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Simpler variational problems for statistical equilibria of the 2D Euler equation and other systems with long range interactions

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Abstract

The Robert–Sommeria–Miller equilibrium statistical mechanics predicts the final organization of two dimensional flows. This powerful theory is difficult to handle practically, due to the complexity associated with an infinite number of constraints. Several alternative simpler variational problems, based on Casimir's or stream function functionals, have been considered recently. We establish the relations between all these variational problems, justifying the use of simpler formulations.

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We consider the 2D Euler equations, on a domain $\mathcal{D}$

$$\frac{\partial \omega}{\partial t} + \mathbf{v}\cdot\nabla\omega = 0; \quad \mathbf{v} = \mathbf{e}_z \times \nabla\psi; \quad \omega = \Delta\psi \tag{1}$$

where ω is the vorticity, $\mathbf{v}$ the velocity and ψ the stream function (with $\psi = 0$ on $\partial\mathcal{D}$, $\mathcal{D}$ is simply connected).

The equilibrium statistical mechanics of the 2D Euler equation (the Robert–Sommeria–Miller (RSM) theory [1–3]), assuming ergodicity, predicts the final organization of the flow, on a coarse grained level (see [4] for a recent review of Onsager ideas, that inspired the RSM theory, see also [5]). Besides its elegance, this predictive theory is a very interesting and useful scientific tool.

From a mathematical point of view, one has to solve a microcanonical variational problem (MVP): maximizing a mixing entropy $\mathcal{S}[\rho] = -\int_{\mathcal{D}} \mathrm{d}^2x \int \mathrm{d}\sigma \rho \log \rho$, with constraints on energy E and vorticity distribution γ

$$S(E_0, \gamma) = \sup_{\{\rho | N[\rho]=1\}} \{\mathcal{S}[\rho] | E[\overline{\omega}] = E_0, \\ D[\rho] = \gamma\} \text{ (MVP)}.$$

$\rho(\mathbf{x}, \sigma)$ is normalized ($N[\rho] = 1$, see (6)) and depends on space $\mathbf{x}$ and vorticity σ variables.

The theoretical predictability of RSM theory requires the knowledge of all conserved quantities. The infinite number of Casimir's functionals (this is equivalent to vorticity distribution γ) have then to be considered. This is a huge practical limitation. When faced with real flows, physicists can then either give physical arguments for a given type of distribution γ (modeler approach) or ask whether there exists some distribution γ with RSM equilibria close to the observed flow (inverse problem approach). However, in any case the complexity remains : the class of RSM equilibria is huge.

During recent years, authors have proposed alternative approaches, which led to practical and/or mathematical simplifications in the study of such equilibria. As a first example, Ellis, Haven and Turkington [6] proposed to treat the vorticity distribution canonically (in a canonical statistical ensemble). From a physical point of view, a canonical ensemble for the vorticity distribution would mean that the system is in equilibrium with a bath providing a prior distribution of vorticity. As such a bath does not exist, the physically relevant ensemble remains the one based on the dynamics : the microcanonical one. However, the Ellis–Haven–Turkington approach is extremely interesting as it provides a drastic mathematical and practical simplification to the problem of

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computing equilibrium states. A second example, largely popularized by Chavanis [7,8], is the maximization of generalized entropies. Both the prior distribution approach of Ellis, Haven and Turkington or its generalized thermodynamics interpretation by Chavanis lead to a second variational problem: the maximization of Casimir's functionals, with energy constraint (CVP)

$$C(E_0, s) = \inf_{\omega} \left\{ \mathcal{C}_s[\omega] = \int_{\mathcal{D}} s(\omega) \mathrm{d}^2 x \,|\, E[\omega] = E_0 \right\} \text{ (CVP)}$$

where $\mathcal{C}_s$ are Casimir's functionals, and s a convex function (Energy-Casimir functionals are used in classical works on nonlinear stability of Euler stationary flows [9,10], and have been used to show the nonlinear stability of some of RSM equilibrium states [2,11]).

Another class of variational problems (SFVP), that involve the stream function only (and not the vorticity), has been considered in relation with the RSM theory

$$D(G) = \inf_{\psi} \left\{ \int_{\mathcal{D}} \mathrm{d}^2 x \left[\frac{1}{2} |\nabla \psi|^2 + G(\psi) \right] \right\} \text{ (SFVP).}$$

Such (SFVP) functionals have been used to prove the existence of solutions to the equation describing critical points of (MVP) [11]. Interestingly, for the Quasi-geostrophic model, in the limit of small Rossby deformation radius, such a SFVP functional is similar to the van der Waals-Cahn Hilliard model which describes phase coexistence in usual thermodynamics [12,13]. This physical analogy has been used to make precise predictions in order to model Jovian vortices [12,14]. Moreover (SFVP) functionals are much more regular than (CVP) functionals and thus also very interesting for mathematical purposes.

When we prescribe appropriate relations between the distribution function γ, the functions s and G, the three previous variational problems have the same critical points. This has been one of the motivations for their use in previous works. However, a clear description of the relations between the stability of these critical points is still missing (is a (CVP) minimizer a RSM equilibria, or does a RSM equilibria minimize (CVP)?). This has led to fuzzy discussions in recent papers. Providing an answer is a very important theoretical issue because, as explained previously, it will lead to deep mathematical simplifications and will provide useful physical analogies.

The aim of this short paper is to establish the relation between these three variational problems. The result is that any minimizer (global or local) of (SFVP) minimizes (CVP) and that any minimizer of (CVP) is a RSM equilibria. The opposite statements are wrong in general. For instance (CVP) minimizers may not minimize (SFVP), but be only saddles. Similarly, RSM equilibria may not minimize (CVP) but be only saddles, even if no explicit example has yet been exhibited.

These results have several interesting consequences:

1. As the ensemble of (CVP) minimizers is a sub-ensemble of the ensemble of RSM equilibria, one cannot claim that (CVP) are more relevant for applications than RSM equilibria (for a different point of view, see for instance [15]).
2. The link between (CVP) and RSM equilibria provides a further justification for studying (CVP).
3. Based on statistical mechanics arguments, when looking at the Euler evolution on a coarse-grained level, it may be natural to expect the RSM entropy to increase. There is however no reason to expect such a property to be true for the Casimir's functional. As explained above, it may also happen that entropy extrema are (CVP) saddles.

In order to simplify the discussion, we keep only the energy constraint at the level of the Casimir functional (CVP). Adding other constraints, such as the circulation [15], or even the microscopic enstrophy, does not change the discussion.

We note that all the discussion can be easily generalized to any system with long range interactions (self-gravitating systems, Vlasov Poisson system) [16].

In the first section, we explain the link between a constrained variational problem and its relaxed version. We explain that any minimizer of the second is a minimizer of the first. In the second section, we present the microcanonical variational problem (MVP). We then introduce a mixed grand canonical ensemble by relaxing the vorticity distribution constraint in the RSM formalism. We prove in the third section that this mixed ensemble is equivalent to (CVP). Similarly, in the last section we prove that the (SFVP) variational problem is equivalent to a relaxed version of (CVP).

1. Relations between constrained and relaxed variational problems

We discuss briefly relations between a constrained variational problem and its relaxed version. This situation appears very often in statistical mechanics when passing from one statistical ensemble to another. We assume that the Lagrange's multipliers rule applies. Let us consider the two variational problems

$$G(C) = \inf_x \{ g(x) | c(x) = C \} \quad \text{and}$$

$$H(\gamma) = \inf_x \left\{ h_\gamma(x) = g(x) - \gamma c(x) \right\}.$$

G is the constrained variational problem and H is the relaxed one, γ is the Lagrange multiplier (or the dual variable) associated to C. We have the results:

1. $H(\gamma) = \inf_C \{G(C) - \gamma C\}$ and $G(C) \geq \sup_\gamma \{\gamma C + H(\gamma)\}$.
2. If x_m is a minimizer of h_γ then x_m is also a minimizer of $G(C)$ with the constraint $C = c(x_m)$.
3. If x_m is a minimizer of $G(C)$, then it exists a value of γ such that x_m is a critical point of h_γ, but x_m may not be a minimizer of h_γ but just a saddle. Then x_m is a minimizer of h_γ if and only if $G(C) = \sup_\gamma \{H(\gamma) + \gamma C\}$ if and only if $G(C)$ coincides with the convex hull of G in C. In this last situation the two variational problems are called *equivalent*.

Such results are classical. More detailed results in this context may be found in [15]. Situations of ensemble inequivalence have been classified, in relation with phase transitions [17].

Equality in point 1. follows from the remark that

$$H(\gamma) = \inf_C \left\{ \inf_x \{g(x) - \gamma c(x) | c(x) = C\} \right\}$$
$$= \inf_C \left\{ \inf_x \{g(x) | c(x) = C\} - \gamma C \right\}.$$

We remark that $-H$ is the Legendre–Fenchel transform of G. The inequality of point 1 is then a classical convex analysis result. We have for any value of γ,

$$\begin{aligned} G(C) &= \inf_x \{g(x) | c(x) = C\} \\ &= \inf_x \{g(x) - \gamma c(x) | c(x) = C\} + \gamma C \\ &\geq \inf_x \{g(x) - \gamma c(x)\} + \gamma C = H(\gamma) + \gamma C. \end{aligned} \quad (2)$$

This is a direct proof of the inequality of point 1.

Point 2: for x_m a minimizer of h_γ and x with $c(x) = c(x_m)$, we have $g(x_m) = h_\gamma(x_m) + \gamma c(x_m) \leq h_\gamma(x) + \gamma c(x_m) = g(x)$. This proves 2 First assertion of 3. is Lagrange's multipliers rule. Clearly, x_m is a minimizer of h_γ if and only if equality occurs in (2). It is a classical result of convex analysis that the convex hull of G is the Legendre–Fenchel transform of $-H$. This concludes the proof of 3. Many examples where x_m is a saddle may be found in the literature (see [17], or examples in the context of Euler equation in [18–20]).

2. RSM statistical mechanics

Euler's equations (1) conserve the kinetic energy

$$E[\omega] = \frac{1}{2} \int_{\mathcal{D}} d^2x \, (\nabla\psi)^2 = -\frac{1}{2} \int_{\mathcal{D}} d^2x \omega\psi = E_0 \quad (3)$$

and for integrable s, Casimirs' functional

$$\mathcal{C}_s[\omega] = \int_{\mathcal{D}} d^2x s(\omega). \quad (4)$$

Let us define $A(\sigma)$ the area of $\mathcal{D}$ with vorticity values lower than σ, and $\gamma(\sigma)$ the vorticity distribution

$$\gamma(\sigma) = \frac{1}{|\mathcal{D}|} \frac{dA}{d\sigma} \quad \text{with } A(\sigma) = \int_{\mathcal{D}} d^2x \, \chi_{\{\omega(\mathbf{x}) \leq \sigma\}}, \quad (5)$$

where χ_B is the characteristic function of the set $B \subset D$, and $|\mathcal{D}|$ is the area of $\mathcal{D}$. As Euler's Eq. (1) is a transport equation by an incompressible flow, $\gamma(\sigma)$ (or equivalently $A(\sigma)$) is conserved by the dynamics. Conservation of distribution $\gamma(\sigma)$ and of all Casimir's functionals (4) is equivalent.

2.1. *RSM microcanonical equilibria (MVP)*

We present the classical derivation [2] of the microcanonical variational problem which describes RSM equilibria. Such equilibria describe the most probable mixing of the vorticity ω, constrained by the vorticity distribution (5) and energy (3) (other conservation laws could be considered, for instance if the domain D has symmetries).

We make a probabilistic description of the flow. We define $\rho(\sigma, \mathbf{x})$ the local probability that the microscopic vorticity ω takes a value $\omega(\mathbf{x}) = \sigma$ at position $\mathbf{x}$. As ρ is a local probability, it satisfies a local normalization

$$N[\rho](\mathbf{x}) \equiv \int_{-\infty}^{+\infty} d\sigma \rho(\sigma, \mathbf{x}) = 1. \quad (6)$$

The known vorticity distribution (5) imposes

$$D[\rho](\sigma) \equiv \int_{\mathcal{D}} d\mathbf{x} \rho(\sigma, \mathbf{x}) = \gamma(\sigma). \quad (7)$$

We are interested on a locally averaged, coarse-grained description of the flow. The averaged vorticity is

$$\overline{\omega}(\mathbf{x}) = \int_{-\infty}^{+\infty} d\sigma \sigma \rho(\sigma, \mathbf{x}). \quad (8)$$

$\overline{\psi} = \Delta\bar{\omega}$ is the averaged stream function. The energy may be expressed in terms of the averaged vorticity distribution as

$$E[\overline{\omega}] \equiv -\frac{1}{2} \int_{\mathcal{D}} \overline{\psi}\overline{\omega} d\mathbf{x} \simeq E_0. \quad (9)$$

The entropy is a measure of the number of microscopic vorticity fields which are compatible with a distribution ρ. By classical arguments, such a measure is given by the entropy

$$\mathcal{S}[\rho] \equiv -\int_{\mathcal{D}} d^2x \int_{-\infty}^{+\infty} d\sigma \rho \log \rho. \quad (10)$$

The most probable mixing for the potential vorticity is thus given by the probability ρ_{eq} which maximizes the entropy (10), subject to the three constraints (6), (7) and (9). The equilibrium entropy $S(E_0, \gamma)$, the value of the constrained entropy maxima, is then given by the microcanonical variational problem (MVP) (see the introduction).

Using the Lagrange multipliers rule, there exists β and $\alpha(\sigma)$ (the Lagrange parameters associated to the energy and vorticity distribution, respectively) such that the critical points of (MVP) satisfy

$$\rho_{eq}(\mathbf{x}, \sigma) = \frac{1}{z_\alpha(\beta\psi_{eq})} \exp\left[\sigma\beta\psi_{eq} - \alpha(\sigma)\right], \quad (11)$$

where

$$z_\alpha(u) = \int_{-\infty}^{+\infty} d\sigma \exp[\sigma u - \alpha(\sigma)] \quad \text{and}$$
$$f_\alpha(u) = \frac{d}{du} \log z_\alpha. \quad (12)$$

We note that z_α is positive, $\log z_\alpha$ is convex, and thus f_α is strictly increasing.

From (11), using (8), the equilibrium vorticity is

$$\omega_{eq} = f_\alpha(\beta\psi_{eq}) \quad \text{or equivalently} \quad g_\alpha(\omega_{eq}) = \beta\psi_{eq}, \quad (13)$$

where g_α is the inverse of f_α. The actual equilibrium ω_{eq} is the minimizer of the entropy while satisfying the constraints, between all critical points for any possible values of β and α.

We note that solutions to (13) are stationary flows.

2.2. RSM constrained grand canonical ensemble

We consider the statistical equilibrium variational problem (MVP), but we relax the vorticity distribution constraint. This constrained (or mixed) grand canonical variational problem is

$$G(E_0,\alpha)=\inf_{\{\rho|N[\rho]=1\}}\left\{\mathcal{G}_\alpha[\rho]|E\left[\overline{\omega}\right]=E_0\right\}, \tag{14}$$

with the Gibbs potential functional defined as

$$\mathcal{G}_\alpha\left[\rho\right]\equiv-S\left[\rho\right]+\int_{\mathcal{D}}\mathrm{d}^2x\int_{-\infty}^{+\infty}\mathrm{d}\sigma\alpha\left(\sigma\right)\rho\left(\mathbf{x},\sigma\right).$$

In the following section, we prove that (14) is equivalent to the constraint Casimir V.P. (CVP). Using the results of the first section, relating constrained and relaxed variational problems, we can thus conclude that minimizers of (CVP) are RSM equilibria, but the converse is wrong in general, as stated in the introduction.

3. Constrained Casimir (CVP) and grand canonical ensembles are equivalent

3.1. Equivalence

We consider a Casimir's functional (4), where s is assumed to be convex. The critical points of the constrained Casimir variational problem (CVP, see introduction) satisfy

$$\frac{\mathrm{d}s}{\mathrm{d}\omega}\left(\omega_{eq}\right)=\beta\psi_{eq}, \tag{15}$$

where β is the Lagrange's multiplier for the energy. Solutions to this equation are stationary states for the Euler equation. Moreover, with suitable assumptions for the function s, such flows are proved to be nonlinearly stable [9].

This last equation is very similar to the one satisfied by RSM equilibria (13). Indeed let us define s_α the Legendre–Fenchel transform of $\log z_\alpha$

$$s_\alpha\left(\omega\right)=\sup_u\left\{u\omega-\log z_\alpha\left(u\right)\right\}. \tag{16}$$

Then s_α is convex. Moreover, if $\log z_\alpha$ is differentiable, then direct computations lead to

$$s_\alpha\left(\omega\right)=\omega g_\alpha\left(\omega\right)-\log\left(z_\alpha\left(g_\alpha\left(\omega\right)\right)\right) \tag{17}$$

and to $\mathrm{d}s/\mathrm{d}\omega=g_\alpha$. The equilibrium relations (13) and (15) with $s=s_\alpha$, are the same ones. It been observed in the past by a number of authors (see for instance [2]).

Let us prove that (14) and (CVP) are equivalent if $s=s_\alpha$. More precisely, we assume that Lagrange's multipliers rule applies, and we prove that minimizers of both variational problems have the same ω_{eq} and that $C(E_0,s_\alpha)=G(E_0,\alpha)$.

We consider a minimizer ρ_{eq} of (14) and $\omega_{eq}=\int\mathrm{d}\sigma\sigma\rho_{eq}$. Then $E\left[\omega_{eq}\right]=E_0$ and $G(E_0,\alpha)=\mathcal{G}_\alpha\left[\rho_{eq}\right]$. A Lagrange multiplier β then exists such that ρ_{eq} satisfies Eq. (11). Direct computation gives $\rho_{eq}\log\rho_{eq}+\alpha\rho_{eq}=\exp\left(\beta\sigma\psi_{eq}-\alpha\left(\sigma\right)\right)\left[-\log z_\alpha\left(\beta\psi_{eq}\right)+\beta\sigma\psi_{eq}\right]/z_\alpha\left(\beta\psi_{eq}\right)$. Using $\omega_{eq}=\int\mathrm{d}\sigma\sigma\rho_{eq}$, (13) and (17), we obtain

$$\begin{aligned}\int_{-\infty}^{+\infty}\mathrm{d}\sigma\left(\rho_{eq}\log\rho_{eq}+\alpha\rho_{eq}\right)&=-\log z_\alpha\left(\beta\psi_{eq}\right)+\beta\psi_{eq}\omega_{eq}\\&=s_\alpha\left(\omega_{eq}\right).\end{aligned} \tag{18}$$

From the definitions of $\mathcal{G}$ and $\mathcal{C}$, we obtain $G(E_0,\alpha)=\mathcal{G}_\alpha\left[\rho_{eq}\right]=\mathcal{C}_{s_\alpha}\left[\omega_{eq}\right]$. Now, as C is an infimum, $C_{s_\alpha}\left[\omega_{eq}\right]\geq C(E_0,s_\alpha)$ and

$$G(E_0,\alpha)\geq C(E_0,s_\alpha).$$

We now prove the opposite inequality. Let $\omega_{eq,2}$ be a minimizer of (CVP) with $s=s_\alpha$. Then there exists β_2 such that (15) is satisfied with $\mathrm{d}s_\alpha/\mathrm{d}\omega=g_\alpha$. We then define $\rho_{eq,2}\equiv\exp\left[\sigma\beta_2\psi_{eq,2}-\alpha\left(\sigma\right)\right]/z_\alpha\left(\beta_2\psi_{eq,2}\right)$. Following the same computations as in (18), we conclude that $\mathcal{G}_\alpha\left[\rho_{eq,2}\right]=\mathcal{C}_{s_\alpha}\left[\omega_{eq,2}\right]=C(E_0,s_\alpha)$. Then using that G is an infimum we have $G(E_0,\alpha)\leq C(E_0,s_\alpha)$ and thus

$$G(E_0,\alpha)=C(E_0,s_\alpha).$$

Then $C_{s_\alpha}\left[\omega_{eq}\right]=C(E_0,s_\alpha)=G(E_0,\alpha)=\mathcal{G}_\alpha\left[\rho_{eq,2}\right]$. Thus ω_{eq} and $\rho_{eq,2}$ are minimizers of (CVP) and of (14) respectively. But as such minimizers are in general not unique, ω_{eq} may be different from $\omega_{eq,2}$ and β may be different from β_2.

A formal, but very instructive, alternative way to obtain equivalence between (CVP) and (14) is to note that

$$C_{s_\alpha}\left[\omega\right]=\inf_{\{\rho|N[\rho]=1\}}\left\{\mathcal{G}_\alpha\left[\rho\right]|\int_{-\infty}^{+\infty}\mathrm{d}\sigma\sigma\rho=\omega(\mathbf{x})\right\}. \tag{19}$$

We do not detail the computation. A proof of this result is easy as we minimize a convex functional with linear constraints. Then, from (14), using (19), we obtain

$$\begin{aligned}&G(E_0,\alpha)\\&=\inf_\omega\left\{\inf_{\{\rho|N[\rho]=1\}}\left\{\mathcal{G}_\alpha[\rho]|\int\mathrm{d}\sigma\sigma\rho=\omega(\mathbf{x})\right\}|E\left[\overline{\omega}\right]=E_0\right\}\\&=C(E_0,s_\alpha).\end{aligned}$$

3.2. Second variations and local stability equivalence

In the previous section, we have proved that the constrained Casimir (CVP) and mixed ensemble (14) variational problems are equivalent, for global minimization. Does this equivalence also hold for local minima ? We now prove that the reply is positive.

We say that a critical point ρ_{eq} of the constrained mixed ensemble variational problem (14) is locally stable iff the second variations $\delta^2\mathcal{J}_\alpha$, of the associated free energy $\mathcal{J}_\alpha=\mathcal{G}_\alpha+\beta E$, are positive for perturbations $\delta\rho$ that respect the linearized energy constraints $\int_{\mathcal{D}}\psi_{eq}\delta\omega=0$, where $\delta\omega=\int\mathrm{d}\sigma\sigma\delta\rho$. Similarly, the second variations $\delta^2\mathcal{D}_s$ of the free energy $\mathcal{D}_s=\mathcal{C}_s+\beta E$ define the local stability of the Casimir maximization.

By a direct computation, we have $\delta^2\mathcal{G}_\alpha\left[\delta\rho\right]=-\delta^2\mathcal{S}_\alpha\left[\delta\rho\right]=\int_{\mathcal{D}}\mathrm{d}\mathbf{x}\int\mathrm{d}\sigma\frac{1}{\rho_{eq}}\left(\delta\rho\right)^2$ and $\delta^2\mathcal{C}_{s_\alpha}\left[\delta\omega\right]=\int_{\mathcal{D}}\mathrm{d}\mathbf{x}s_\alpha''\left(\omega_{eq}\right)\left(\delta\omega\right)^2$.

We decompose any $\delta\rho$ as

$$\delta\rho = \delta\rho^{\parallel} + \delta\rho^{\perp} \quad \text{with } \delta\rho^{\parallel} = \frac{\delta\omega}{f'_\alpha}\left(\frac{-z'_\alpha + \sigma z_\alpha}{z_\alpha^2}\right)\exp\left[\sigma\beta\psi_{eq} - \alpha(\sigma)\right].$$

In this expression, the functions f'_α, z_α and z'_α are evaluated at the point $\beta\psi_{eq}$. Using the definition of f_α and of z_α (12), and the fact that $f'_\alpha = \left(-z'^2_\alpha + z_\alpha z''_\alpha\right)/z_\alpha^2$ we easily verify that the above expression is consistent with the relation $\delta\omega = \int \mathrm{d}\sigma\sigma\delta\rho$.

Moreover by lengthy but straightforward computations, we verify that $\int \mathrm{d}\sigma\delta\rho^{\parallel}\delta\rho^{\perp}/\rho_{eq} = 0$. In this sense, the decomposition $\delta\rho = \delta\rho^{\parallel} + \delta\rho^{\perp}$ distinguishes the variations of ρ that are normal to equilibrium relation (11) from the tangential ones.

From $s'_\alpha = g_\alpha$ and using that $(g_\alpha)^{-1} = f_\alpha$, we obtain $s''_\alpha = \left(f'_\alpha\right)^{-1}$. Using this relation we obtain $\int \mathrm{d}\sigma \left(\delta\rho^{\parallel}\right)^2/\rho_{eq} = s''_\alpha\left(\omega_{eq}\right)(\delta\omega)^2$. We thus conclude

$$\delta^2\mathcal{J}_\alpha[\delta\rho] = \int_{\mathcal{D}} \mathrm{d}^2x \int_{-\infty}^{+\infty} \mathrm{d}\sigma \frac{1}{\rho_{eq}}\left(\delta\rho^{\perp}\right)^2 + \delta^2\mathcal{D}_{s_\alpha}[\delta\omega]. \quad (20)$$

To the best of our knowledge, this equality has never been derived before in this context, see [21] in plasma physics (information provided by one of the referee). It may be very useful as second variations are involved in many stability discussions.

From equality (20), it is obvious that the second variations of $\mathcal{J}_\alpha$ are positive iff the second variations of $\mathcal{D}_{s_\alpha}$ are positive. If we also note that perturbations which respect the linearized energy constraint are the same for both functionals, we conclude that the local stabilities of the two variational problems are equivalent.

4. Relation between RSM equilibria and stream function functionals

In this section, we establish the relation between stream function functionals and RSM equilibria. For this we consider the constrained Casimir variational problem (CVP). However, we relax the energy constraint. We thus consider the free energy associated to CVP

$$F(\beta, s) = \inf_{\omega}\left\{\mathcal{F}_s[\omega] = \mathcal{C}_s[\omega] + \beta E[\omega]\right\}. \quad (21)$$

This is an Energy-Casimir functional [9]. As previously explained, minima of this relaxed variational problem are also minimum (CVP). It is thus also a RSM equilibria.

Let $\tilde{G}$ be the Legendre–Fenchel transform of the function s: $\tilde{G}(z) = \sup_y\{zy - s(y)\}$. $\tilde{G}$ is thus convex. Let us define $G_\beta(\psi) = \tilde{G}(\beta\psi)/\beta$. G_β is thus convex for positive β and concave for negative β. In the following, we will show that the variational problem (21) is equivalent to the SFVP

$$D\left(G_\beta\right) = \inf_{\psi}\left\{\mathcal{D}_{G_\beta}[\psi] = \int_{\mathcal{D}} \mathrm{d}^2x\left[|\nabla\psi|^2 + G_\beta(\psi)\right]\right\}.$$

More precisely in the following discussion we prove that

1. $F(\beta, s) = -\beta D(G_\beta)$.
2. If ψ_{eq} is a local minimizer of $\mathcal{D}_{G_\beta}$ then it is a local minimizer of $\mathcal{F}_s$.
3. If we assume that a global minimizer of $\mathcal{D}_{G_\beta}$ exists, then $\omega_{eq} = \Delta\psi_{eq}$ is a global minimizer of $\mathcal{F}_s$ if and only if ψ_{eq} is a global minimizer of $\mathcal{D}_{G_\beta}$.

When $\mathcal{D}_{G_\beta}[\psi]$ and $\mathcal{F}_s[\omega]$ are strictly convex, both variational problems have a single minimizer. As the equations for the critical points of the variational problems coincide, points 2. and 3. above are thus easily verified [11]. Conditions for $\mathcal{D}_{G_\beta}[\psi]$ and $\mathcal{F}_s[\omega]$ to be strictly convex are given, for instance in [11], or [9] for $\mathcal{F}_s[\omega]$. This is obvious for positive temperature $\beta > 0$, as G_β is convex in this case. For negative temperature, G_β is concave. However, if we assume that $\tilde{G}''$ is bounded $0 \le \tilde{G}''(z) \le g$, then it can be proven that $\mathcal{D}_{G_\beta}[\psi]$ is strictly convex for $\beta_c \le \beta \le 0$, with $\beta_c \le \lambda_1/g$, where λ_1 is the opposite of the first eigenvalue of the Laplacian over the domain $\mathcal{D}$ (this follows from the Poincaré inequality, see [9,11]). ($\tilde{G}''$ is actually bounded, for instance if the vorticity distribution $\gamma(\sigma)$ (5) has a compact support, or for the point vortex model). In the following we prove that results 1., 2. and 3. are valid also when $\mathcal{D}_{G_\beta}[\psi]$ and $\mathcal{F}_s[\omega]$ are no longer convex.

In order to prove these results for negative temperature $\beta < 0$, it is sufficient to prove:

(a) $\omega_c = \Delta\psi_c$ is a critical points of $\mathcal{F}_s$ if and only if ψ_c is a critical point of $\mathcal{D}_G$, and then $\mathcal{F}_s[\omega_c] = -\beta\mathcal{D}_{G_\beta}[\psi_c]$.
(b) For any $\omega = \Delta\psi$, $\mathcal{F}_s[\omega] \ge -\beta\mathcal{D}_{G_\beta}[\psi]$.

Point **(a)** has been noticed in [13], and is actually sufficient to prove points 1 and 2. The inequality **(b)** [22] proves that $\mathcal{D}_{G_\beta}$ is a support functional to $\mathcal{F}_s$ [22]. Let us prove points **(a)** and **(b)**. First, the critical points of $\mathcal{F}_s$ and $\mathcal{D}_{G_\beta}$ verify $s'(\omega_c) = \beta\psi_c$ and $\omega_c = G'(\beta\psi_c)$. Now using that G is the Legendre–Fenchel transform of s, if s is differentiable, we have $\left(s'\right)^{-1} = G'$. Thus the critical points of both functionals are the same.

Let us prove point **(b)**

$$\mathcal{F}_s[\omega] = -\int_{\mathcal{D}} \mathrm{d}^2x\left[-s(\omega) + \beta\omega\psi\right] + \int_{\mathcal{D}} \mathrm{d}^2x \frac{\beta}{2}\omega\psi \ge \int_{\mathcal{D}} \mathrm{d}^2x\left[-G(\beta\psi) + \frac{\beta}{2}\omega\psi\right] = -\beta\mathcal{D}_{G_\beta}[\psi]$$

where we have used the definition of G, as the Legendre–Fenchel transform of s, in order to prove the inequality. We now conclude the proof of point **(a)**. A direct computation gives $G(x) = x\left(s'\right)^{-1}(x) - s\left[\left(s'\right)^{-1}(x)\right]$. Thus $G(\beta\psi_c) = \beta\psi_c\omega_c - s(\omega_c)$. This proves that in the preceding inequality, an equality actually occurs for the critical points: $\mathcal{F}_s[\omega_c] = -\beta\mathcal{D}_{G_\beta}[\psi_c]$.

We have thus established the relations between RSM equilibria and the simpler Casimirs (CVP) and stream function (SFVP) variational problems.

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