

## Out-of-Equilibrium States as Statistical Equilibria of an Effective Dynamics in a System with Long-Range Interactions

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We study the formation of coherent structures in a system with long-range interactions where particles moving on a circle interact through a repulsive cosine potential. Nonequilibrium structures are shown to correspond to statistical equilibria of an effective dynamics, which is derived using averaging techniques. This simple behavior might be a prototype of others observed in more complicated systems with long-range interactions, such as two-dimensional incompressible fluids and wave-particle interaction in a plasma.

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The behavior of many complex nonlinear dynamical systems results in the formation of spatially ordered structures. Examples include two-dimensional incompressible weakly dissipative fluids [1], two- and three-dimensional magnetohydrodynamics [2], planet atmospheres [3], electrostatic potential in inhomogeneous magnetized plasmas, galaxy cluster formation in self-gravitating astrophysical systems [4], and vortices in rotating Bose-Einstein condensates [5]. Similar effects were also numerically revealed [6] in discrete lattices, in which the modulational instability of a linear wave was shown to be the first step towards energy localization, followed by the nonlinear interaction among breatherlike excitations. Related phenomena have been experimentally reported, such as the appearance of oscillons in vibrated granular media [7].

Besides all analogies, the reasons for this *self-organization* can be indeed quite different. In systems for which a structure originates from any randomly chosen initial condition, a natural explanation is of a statistical nature: see, for instance, Ref. [8] for self-gravitating systems or Ref. [9] for two-dimensional conservative fluids. In other systems, the structures arise and persist as a direct consequence of the nonlinear dynamics as, for instance, in Ref. [6]. We present in this Letter a simple and fully tractable model for which both reasons for self-organization, statistical mechanics, and nonlinear dynamics, have to be invoked. Indeed, we will show that Gibbsian statistical mechanics can be safely applied *only* when the correct dynamical variables are singled out [10].

The model, with long-range forces, exhibits long lived out of equilibrium structures and related unusual relaxation phenomena [11], such as those encountered in self-gravitating systems, plasmas, or two-dimensional fluids. Despite the importance of such systems, a clear understanding of their peculiarities is still an open problem.

We study the repulsive Hamiltonian mean-field (HMF) model [12], that describes the motion of  $N$  particles on a circle. Its Lagrangian is

$$L(\theta_i, \dot{\theta}_i) = \sum_{i=1}^N \frac{\dot{\theta}_i^2}{2} - \frac{1}{2N} \sum_{i,j=1}^N \cos(\theta_i - \theta_j). \quad (1)$$

This model is an archetype of mean-field models for which interactions are infinite range with size dependent coupling [13]. For a special but wide class of initial conditions, this model [14,15] has a very interesting dynamics since, contrary to statistical mechanics expectations, a localized structure (bicluster) appears (see Fig. 1). Its initial formation results from a fast oscillation of the medium, that nonlinearly drives, on a longer time scale, an average

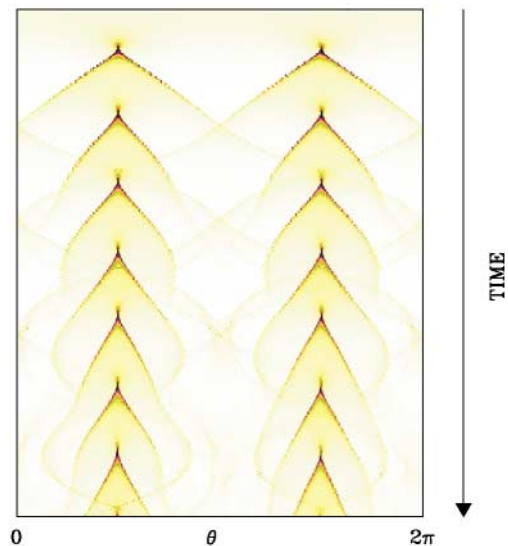


FIG. 1 (color online). Bicluster formation: short-time evolution of the particle density in gray scale; the darker the gray, the higher the density. Starting from an initial condition with all the particles evenly distributed on the circle, one observes a very rapid concentration of particles, followed by the quasiperiodic appearance of “chevrons,” that shrink as time increases. The structure later stabilizes and form two well-defined clusters.

motion of the particles in an effective double-well potential. The mechanism is qualitatively reminiscent of the parametrically forced pendulum, first analyzed by Kapitza [16]: the role of the harmonic forcing of the pivot is here played, by the *self-induced* fast oscillation of the original dynamics. As for the Kapitza pendulum, we average over the fast oscillation and obtain an effective Hamiltonian for the slow motion. We will show that the final bicluster structure is a statistical equilibrium of such an effective Hamiltonian.

Let's consider the evolution of particles for small initial energy and velocity dispersion [14]. The analysis relies on the existence of two time scales in the system: the first one, defined by the linearized dynamics, is intrinsic and corresponds to the inverse of the "plasma frequency"  $\omega = \sqrt{2}/2$ . The second one depends on the energy per particle  $e = H/N$  via the relation  $\tau = \varepsilon t$ , with  $\varepsilon = \sqrt{2}e$ . As a consequence, we use the following ansatz  $\theta_i = \theta_i^0(\tau) + \varepsilon f_i(t, \tau)$ . A multiscale analysis shows that the correct choice to describe the small and rapid oscillations is  $f_i = \{\sqrt{2}A_+(\tau) \sin[\phi_+(t)] + a_+(\tau)\} \cos(\theta_i^0 + \psi) + \{\sqrt{2}A_-(\tau) \sin[\phi_-(t)] + a_-(\tau)\} \sin(\theta_i^0 + \psi)$ , where  $\phi_{\pm}(t)$  are fast variables, and  $d\phi_{\pm}/dt$ ,  $A_{\pm}(\tau)$  and  $a_{\pm}(\tau)$  are slow variables. The phase  $\psi$  is defined via the complex number  $M_2 = N^{-1} \sum_{j=1}^N e^{2i\theta_j^0} \equiv |M_2| e^{-2i\psi}$ . We introduce this expression in Lagrangian (1) and keep terms up to order  $\varepsilon^2$ , averaging over the fast time  $t$ , as described, e.g., in Ref. [17]. Dropping the  $\varepsilon^2$  overall factor, the averaged Lagrangian reads

$$L_{\text{slow}} = \frac{1}{2} \sum_{i=1}^N \left( \frac{d\theta_i^0}{d\tau} \right)^2 + L_+ + L_-, \quad (2)$$

where

$$L_{\pm} = \frac{N}{2} [A_{\pm}^2 \dot{\phi}_{\pm}^2 \omega_{\pm}^2 - (M_{1\pm}^0)^2 \mp 2M_{1\pm}^0 a_{\pm} \omega_{\pm}^2 - (A_{\pm}^2 + a_{\pm}^2) \omega_{\pm}^4], \quad (3)$$

$M_1^0 \equiv \varepsilon N^{-1} \sum_{j=1}^N e^{i\theta_j^0} = M_{1-}^0 + iM_{1+}^0$  is the first moment associated to the angles and  $\omega_{\pm} \equiv \sqrt{(1 \pm |M_2|)}/2$  will turn out to be the fast frequencies (for details, see Ref. [18]).  $\phi_{\pm}$  are now cyclic variables and correspond to two quasiconserved quantities,  $P_{\pm} = NA_{\pm}^2 \omega_{\pm}^2 \dot{\phi}_{\pm}$ , which are adiabatic invariants of the dynamics. Going from the Lagrangian to the Hamiltonian formalism, one must first remark that, having eliminated the dependence on the time derivatives of  $A_{\pm}$ ,  $a_{\pm}$ , there is no Legendre transform over these variables. In the absence of conjugate variables to the amplitudes  $A_{\pm}$ , the corresponding Hamilton's equation are simply given, from the least action principle, by  $\partial_{A_{\pm}} H_{\text{slow}} = 0$ . This leads to the expression for the amplitudes  $NA_{\pm}^2 = P_{\pm} \omega_{\pm}^{-3}$ . Finally, from the equations  $\partial_{a_{\pm}} H_{\text{slow}} = 0$  and  $\dot{\phi}_{\pm} = \partial_{P_{\pm}} H_{\text{slow}}$ , we determine the shifts  $a_{\pm}$  and the phases  $\phi_{\pm}$ .

Rewriting the action, taking into account the relations for  $A_{\pm}$  and  $a_{\pm}$ , we end up with a very simple *effective*

Hamiltonian that retains only the slow motion

$$H_{\text{eff}} = \sum_i \frac{p_i^{02}}{2} + P_+ \sqrt{\frac{1 + |M_2|}{2}} + P_- \sqrt{\frac{1 - |M_2|}{2}}, \quad (4)$$

and which describes the evolution of the full system. The constant  $P_+$  and  $P_-$  are determined by the initial condition.

For the sake of simplicity, we restrict here to the case in which only one mode,  $P_-$ , is initially excited ( $P_+ = 0$ ). Dropping the subscript for  $P$ , we then consider the Hamiltonian

$$H_{\text{eff}} = \sum_i \frac{p_i^{02}}{2} + P \sqrt{\frac{1 - |M_2|}{2}}. \quad (5)$$

The particle dynamics determined by this Hamiltonian perfectly compares with the one given by Lagrangian (1), as shown in Fig. 2.

In order to explain the formation of the bicluster and the chevrons observed in Fig. 1, let us first remark that the equations of motion of the effective Hamiltonian  $H_{\text{eff}}$

$$\ddot{\theta}_i^0 = - \frac{P}{N\sqrt{2(1 - |M_2|)}} \sin 2(\theta_i^0 + \psi) \quad (6)$$

are pendulumlike. The evolution of the system thus consists of  $N$  pendulum trajectories in a double-well potential, which is *self-consistently* modified since  $M_2$  depends on the  $\theta_i^0$ . If the initial velocity dispersion is small,

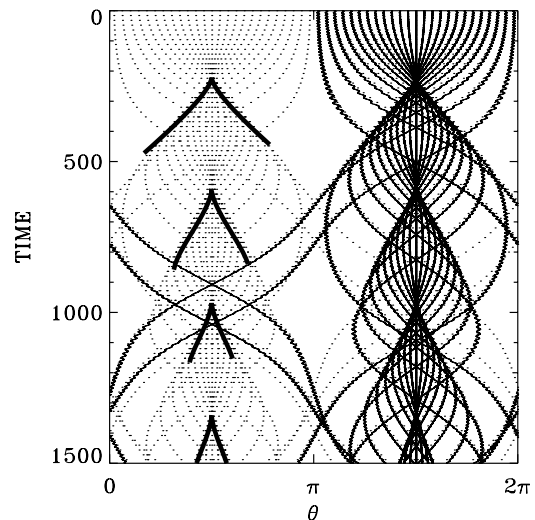


FIG. 2. Spatiotemporal evolution of 50 particles with initial energy per particle  $e = 2.5 \times 10^{-5}$ . The dotted lines present the results given by the effective Hamiltonian (5), whereas the solid lines show the evolution of particles initially between  $\pi$  and  $2\pi$  according to the original Lagrangian (1): except for the fast small scales oscillations, they are almost indistinguishable. On the left (between 0 and  $\pi$ ), the caustics (thick lines) given by formula (7) reproduce the "chevrons."

singularities appear in the density profile which becomes infinite along the envelope (or caustics) of the trajectories; the chevrons of Fig. 1 are a manifestation of these singularities [19]. This picture allows us also to make quantitative predictions. To simplify the calculations, we assume that position and depth of the double-well potential are fixed; this amounts to take  $\psi = \text{const}$  in Eq. (6) (as suggested by Fig. 1) and  $|M_2| = \text{const}$  which is the simplest hypothesis we can make in order to characterize the chevrons. We can compute the time  $t_s$  of the first divergence of the density: it is the time when the particles in the bottom of the two wells of the potential, that perform a quasiharmonic motion, cross. We get  $t_s = \pi\sqrt{N\omega_-/(8Pe)}$ , where  $P$  and  $e$  depend on the initial condition, and  $\omega_-$  depends on  $|M_2|$ . Approximating the trajectories by straight lines near the bottom of the wells, we obtain the shape of the “chevrons,” resorting to standard methods [20] of curve envelope calculation (see details in [18]). The lowest order approximation of the  $n$ th shock is

$$\theta \propto \frac{[t - (2n - 1)t_s]^{3/2}}{\sqrt{2n - 1}}, \quad (7)$$

where the  $1/\sqrt{n}$  factor explains the shrinking of the chevrons, whereas the  $3/2$  scaling law is generic according to catastrophe’s theory [19]. Taking for  $|M_2|$  its mean value (0.51 for the present initial condition), Fig. 2 emphasizes that the agreement with the numerics is quantitatively excellent.

Since the short-time dynamics and the “chevrons” formation is now clarified, let us next consider the stabilization of this outof equilibrium state: this is where statistical mechanics comes into play.

Because of the special form of the effective potential, which depends only on the global variable  $|M_2|$ , the statistical mechanics of the effective Hamiltonian (5) is exactly tractable. Indeed, the density of states  $\Omega(E)$  can be expressed in terms of the density of states corresponding to the kinetic part of the Hamiltonian  $\Omega_{\text{kin}}$  and of the angular configurational part  $\Omega_{\text{conf}}$  as follows:

$$\Omega(E) \propto \int d|M_2| \Omega_{\text{kin}}[E - V(|M_2|)] \Omega_{\text{conf}}(|M_2|), \quad (8)$$

where  $V(|M_2|) = P\sqrt{(1 - |M_2|)/2}$ . Using an inverse Laplace transform and the Hubbard-Stratonovitch trick, one also obtains  $\Omega_{\text{conf}}$  after some calculations.  $\Omega(E)$  in Eq. (8) can thus be evaluated [18] by the saddle point method, which leads to the equilibrium value  $|M_2^*|$  as a function of  $E/P$ . This explains the numerical result  $|M_2| \simeq 0.5$  found by the authors of Ref. [14] (their initial conditions correspond to  $E/P = \sqrt{2}/2$  and  $|M_2^*| = 0.51$ ). In addition, this shows that other initial conditions lead to other values of  $|M_2^*|$ , which are in excellent agreement with numerical simulations (see Fig. 3). The long lifetime of these out-of-equilibrium states is therefore fully under-

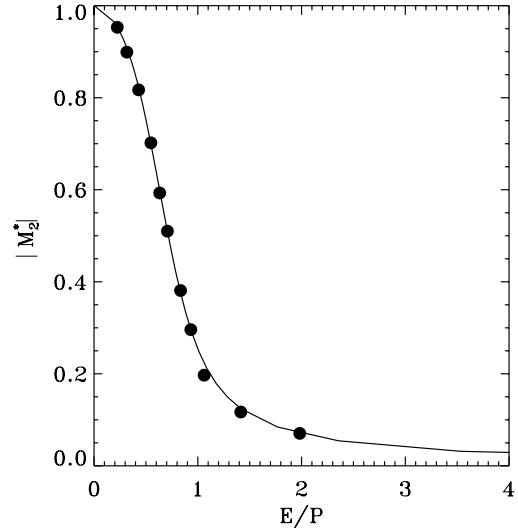


FIG. 3. The solid curve shows the theoretical prediction of  $|M_2^*|$  as a function of the ratio between the energy  $E$  and the adiabatic invariant  $P$ , whereas the filled circles correspond to numerical results for the full Lagrangian (1).

stood, since they appear as equilibrium states of an effective Hamiltonian that well represents the long-time motion.

Once  $|M_2^*|$  is known, it is easy to show that the equilibrium velocity distribution is Maxwellian with temperature  $T = 2\langle E_c \rangle/N$ , and that the distribution of angles has a Gibbsian shape  $\rho(\theta) \propto e^{-V(\theta)/T}$ , with the potential inferred from the equations of motion. However, whereas  $|M_2|$  quickly reaches its equilibrium value, the distribution  $\rho(\theta)$  relaxes very slowly, in the numerical experiments. Moreover, the relaxation time increases with  $N$ . Nevertheless, the density profile  $\rho(\theta)$  obtained in long-time simulations with the full system (1) fully agrees with the one obtained with the effective dynamics (5), as shown in Fig. 4. This makes this model a good candidate to study the unusual relaxation properties and the non equilibrium states observed in other systems with long-range interactions [11].

We thus conclude that the dynamics of the effective Hamiltonian (5) parametrizes very well the one of the full Lagrangian (1), for short as well as for long-time. The variational multiscale analysis we used allows us to exhibit naturally the adiabatic invariants and to preserve the Hamiltonian structure of the problem (to leading order), making it well suited for a statistical treatment. This leads us to predict statistical properties of the full system, as the asymptotic value of  $|M_2|$ . Moreover, the effective Hamiltonian gives the opportunity to study numerically the relaxation towards equilibrium of the bicluster, whereas it was not possible in the real dynamics, because of computational limitations. Indeed, let us observe that the ratio of the typical time scale of the two dynamics is of order 100 or even larger at smaller energies.

We have thus described a simple mechanism to explain the existence of a stable out-of-equilibrium structure in a

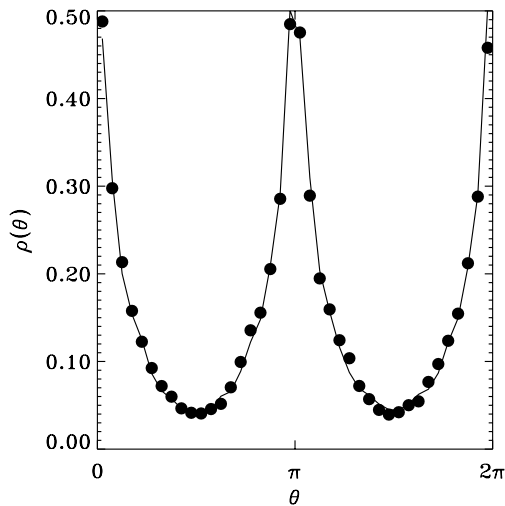


FIG. 4. Comparison of the non-Gibbsian density distributions  $\rho(\theta)$  of particles obtained with the original Lagrangian (1) (circles) and with the effective one (5) (solid line). Both results have been obtained for  $N = 10^3$  particles and are averaged over times corresponding to  $\tau = 10^3 \rightarrow 10^4$ . The energy per particle is  $e = 2.5 \times 10^{-5}$ .

Hamiltonian mean-field model. This model deserves special attention for different reasons. It is probably the simplest  $N$ -particles solvable model in one dimension which exhibits such a stabilization effect, corresponding to the coupling of very fast oscillations self-interacting with a slower motion. This model is moreover a simple analogue of other examples of nonlinear interactions of rapid oscillations with a slower global motion like the piston problem [21]: averaging techniques could be applied to the fast motion of gas particles in a piston which itself has a slow motion [22]. Examples can also be found in applied physics as for instance wave-particle interaction in plasma physics [23], or the interaction of fast inertia gravity waves with the vortical motion for the rotating shallow water model [24].

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