V(0,-1)\rangle = \frac{\pi^2 f_0(p)}{|\varepsilon(p,1)|^2}. \qquad (10)$$

These results are the exact leading order terms in an expansion where $1/N$ is the small parameter. We obtained previously equivalent results, but restricted to equilibrium (Gaussian) f_0 [9], by a rather different and instructive approach. In this last paper, a comparison of the diffusion coefficient $D(p)$ with N particle numerical simulations is presented, illustrating that both results are undistinguishable. Let us carry on by explicitly evaluating the diffusion coefficient for a homogenous Gaussian distribution function $f_g(\theta,p) = \sqrt{\beta/(2\pi)^3}e^{-\beta p^2/2}$. In that case, after straightforward calculations, one gets the expression derived in Ref. [9]. The dif-

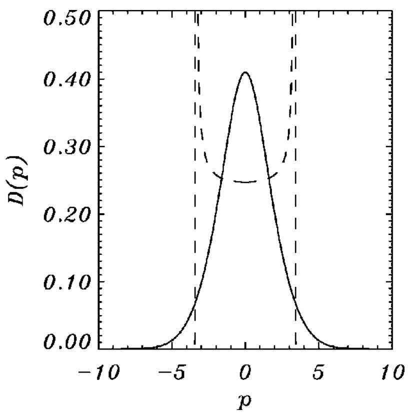

FIG. 1. Diffusion coefficient $D(p)$ in the case $H_N/N=2$ for a Boltzmann thermal bath (solid line) and a waterbag distribution (dashed line).

fusion coefficient $D(p)$ has the asymptotic expression $\sqrt{\pi\beta/2}e^{-\beta p^2/2}$ for large p values (see Fig. 1).

Interestingly, the method presented here, can be used for *any Vlasov-stable out-of-equilibrium distributions*. For instance, in Fig. 1, we present the result for the waterbag distribution.

Using time variable $\tau=t/N$ as suggested by Eqs. (8) and (9), the Fokker-Planck equation describing the time evolution of the distribution of the test particle is

$$\frac{\partial f_1(\tau,p)}{\partial \tau}=\frac{\partial}{\partial p}\left[D(p)\left(\frac{\partial f_1(\tau,p)}{\partial p}-\frac{1}{f_0}\frac{\partial f_0}{\partial p}f_1(\tau,p)\right)\right]. \tag{11}$$

We stress that this equation depends on the bath distribution f_0. It is valid both for equilibrium and out-of-equilibrium f_0, provided that f_0 is a stable stationary solution of the Vlasov equation. In the limit $\tau\to\infty$ (more precisely $1\ll\tau\ll N$), the bracket vanishes: the pdf f_1 of the test particle converges toward the quasistationary distribution f_0 of the surrounding bath. This is in complete agreement with the result that f_0 is stationary for times scales of order N.

For a large class of quasistationary distributions f_0, using Eq. (11), we can compute the momenta autocorrelation $\langle p(\tau)p(0)\rangle$, fitted numerically [5,7,18] with power laws, stretched exponentials, or q exponential. The second result of this paper is that *the time dependence of the momentum autocorrelation function scale with N as* $\langle p(t)p(0)\rangle=C(t/N)$ where $C(.)$ is a function. Let us first present the particular but very important case of a test particle in contact with a *gaussian distribution* f_0 (equilibrium bath). Figure 2 shows the momenta autocorrelation $\langle p(\tau)p(0)\rangle$, numerically computed from the Fokker-Planck Eq. (11); it presents an unexpected very slow relaxation which can, numerically, hardly be distinguished from a $1/\tau$ law. As shown below, the very fast decrease of the diffusion coefficient shown in Fig. 1 is actually the key point in these interesting and unusual properties of the momenta autocorrelation.

One derive analytically the large time behavior of the autocorrelation function, for any Vlasov stable distribution f_0, using an asymptotic expansion inspired by Ref. [19] (see also Ref. [20]). By introducing the appropriate change of

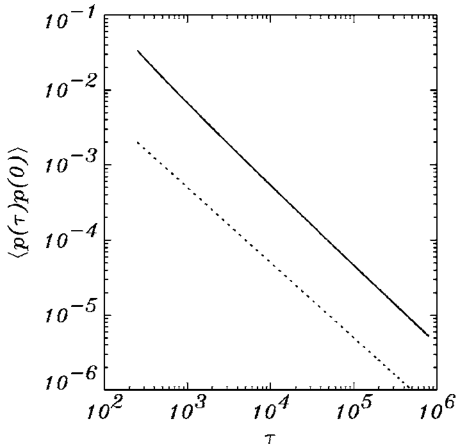

FIG. 2. The solid line represents the time evolution of the momentum autocorrelations obtained using the numerical integration of the Fokker-Planck equation for a test particle in a surrounding gaussian reservoir ($\beta=0.25$). Its slope can hardly be distinguished from a $1/\tau$ law (dotted line).

variable $x=x(p)$, defined by $dx/dp=1/\sqrt{D(p)}$, one ends up with the constant diffusion coefficient Fokker-Planck equation

$$\frac{\partial \hat{f}_1(\tau,x)}{\partial \tau}=\frac{\partial}{\partial x}\left(\frac{\partial \hat{f}_1(\tau,x)}{\partial x}+\frac{\partial \psi}{\partial x}\hat{f}_1(\tau,x)\right), \tag{12}$$

where $\psi(x)=-\ln[\sqrt{D(p)}f_0(p)]$. Using $\varepsilon(p,1)\overset{|p|\to\infty}{\sim}1$, Eq. (10) shows that $D(p)\overset{|p|\to\infty}{\sim}\pi^2 f_0(p)$.

Let us first consider distribution functions so that $f_0(p)\overset{|p|\to\infty}{\sim}C\exp(-\gamma p^{\delta})$, which includes not only the Gaussian ($\delta=2$) and exponential tails ($\delta=1$), but also stretched-exponential ones with $\delta>0$. Asymptotic analysis leads to $p(x)\overset{|x|\to\infty}{\sim}2/\gamma[\ln(x)]^{1/\delta}$ whereas

$$\psi(x)\overset{x\to\pm\infty}{\sim}\alpha\ln|x|,\quad \text{with } \alpha=3. \tag{13}$$

Interestingly, this log potential is a limiting case between a pure diffusive process leading to self-similar solutions and exponentially decreasing $\hat{f}_1$ solutions in a strongly confining potential. Asymptotic behavior (13) is still valid for distribution functions with algebraic tails, i.e., $f_0(p)\overset{|p|\to\infty}{\sim}Cp^{-\nu}$ where $\nu>3$. However, $\alpha=3\nu/(2+\nu)$ and $p(x)\overset{|x|\to\infty}{\sim}C'x^{2/(2+\nu)}$.

By considering the ansatz $\hat{f}_1(x,t)=\varphi_\lambda(x)e^{-\lambda\tau}$, one gets that the spectrum of the Fokker-Planck operator corresponds to only one bound state $\hat{f}_1^0(x)=f_0$, normalizable for $\alpha>1$, with an associated eigenvalue located at the bottom of the continuum: the absence of gap forbids *a priori* any exponential relaxation.

As we are interested in the asymptotic large-τ limit, we will restrict the analysis to the small-λ regime. A matched asymptotic procedure in the two distinct regions, $|x|>\ell$ and $|x|<\ell$, solves the equation. In the first domain, introducing $z=\sqrt{\lambda}x$ and $g_\lambda(z)=z\varphi_\lambda$, and using the asymptotic result (13), one ends up with $z^2 g_\lambda'' + g_\lambda'(\alpha-2)z + g_\lambda(2-2\alpha+z^2)=0$. The solutions can be expressed in terms of Bessel functions of order ν, J_ν, and Y_ν, as $g_\lambda(z)=A_{\lambda,\ell} z^{(3-\alpha)/2} J_{(\alpha+1)/2}(z) + B_{\lambda,\ell} z^{(3-\alpha)/2} Y_{(\alpha+1)/2}(z)$. In the domain $|x|<\ell$, where one neglects the term proportional to the vanishing eigenvalue λ, the solution is $\varphi_\lambda(x)=D_{\lambda,\ell} e^{-\psi(x)} + C_{\lambda,\ell} e^{-\psi(x)} \int_0^x du\, e^{\psi(x)}$. In order to compute the momenta autocorrelation function, noting that $p(x)$ is an odd function of x, we focus on odd eigenstates, obtained by considering $D_{\lambda,\ell}=0$. By taking care of the matching condition in $x=\ell$ and of the normalization condition, one ends up with the scaling: $A_{\lambda,\ell} \overset{\lambda\to 0}{\sim} \sqrt{\lambda}/2$, $B_{\lambda,\ell} \overset{\lambda\to 0}{\sim} \lambda/4$ and $C_{\lambda,\ell} \overset{\lambda\to 0}{\sim} C(l)\lambda^{(5/2)}$.

All these results are finally useful to derive the momentum auto-correlation. Indeed using as initial condition $\hat{f}_1(x,0) \equiv N(\ell)p(x)e^{-\psi(x)} = \int_0^{+\infty} d\lambda\, \mu(\lambda)\varphi_\lambda(x)$ where $N(l)$ is the ground state normalization factor and $\mu(\lambda) = N(\ell)\int dx\, p(x)\varphi_\lambda(x)$, one gets

$$\langle p(\tau)p(0)\rangle = \frac{1}{N(\ell)}\int_0^{+\infty} d\lambda\, \mu(\lambda)^2 e^{-\lambda\tau}. \tag{14}$$

Its limiting behavior in the $\tau\to\infty$ limit will be given by the behavior of $\mu(\lambda)$ when $\lambda\to 0$, which is itself determined by the large $|x|$ behavior of $p(x)$.

Let us be more specific in several important cases. The choice $p(x)\propto(\ln x)^{1/\delta}$ which corresponds to distribution functions with gaussian, exponential, or stretched-exponential tails, leads to the result

$$\langle p(\tau)p(0)\rangle \overset{\tau\to+\infty}{\propto} \frac{(\ln \tau)^{2/\delta}}{\tau}, \tag{15}$$

independent on α. This very slow algebraic relaxation, with logarithmic corrections, agrees with numerical simulation shown in Fig. 2, for the Gaussian case $\delta=2$.

An extension of this approach to distribution function $f_0(p)$ with algebraic tails is of prime interest. Following the same steps in the case $p(x)=x^\eta(\eta<(\alpha-1)/2)$ leads to $\mu(\lambda) \overset{\lambda\to 0}{\propto} \lambda^{(\alpha-2\eta-3)/4}$ and to the limiting behavior $\langle x^\eta(\tau)x^\eta(0)\rangle \overset{\tau\to+\infty}{\propto} \tau^{\eta-(\alpha-1)/2}$. Applying this result to the algebraic decay of the distribution function $f_0(p) \overset{|p|\to\infty}{\sim} Cp^{-\nu}$, with $\nu>2$ and using the corresponding asymptotic expression for $p(x)$, discussed above, we get $\eta=2/(2+\nu)$ and $\alpha=3\nu/(2+\nu)$. We thus obtain the *algebraic decay*

$$\langle p(\tau)p(0)\rangle \overset{\tau\to+\infty}{\propto} \tau^{-(\nu-3)/(2+\nu)}. \tag{16}$$

This is the third result: the equilibrium distribution and *generic out of equilibrium distributions lead to algebraic large time behaviors of the momentum autocorrelation functions*.

Finally, from the momenta correlations, one usually derives the angle diffusion $\langle(\theta(\tau)-\theta(0))^2\rangle=2D_\theta\tau$ where D_θ is defined via Kubo formula $D_\theta=\int_0^{+\infty} d\tau \langle p(\tau)p(0)\rangle$. However, asymptotic result (15) shows that this integral diverges leading to small anomalous diffusion for stretched exponential bath distributions. By contrast, for distributions with algebraic tails, since the exponent $(\nu-3)/(2+\nu)$ in Eq. (16) is smaller than one, we predict *strong anomalous diffusion*. This is the fourth result.

In summary, this work confirms and explains the unexpected kinetic behavior of long-range interacting systems: ubiquity of non-Gaussian distribution and algebraic behavior for momenta autocorrelation functions. Using traditional kinetic theory, we also predict strong anomalous diffusion for angles for a large class of initial distributions. Finally, let us anticipate that several physical systems with long-range interactions should exhibit similar features. One would in particular quote dynamics of vortices [21] and cold atoms physics [22].

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