ORIENTABLE HOMOTOPY MODULES

FRÉDÉRIC DÉGLISE

Abstract. We prove a conjecture of Morel identifying Voevodsky's homotopy invariant sheaves with transfers with spectra in the stable homotopy category which are concentrated in degree zero for the homotopy t-structure and have a trivial action of the Hopf map. This is done by relating these two kind of objects to Rost's cycle modules. Applications to algebraic cobordism and construction of cycle classes are given.

Contents

Introduction

In [Mor04] and [Mor06], F. Morel started a thorough analysis of the stable homotopy category of schemes over a field culminating in the computation of the zero-th stable homotopy groups of the zero sphere $\pi_0(S^0)_*$ in a joint work with M. Hopkins. The result involves a mixture of the Witt ring of the field and its Milnor K-theory. This paper is built on the idea of Morel that the Witt part contains the obstruction to orientation in stable homotopy. Indeed, let us recall that

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$\pi_0(S^0)_*$ is generated as an algebra by the units of the field k and the Hopf map[1] η. As in topology, the Hopf map η is the first obstruction for a ring spectrum to be oriented.[2]

On the other hand the units of k generate a subring of $\pi_0(S^0)_*$ which turns out to be exactly the Milnor K-theory of k, or, in modern terms, the part of motivic cohomology where the degree is equal to the twist. It is understood now, though still conjectural, that motivic cohomology is the universal oriented cohomology with additive formal group law.

To build something out of these general remarks, one has to go more deeply into motivic homotopy theory. The category $DM^{eff}(k)$ of V. Voevodsky's motivic complexes over a perfect field k is built out of the so called homotopy invariant sheaves with transfers which have the distinctive property that their cohomology is homotopy invariant.

Starting from the point that these sheaves form the heart of a natural t-structure on $DM^{eff}(k)$, Morel introduced the homotopy t-structure on the stable homotopy category $SH(k)$, analog of the previous one, and identified its heart as some graded sheaves without transfers but still with homotopy invariant cohomology. They are called *homotopy modules* by Morel (see Definition 1.2.2) and we denote their category by $\Pi_*(k)$. These sheaves have to be thought as *stable homotopy groups*: in fact, taking the zero-th homology of S^0 with respect to the homotopy t-structure gives a homotopy module $\underline{\pi}_0(S^0)_*$ whose fiber at the point k is precisely the graded abelian group $\pi_0(S^0)_*$. Note that, as a consequence, any homotopy module has a natural action of the Hopf map η seen as an element of $\underline{\pi}_0(S^0)_*(k)$.

To relate these two kinds of sheaves, it is more accurate to introduce the non effective (*i.e.* stable) version of $DM^{eff}(k)$, simply denoted by $DM(k)$. The canonical t-structure on $DM^{eff}(k)$ extends to a t-structure on $DM(k)$ whose heart $\Pi_*^{tr}(k)$ now consists of certain graded homotopy invariant sheaves with transfers which we call *homotopy modules with transfers* (see Definition 1.3.2).

Then the natural map $DM(k) \to SH(k)$ induces a natural functor

$$\gamma_* : \Pi_*^{tr}(k) \to \Pi_*(k)$$

which basically does nothing more than forgetting the transfers. In this article, we prove the following conjecture of Morel:

Theorem. (1) *The functor γ_* is fully faithful and its essential image consists of the homotopy modules on which η acts as* 0.
(2) *The functor γ_* is monoidal and homotopy modules with transfers can be described as homotopy modules with an action of the unramified Milnor K-theory sheaf.*

Actually the first part was conjectured by Morel ([Mor04, conj. 3, p. 71]) and the second one is a remark we made, deduced from the first one.

Let us now come back to the opening point of this introduction: this theorem implies that, for homotopy modules seen as objects of $SH(k)$, η is precisely the obstruction to be orientable (see Corollary 4.1.7 for the precise statement). Moreover, the Milnor part of $\underline{\pi}_0(S^0)_*$ – *i.e.* the unramified Milnor K-theory sheaf – is the universal homotopy module on which η acts as 0.

This theorem relies on our previous work on homotopy modules with transfers where we showed that they are completely determined by the system of their fibers over finitely generated extensions of k, which can be described precisely as a cycle module in the sense of M. Rost. Actually, we obtained in [Dég09, th. 3.4] an equivalence of categories between homotopy modules with transfers and cycles modules: this gives a way to construct transfers on sheaves out of a more elementary algebraic structure on their fibers. The proof of our theorem thus consists in showing that given a homotopy module with trivial action of η, taking fibers gives rise to a cycle module; this was actually the original form of the conjecture of Morel.

The construction of the cycle module structure is obtained by appealing to our work on the Gysin triangle [Dég08b] in the framework of modules over ring spectra: indeed, the main idea of

[1]See Paragraph 1.2.6.
[2]See Remark 1.2.8.

the proof is that a homotopy module with trivial action of η has a (weak) structure of a module over the motivic cohomology ring spectrum $\mathbf{H}$. After recalling the central definitions involved in the formulation of the previous theorem in Section 1, we have dedicated Section 2 to recall the main technical tools which will be involved in the proof: cycle modules, modules over ring spectra, Gysin triangles and morphisms, the coniveau spectral sequence and the computation of its differentials. This enables us, in Section 3, to give a neat proof of the first part of our theorem though it uses all the previous technical tools. In the last section, we prove point (2) above and we collect some other results illustrate our techniques. Notably, we use the coniveau spectral sequence to obtain the following results:

- (Cor. 4.3.4) Let $\mathbf{MGL}$ be Voevodsky's algebraic cobordism spectrum. Then for any smooth connected scheme of dimension d, the canonical map
$$\mathbf{MGL}^{2d,d}(X) \to CH_0(X),$$
with target the 0-cycles modulo rational equivalence, is an isomorphism.
- (Prop. 4.4.5) Let $\mathbb{E}$ be a monoid in $SH(k)$ satisfying the following assumptions:
  (a) η acts trivially on $\underline{\pi}_0(\mathbb{E})_*$.
  (b) For any negative integers n, m, any smooth connected scheme X, and any cohomology class $\rho \in \mathbb{E}^{n,m}(X)$, there exists a non empty open $U \subset X$ such that $\rho|_U = 0$.

  Then $\mathbb{E}$ admits an orientation whose associated formal group law is additive. Moreover, for any integer $n \geq 0$ and any smooth scheme X, there exists a canonical cycle class map
$$CH^n(X) \to \mathbb{E}^{2n,n}(X)$$
which satisfies all the usual properties.

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Notations and conventions

In this article, k is a perfect field. Any scheme is assumed to be a *separated k-scheme* unless stated otherwise. Such a scheme will be called a smooth scheme if it is *smooth of finite type*. We denote by Sm_k (resp. $Sm_{k,\bullet}$) the category of smooth schemes (resp. pointed smooth schemes). Given a smooth scheme X, we put $X_+ = X \sqcup \operatorname{Spec}(k)$ considered as a pointed scheme by the obvious k-point.

Following Morel, our convention for t-structures in triangulated categories is *homological* – see in particular Definition 1.1.3.

Our monoidal categories, as well as functors between them, are always assumed to be symmetric monoidal. Similarly, monoids are always assumed to be commutative.

In diagrams involving shifts or twists of morphisms, we do not indicate them in the label of the arrows – this does not lead to any confusion as they are indicated in the source and target of the arrows.

Graded (resp. bigraded) means $\mathbb{Z}$-graded (resp. $\mathbb{Z}^2$-graded).

1. The conjecture of Morel

1.1. The homotopy t-structure.

1.1.1. We will denote by $SH(k)$ the *stable homotopy category over k* of Morel and Voevodsky (see [Mor04] and [Jar00]). Objects of $SH(k)$ will be simply called *spectra.*[3] It is a triangulated category with a canonical functor:
$$Sm_{k,\bullet} \to SH(k), X \mapsto \Sigma^\infty X.$$

[3] They are called $\mathbb{P}^1$-spectrum in [Mor04] because they have to be distinguished from S^1-spectra. As we will not use S^1-spectra in this work, there will be no risk of confusion.

It has a closed monoidal structure; the tensor product denoted by $\wedge$ is characterized by the property: $\Sigma^\infty(X \times_k Y)_+ = \Sigma^\infty X_+ \wedge \Sigma^\infty Y_+$. We denote by $S^0 = \Sigma^\infty \operatorname{Spec}(k)_+$ the unit of this tensor product. By construction of $SH(k)$, the object $\Sigma^\infty \mathbb{G}_m$, where $\mathbb{G}_m$ is pointed by its unit section, is invertible for the tensor product. For any integer $n \in \mathbb{Z}$, we will denote by $S^{n,n}$ its n-th tensor power. Moreover, for any couple $(i,n) \in \mathbb{Z}^2$, we put $S^{i,n} = S^{n,n}[i-n]$.

1.1.2. Consider a spectrum $\mathbb{E}$. For any smooth k-scheme X and any couple $(i,n) \in \mathbb{Z}^2$, we put $\mathbb{E}^{i,n}(X) = \operatorname{Hom}_{SH(k)}(\Sigma^\infty X_+, S^{i,n} \wedge \mathbb{E})$. This defines a bigraded cohomology theory on Sm_k. We let $\underline{\pi}_i(\mathbb{E})_n$ be the Nisnevich sheaf of abelian groups on Sm_k associated with the presheaf:

$$X \mapsto \mathbb{E}^{n-i,n}(X).$$

For a fixed integer $i \in \mathbb{Z}$, $\underline{\pi}_i(\mathbb{E})_*$ will be considered as an abelian $\mathbb{Z}$-graded sheaf. Recall Definition 5.2.1 of [Mor04]:

Definition 1.1.3. A spectrum $\mathbb{E}$ will be said to be *non-negative* (resp. *non-positive*) if for any $i < 0$ (resp. $i > 0$) $\underline{\pi}_i(\mathbb{E})_* = 0$. We let $SH(k)_{\geq 0}$ (resp. $SH(k)_{\leq 0}$) be the full subcategory of $SH(k)$ consisting of non-negative (resp. non-positive) objects of $SH(k)$.

1.1.4. As proved in [Mor04, 5.2.3], this definition gives rise to a t-structure on the triangulated category $SH(k)$ with *homological conventions*. More precisely, the following properties are satisfied:

(1) The inclusion functor $o_+ : SH(k)_{\geq 0} \to SH(k)$ (resp. $o_- : SH(k)_{\leq 0} \to SH(k)$) admits a right adjoint t_+ (resp. left adjoint t_-). For any spectrum $\mathbb{E}$ and any integer $i \in \mathbb{Z}$, we put:

$$\begin{array}{ll} \mathbb{E}_{\geq 0} = o_+ t_+(\mathbb{E}), & \mathbb{E}_{\leq 0} = o_- t_-(\mathbb{E}), \\ \mathbb{E}_{\geq i} = (E[-i])_{\geq 0}[i], & \mathbb{E}_{\leq i} = (\mathbb{E}[-i])_{\leq 0}[i], \\ \mathbb{E}_{> i} = \mathbb{E}_{\geq i+1}, & \mathbb{E}_{< i} = (\mathbb{E})_{\leq i-1}. \end{array}$$

(2) For any spectra $\mathbb{E}, \mathbb{F}$,

$$\operatorname{Hom}_{SH(k)}(\mathbb{E}_{>0}, \mathbb{F}_{\leq 0}) = 0. \tag{1.1.4.a}$$

(3) For any object $\mathbb{E}$ in $SH(k)$, there is a unique distinguished triangle in $SH(k)$:

$$\mathbb{E}_{\geq 0} \to \mathbb{E} \to \mathbb{E}_{<0} \to \mathbb{E}_{\geq 0}[1] \tag{1.1.4.b}$$

where the first two maps are given by the adjunctions of point (1).

Remark 1.1.5. The key point for the previous results is the stable $\mathbb{A}^1$-connectivity theorem of Morel (see [Mor04, th. 4.2.10]). Recall this theorem also implies that for any smooth k-scheme X and any pair $(i,n) \in \mathbb{N} \times \mathbb{Z}$, the spectrum $S^{i+n,n} \wedge \Sigma^\infty X_+$ is non negative. Then, according to the previous definition, the spectra of this form constitute an essentially small family of generators for the localizing subcategory $SH(k)_{\geq 0}$ of $SH(k)$.

1.2. **Homotopy modules.**

1.2.1. Given an abelian Nisnevich sheaf F on Sm_k, we denote by $F_{-1}(X)$ the kernel of the morphism $F(X \times_k \mathbb{G}_m) \to F(X)$ induced by the unit section of $\mathbb{G}_m$. Following the terminology of Morel, we will say that F is *strictly homotopy invariant* if the Nisnevich cohomology presheaf $H^*_{\mathrm{Nis}}(-,F)$ is homotopy invariant.

Definition 1.2.2 (Morel)**.** A *homotopy module* is a pair (F_*, ϵ_*) where F_* is a $\mathbb{Z}$-graded abelian Nisnevich sheaf on Sm_k which is strictly homotopy invariant and for which $\epsilon_n : F_n \to (F_{n+1})_{-1}$ is an isomorphism. A morphism of homotopy modules is a homogeneous natural transformation of $\mathbb{Z}$-graded sheaves compatible with the given isomorphisms.

We denote by $\Pi_*(k)$ the category of homotopy modules.

1.2.3. For any spectrum $\mathbb{E}$, $\underline{\pi}_0(\mathbb{E})_*$ has a canonical structure of a homotopy module. Moreover, the functor $\mathbb{E} \mapsto \underline{\pi}_0(\mathbb{E})_*$ induces an equivalence of categories between the heart of $SH(k)$ for the homotopy t-structure and the category $\Pi_*(k)$ (see [Mor04, 5.2.6]). As in *loc. cit.* we will denote its quasi-inverse by:

$$H : \Pi_*(k) \to \underline{\pi}_0(SH(k)). \tag{1.2.3.a}$$

Note this result implies that $\Pi_*(k)$ is a Grothendieck abelian category with generators of the form

(1.2.3.b) $$\underline{\pi}_0(X)_*\{n\} = \underline{\pi}_0(S^{n,n} \wedge \Sigma^\infty X_+)_*.$$

for a smooth k-scheme and an integer $n \in \mathbb{Z}$. It admits a monoidal structure defined by

(1.2.3.c) $$F_* \otimes G_* := \underline{\pi}_0(H(F_*) \wedge H(G_*))_* \simeq (H(F_*) \wedge H(G_*))_{\geq 0}.$$

The isomorphism follows from the fact that the tensor product on $SH(k)$ preserves non negative spectra (according to Remark 1.1.5).

Note that $\underline{\pi}_0(S^0)_*$ is the unit object for this monoidal structure. Given a homotopy module $F_* = (F_*, \epsilon_*)$ and an integer $n \in \mathbb{Z}$, we will denote by $F_*\{n\}$ the homotopy module whose i-th graded term is $(F_{i+n}, \epsilon_{i+n})$. Then:

(1.2.3.d) $$F_*\{n\} = \underline{\pi}_0(\mathbb{G}_m^{\wedge,n})_* \otimes F_*$$

so that this notation agrees with that of (1.2.3.b).

Remark 1.2.4. Let F_* be a homotopy module. Then according to the construction of the spectrum $\mathbb{F} := H(F_*)$, for any couple $(i, n) \in \mathbb{Z}^2$, we get an isomorphism:

(1.2.4.a) $$\mathbb{F}^{i,n}(X) \simeq H^{i-n}_{\mathrm{Nis}}(X, F_n),$$

natural in X.

Example 1.2.5. Let $\mathbf{H}$ be the spectrum representing motivic cohomology.[4] The homotopy module $\underline{\pi}_0(\mathbf{H})_*$ is the sheaf of unramified Milnor K-theory $\underline{K}^M_*$ on Sm_k.[5]

1.2.6. Define the *Hopf map* in $SH(k)$ as the morphism $\eta : \mathbb{G}_m \to S^0$ obtained by applying the tensor product with $S^{-2,-1}$ to the map induced by

$$\mathbb{A}^2_k - \{0\} \to \mathbb{P}^1_k, (x, y) \mapsto [x, y].$$

According to [Mor04, 6.2.4],[6] there exists a canonical exact sequence in $\Pi_*(k)$ of the form:

(1.2.6.a) $$\underline{\pi}_0(\mathbb{G}_m)_* \xrightarrow{\eta_*} \underline{\pi}_0(S^0)_* \to \underline{K}^M_* \to 0.$$

Definition 1.2.7. A homotopy module F_* is said to be *orientable* if the map induced by η:

$$1 \otimes \eta_* : F_*\{1\} \to F_*$$

is zero. We will denote by $\Pi^{\eta=0}_*(k)$ the full subcategory of $\Pi_*(k)$ consisting of the orientable homotopy modules.

Note that η is in fact an element of $\underline{\pi}_0(S^0)_{-1}(k)$. Given a homotopy module F_*, a smooth scheme X and an integer $n \geq 0$, $H^n_{\mathrm{Nis}}(X, F_*)$ has a canonical structure of a graded $\underline{\pi}_0(S^0)_*(k)$-module. Then the following conditions are equivalent:

(i) F_* is orientable.
(ii) For any smooth scheme X and any integer $n \geq 0$, the action of η on $H^n_{\mathrm{Nis}}(X, F_*)$ is zero.

Remark 1.2.8. (1) Consider a linear embedding $i : \mathbb{P}^1 \to \mathbb{P}^2$. Then from [Mor04, 6.2.1], the sequence

$$\Sigma^\infty(\mathbb{A}^2 - \{0\}) \xrightarrow{S^{2,1} \wedge \eta} \Sigma^\infty \mathbb{P}^1 \xrightarrow{i_*} \Sigma^\infty \mathbb{P}^2$$

is homotopy exact in $SH(k)$. In particular, one sees that a homotopy module F_* is orientable if $i^* : H^*_{\mathrm{Nis}}(\mathbb{P}^2, F_*) \to H^*_{\mathrm{Nis}}(\mathbb{P}^1, F_*)$ is split – actually the reciprocal statement holds (see Corollary 4.1.7).

(2) Recall that modulo 2-torsion, if -1 is a sum of squares in k, any homotopy module is orientable (see [Mor04, 6.3.5]).

[4]See [Voe98] or example 2.2.6.

[5]This follows from two arguments: the identification of the unstable motivic cohomology of bidegree (n, n) of a field E with the n-th Milnor ring $K^M_n(E)$; the cancellation theorem of Voevodsky to identify unstable motivic cohomology with the stable one.

[6]See [Mor06] for a proof (stated as Corollary 21 in the introduction, which can be deduced from the results of section 2.3).

1.3. **Homotopy modules with transfers.**

1.3.1. Recall Voevodsky has introduced the category Sm_k^{cor} whose objects are the smooth schemes and morphisms are the *finite correspondences*. Taking the graph of a morphism of smooth schemes induces a functor $\gamma : Sm_k \to Sm_k^{cor}$. A Nisnevich sheaf with transfers is an abelian presheaf F on Sm_k^{cor} such that $F \circ \gamma$ is a Nisnevich sheaf. Note that the construction F_{-1} of Paragraph 1.2.1 applied to a sheaf with transfers F gives in fact a sheaf with transfers, still denoted by F_{-1}. In [Dég09, 1.15], we have introduced the following definition:

Definition 1.3.2. A *homotopy module with transfers* is a pair (F_*, ϵ_*) where F_* is a $\mathbb{Z}$-graded abelian Nisnevich sheaf with transfers which is strictly homotopy invariant and $\epsilon_n : F_n \to (F_{n+1})_{-1}$ is an isomorphism of sheaves with transfers. A morphism of homotopy modules with transfers is a homogeneous natural transformation of $\mathbb{Z}$-graded sheaves with transfers compatible with the given isomorphisms.

We denote by $\Pi_*^{tr}(k)$ the category of homotopy modules with transfers.

1.3.3. Let (F_*, ϵ_*) be a homotopy module with transfers. Obviously, the functor $\gamma_*(F_*) = F_* \circ \gamma$ together with the natural isomorphism $\epsilon_*.\gamma$ is a homotopy module.
Recall from [Dég09, 1.3] that one can attach to F_* a cohomological functor:

$$\varphi : DM_{gm}(k)^{op} \to \mathscr{A}b$$

from Voevodsky's category of geometric motives to the category of abelian groups such that:

$$\varphi(M(X)(n)[n+i]) = H^i_{\mathrm{Nis}}(X, F_{-n}).$$

According to Remark 1.2.8, the projective bundle theorem in $DM_{gm}(k)$ implies that $\gamma_*(F_*)$ is orientable. Thus we have obtained a canonical functor:

$$\gamma'_* : \Pi_*^{tr}(k) \to \Pi_*^{\eta=0}(k). \tag{1.3.3.a}$$

One can see that γ'_* is faithful. The main point of this note is the following theorem:

Theorem 1.3.4. *The functor γ'_* introduced above is an equivalence of categories.*

1.3.5. This theorem is an equivalent form of the conjecture [Mor04, conj. 3, p. 160] of Morel. Indeed, recall the main theorem of [Dég09] establishes an equivalence of categories between the category of homotopy modules with transfers and the category $\mathscr{M}Cycl(k)$ of cycle modules defined by M. Rost in [Ros96]. Indeed, from [Dég09, 3.3], we get quasi-inverse equivalences of categories:

$$\rho : \Pi_*^{tr}(k) \leftrightarrows \mathscr{M}Cycl(k) : A^0. \tag{1.3.5.a}$$

In fact, we will prove the previous theorem by proving the following equivalent form:

Theorem 1.3.6. *There exists a canonical functor*

$$\tilde{\rho} : \Pi_*^{\eta=0}(k) \to \mathscr{M}Cycl(k)$$

and a natural isomorphism of endofunctors of $\Pi_*^{\eta=0}(k)$:

$$\epsilon : 1 \to (\gamma'_* \circ A^0 \circ \tilde{\rho}).$$

The functor $\tilde{\rho}$ and the natural isomorphism ϵ will be constructed respectively in Corollary 3.3.6 – point (4) – and in Section 3.4.

2. Preparations

2.1. **The theory of cycle modules.** We briefly recall the theory of cycle modules by M. Rost (see [Ros96]) in a way that will facilitate the proof of our main result.

Cycle premodules.

2.1.1. A *function field* will be an extension field E of k having finite transcendental degree. A *valued function field* (E, v) will be a function field E together with a discrete valuation $v : E^\times \to \mathbb{Z}$ whose ring of integers $\mathcal{O}_v$ is essentially of finite type over k. In this latter situation, we will denote by $\kappa(v)$ the residue field of $\mathcal{O}_v$.

Given a function field E, we will denote by $K^M_*(E)$ the Milnor K-theory of E. Expanding a remark of Rost ([Ros96, (1.10)]), we introduce the additive category $\tilde{\mathscr{E}}_k$ as follows: its objects are the pairs (E, n) where E is a function field and $n \in \mathbb{Z}$ an integer; the morphisms are defined by the following generators and relations:

Generators :

- **(D1)** $\varphi_* : (E, n) \to (L, n)$ for a field extension $\varphi : E \to L$, $n \in \mathbb{Z}$.
- **(D2)** $\varphi^* : (L, n) \to (E, n)$ for a finite field extension $\varphi : E \to L$, $n \in \mathbb{Z}$.
- **(D3)** $\gamma_x : (E, n) \to (E, n + r)$, for any $x \in K^M_r(E)$, $n \in \mathbb{Z}$.
- **(D4)** $\partial_v : (E, n) \to (\kappa(v), n - 1)$, for any valued function field (E, v), $n \in \mathbb{Z}$.

Relations :

- **(R0)** For all $x \in K^M_*(E)$, $y \in K^M_*(E)$, $\gamma_x \circ \gamma_y = \gamma_{x.y}$.
- **(R1a)** $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.
- **(R1b)** $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.
- **(R1c)** Consider finite field extensions $\varphi : K \to E$, $\psi : K \to L$ and put $R = E \otimes_K L$. For any point $z \in \mathrm{Spec}(R)$, let $\bar{\varphi}_z : L \to R/z$ and $\bar{\psi}_z : E \to R/z$ be the induced morphisms. Under these notations, the relation (R1c) is:
  $\psi_*\varphi^* = \sum_{z \in \mathrm{Spec}(R)} \lg\big(R_z\big).(\bar{\varphi}_z)^*(\bar{\psi}_z)_*$,
  where lg stands for the length of an Artinian ring.
- **(R2a)** For a field extension $\varphi : E \to L$, $x \in K^M_*(E)$, $\varphi_* \circ \gamma_x = \gamma_{\varphi_*(x)} \circ \varphi_*$.
- **(R2b)** For a finite field extension $\varphi : E \to L$, $x \in K^M_*(E)$, $\varphi^* \circ \gamma_{\varphi_*(x)} = \gamma_x \circ \varphi^*$.
- **(R2c)** For a finite field extension $\varphi : E \to L$, $y \in K^M_*(L)$, $\varphi^* \circ \gamma_y \circ \varphi_* = \gamma_{\varphi^*(y)}$.
- **(R3a)** Let $\varphi : E \to L$ be a morphism, where (L, v) and (E, w) are valued function fields such that $v|_{E^\times} = e.w$ for a positive integer e. Let $\bar{\varphi} : \kappa(w) \to \kappa(v)$ be the induced morphism. Under these notations, relation (R3a) is: $\partial_v \circ \varphi_* = e.\bar{\varphi}_* \circ \partial_w$.
- **(R3b)** Let $\varphi : E \to L$ be a finite field extension where (E, v) is a valued function field. For any valuation w on L, we let $\bar{\varphi}_w : \kappa(v) \to \kappa(w)$ be the induced morphism. Then relation (R3b) is: $\partial_v \circ \varphi^* = \sum_{w/v} \bar{\varphi}^*_w \circ \partial_w$.
- **(R3c)** For any $\varphi : E \to L$, any valuation v on L, trivial on $E^\times$: $\partial_v \circ \varphi_* = 0$.
- **(R3d)** For a valued function field (E, v), a prime π of v : $\partial_v \circ \gamma_{\{-\pi\}} \circ \varphi_* = \bar{\varphi}_*$.
- **(R3e)** For a valued field (E, v) and a unit $u \in E^\times$: $\partial_v \circ \gamma_{\{u\}} = -\gamma_{\{\bar{u}\}} \circ \partial_v$.

Recall the following definition – [Ros96, (1.1)]:

Definition 2.1.2. A cycle premodule is an additive covariant functor $M : \tilde{\mathscr{E}}_k \to \mathscr{A}b$. We denote by $\mathscr{A}b^{\tilde{\mathscr{E}}_k}$ the category of such functors with natural transformations as morphisms.

Cycle modules.

2.1.3. Consider a cycle premodule M, a scheme X essentially of finite type over k, and an integer $p \in \mathbb{Z}$. According to *op. cit.*, §5, we set:

$$C^p(X; M) = \bigoplus_{x \in X^{(p)}} M(\kappa_x) \tag{2.1.3.a}$$

where $X^{(p)}$ denotes the set of codimension p points in X and κ_x denotes the residue field of a point x in X. This is a graded abelian group and we put: $C^p(X; M)_n = \bigoplus_{x \in X^p} M_{n-p}(\kappa_x)$.[7]

Consider a pair $(x, y) \in X^{(p)} \times X^{(p+1)}$. Assume that y is a specialization of x. Let Z be the reduced closure of x in X and $\tilde{Z} \xrightarrow{f} Z$ be its normalization. Each point $t \in f^{-1}(y)$ corresponds to a discrete valuation v_t on κ_x with residue field κ_t. We denote by $\varphi_t : \kappa_y \to \kappa_t$ the morphism

[7]It is denoted by $C^p(X; M, n)$ in *loc. cit.*

induced by f. Then, we define according to [Ros96, (2.1.0)] the following homogeneous morphism of graded abelian groups:

$$\partial_y^x = \sum_{t \in f^{-1}(y)} \varphi_t^* \circ \partial_{v_t} : C^p(X;M)_n \to C^{p+1}(X;M)_n \tag{2.1.3.b}$$

If y is not a specialization of x, we put: $\partial_y^x = 0$.

Definition 2.1.4. Consider the hypothesis and notations above. We introduce the following property of the cycle premodule M:

($\mathbf{FD}_X$) For any $x \in X$ and any $\rho \in M(\kappa_x)$, $\partial_y^x(\rho) = 0$ for all but finitely many $y \in X$.

Assume ($\mathbf{FD}_X$) is satisfied. Then for any integer p, we define according to [Ros96, (3.2)] the following morphism:

$$d_X^p = \sum_{(x,y) \in X^{(p)} \times X^{(p+1)}} \partial_y^x : C^p(X;M)_n \to C^{p+1}(X;M)_n, \tag{2.1.4.a}$$

and introduce the following further property of M:

($\mathbf{C}_X$) For any integer $p \geq 0$, $d_X^{p+1} \circ d_X^p = 0$.

2.1.5. Thus, under ($\mathbf{FD}_X$) and ($\mathbf{C}_X$), the map d_X^* is a differential on $C^*(X;M)$ and we get the complex of cycles in X with coefficients in M defined by Rost in *loc. cit.* In fact, the complex $C^*(X;M)$ is graded according to (2.1.4.a).

In the conditions of the above definition, the following properties are clear from the definition:

- Let Y be a closed subscheme (resp. open subscheme, localized scheme) of X. Then ($\mathbf{FD}_X$) implies ($\mathbf{FD}_Y$) and ($\mathbf{C}_X$) implies ($\mathbf{C}_Y$).
- Let $X = \cup_{i \in I} U_i$ be an open cover of X. Then ($\mathbf{FD}_X$) is equivalent to ($\mathbf{FD}_{U_i}$) for any $i \in I$.

As any affine algebraic k-scheme X can be embedded into $\mathbb{A}_k^n$ for n sufficiently large, we deduce easily from these facts the following lemma:

Lemma 2.1.6. *Let M be a cycle premodule. Then the following conditions are equivalent:*

(i) *M satisfies ($\mathbf{FD}_X$) for any X.*
(ii) *M satisfies ($\mathbf{FD}_{\mathbb{A}^n}$) for any integer $n \geq 0$.*

Assume these equivalent conditions are satisfied. Then, the following conditions are equivalent:

(iii) *M satisfies ($\mathbf{C}_X$) for any X.*
(iv) *M satisfies ($\mathbf{C}_{\mathbb{A}^n}$) for any integer $n \geq 0$.*

The following definition is equivalent to that of [Ros96, (2.1)] (see in particular [Ros96, (3.3)]):

Definition 2.1.7. A cycle premodule M which satisfies the conditions (i)-(iv) of the previous lemma is called a cycle module. We denote by $\mathscr{M}Cycl(k)$ the full subcategory of $\mathscr{A}b^{\tilde{\mathscr{E}}_k}$ which consists of cycle modules.

2.1.8. Given a cycle module M and a scheme X essentially of finite type over k, we denote by $A^p(X;M)$ the p-th cohomology group of the complex $C^*(X;M)$. This group is graded according to the graduation of $C^*(X;M)$.

Recall that according to one of the main constructions of Rost, $A^*(X;M)$ is contravariant with respect to morphisms of smooth schemes. Moreover, according to [Dég08c, Prop. 6.9], the presheaf

$$X \mapsto A^0(X;M)$$

has a canonical structure of a homotopy module with transfers (Definition 1.3.2) which actually defines the functor A^0 of (1.3.5.a).

2.2. Modules and ring spectra.

2.2.1. Recall that a *ring spectrum* is a (commutative) monoid $\mathbf{R}$ of the monoidal category $SH(k)$.

A module over the ring spectrum $\mathbf{R}$ is an $\mathbf{R}$-module in $SH(k)$ in the classical sense: a spectrum $\mathbb{E}$ equipped with a multiplication map $\gamma_{\mathbb{E}} : \mathbf{R} \wedge \mathbb{E} \to \mathbb{E}$ satisfying the usual identities – see [ML98].[8]

Given two $\mathbf{R}$-modules $\mathbb{E}$ and $\mathbb{F}$, a morphism of $\mathbf{R}$-modules is a morphism $f : \mathbb{E} \to \mathbb{F}$ in $SH(k)$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathbf{R} \wedge \mathbb{E} & \xrightarrow{1_{\mathbf{R}} \wedge f} & \mathbf{R} \wedge \mathbb{F} \\ {\scriptstyle \gamma_{\mathbb{E}}} \downarrow & & \downarrow {\scriptstyle \gamma_{\mathbb{F}}} \\ \mathbb{E} & \xrightarrow{\quad f \quad} & \mathbb{F}. \end{array}$$

We will denote by $\mathbf{R}-mod^w$ the additive category of $\mathbf{R}$-modules.

Given any spectrum $\mathbb{E}$, $\mathbf{R} \wedge \mathbb{E}$ has an obvious structure of an $\mathbf{R}$-module. The assignment $L^w_{\mathbf{R}} : \mathbb{E} \mapsto \mathbf{R} \wedge \mathbb{E}$ defines a functor left adjoint to the inclusion functor $\mathcal{O}^w_{\mathbf{R}}$ and we get an adjunction of categories:

(2.2.1.a) $$L^w_{\mathbf{R}} : SH(k) \leftrightarrows \mathbf{R}-mod^w : \mathcal{O}^w_{\mathbf{R}}.$$

Remark 2.2.2. The category $\mathbf{R}-mod^w$ is not well behaved. For example, it is not possible in general to define a tensor product or a triangulated structure on $\mathbf{R}-mod^w$. This motivates the definitions which follow.

2.2.3. Recall also from [Jar00] that $SH(k)$ is the homotopy category of a monoidal model category: the category of symmetric spectra denoted by $Sp^{\Sigma}(k)$. A *strict ring spectrum* $\mathbf{R}$ is a commutative monoid object in the category $Sp^{\Sigma}(k)$. Given such an object we can form the category $\mathbf{R}-mod^{Sp}$ of $\mathbf{R}$-modules with respect to the monoidal category $Sp^{\Sigma}(k)$ and consider the natural adjunction

(2.2.3.a) $$Sp^{\Sigma}(k) \leftrightarrows \mathbf{R}-mod^{Sp}$$

where the right adjoint is the forgetful functor. An object of $\mathbf{R}-mod^{Sp}$ will be called a *strict module over* $\mathbf{R}$.

The monoidal model category $Sp^{\Sigma}(k)$ satisfies the monoid axiom of Schwede and Shipley: this implies that the category of strict $\mathbf{R}$-modules admits a (symmetric) monoidal model structure such that the right adjoint of (2.2.3.a) preserves and detects fibrations and weak equivalences (*cf.* [SS00, 4.1]).

Definition 2.2.4. We denote by $\mathbf{R}-mod$ the homotopy category associated with the model category $\mathbf{R}-mod^{Sp}$ described above.

2.2.5. Note that $\mathbf{R}-mod$ is a monoidal triangulated category. We denote by $\otimes_{\mathbf{R}}$ its tensor product. The Quillen adjunction (2.2.3.a) induces a canonical adjunction:

(2.2.5.a) $$L_{\mathbf{R}} : SH(k) \leftrightarrows \mathbf{R}-mod : \mathcal{O}_{\mathbf{R}}.$$

By construction, the functor $L_{\mathbf{R}}$ is triangulated and monoidal. Given a smooth scheme X and a couple $(i,n) \in \mathbb{Z}^2$, we will put:

(2.2.5.b) $$\underline{\mathbf{R}}(X)(n)[i] := L_{\mathbf{R}}(S^{i,n} \wedge \Sigma^\infty X_+).$$

The $\mathbf{R}$-modules of the above form are compact and constitute a family of generators for the triangulated category $\mathbf{R}-mod$.

The functor $\mathcal{O}_{\mathbf{R}}$ is triangulated and conservative. Because it is the right adjoint of a monoidal functor, it is weakly monoidal. In other words, for any strict $\mathbf{R}$-modules M and N, we get a canonical map in $SH(k)$:

$$\mathcal{O}_{\mathbf{R}}(M \otimes_{\mathbf{R}} N) \to \mathcal{O}_{\mathbf{R}}(M) \wedge \mathcal{O}_{\mathbf{R}}(N).$$

This implies that $\mathcal{O}_{\mathbf{R}}$ maps into the subcategory $\mathbf{R}-mod^w$ giving a functor

(2.2.5.c) $$\mathcal{O}'_{\mathbf{R}} : \mathbf{R}-mod \to \mathbf{R}-mod^w.$$

[8] As $\mathbf{R}$ is commutative, we will not distinguish the left and right $\mathbf{R}$-modules.

For any spectrum $\mathbb{E}$, $\mathscr{O}_{\mathbf{R}} L_{\mathbf{R}}(\mathbb{E}) = \mathbf{R} \wedge \mathbb{E}$. Thus the following diagram commutes:

(2.2.5.d)
$$\begin{array}{ccc} & SH(k) & \\ {}^{L_{\mathbf{R}}}\swarrow & & \searrow^{L^w_{\mathbf{R}}} \\ \mathbf{R}{-}mod & \xrightarrow{\mathscr{O}'_{\mathbf{R}}} & \mathbf{R}{-}mod^w. \end{array}$$

Example 2.2.6. The spectrum $\mathbf{H}$ representing motivic cohomology has a canonical structure of a strict ring spectrum. This follows from the adjunction of triangulated categories

(2.2.6.a) $$\gamma^* : SH(k) \leftrightarrows DM(k) : \gamma_*$$

where $DM(k)$ is the category of stable motivic complexes over k. In fact, by the very definition,

$$\mathbf{H} = \gamma_*(\mathbb{1})$$

where $\mathbb{1}$ is the unit object for the monoidal structure on $DM(k)$. To prove that $\mathbf{H}$ is a strict ring spectrum, the argument is that γ^* is induced by a monoidal left Quillen functor between the underlying monoidal model categories (see [CD09] for details).

Note also that any object of $DM(k)$ admits a canonical structure of a strict $\mathbf{H}$-module. In fact, the previous adjunction induces a canonical adjunction

(2.2.6.b) $$\tilde{\gamma}^* : \mathbf{H}{-}mod \leftrightarrows DM(k) : \tilde{\gamma}_*$$

(see again [CD09]).

2.3. **On the Gysin triangle and morphism.**

2.3.1. The proof of Theorem 3.2.2 requires some of the results of [Dég08b]. We recall them to the reader to make the proof more intelligible.

Recall that for any smooth scheme X, we have[9] an isomorphism of abelian groups:

(2.3.1.a) $$c_1 : \mathrm{Pic}(X) \to \mathbf{H}^{2,1}(X).$$

Considering the left adjoint $L_{\mathbf{R}}$ of (2.2.5.a), we get a monoidal functor:

(2.3.1.b) $$SH(k) \to \mathbf{H}{-}mod$$

and this functor shows the category $\mathbf{H}{-}mod$ satisfies the axioms of Paragraph 2.1 in *op. cit.* – see Example 2.12 of *op. cit.*

Remark 2.3.2. Note that, because the morphism c_1 above is a morphism of groups, the formal group law attached to the category $\mathbf{H}{-}mod$ in [Dég08b, 3.7] is just the additive formal group law $F(x, y) = x + y$. In particular, the power series in the indeterminate x

$$[2]_F = F(x, x), [3]_F = F(x, F(x, x)), \ldots$$

are simply given by the formula: $[r]_F = r.x$ for any integer r.

2.3.3. *The Gysin triangle*: For any smooth schemes X, Z and any closed immersion $i : Z \to X$ of pure codimension c, with complementary open immersion j, there exists a canonical distinguished triangle (see [Dég08b, def. 4.6]):

(2.3.3.a) $$\underline{\mathbf{H}}(Z - X) \xrightarrow{j_*} \underline{\mathbf{H}}(X) \xrightarrow{i^*} \underline{\mathbf{H}}(Z)(c)[2c] \xrightarrow{\partial_{X,Z}} \underline{\mathbf{H}}(X - Z)[1].$$

The map i^* (resp. $\partial_{X,Z}$) is called the Gysin (resp. residue) morphism associated with i. Consider moreover a commutative square of smooth schemes

(2.3.3.b)
$$\begin{array}{ccc} T & \xrightarrow{k} & Y \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{i} & X \end{array}$$

such that i and k are closed immersions of pure codimension c and T is the reduced scheme associated with $Z \times_X Y$. Let $h : (Y - T) \to (X - Z)$ be the morphism induced by p. Then the following formulas hold:

[9] Note that, as in example 1.2.5, one uses the cancellation theorem of Voevodsky to get this isomorphism.

(G1a) Assume $T = Z \times_X Y$. Then, according to [Dég08b, prop. 4.10], the following diagram is commutative:

$$\begin{CD}
\underline{\mathbf{H}}(Y) @>{k^*}>> \underline{\mathbf{H}}(T)(c)[2c] @>{\partial_{Y,T}}>> \underline{\mathbf{H}}(Y-T)[1] \\
@V{p_*}VV @V{q_*}VV @VV{h_*}V \\
\underline{\mathbf{H}}(X) @>{i^*}>> \underline{\mathbf{H}}(Z)(c)[2c] @>{\partial_{X,Z}}>> \underline{\mathbf{H}}(X-Z)[1].
\end{CD}$$

(G1b) Assume $Z \times_X Y$ is irreducible and let e be its geometric multiplicity. Then according to [Dég08b, 4.26] combined with Remark 2.3.2, the following diagram is commutative:

$$\begin{CD}
\underline{\mathbf{H}}(Y) @>{k^*}>> \underline{\mathbf{H}}(T)(c)[2c] @>{\partial_{Y,T}}>> \underline{\mathbf{H}}(Y-T)[1] \\
@V{p_*}VV @V{e.q_*}VV @VV{h_*}V \\
\underline{\mathbf{H}}(X) @>{i^*}>> \underline{\mathbf{H}}(Z)(c)[2c] @>{\partial_{X,Z}}>> \underline{\mathbf{H}}(X-Z)[1].
\end{CD}$$

Remark 2.3.4. The Gysin morphism i^* in the triangle (2.3.3.a) induces a pushforward in motivic cohomology:

$$i_* : \mathbf{H}^{n,m}(Z) \to \mathbf{H}^{n+2c,m+c}(X).$$

Considering the unit element of the graded algebra $\mathbf{H}^{*,*}(Z)$, we define the fundamental class associated with i as the element $\eta_X(Z) := i_*(1) \in \mathbf{H}^{2c,c}(X)$.

Assume $c = 1$. Then Z defines an effective Cartier divisor of X which corresponds uniquely to a line bundle L over X together with a section $s : X \to L$. Moreover, s is transversal to the zero section s_0 and the following square is cartesian:

$$\begin{CD}
Z @>{i}>> X \\
@V{i}VV @VV{s}V \\
X @>{s_0}>> L
\end{CD}$$

Then according to [Dég08b, 4.21], $\eta_X(Z) = c_1(L)$ with the notation of (2.3.1.a).[10]

2.3.5. Consider a smooth scheme X such that $X = X_1 \sqcup X_2$ and let $x_1 : X_1 \to X$ be the canonical open and closed immersion. By additivity, $\underline{\mathbf{H}}(X) = \underline{\mathbf{H}}(X_1) \oplus \underline{\mathbf{H}}(X_2)$. Moreover, x_{1*} is the obvious split inclusion which corresponds to this decomposition. Correspondingly, it follows from (G1a) that x_1^* is the obvious split epimorphism. One can complement the results of [Dég08b] by the following lemma which describes the additivity properties of the Gysin triangle:

Lemma 2.3.6. *Consider smooth schemes X, Z and a closed immersion $\nu : Z \to X$ of pure codimension n.*

Consider the canonical decompositions $Z = \sqcup_{i \in I} Z_i$ and $X = \sqcup_{j \in J} X_j$ into connected components and put $\hat{Z}_j = Z \times_X X_j$. We also consider the obvious inclusions:

$$z_i : Z_i \to Z, x_j : X_j \to X, u_j : (X_j - \hat{Z}_j) \to (X - Z)$$

and the morphisms ν_{ji}, ∂_{ij} uniquely defined by the following commutative diagram:

$$\begin{CD}
\underline{\mathbf{H}}(X) @>{\nu^*}>> \underline{\mathbf{H}}(Z)(n)[2n] @>{\partial_{X,Z}}>> \underline{\mathbf{H}}(X-Z)[1] \\
@A{\sum_j x_{j*}}A{\sim}A @A{\sim}A{\sum_i z_{i*}}A @A{\sim}A{\sum_j u_{j*}}A \\
\oplus_{j \in J} \underline{\mathbf{H}}(X_j) @>>{(\nu_{ji})_{j \in J, i \in I}}> \oplus_{i \in I} \underline{\mathbf{H}}(Z_i)(n)[2n] @>>{(\partial_{ij})_{i \in I, j \in J}}> \oplus_{j \in J} \underline{\mathbf{H}}(X_j - \hat{Z}_j)[1].
\end{CD}$$

Then, for any pair $(i,j) \in I \times J$:

1. *if $Z_i \subset X_j$, we let $\nu_i^j : Z_i \to X_j$ be the obvious inclusion and we put: $Z'_i = \hat{Z}_j - Z_i$. Then $\nu_{ji} = (\nu_i^j)^*$ and $\partial_{ij} = \partial_{X_j - Z'_i, Z_i}$.*

[10] One can prove more generally that, through the isomorphism $\mathbf{H}^{2c,c}(X) \simeq CH^c(X)$ where the left hand side denotes the Chow group of codimension c cycles, i_* agrees with the usual pushout on Chow groups (see [Dég09, Prop. 3.11]). Thus $\eta_X(Z)$ simply corresponds to the cycle defined by Z in X.

(2) *Otherwise,* $\nu_{ji} = 0$ *and* $\partial_{ij} = 0$.

Proof. According to the preamble 2.3.5, we get: $\nu_{ji} = z_i^* \nu^* x_{j*}$, $\partial_{i,j} = u_j^* \partial_{X,Z} z_{i*}$.

In cases (1) and (2), we consider respectively the following cartesian squares:

(1) If $Z_i \subset X_j$,

$$\begin{array}{ccccccc}
Z_i & \xrightarrow{\nu_i^j} & X_j & \xleftarrow{\hat{z}_j} & \hat{Z}_j & \longleftarrow & Z_i \\
\| & \Delta_1 & \downarrow x_j & \Delta_2 & \downarrow \nu_i^j & \Delta_3 & \| \\
Z_i & \xrightarrow[\nu_i]{} & X & \xleftarrow[\nu]{} & Z & \xleftarrow[z_i]{} & Z_i
\end{array}$$

(2) otherwise,

$$\begin{array}{ccccccc}
\emptyset & \longrightarrow & X_j & \xleftarrow{\hat{z}_j} & \hat{Z}_j & \longleftarrow & \emptyset \\
\downarrow & \Delta_1 & \downarrow x_j & \Delta_2 & \downarrow \nu_i^j & \Delta_3 & \downarrow \\
Z_i & \xrightarrow[\nu_i]{} & X & \xleftarrow[\nu]{} & Z & \xleftarrow[z_i]{} & Z_i
\end{array}$$

Then the result follows from the following computations:

$$z_i^* \nu^* x_{j*} \overset{(a)}{=} \nu_i^* x_{j*} \overset{(b)}{=} \begin{cases} (\nu_i^j)^* & \text{if } Z_i \subset X_j, \\ 0 & \text{otherwise.} \end{cases}$$

$$u_j^* \partial_{X,Z} z_{i*} \overset{(c)}{=} \partial_{X_j, \hat{Z}_j} \hat{z}_j^* z_{i*} \overset{(d)}{=} \begin{cases} \partial_{X_j, \hat{Z}_j} (z_i^j)_* \overset{(e)}{=} \partial_{X_j - Z_i', Z_i} & \text{if } Z_i \subset X_j, \\ 0 & \text{otherwise.} \end{cases}$$

We give the following justifications for each equality: (a) use (G2a) ($\nu_i = \nu \circ z_i$), (b) use (G1a) applied to Δ_1, (c) use (G2b) applied to Δ_2, (d) use (G1a) applied to Δ_3, (e) use (G1a). □

2.3.7. *The Gysin morphism*: For any smooth schemes X and Y and any projective equidimensional morphism $f : Y \to X$ of dimension n, there exists (see [Dég08b, def. 5.12]) a canonical morphism in $\mathbf{H}-mod$ of the form

$$f^* : \underline{\mathbf{H}}(X)(n)[2n] \to \underline{\mathbf{H}}(Y) \tag{2.3.7.a}$$

which coincides with the morphism of (2.3.3.a) in the case where f is a closed immersion. It satisfies the following properties:

(G2a) Whenever it makes sense, we get (see [Dég08b, prop. 5.14]): $f^* g^* = (gf)^*$.

(G2b) Consider a cartesian square of smooth schemes of form (2.3.3.b) such that i and k (resp. p and q) are closed immersions (resp. projective equidimensional morphisms). Assume i, k, f, g have constant relative codimension n, m, s, t respectively. Put $d = n + t = m + s$. Let $h : (Y - T) \to (X - Z)$ be the morphism induced by f. Then the following square is commutative (see [Dég08b, prop. 5.15][11]):

$$\begin{array}{ccc}
\underline{\mathbf{H}}(Z)(n)[2n] & \xrightarrow{\partial_{X,Z}} & \underline{\mathbf{H}}(X - Z)[1] \\
{\scriptstyle g^*} \downarrow & & \downarrow {\scriptstyle h^*} \\
\underline{\mathbf{H}}(T)(d)[2d] & \xrightarrow{\partial_{Y,T}} & \underline{\mathbf{H}}(Y - T)(s)[2s+1].
\end{array}$$

(G2c) For any cartesian square of smooth schemes

$$\begin{array}{ccc}
T & \xrightarrow{q} & Y \\
{\scriptstyle g} \downarrow & \Delta & \downarrow {\scriptstyle f} \\
Z & \xrightarrow{p} & X
\end{array} \tag{2.3.7.b}$$

such that p and q are projective equidimensional of the same dimension, we get (see [Dég08b, 5.17(i)]):

$$p^* f_* = g_* q^*.$$

(G2d) Consider a commutative square of smooth schemes of shape (2.3.7.b) such that:

- p, q, f and g are finite equidimensional morphisms.
- T is equal to the reduced scheme associated with $T' = Z \times_X Y$.

[11] There is a misprint in *loc. cit.*: one should read $n + t = m + s$, $d = n + t$.

Let $(T'_i)_{i\in I}$ be the connected components of T'. For any index $i \in I$, we let $q_i : T'_i \to X'$ (resp. $g_i : T'_i \to Y$) be the morphism induced by q (resp. g) and we denote by r_i the geometric multiplicity of T'_i. Then according to [Dég08b, prop. 5.22] combined with Remark 2.3.2, we get the following formula:

$$p^* f_* = \sum_{i \in I} r_i.g_{i*}q_i^*.$$

Remark 2.3.8. In the proof of Proposition 3.3.4, we will need property (G2b) when we only require that the square (2.3.3.b) is topologically cartesian (*i.e.* $T = (Y \times_X Z)_{red}$). In fact, the proof given in [Dég08b] requires only this assumption: in *op. cit.*, one proves Proposition 5.15 by reducing to Theorem 4.32 (the case where p and q are closed immersions), and the proof uses only the fact that the square is topologically cartesian (in the construction of the isomorphism denoted by $\mathfrak{p}_{(X;Y,Y')}$).

2.3.9. *Products*: Let X be a smooth scheme and $\delta : X \to X \times X$ the diagonal embedding. For any morphisms $a : \underline{\mathbf{H}}(X) \to \mathbb{E}$ and $b : \underline{\mathbf{H}}(X) \to \mathbb{F}$, we denote by $a \boxtimes_X b$ the composite morphism:

(2.3.9.a) $$\underline{\mathbf{H}}(X) \xrightarrow{\delta_*} \underline{\mathbf{H}}(X) \otimes_{\mathbf{H}} \underline{\mathbf{H}}(X) \xrightarrow{a \otimes_{\mathbf{H}} b} \mathbb{E} \otimes_{\mathbf{H}} \mathbb{F}.$$

(G3a) For any morphism $f : Y \to X$ of smooth schemes and any couple (a, b) of morphisms in $\mathbf{H}-mod$ with source $\underline{\mathbf{H}}(Y)$, we get (obviously):

$$(a \boxtimes_X b) \circ f_* = (a \circ f_*) \boxtimes_Y (b \circ f_*)$$

(G3b) For any projective morphism $f : Y \to X$ of smooth schemes and any morphism a (resp. b) in $\mathbf{H}-mod$ with source $\underline{\mathbf{H}}(X)$ (resp. $\underline{\mathbf{H}}(Y)$), we get (see [Dég08b, 5.18]):

$$[a \boxtimes_X (b \circ f_*)] \circ f^* = (a \circ f^*) \boxtimes_Y b.$$
$$[(b \circ f_*) \boxtimes_X a] \circ f^* = b \boxtimes_Y (a \circ f^*).$$

Remark 2.3.10. Consider a smooth scheme X and a vector bundle E/X of rank n. Let $P = \mathbb{P}(E)$ be the associated projective bundle with projection $p : P \to X$ and λ the canonical line bundle such that $\lambda \subset p^{-1}(E)$. Then, for any integer $r \geq 0$, the r-th power of the class $c = c_1(\lambda^\vee)$ (see (2.3.1.a)) in motivic cohomology corresponds to a morphism in $\mathbf{H}-mod$ which we denote by: $c^r : \underline{\mathbf{H}}(P) \to \underline{\mathbf{H}}(r)[2r]$. The projective bundle theorem ([Dég08b, 3.2]) says that the map

$$\sum_{0 \leq r \leq n} p_* \boxtimes_P c^r : \underline{\mathbf{H}}(P) \to \oplus_{0 \leq r \leq n} \underline{\mathbf{H}}(X)(r)[2r].$$

is an isomorphism. Thus we get a canonical map $\mathfrak{l}_n(P) : \underline{\mathbf{H}}(X)(n)[2n] \to \underline{\mathbf{H}}(P)$.

Given a finite epimorphism $f : Y \to X$ which admits a factorization as

$$Y \xrightarrow{i} P \xrightarrow{p} X$$

where i is a closed immersion and P/X is the projective bundle considered above, we recall that the Gysin morphism f^* of (2.3.7.a) is defined as the composite map:

$$\underline{\mathbf{H}}(X)(n)[2n] \xrightarrow{\mathfrak{l}_n(P)} \underline{\mathbf{H}}(P) \xrightarrow{i^*} \underline{\mathbf{H}}(Y)(n)[2n]$$

after taking the tensor product with $\mathbf{H}(-n)[-2n]$.

Assume Y (thus X) is connected and P/X has relative dimension 1. The Gysin morphism f^* induces a pushforward in motivic cohomology which we denote by $f_* : \mathbf{H}^{n,m}(Y) \to \mathbf{H}^{n,m}(X)$. According to the above description, we get a factorization of f_* as:

$$\mathbf{H}^{n,m}(Y) \xrightarrow{i_*} \mathbf{H}^{n+2,m+1}(P) \xrightarrow{p_*} \mathbf{H}^{n,m}(X).$$

where i_* corresponds to the morphism of Remark 2.3.4. To describe the second map, recall that any class $\alpha \in \mathbf{H}^{n+2,m+1}(P)$ can be written uniquely as $\alpha = p^*(\alpha_0) + p^*(\alpha_1).c_1(\lambda^\vee)$: then $p_*(\alpha) = \alpha_1$. Thus, we now deduce from this description the following *trace formula*:

(2.3.10.a) $$f_*(1) = d$$

where d is the generic degree of f. In fact, according to Remark 2.3.4 and its notations, $i_*(1) = c_1(L)$. Thus the formula follows from the equality $c_1(L) = d.c_1(\lambda^\vee)$ in $\mathrm{Pic}(P)$ modulo $\mathrm{Pic}(X)$.

2.3.11. Consider a regular invertible function $x : X \to \mathbb{G}_m$ on a smooth scheme X. According to the canonical decomposition $\underline{\mathbf{H}}(\mathbb{G}_m) = \underline{\mathbf{H}} \oplus \underline{\mathbf{H}}(1)[1]$, it induces a morphism: $x' : \underline{\mathbf{H}}(X) \to \underline{\mathbf{H}}(1)[1]$.

Using the product (2.3.9.a), we then deduce the following morphism:

$$\gamma_x = 1_{X*} \boxtimes_X x' : \underline{\mathbf{H}}(X) \to \underline{\mathbf{H}}(X)(1)[1]. \tag{2.3.11.a}$$

If $\nu_x : X \to X \times \mathbb{G}_m$ denotes the graph of x, then γ_x is also equal to the following composite map:

$$\underline{\mathbf{H}}(X) \xrightarrow{\nu_{X*}} \underline{\mathbf{H}}(X \times \mathbb{G}_m) = \underline{\mathbf{H}}(X) \otimes_{\mathbf{H}} \underline{\mathbf{H}}(\mathbb{G}_m) \to \underline{\mathbf{H}}(X)(1)[1]. \tag{2.3.11.b}$$

We will need the following properties of this particular kind of product:

Proposition 2.3.12. *Let X be a smooth scheme and $i : Z \to X$ be the immersion of a smooth divisor. Put $U = X - Z$ and let $j : U \to X$ be the canonical open immersion.*

1. *Let $x : X \to \mathbb{G}_m$ be a regular invertible function, $\bar{u}$ (resp. u) its restriction to Z (resp. U). Then the following diagram is commutative:*

$$\begin{array}{ccc} \underline{\mathbf{H}}(Z)(1)[1] & \xrightarrow{\partial_{X,Z}} & \underline{\mathbf{H}}(U) \\ \downarrow{\scriptstyle \gamma_{\bar{u}} \otimes_{\mathbf{H}} Id} & & \downarrow{\scriptstyle \gamma_u} \\ \underline{\mathbf{H}}(Z)(2)[2] & \xrightarrow{-\partial_{X,Z} \otimes_{\mathbf{H}} Id} & \underline{\mathbf{H}}(U)(1)[1] \end{array}$$

2. *Consider a regular function $\pi : X \to \mathbb{A}^1$ such that $Z = \pi^{-1}(0)$. Write again $\pi : U \to \mathbb{G}_m$ for the obvious restriction of π. Then the following diagram is commutative:*

$$\begin{array}{ccccc} \underline{\mathbf{H}}(Z)(1)[1] & \xrightarrow{\partial_{X,Z}} & \underline{\mathbf{H}}(U) & \xrightarrow{\gamma_\pi} & \underline{\mathbf{H}}(U)(1)[1] \\ & \searrow^{i_*} & & \swarrow^{j_*} & \\ & & \underline{\mathbf{H}}(Z)(1)[1] & & \end{array}$$

Proof. (1) Let ν_x, ν_u, $\nu_{\bar{u}}$ be the respective graphs of x, u and $\bar{u}$. Applying property (G1a), we get a commutative square:

$$\begin{array}{ccc} \underline{\mathbf{H}}(Z)(1)[1] & \xrightarrow{\partial_{X,Z}} & \underline{\mathbf{H}}(U) \\ \downarrow{\scriptstyle \nu_{\bar{u}*} \otimes_{\mathbf{H}} Id} & & \downarrow{\scriptstyle \nu_{u*}} \\ \underline{\mathbf{H}}(Z \times \mathbb{G}_m)(1)[1] & \xrightarrow{\partial_{X \times \mathbb{G}_m, Z \times \mathbb{G}_m}} & \underline{\mathbf{H}}(U \times \mathbb{G}_m) \end{array}$$

According to [Dég08b, 4.12], we get a commutative diagram:

$$\begin{array}{ccccc} \underline{\mathbf{H}}(Z \times \mathbb{G}_m)(1)[1] & & \xrightarrow{\partial_{X \times \mathbb{G}_m, Z \times \mathbb{G}_m}} & & \underline{\mathbf{H}}(U \times \mathbb{G}_m) \\ \| & & & & \| \\ \underline{\mathbf{H}}(Z) \otimes_{\mathbf{H}} \underline{\mathbf{H}}(\mathbb{G}_m)(1)[1] & \xrightarrow[\sim]{\epsilon} & \underline{\mathbf{H}}(Z)(1)[1] \otimes_{\mathbf{H}} \underline{\mathbf{H}}(\mathbb{G}_m) & \xrightarrow{\partial_{X,Z} \otimes Id} & \underline{\mathbf{H}}(U) \otimes_{\mathbf{H}} \underline{\mathbf{H}}(\mathbb{G}_m) \end{array}$$

where ϵ is the symmetry isomorphism for the tensor product of $\mathbf{H}-mod$. The result then follows from the fact $\epsilon = -1$.[12]

(2) For this point, we refer the reader to the proof of [Dég08c, 2.6.5]. □

2.4. **Coniveau spectral sequence.**

2.4.1. Recall from [Dég08a, sec. 3.1.1] that a *triangulated exact couple* in a triangulated category $\mathscr{T}$ consists of bigraded objects D and E of $\mathscr{T}$ and homogeneous morphisms between them

$$\begin{array}{ccc} D & \xrightarrow[\alpha]{(1,-1)} & D \\ {\scriptstyle (-1,0)}\nwarrow^{\gamma} & & \swarrow^{\beta}{\scriptstyle (0,0)} \\ & E & \end{array} \tag{2.4.1.a}$$

[12] Indeed it is well known that for any classes $x, y \in \mathbf{H}^{n,m}(X) \times \mathbf{H}^{s,t}(X)$, $xy = (-1)^{n+s} yx$.

with the bidegrees of each morphism indicated in the diagram and such that for any pair of integers (p,q):

(1) $D_{p,q+1} = D_{p,q}[-1]$,
(2) the following sequence is a distinguished triangle:

$$D_{p-1,q+1} \xrightarrow{\alpha_{p-1,q+1}} D_{p,q} \xrightarrow{\beta_{p,q}} E_{p,q} \xrightarrow{\gamma_{p,q}} D_{p-1,q} = D_{p-1,q+1}[1].$$

We can associate with such a triangulated exact couple a *differential* according to the formula: $d = \beta \circ \gamma$. According to point (2) above, $d^2 = 0$ and we have defined a complex in $\mathscr{T}$:

$$\ldots \to E_{p+1,q} \xrightarrow{d_{p,q}} E_{p,q} \to \ldots$$

Recall also from [Dég08a, def. 3.3] that, given a smooth scheme X, a *flag* of X is a decreasing sequence $(Z^p)_{p\in\mathbb{Z}}$ of closed subschemes of X such that for all integer $p \in \mathbb{Z}$, Z^p is of codimension greater than p in X if $p \geq 0$ and $Z^p = X$ otherwise. We denote by $\mathcal{D}(X)$ the set of flags of X, ordered by termwise inclusion.

Consider such a flag $(Z^p)_{p\in\mathbb{Z}}$. Put $U_p = X - Z^p$. By hypothesis, we get an open immersion $j_p : U_{p-1} \to U_p$. In the category of pointed Nisnevich sheaves of sets on Sm_k, we get an exact sequence:

$$0 \to (U_{p-1})_+ \xrightarrow{j_p} (U_p)_+ \to U_p/U_{p-1} \to 0$$

because j_p is a monomorphism (in the category of schemes). Then, for any pair of integers (p,q), we deduce from that exact sequence a distinguished triangle in $SH(k)$:

(2.4.1.b)
$$\Sigma^\infty(U_{p-1})_+[-p-q] \xrightarrow{j_{p*}} \Sigma^\infty(U_p)_+[-p-q] \to \Sigma^\infty(U_p/U_{p-1})[-p-q] \to \Sigma^\infty(U_{p-1})_+[-p-q+1],$$

which in turn gives a triangulated exact couple according to the above definition.

2.4.2. In what follows, we will not consider the preceding exact couple for only one flag. Rather, we remark that the triangle (2.4.1.b) is natural with respect to the inclusion of flags: thus we really get a projective system of triangles and then a projective system of triangulated exact couples.

Recall that a pro-object of a category $\mathcal{C}$ is a (*covariant*) functor F from a left filtering category $\mathcal{I}$ to the category $\mathcal{C}$. Usually, we will denote such a pro-object F by the intuitive notation $\text{“}\varprojlim_{i\in\mathcal{I}}\text{”}\, F_i$ and call it the *formal projective limit* of the projective system $(F_i)_{i\in\mathcal{I}}$.

For any integer $p \in \mathbb{Z}$, we introduce the following pro-objects of $SH(k)$:

(2.4.2.a)
$$F_p(X) = \text{“}\varprojlim_{Z^*\in\mathcal{D}(X)}\text{”}\ \Sigma^\infty(X - Z^p)_+$$

(2.4.2.b)
$$Gr_p(X) = \text{“}\varprojlim_{Z^*\in\mathcal{D}(X)}\text{”}\ \Sigma^\infty(X - Z^p/X - Z^{p-1})$$

Taking the formal projective limit of the triangles (2.4.1.b) where Z^* runs over $\mathcal{D}(X)$, we get a pro-distinguished triangle[13]:

(2.4.2.c)
$$F_{p-1}(X)[-p-q] \xrightarrow{\alpha_{p-1,q+1}} F_p(X)[-p-q] \xrightarrow{\beta_{p,q}} Gr_p(X)[-p-q] \xrightarrow{\gamma_{p,q}} F_{p-1}(X)[-p-q+1],$$

Definition 2.4.3. Considering the above notations, we define the *homotopy coniveau exact couple* as data for any couple of integers (p,q) of the pro-spectra:

$$D_{p,q} = F_p(X)[-p-q], \qquad E_{p,q} = Gr_p(X)[-p-q]$$

and that of the homogeneous morphisms of pro-objects α, β, γ appearing in the pro-distinguished triangle (2.4.2.c).

For short, a projective system of triangulated exact couples will be called a *pro-triangulated exact couple.*

[13] *i.e.* the formal projective limit of a projective system of distinguished triangles.

Example 2.4.4. Consider a spectrum $\mathbb{E}$. We can associate to $\mathbb{E}$ a functor with source the category of pro-spectra as follows:

$$\bar{\varphi}_{\mathbb{E}} : (\mathbb{F}_i)_{i \in \mathcal{I}} \mapsto \varinjlim_{i \in \mathcal{I}^{op}} \mathrm{Hom}_{SH(k)}(\mathbb{F}_i, \mathbb{E}).$$

Applying the functor $\bar{\varphi}_{\mathbb{E}}$ to the homotopy coniveau exact couple gives an exact couple of abelian groups in the classical sense (with the conventions of [McC01, th. 2.8]) whose associated spectral sequence is:

$$E_1^{p,q}(X, \mathbb{E}) = \varinjlim_{Z^* \in \mathcal{D}(X)} \mathbb{E}^{p+q}(X - Z^p / X - Z^{p-1}) \Rightarrow \mathbb{E}^{p+q}(X). \tag{2.4.4.a}$$

This is the usual coniveau spectral sequence associated with $\mathbb{E}$ (see [BO74], [CTHK97]). Note that it is concentrated in the band $0 \leq p \leq \dim(X)$; thus it is convergent.

Remark 2.4.5. The canonical functor

$$SH(k) \to DM(k),$$

once extended to pro-objects, sends the data defined above to the motivic coniveau exact couple considered in [Dég08a, def. 3.5].

3. The proof

3.1. The (weak) H-module structure.

Proposition 3.1.1. *Let F_* be an orientable homotopy module.*

Then the spectrum $H(F_*)$ (see (1.2.3.b)*) admits a canonical structure of* $\mathbf{H}$*-module in* $SH(k)$*.*

Proof. This follows from the exact sequence (1.2.6.a), Example 1.2.5 and the fact that the tensor product on $SH(k)$ preserves positive objects for the homotopy t-structure. □

3.1.2. In particular, the presheaf represented by the (weak) $\mathbf{H}$-module $H(F_*)$ precomposed with the functor $\mathcal{O}'_{\mathbf{H}}$ of (2.2.5.c) induces a canonical functor:

$$\varphi_F : (\mathbf{H}{-}mod)^{op} \to \mathscr{A}b. \tag{3.1.2.a}$$

According to the commutative diagram (2.2.5.d), we get a commutative diagram:

$$\begin{array}{ccc} SH(k)^{op} & \xrightarrow{\varphi_F^0} & \\ {\scriptstyle (L_{\mathbf{H}})^{op}} \downarrow & & \mathscr{A}b \\ (\mathbf{H}{-}mod)^{op} & \xrightarrow{\varphi_F} & \end{array} \tag{3.1.2.b}$$

where φ_F^0 is the presheaf represented by the spectrum $H(F_*)$. According to the isomorphism (1.2.4.a), this implies that for any smooth scheme X and any integer $n \in \mathbb{Z}$, $\varphi_F(\underline{\mathbf{H}}(X)(n)[n]) = F_{-n}(X)$.

3.2. The associated cycle premodule.

3.2.1. Let $\mathcal{O}$ be a formally smooth essentially of finite type k-algebra. A *smooth model* of $\mathcal{O}$ will be an affine smooth scheme $X = \mathrm{Spec}(A)$ (of finite type) such that A is a sub-k-algebra of $\mathcal{O}$. Let $\mathcal{M}^{sm}(\mathcal{O}/k)$ be the set of such smooth models, ordered by the relation: $\mathrm{Spec}(B) \leq \mathrm{Spec}(A)$ if $A \subset B$. As k is perfect, $\mathcal{M}^{sm}(\mathcal{O}/k)$ is a non empty left filtering ordered set. We will denote by $(\mathcal{O})$ the pro-scheme $(X)_{X \in \mathcal{M}^{sm}(\mathcal{O}/k)}$.

Theorem 3.2.2. *Consider the above notations and the category $\tilde{\mathscr{E}}_k$ introduced in 2.1.1. There exists a canonical additive functor*

$$\underline{\mathbf{H}}^{(0)} : \tilde{\mathscr{E}}_k^{op} \to \text{pro-}(\mathbf{H}{-}mod)$$

defined on an object (E, n) of $\tilde{\mathscr{E}}_k$ by the formula:

$$\underline{\mathbf{H}}^{(0)}(E, n) = \text{“}\varprojlim_{X \in \mathcal{M}^{sm}(E/k)}\text{”} \, \underline{\mathbf{H}}(X)(-n)[-n].$$

Remark 3.2.3. Note this theorem follows directly from our previous work on generic motives [Dég08c, Th. 5.1.1] when k admits resolution of singularities because the adjunction (2.2.6.b) is then an isomorphism. However, we give a proof (see Paragraphs 3.2.5 and 3.2.10) which avoids this assumption. The proof given here uses the same arguments as in the proof of *loc. cit.* after suitable generalization of the geometric constructions therein to the category $\mathbf{H}-mod$ (see Section 2.3).

Before giving the proof, we state the corollary which motivates the previous result:

Corollary 3.2.4. *Let F_* be an orientable homotopy module.*

Then one associates with F_* a canonical cycle premodule $\hat{F}_* : \tilde{\mathscr{E}}_k \to \mathscr{A}b$ defined on an object (E,n) of $\tilde{\mathscr{E}}_k$ by the formula:

$$\hat{F}_*(E,n) = \varinjlim_{X \in \mathcal{M}^{sm}(E/k)} F_{-n}(X).$$

This defines a functor: $\tilde{\rho}' : \Pi_*^{\eta=0}(k) \to \mathscr{A}b^{\tilde{\mathscr{E}}_k}, F_* \mapsto \hat{F}_*$.

Proof. In fact, the functor φ_F associated with F_* in (3.1.2.a) admits an obvious extension $\bar{\varphi}_F$ to pro-objects of $\mathbf{H}-mod$. One simply puts: $\hat{F}_* = \bar{\varphi}_F \circ \underline{\mathbf{H}}^{(0)}$. This is obviously functorial in F_*. □

The remaining of this section is devoted to the proof of Theorem 3.2.2. It will be completed in several steps trough the paragraphs which follow.

3.2.5. Functorialities:
(D1) is induced by the natural functoriality of $X \mapsto \underline{\mathbf{H}}(X)$.
(D2): A finite field extension $\varphi : E \to L$ induces a morphism of pro-schemes $(\varphi) : (L) \to (E)$ such that

$$(\varphi) = \text{“}\varprojlim_{i \in I}\text{”} \ (f_i : Y_i \to X_i)$$

where the f_i are finite surjective morphisms, whose associated generic residual extension is L/E, and the transition morphisms in the previous formal projective limit are made by transversal squares (see [Dég08c, 5.2] for details). One defines the map $\varphi_* : \underline{\mathbf{H}}(E) \to \underline{\mathbf{H}}(L)$ corresponding to (D2) as the formal projective limit:

$$\text{“}\varprojlim_{i \in I}\text{”} \ \big(f_i^* : \underline{\mathbf{H}}(X_i) \to \underline{\mathbf{H}}(Y_i)\big)$$

using the Gysin morphism (2.3.7.a) and property (G2c).
(D3): According to Example 1.2.5, for any function field E/k and any integer $n \geq 0$,

$$\left(\varinjlim_{X \in \mathcal{M}^{sm}(E/k)} \mathbf{H}^n(X)\right) \simeq K_n^M(E). \tag{3.2.5.a}$$

Thus any symbol $\sigma \in K_n^M(E)$ corresponds to a morphism of pro-objects

$$\underline{\mathbf{H}}^{(0)}(E) \to \underline{\mathbf{H}}(n)[n]$$

still denoted by σ. For any smooth model X of E/k, we let $\sigma_X : \underline{\mathbf{H}}^{(0)}(X) \to \underline{\mathbf{H}}(n)[n]$ the component of σ corresponding to X. We define γ_σ as the formal projective limit:

$$\text{“}\varprojlim_{X \in \mathcal{M}^{sm}(E/k)}\text{”} \ \big(\sigma_X \boxtimes_X 1_{X*}\big)$$

with the definition given by formula (2.3.9.a).
(D4): Let (E,v) be a valued function field with ring of integers $\mathcal{O}_v$ and residue field κ_v. There exists a smooth model X of $\mathcal{O}_v$ and a point $x \in X$ of codimension 1 corresponding to the valuation v. Let Z be the reduced closure of x in X. Given an open neighborhood U of x in X such that $Z \cap U$ is smooth, we can write the corresponding Gysin triangle (2.3.3.a) as follows:

$$\underline{\mathbf{H}}(Z \cap U)(1)[1] \xrightarrow{\partial_{U, Z\cap U}} \underline{\mathbf{H}}(U - Z \cap U) \xrightarrow{j_*} \underline{\mathbf{H}}(U) \to \underline{\mathbf{H}}(Z \cap U)(1)[2] \tag{3.2.5.b}$$

According to property (G1a), the morphism $\partial_{U,Z\cap U}$ is functorial with respect to the open subscheme U. Taking the formal projective limit of this morphism with respect to the neighborhoods U as above, we obtain the desired map:

$$\partial_v : \underline{\mathbf{H}}(\kappa_v)(1)[1] \to \underline{\mathbf{H}}(E).$$

Remark 3.2.6. Note for future reference that the triangle (3.2.5.b) being distinguished, we get with the above notations: $j_* \circ \partial_{U,Z\cap U} = 0$.

3.2.7. Before proving the relations, we note that one can apply the preceding construction to obtain maps in motivic cohomology. More precisely, given any function field E/k, we put (as in Corollary 3.2.4):

$$\hat{\underline{\pi}}_0(\mathbf{H})_n(E) := \varinjlim_{X \in \mathcal{M}^{sm}(E/k)} \mathbf{H}^n(X).$$

As $\mathbf{H}^n(X) = \mathrm{Hom}_{\mathbf{H}\text{–}mod}(\mathbf{H}(X)(-n)[-2n], \mathbf{H})$, we thus obtain that for any extension (resp. finite extension) of function fields $\varphi : K \to E$, the map (D1) (resp. (D2)) induces a canonical morphism:

$$\varphi_* : \hat{\underline{\pi}}_0(\mathbf{H})_n(K) \to \hat{\underline{\pi}}_0(\mathbf{H})_n(E),$$
$$\text{resp. } \varphi^* : \hat{\underline{\pi}}_0(\mathbf{H})_n(E) \to \hat{\underline{\pi}}_0(\mathbf{H})_n(K).$$

We will need the following lemma concerning these maps:

Lemma 3.2.8. *Consider for any function field E/k the isomorphism* (3.2.5.a) *of graded abelian groups:*

$$K_*^M(E) \xrightarrow{\epsilon_E} \hat{\underline{\pi}}_0(\mathbf{H})_*(E).$$

Then, the following properties hold:

(1) *ϵ_E is an isomorphism of graded algebras.*

(2) *For any morphism $\varphi : E \to L$ of function fields, the following square is commutative:*

$$\begin{array}{ccc} K_*^M(E) & \xrightarrow{\epsilon_E} & \hat{\underline{\pi}}_0(\mathbf{H})_*(E) \\ {\scriptstyle \varphi_*}\downarrow & & \downarrow{\scriptstyle \varphi_*} \\ K_*^M(L) & \xrightarrow{\epsilon_L} & \hat{\underline{\pi}}_0(\mathbf{H})_*(L) \end{array}$$

where the left (resp. right) vertical map corresponds to the restriction map in Milnor K-theory (resp. data (D1)).

(3) *For any finite morphism $\varphi : E \to L$ of function fields, the following square is commutative:*

$$\begin{array}{ccc} K_*^M(E) & \xrightarrow{\epsilon_E} & \hat{\underline{\pi}}_0(\mathbf{H})_*(E) \\ {\scriptstyle \varphi^*}\uparrow & & \uparrow{\scriptstyle \varphi^*} \\ K_*^M(L) & \xrightarrow{\epsilon_L} & \hat{\underline{\pi}}_0(\mathbf{H})_*(L) \end{array}$$

where the left (resp. right) vertical map corresponds to the corestriction map in Milnor K-theory (resp. data (D2)).

Proof. Assertions (1) and (2) follow immediately from [SV00, th. 3.4]. The verification of (3) is not easy due to the abstract nature of the Gysin morphism used to define (D2). However, arguing as in [SV00, lm. 3.4.1, 3.4.4], it suffices to prove the following lemma:

Lemma 3.2.9. *Let $\varphi : K \to E$ and $\psi : K \to L$ be finite extensions of fields. Let $[E : K]$ (resp. $[E : K]_i$) be the degree (resp. inseparable degree) of E/K. Then for any elements $x \in \hat{\underline{\pi}}_0(\mathbf{H})_n(E)$ and $y \in \hat{\underline{\pi}}_0(\mathbf{H})_n(K)$, the following formulas hold:*

(4) *$\varphi^*(x.\varphi_*(y)) = \varphi^*(x).y$, $\varphi^*(\varphi_*(y).x) = y.\varphi^*(x)$.*

(5) *$\varphi^*\varphi_*(x) = [E : K].x$.*

(6) *Put $R = E \otimes_K L$. Then, $\psi_*\varphi^*(x) = \sum_{z\in\mathrm{Spec}(R)} \lg(R_z).\bar{\varphi}_z^*\bar{\psi}_{z*}(x)$, with the notations of (R1c).*

(7) *Assume L/K is normal. Then, $\psi_*\varphi^*(x) = [E : K]_i. \sum_{j\in\mathrm{Hom}_K(E,L)} j_*(x)$.*

In fact, point (7) implies point (3) for the graded part of degree 1 because according to point (2), $\psi_* : \underline{\pi}_0(\mathbf{H})_1(K) \to \underline{\pi}_0(\mathbf{H})_1(L)$ is injective. Then the proof of [SV00, lem. 3.4.4] allows to deduce point (3): this proof indeed uses only the preceding fact together with points (4) and (5).

Let us now prove the preceding lemma. Points (4) (resp. (6)) follow from the definition of (D2) and property (G3b) (resp. (G2d)). Point (7) is then an easy consequence of (6) using elementary Galois theory. The difficult part is point (5).

According to point (4), it suffices to prove that $\varphi^*(1) = [E : K]$. Because of property (G2a) of the Gysin morphism, we reduce to the case where E/K is generated by a single element. In other words, $E = K[t]/(P)$ where P is a polynomial with coefficients in t. Thus we can find a smooth model X (resp. Y) of K/k (resp. E/k) and a finite surjective morphism $f : Y \to X$ whose generic residual extension is E/K and which factors as:

$$Y \xrightarrow{i} \mathbb{P}^1_X \xrightarrow{p} X$$

where p is the canonical projection and i is a closed immersion of codimension 1. Thus point (5) now follows from the definition of (D2) and the trace formula (2.3.10.a). □

3.2.10. Relations:
The relations follow from the preparatory work done in Section 2.3, according to the following table:

(R0)	3.2.8(1)	(R1a)	Obvious.
(R1b)	(G2a)	(R1c)	(G2d)
(R2a)	(G3a)+lem. 3.2.8(2)	(R2c)	(G3b)+lem. 3.2.8(2)
(R2c)	(G3b)+lem. 3.2.8(3)	(R3a)	(G1b)
(R3b)	(G2b)	(R3c)	rem. 3.2.6
(R3d)	prop. 2.3.12(2)	(R3e)	prop. 2.3.12(1)

This concludes the proof of 3.2.2.

3.3. **The associated cycle module.**

3.3.1. Using the obvious extension of the functor (2.3.1.b) to pro-objects, the homotopy coniveau exact couple induces a pro-triangulated exact couple in pro$-(\mathbf{H}-mod)$ whose graded terms are:

(3.3.1.a)
$$\begin{aligned} \underline{\mathbf{H}}(F_p(X))[-p-q] &= \text{“}\varprojlim_{Z^* \in \mathcal{D}(X)}\text{”}\ \underline{\mathbf{H}}(X - Z_p)[-p-q] \\ \underline{\mathbf{H}}(Gr_p(X))[-p-q] &= \text{“}\varprojlim_{Z^* \in \mathcal{D}(X)}\text{”}\ \underline{\mathbf{H}}(X - Z_p/X - Z_{p-1})[-p-q] \end{aligned}$$

Given a smooth closed subscheme $Z \subset X$ of pure codimension p, if we apply [Dég08b, prop. 4.3] to the closed pair (X, Z) according to Paragraph 2.3.1, we get a canonical isomorphism:

(3.3.1.b)
$$\mathfrak{p}_{X,Z} : \underline{\mathbf{H}}(X/X - Z) \to \underline{\mathbf{H}}(Z)(p)[2p].$$

Using this isomorphism, we can easily obtain the analog of [Dég08a, 3.11] in the setting of **H**-modules:

Proposition 3.3.2. *Consider the above notations. Let $p \in \mathbb{Z}$ be an integer and denote by $X^{(p)}$ the set of codimension p points of X. Then there exists a canonical isomorphism:*

$$\underline{\mathbf{H}}(Gr_p(X)) \xrightarrow{\epsilon_p} \text{“}\prod_{x \in X^{(p)}}\text{”}\ \underline{\mathbf{H}}(\kappa(x))(p)[2p].$$

In particular, for any point $x \in X^{(p)}$ we get a canonical projection map:

(3.3.2.a)
$$\pi_x : \underline{\mathbf{H}}(Gr_p(X)) \to \underline{\mathbf{H}}(\kappa(x))(p)[2p].$$

For the proof, we refer the reader to the proof of [Dég08a, 3.11] — the same proof works in our case if we use the purity isomorphism (3.3.1.b).

3.3.3. Let X be a scheme essentially of finite type over k and consider a couple $(x, y) \in X^{(p)} \times X^{(p+1)}$.

Assume that y is a specialization of x. Let Z be the reduced closure of x in X and $\tilde{Z} \xrightarrow{f} Z$ be its normalization. Each point $t \in f^{-1}(y)$ corresponds to a discrete valuation v_t on κ_x with residue field κ_t. We denote by $\varphi_t : \kappa_y \to \kappa_t$ the morphism induced by f. Then, we define the following morphism of pro-**H**-modules:

$$d_y^x = \sum_{t \in f^{-1}(y)} \partial_{v_t} \circ \varphi_{t*} : \underline{\mathbf{H}}(\kappa_y)(1)[1] \to \underline{\mathbf{H}}(\kappa_x) \tag{3.3.3.a}$$

using the notations of paragraph 3.2.5. If y is not a specialization of x, we put: $\partial_y^x = 0$.

Proposition 3.3.4. *Consider the above hypothesis and notations. If X is smooth then the following diagram is commutative:*

$$\begin{CD}
\underline{\mathbf{H}}(Gr_{p+1}(X)) @>{d_{p+1,-p-1}}>> \underline{\mathbf{H}}(Gr_p(X))[1] \\
@V{\pi_y}VV @VV{\pi_x}V \\
\underline{\mathbf{H}}(\kappa_y)(p+1)[2p+2] @>{d_y^x}>> \underline{\mathbf{H}}(\kappa_x)(p)[2p+1]
\end{CD}$$

where the vertical maps are defined in (3.3.2.a) *and the map $d_{p+1,-p-1}$ in the differential associated with the pro-triangulated exact couple* (3.3.1.a).

The proof is the same as the one of [Dég08a, 3.13] once we use the following correspondence table for the results used in it:

[Dég08a, 3.13]	Proposition 3.3.4
Proposition 1.36	Lemma 2.3.6
Theorem 1.34, relation (2)	(G2b)
Proposition 2.13	Remark 2.3.8
Proposition 2.9	(G2a)

3.3.5. Consider an orientable homotopy module F_*, a smooth scheme X and an integer $n \in \mathbb{Z}$. Put $\mathbb{F} = H(F_*)$ (using the functor H of (1.2.3.a)).

The coniveau spectral sequence (2.4.4.a) associated with $\mathbb{F}(n)[n]$ has the following shape:

$$E_1^{p,q}(X, \mathbb{F}(n)[n]) \Rightarrow H_{\mathrm{Nis}}^{p+q}(X, F_n),$$

where we have used the isomorphism (1.2.4.a) to identify the abutment.

Consider also the commutative diagram (3.1.2.b) and the obvious extension of $\bar{\varphi}_F^0$ (resp. $\bar{\varphi}_F$) to pro-objects:

$$\bar{\varphi}_F^0 : \text{pro}-SH(k)^{op} \to \mathscr{A}b, \text{ resp. } \bar{\varphi}_F : \text{pro}-(\mathbf{H}-mod)^{op} \to \mathscr{A}b.$$

Then the previous spectral sequence is defined by the exact couple

$$\big(\bar{\varphi}_F^0(D_{p,q}), \bar{\varphi}_F^0(E_{p,q})\big) = \big(\bar{\varphi}_F(L_{\mathbf{H}}(D_{p,q})), \bar{\varphi}_F(L_{\mathbf{H}}(E_{p,q}))\big).$$

Thus, Proposition 3.3.2 gives a canonical isomorphism:

$$E_1^{p,q}(X, \mathbb{F}(n)[n]) \xrightarrow{\epsilon_p^*} \begin{cases} C^p(X, \hat{F}_*)_n & \text{if } q = 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\hat{F}_*$ is the cycle premodule associated with F_*. Consider moreover, a couple $(x, y) \in X^{(p)} \times X^{(p+1)}$. Comparing formula (2.1.3.b) with formula (3.3.3.a), proposition 3.3.4 gives the following commutative diagram:

$$\begin{CD}
E_1^{p,0}(X, \mathbb{F}(n)[n]) @>{d_1^{p,0}}>> E_1^{p+1,0}(X, \mathbb{F}(n)[n]) \\
@AAA @AAA \\
\hat{F}_*(\kappa(x)) @>{\partial_y^x}>> \hat{F}_*(\kappa(y))
\end{CD}$$

where the vertical maps are the canonical injections. In particular:

Corollary 3.3.6. *Consider the previous notations.*

(1) *The cycle premodule* $\hat{F}_*$ *satisfies properties* $(\mathbf{FD}_X)$ *and* $(\mathbf{C}_X)$.
(2) *There is a canonical isomorphism of complexes:*
$$E_1^{*,0}(X, \mathbb{F}(n)[n]) \simeq C^*(X; \hat{F}_*)_n.$$
(3) *For any integer* $p \in \mathbb{Z}$*, there is a canonical isomorphism:*
$$H^p_{\mathrm{Nis}}(X, F_n) \simeq A^p(X; \hat{F}_*)_n$$
where the right hand side is the p-th cohomology of the complex $C^*(X; \hat{F}_*)_n$ *(notation of paragraph 2.1.8).*
(4) *The cycle premodule* $\hat{F}_*$ *is a cycle module.*
In other words, the functor $\tilde{\rho}'$ *of Corollary 3.2.4 induces a functor:*
$$\tilde{\rho} : \Pi_*^{\eta=0}(k) \to \mathscr{M}Cycl(k).$$

Note that point (4) follows from point (1) above and Lemma 2.1.6. This corollary establishes the first assertion of Theorem 1.3.6.

3.4. **The remaining isomorphism.** It remains to construct the natural isomorphism which appears in the statement of Theorem 1.3.6. Point (3) of the preceding corollary in the case $p = 0$ gives an isomorphism of $\mathbb{Z}$-graded abelian groups:
$$\epsilon_X : F_*(X) \to A^0(X; \hat{F}_*).$$
According to the definition of the functor A^0 of (1.3.5.a), the target of ϵ_X is the group of sections over X of the homotopy module $(\gamma_* \circ A^0 \circ \tilde{\rho})(F_*)$. We prove that ϵ_X is natural in X. Explicitly, for any morphism $f : Y \to X$ of smooth schemes, we have to prove the following diagram commutes:

(3.4.0.a)
$$\begin{array}{ccc} F_*(X) & \xrightarrow{\epsilon_X} & A^0(X; \hat{F}_*) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ F_*(Y) & \xrightarrow{\epsilon_Y} & A^0(Y; \hat{F}_*) \end{array}$$

where the vertical map on the right hand side refers to the functoriality defined by Rost.

To prove this, we can assume by additivity that X is connected, with function field E. According to our definition,
$$\hat{F}_*(E) = \varinjlim_{j_U : U \subset X} F_*(U)$$
where the colimit runs over the non empty open subschemes of X. In particular, the colimit of the morphism j_U^* induces a canonical map $\rho_X : F_*(X) \to \hat{F}_*(E)$. By definition of the coniveau spectral sequence, the isomorphism ϵ_X is induced by the exact sequence:
$$0 \to F_*(X) \xrightarrow{\rho_X} \hat{F}_*(E) \xrightarrow{d_X^0} C^1(X; \hat{F}_*)_n$$
where d_X^0 is the differential (2.1.4.a) associated with the cycle module $\hat{F}_*$.

(1) *The case of a flat morphism*: Consider a flat morphism $f : Y \to X$ of connected smooth schemes and let $\varphi : E \to L$ be the induced morphism on function fields. According to the definition of (D1) in paragraph 3.2.5, the following square is commutative:
$$\begin{array}{ccc} F_*(X) & \xrightarrow{\rho_X} & \hat{F}_*(E) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle \varphi_*} \\ F_*(Y) & \xrightarrow{\rho_Y} & \hat{F}_*(L) \end{array}$$
Thus, the commutativity of (3.4.0.a) in this case follows from the definition of flat pullbacks on A^0 (see [Ros96, 12.2, 3.4]).

(2) *The general case*: The morphism f can be written as the composite
$$Y \xrightarrow{\gamma} Y \times X \xrightarrow{p} X$$

where γ is the graph[14] of f and p is the canonical projection. To prove the commutativity of (3.4.0.a) in the case of f, we are reduced to the cases of p and γ. The case of the smooth morphism p follows from point (1). Thus we are reduced to the case where $f = i : Z \to X$ is a closed immersion between smooth schemes.

Consider an open subscheme $U \subset X$ such that the induced open immersion $j_Z : Z \cap U \to Z$ is dense. Recall that the map $j_Z^* : A^0(Z; \hat{F}_*) \to A^0(Z \cap U; \hat{F}_*)$ is injective[15]. Thus, according to case (1) when $f = j$ and $f = j_Z$, we are reduced to prove the commutativity of (3.4.0.a) when f is the closed immersion $Z \cap U \to U$.

In particular, we can assume that i is the composition of immersions of a smooth principal divisor. Then we are restricted to the case where Z is a smooth principal divisor.

Because Z is principal, it is parametrized by a regular function $\pi : X \to \mathbb{A}^1$. One can assume Z is connected. Then, its generic point defines a codimension 1 point of X which corresponds to a discrete valuation v on the function field E of X. We denote by $\kappa(v)$ the residue field of v. According to the computation of $i^* : A^0(X; \hat{F}_*) \to A^0(Z; \hat{F}_*)$ of [Ros96, (12.4)], the commutativity of (3.4.0.a) in the case $f = i$ is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc} F_*(X) & \xrightarrow{\rho_X} & \hat{F}_*(E) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle \partial_v \circ \gamma_\pi} \\ F_*(Z) & \xrightarrow{\rho_Z} & \hat{F}_*(\kappa(v)) \end{array}$$

But this follows from point (2) of Proposition 2.3.12 and the definition of data (D3) and (D4) for the cycle module $\hat{F}_*$ (paragraph 3.2.5).

According to the construction of the structural isomorphism

$$A^0(X; \hat{F}_*)_n \to \big(A^0(-; \hat{F}_*)_{n+1}\big)_{-1}(X)$$

given in [Dég09, 2.8], it is now clear that $\epsilon : F_* \to A^0(-; \hat{F}_*)$ is a morphism of homotopy modules. As it is an isomorphism by construction, this concludes the proof of the main theorem 1.3.6.

4. Some further comments

4.1. **Monoidal structure.** The following proposition and its corollary, which we will apply in the context of homotopy modules, are probably well known in category theory. We include a proof for completeness:

Proposition 4.1.1. *Let $\mathscr{M}$ (resp. $\mathscr{M}'$) be a monoidal category with unit e (resp. e′). We consider an adjunction of categories*

$$\varphi^* : \mathscr{M} \leftrightarrows \mathscr{M}' : \varphi_*.$$

and denote by $\mathcal{I}$ the essential image of φ_*. We assume the following:

(a) *φ^* is monoidal.*
(b) *φ_* is fully faithful.*
(c) *For any objects M, N of $\mathcal{I}$, $M \otimes N$ is an object of $\mathcal{I}$.*

Then the following conditions hold:

(1) *For any objects M' of $\mathscr{M}'$ and N of $\mathscr{M}$, there exists a canonical isomorphism:*

$$\varphi_*(M') \otimes N \xrightarrow{\epsilon_{M',N}} \varphi_*(M' \otimes \varphi^*(N))$$

which is functorial in M' and N.

(2) *For any objects M', N' of $\mathscr{M}'$, there exists a canonical isomorphism:*

$$\varphi_*(M') \otimes \varphi_*(N') \xrightarrow{\phi_{M',N'}} \varphi_*(M' \otimes N')$$

which is functorial in M' and N'.

---

[14] As X/k is separated by our general assumption, γ is a closed immersion.

[15] This follows for example from the localization long exact sequence for Chow groups with coefficients (*cf.* [Ros96, 3.10]).

Proof. Let us consider the unit and counit of the adjunction (φ^*, φ_*):

$$\alpha : 1 \to \varphi_*\varphi^*, \ \alpha' : \varphi^*\varphi_* \to 1. \tag{4.1.1.a}$$

For the first point, define $\epsilon_{M',N}$ as the composite:

$$\varphi_*(M') \otimes N \xrightarrow{(1)} \varphi_*\varphi^*(\varphi_*(M') \otimes N) \xrightarrow{(2)} \varphi_*(\varphi^*\varphi_*(M') \otimes \varphi^*(N)) \xrightarrow{(3)} \varphi_*(M' \otimes \varphi^*(N))$$

where (1) is equal to $\alpha_{\varphi_*(M')\otimes N}$, (2) is the isomorphism of assumption (a) and (3) is induced by $\alpha'_{M'}$. We have to check $\epsilon_{M',N}$ is an isomorphism. According to assumption (c), $\varphi_*(M')\otimes N$ belongs to $\mathcal{I}$. Thus, assumption (b) tells us it is enough to prove that $\varphi^*(\epsilon_{M',N})$ is an isomorphism. This last assertion follows from assertion (b) and the straightforward commutativity of the following diagram:

$$\begin{array}{ccc}
\varphi^*(\varphi_*(M') \otimes N) & \xrightarrow{\varphi^*(\epsilon_{M',N})} & \varphi^*\varphi_*(M' \otimes \varphi^*(N)) \\
\sim\downarrow & & \\
\varphi^*\varphi_*(M') \otimes \varphi^*(N) & & \uparrow \alpha'_{M'\otimes\varphi^*(N)} \\
\alpha'_{M'}\otimes Id \uparrow & & \\
M' \otimes \varphi^*(N) & = = = = & M' \otimes \varphi^*(N).
\end{array}$$

Then, the second point follows: we define $\phi_{M',N'}$ as the composite:

$$\varphi_*(M') \otimes \varphi_*(N') \xrightarrow{\epsilon_{M',\varphi_*(N')}} \varphi_*(M' \otimes \varphi^*(\varphi_*(N'))) \xrightarrow{(4)} \varphi_*(M' \otimes N'),$$

where the map (4) is induced by the isomorphism $\alpha'_{N'}$. □

4.1.2. Consider again the hypothesis and notations of the above proposition, together with the notation of (4.1.1.a). We put $A = \varphi_*(e')$. One checks easily that A admits a monoid structure in $\mathscr{M}$ whose multiplication and unit maps are given by:

$$\begin{aligned} \mu = \phi_{e',e'} &: A \otimes A \to A, \\ u = \alpha_e &: e \to A. \end{aligned} \tag{4.1.2.a}$$

We denote by $A{-}mod$ the category of A-modules in $\mathscr{M}$, and by

$$L_A : A{-}mod \to \mathscr{M} : \mathcal{O}_A$$

the adjunction of categories such that $\mathcal{O}_A$ is the obvious forgetful functor.

Corollary 4.1.3. *Consider the hypothesis and notations above. We can add the following assertion to that of the previous proposition:*

(3) *For any object M of $\mathscr{M}$, there exists a canonical isomorphism in $\mathscr{M}$:*

$$\epsilon^0_M : \varphi_*\varphi^*(M) \to A \otimes M$$

natural in M. Moreover, the following diagram is commutative:

$$\begin{array}{ccc}
 & \overset{\alpha_M}{\nearrow} & \varphi_*\varphi^*(M) \\
M & & \downarrow \epsilon^0_M \\
 & \underset{u\otimes Id_M}{\searrow} & A \otimes M.
\end{array}$$

(4) *Let M be an object of $\mathscr{M}$. Then the following conditions are equivalent:*
   (i) *M belongs to $\mathcal{I}$.*
   (ii) *The map $u \otimes Id_M : M \to A \otimes M$ is an isomorphism.*
   (iii) *M admits a structure of an A-module.*
   Moreover, when these conditions hold, the structure of an A-module on M is unique.

(5) *The category $A{-}mod$ admits a unique structure of a monoidal category such that A is the unit object and $\mathcal{O}_A$ is monoidal. Moreover, there exists a unique equivalence of monoidal categories $\sigma : A{-}mod \longrightarrow \mathscr{M}'$ such that φ^* is equal to the composite map:*

$$\mathscr{M} \xrightarrow{L_A} A{-}mod \xrightarrow{\sigma} \mathscr{M}'.$$

Proof. For point (3), we define ϵ_0 as the composite map:

$$\varphi_*\varphi^*(M) = \varphi_*(e' \otimes \varphi^*(M)) \xrightarrow{\epsilon_{e',M}} \varphi_*(e') \otimes M = A \otimes M.$$

The commutativity of the diagram is then straightforward.

Consider point (4):
(i) implies (ii) according to the commutative diagram of point (3).
(ii) implies (iii): obvious.
(iii) implies (i): Considering the map $\gamma : A \otimes M \to M$ of a structure of an A-module on M, the following composite

$$M \xrightarrow{u \otimes Id_M} A \otimes M \xrightarrow{\gamma} M$$

is by definition the identity. Thus point (3) and the fact $\mathcal{I}$ is stable by retract gives the implication. The remaining assertion of point (4) is obvious: indeed, with the notations above, we obtain: $\gamma = (u \otimes Id_M)^{-1}$.

Point (5): the first assertion holds because the multiplication map μ of A is an isomorphism. Note this last fact also implies that $\mathcal{O}_A$ is fully faithful.
Then, given any object M' of $\mathscr{M}'$, we let $\sigma(M')$ be the A-module $\varphi_*(M)$ with its unique structure of an A-module given by (4). From what we just said, this defines the functor σ and the remaining properties of point (5) are clear. □

4.1.4. Recall we have defined a t-structure on the category $DM(k)$ of motivic spectra (*cf.* [Dég09, section 5.2]) called the *homotopy t-structure* whose heart is the category $\Pi_*^{tr}(k)$ of definition 1.3.2. By its very construction, the right adjoint functor γ_* of (2.2.6.a) is t-exact with respect to the homotopy t-structures on $SH(k)$ and $DM(k)$; thus it preserves the objects of the hearts. It also follows that its left adjoint γ^* preserves homologically positive objects and (2.2.6.a) induces an adjunction of abelian categories:

$$\gamma^*_{\leq 0} : \Pi_*(k) \leftrightarrows \Pi_*^{tr}(k) : \gamma_* \tag{4.1.4.a}$$

where $\gamma^*_{\leq 0} = t_-\gamma^*$ using the notation of Paragraph 1.1.4. In particular, $\gamma^*_{\leq 0}$ is monoidal.

According to the above definition the functor γ_* is equal to the composite map:

$$\Pi_*^{tr}(k) \xrightarrow{\gamma'_*} \Pi_*^{\eta=0}(k) \to \Pi_*(k)$$

of the equivalence γ'_* of Theorem 1.3.4 followed by the natural inclusion. Thus, it follows from this later theorem that γ_* is fully faithful. Its essential image is made by the orientable homotopy modules. Obviously, given definition 1.2.7, the tensor product of two orientable homotopy modules is again oriented.

Thus, the adjunction (4.1.4.a) satisfies all the conditions of Proposition 4.1.1. Thus we can apply this proposition and its corollary in our situation.
Recall that $\underline{K}_*^M$, equipped with its canonical structure of a graded sheaf with transfers, is the unit object of the monoidal category $\Pi_*^{tr}(k)$ (*cf.* [Dég09, 1.15, 3.8]). To simplify notations, we will identify $\underline{K}_*^M$ and the object $\gamma_*(\underline{K}_*^M)$ of $\Pi_*(k)$ it determines. Thus $\underline{K}_*^M$ corresponds to the ring A of Paragraph 4.1.2. We denote simply by $(L, \mathcal{O})$ the adjunction $(L_A, \mathcal{O}_A)$ of *loc. cit.*
Note also that, according to Example 1.2.5, $\underline{K}_*^M$ is identified with the homotopy module $\pi_0(\mathbf{H})_*$. Then the structural maps (4.1.2.a) of the monoid $\underline{K}_*^M$ coincide with the following ones:

$$\mu = \underline{\pi}_0(\mu_{\mathbf{H}})_* : \underline{K}_*^M \otimes \underline{K}_*^M \to \underline{K}_*^M, \ u = \underline{\pi}_0(u_{\mathbf{H}})_* : \underline{\pi}_0(S^0)_* \to \underline{K}_*^M$$

where $\mu_{\mathbf{H}}$ (resp. $u_{\mathbf{H}}$) is the multiplication (resp. unit) of the ringed spectrum $\mathbf{H}$.

Point (4) and (5) of Corollary 4.1.3 applied to the situation of this paragraph thus gives:

Corollary 4.1.5. *Consider the above notations.*

1. *The multiplication map μ is an isomorphism.*
2. *Given a homotopy module F_*, the following conditions are equivalent:*
   - (i) *F_* is an orientable homotopy module (Definition 1.2.7).*
   - (ii) *F_* as a Nisnevich sheaf admits a structure of a sheaf with transfers.*
   - (iii) *The map $u \otimes 1_{F_*} : F_* \to \underline{K}_*^M \otimes F_*$ is an isomorphism.*

(iv) F_* *admits a structure of a* $\underline{K}_*^M$*-module in* $\Pi_*(k)$.

Moreover, when these conditions hold, the structures in conditions (ii) and (iv) are unique.

(3) *The category* $\underline{K}_*^M-mod$ *admits a unique structure of a monoidal category such that* $\underline{K}_*^M$ *is the unit object and* $\mathcal{O}$ *is monoidal.*

Moreover, there exists a unique equivalence $\sigma : \underline{K}_*^M-mod \longrightarrow \Pi_*^{tr}(k)$ *of monoidal categories such that* $\gamma_{\leq 0}^*$ *is equal to the composite map:*

$$\Pi_*(k) \xrightarrow{L} \underline{K}_*^M-mod \xrightarrow{\sigma} \Pi_*^{tr}(k).$$

Remark 4.1.6. It is not true in general that the multiplication map $\mu : \mathbf{H} \wedge \mathbf{H} \to \mathbf{H}$ is an isomorphism according to the non triviality of the Steenrod operations.[16]

Using Example 2.2.6, we can add the following conditions to the caracterisation of a orientable homotopy module constituted by the point (2) above:

Corollary 4.1.7. *Let* F_* *be a homotopy module with associated spectrum* $H(F_*)$ *using the notation of* (1.2.3.a)*. Then the conditions (i)-(iv) of point (2) in the previous corollary are equivalent to the following ones:*

(v) *The spectrum* $H(F_*)$ *has a structure of a strict* **H***-module.*

(vi) *The spectrum* $H(F_*)$ *has a structure of an* **MGL***-module.*[17]

4.2. **Weakly orientable spectra.**

4.2.1. Consider a spectrum $\mathbb{E}$. The Hopf map η obviously acts on the $\mathbb{E}$-cohomology of a smooth scheme as:

$$\gamma_\eta : \mathbb{E}^{n,p}(X) \to \mathbb{E}^{n-1,p-1}(X), \rho \mapsto \eta \wedge \rho.$$

The following lemma is obvious:

Lemma 4.2.2. *Given a spectrum* $\mathbb{E}$*, the following conditions are equivalent:*

(i) *For any integer* $n \in \mathbb{Z}$, $\underline{\pi}_n(\mathbb{E})_*$ *is an orientable.*

(ii) *The map* $\eta \wedge 1_{\mathbb{E}}$ *is zero.*

(ii') *The map* $\underline{\mathrm{Hom}}(\eta, \mathbb{E})$ *is zero.*

(iii) *The operation* γ_η *on* $\mathbb{E}$*-cohomology defined above is zero.*

Definition 4.2.3. A spectrum $\mathbb{E}$ satisfying the equivalent conditions of the preceding lemma will be said to be *weakly orientable.*

Remark 4.2.4. From the second point of remark 1.2.8, if 2 is invertible in $E^{**} = E^{**}(k)$ and (-1) is a sum of squares in k, then $\mathbb{E}$ is weakly orientable.

As a corollary of our main Theorem 1.3.4, we get the following way to construct cycle modules:

Proposition 4.2.5. *Let* $\mathbb{E}$ *be a weakly orientable spectrum.*

Then for any integer $i \in \mathbb{Z}$*, there exists a canonical cycle module whose values on a couple* (L, n) *of* $\tilde{\mathscr{E}}_k$ *is the abelian group:*

$$\mathbb{E}^{i+n,n}(L) = \varinjlim_{X \in \mathcal{M}^{sm}(E/k)^{op}} \mathbb{E}^{i+n,n}(X),$$

with the notations of paragraph 3.2.1.

The new example here is given by the case where $\mathbb{E} = \mathbf{MGL}$ is Voevodsky's algebraic cobordism spectrum.

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[16]However, we proved in [CD09, 13.1.6] that $\mu \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism (in $SH(k)_{\mathbb{Q}}$).

[17]On can check the spectrum that **H** admits a strict structure of an **MGL**-algebra. Thus one can conclude in point (vi) that $H(F_*)$ admits a structure of a strict **MGL**-module.

4.3. **Coniveau spectral sequence.** As a corollary of the detailed analysis of Proposition 3.3.4, we get the following statement:

Proposition 4.3.1. *Let $\mathbb{E}$ be a spectrum and $q \in \mathbb{Z}$ be an integer such that the homotopy module $\underline{\pi}_{-q}(\mathbb{E})_*$ is orientable.*

Then for any smooth scheme X, there exists a canonical isomorphism of complexes of abelian groups:

$$E_1^{*,q}(X,\mathbb{E}) \simeq C^*(X, \hat{\underline{\pi}}_{-q}(\mathbb{E})_*)_0$$

where the left hand sides refers to the q-th line of the first page of the coniveau spectral sequence associated with $\mathbb{E}$ — cf. *Example 2.4.4.*

Proof. We can obviously assume $q = 0$. The complex $E_1^{*,0}(X,\mathbb{E})$ is natural in $\mathbb{E}$. Moreover, for any spectrum $\mathbb{F}$, one checks using the purity isomorphism in $SH(k)$ ([MV99, §4, 2.23]) and the same argument as in the proof of Proposition 3.3.2 that the induced maps:

$$E_1^{*,0}(X,\mathbb{F}_{\geq 0}) \to E_1^{*,0}(X,\mathbb{F}), \quad E_1^{*,0}(X,\mathbb{F}) \to E_1^{*,0}(X,\mathbb{F}_{\leq 0})$$

are isomorphisms. This yields an isomorphism of complexes:

$$E_1^{*,0}(X,\mathbb{E}) \simeq E_1^{*,0}(X,\underline{\pi}_0(\mathbb{E})_*).$$

so that we are reduced to the case where E is an orientable homotopy module. This case is precisely Corollary 3.3.6. □

Remark 4.3.2. The previous corollary can be applied to any weakly orientable spectrum $\mathbb{E}$: we get for any smooth scheme X and any integer $n \in \mathbb{Z}$ a convergent spectral sequence of the form:

$$E_2^{p,q}(X,\mathbb{E}(n)) = A^p(X, \hat{\underline{\pi}}_{q-n}(\mathbb{E})_*)_n \Rightarrow \mathbb{E}^{p+q,n}(X). \tag{4.3.2.a}$$

4.3.3. Let **MGL** be Voevodsky's algebraic cobordism spectrum. According to a theorem of Morel, there exists a canonical isomorphism of homotopy modules: $\underline{\pi}_0(\mathbf{MGL})_* \simeq K_*^M$.[18] One deduces from this result that the ring morphism $\varphi : \mathbf{MGL} \to \mathbf{H}$ corresponding to the orientation of $\mathbf{H}$ given by (2.3.1.a) induces an isomorphism:

$$\underline{\pi}_0(\mathbf{MGL})_* \to \underline{\pi}_0(\mathbf{H})_*.$$

Moreover, given a smooth connected scheme X and an integer $d \in \mathbb{Z}$, the morphism φ induces a morphism of the spectral sequences of type (4.3.2.a) which, on the E_2-term is equal to:

$$A^p(X, \hat{\underline{\pi}}_{q-d}(\mathbf{MGL})_*)_d \to A^p(X, \hat{\underline{\pi}}_{q-d}(\mathbf{H})_*)_d.$$

In case d is the dimension of X, the terms $E_2^{p,q}$ of the respective spectral sequences for **MGL** and **H** are concentrated in the region $0 \leq p \leq d$, $g \leq d$. Thus we get the following:

Corollary 4.3.4. *Let X be a smooth scheme of pure dimension d. Then the morphism φ induces an isomorphism:*

$$\mathbf{MGL}^{2d,d}(X) \xrightarrow{\varphi_*} \mathbf{H}^{2d,d}(X) = A^d(X, K_*^M)_d = CH^d(X).$$

Obviously, this isomorphism is natural with respect to pullbacks. It is also compatible with Gysin pushforwards associated with a projective morphism $f : Y \to X$ of smooth connected schemes.

Remark 4.3.5. If X is projective and smooth, the corollary can be reformulated using duality (see [Dég08b, th. 5.23]) by saying that φ induces an isomorphism in homology:

$$\mathbf{MGL}_0(X) \xrightarrow{\sim} CH_0(X).$$

Assume k admits resolution of singularities. If X is only smooth, under the assumption of resolution of singularities, one can replace the left hand side in the previous isomorphism by the Borel Moore algebraic cobordism (obtained by duality):

$$\mathbf{MGL}_0^c(X) \xrightarrow{\sim} CH_0(X).$$

[18] Indeed, this follows from the exact sequence (1.2.6.a).

4.4. Cohomology spectral sequence and cycle classes.

4.4.1. Consider a spectrum $\mathbb{E}$. The truncation functor for the homotopy t-structure gives a canonical (functorial) tower in $SH(k)(k)$:

$$\ldots \to \mathbb{E}_{\geq p} \to \mathbb{E}_{\geq p+1} \to \ldots$$

called the *Postnikov tower* of $\mathbb{E}$.

Then for any smooth scheme X and any integer $n \in \mathbb{Z}$, the Postnikov tower of $\mathbb{E}(n)$ gives the following spectral sequence:

$$E_{2,t}^{p,q}(X, \mathbb{E}) = H^q_{\mathrm{Nis}}(X, \underline{\pi}_{n-p}(\mathbb{E})_n) \Rightarrow \mathbb{E}^{p+q,n}(X) \tag{4.4.1.a}$$

which we simply call the *cohomology spectral sequence*.

Note that, when $\mathbb{E}$ is weakly orientable, Corollary 3.3.6 gives a canonical isomorphism between the E_2-term of the coniveau spectral sequence (4.3.2.a) and the E_2-term of the cohomology spectral sequence.

Remark 4.4.2. Using the same argument as in the proof of [Dég09, th. 6.4] and a construction of Gillet and Soulé[19], one can show that this isomorphism is compatible with the differential on each E_2-term and that they induces an isomorphism of spectral sequences. However, in the sequel, we will only need to be able to compute the E_2-term using the isomorphism of Corollary 3.3.6.

4.4.3. The advantage of the spectral sequence (4.4.1.a) compared to the coniveau spectral sequence is that it is obviously functorial in X (contravariantly).

Assume that $\mathbb{E}$ has the structure of a weak **MGL**-module. Consider smooth schemes X, Y and a projective morphism $f : Y \to X$ equidimensional of dimension n. Then according to the construction of [Dég08b, def. 5.12], we get a canonical morphism of **MGL**-modules:

$$f^* : \underline{\mathbf{MGL}}(X)(n)[2n] \to \underline{\mathbf{MGL}}(Y).$$

Using diagram (2.2.5.d) in the case $\mathbf{R} = \mathbf{MGL}$, this map induces after applying the functor $\mathbb{E}^{i,j}$ a canonical pushforward:

$$\mathbb{E}^{i,j}(Y) \xrightarrow{f_*} \mathbb{E}^{i-2n,j-n}(X). \tag{4.4.3.a}$$

Lemma 4.4.4. *Consider the notations above. Then the Gysin map f^* induces a morphism of spectral sequences:*

$$E_{2,t}^{p,q}(\underline{\mathbf{MGL}}(Y), \mathbb{E}) = H^q_{\mathrm{Nis}}(Y, \underline{\pi}_{-p}(\mathbb{E})_0) \to H^{q-n}_{\mathrm{Nis}}(Y, \underline{\pi}_{-p}(\mathbb{E})_{-n}) = E_{2,t}^{p,q}(\underline{\mathbf{MGL}}(X)(n)[2n], \mathbb{E})$$

which converges to the morphism (4.4.3.a)*.*

Using the cohomology spectral sequence, we get the following proposition which gives mild conditions on a spectrum $\mathbb{E}$ for the existence of cycle classes in $\mathbb{E}$-cohomology satisfying the usual properties:

Proposition 4.4.5. *Let $\mathbb{E}$ be a (weak) ring spectrum such that:*

(a) *The homotopy module $\underline{\pi}_0(\mathbb{E})_*$ is orientable.*
(b) *For any function field K/k, $\mathbb{E}^{n,m}(K) = 0$ if $n < 0$ and $m < 0$.*

Then the following conditions hold:

(1) *The spectrum $\mathbb{E}$ admits an orientation whose associated formal group law is additive.*
(2) *For any smooth scheme X and any integer $n \geq 0$, there exists a canonical morphism of abelian groups:*

$$\sigma_X : CH^n(X) \to \mathbb{E}^{2n,n}(X)$$

which is natural with respect to pullbacks, projective pushforwards and compatible with products.

[19]More precisely, ore replaces the use of the shifted filtration of Deligne on complexes by its generalization for spectra given in [GS99].

Proof. According to assumption (a) and Corollary 4.1.7, the unit map $S^0 \to \mathbb{E}$ induces a canonical map

$$\underline{K}^M_* = \underline{K}^M_* \otimes \underline{\pi}_0(S^0)_* \to \underline{K}^M_* \otimes \underline{\pi}_0(\mathbb{E})_* = \underline{\pi}_0(\mathbb{E})_*$$

which is a morphism of monoids in $\Pi^{\eta=0}_*(k)$. In particular, according to Corollary 3.3.6, we get for any smooth scheme X and any integer $n \geq 0$ a canonical morphism:

$$CH^n(X) = A^n(X, K_*)_n \to A^n(X, \hat{\underline{\pi}}_0(\mathbb{E})_*)_n = H^n_{\mathrm{Nis}}(X, \underline{\pi}_0(\mathbb{E})_n) \tag{4.4.5.a}$$

compatible with pullbacks, pushforwards and products.

Applying again Corollary 3.3.6, assertion (b) implies that the term $E^{p,q}_{2,t}(X, \mathbb{E}(n))$ of the cohomology spectral sequence is zero if $p > min(q, n)$. Thus, we get a canonical composite map:

$$E^{n,n}_{2,t}(X, \mathbb{E}(n)) \xrightarrow{(1)} E^{n,n}_{\infty,t}(X, \mathbb{E}(n)) \xrightarrow{(2)} \mathbb{E}^{2n,n}(X) \tag{4.4.5.b}$$

where the map (1) is obtain as the sequence of epimorphism deduced from the spectral sequence (4.4.1.a) and the map (2) is the edge morphism (which is a monomorphism). This composite map is compatible with products and pushforwards according to paragraph 4.4.3. The fact it is compatible with products follows from the construction of products on the spectral sequence of the type (4.4.1.a) (see [McC01]).

The composition of (4.4.5.a) and (4.4.5.b) gives the map of the point (2). In the case $n = 1$, we get an orientation of $\mathbb{E}$ whose associated formal group law is additive because σ_X is a morphism of groups. □

Remark 4.4.6. According to a result announced by Hopkins and Morel, at least over a field of characteristic 0, the map induced by the morphism φ of Paragraph 4.3.3 induces an isomorphism of ring spectra

$$\mathbf{MGL}/\{a_{ij}, (i,j) \in \mathbb{N}^2\} \to \mathbf{H}$$

where $a_{ij} : S^{2(i+j),i+j} \to \mathbf{MGL}$ denotes the coefficients of the formal group law of $\mathbf{MGL}$ equipped with its obvious orientation.

The orientation of the first point of the previous proposition corresponds to a morphism of ring spectra

$$\psi : \mathbf{MGL} \to \mathbb{E}$$

such that $\psi \circ a_{ij}$ is zero. In particular, ψ induces a canonical morphism of ring spectra:

$$\sigma : \mathbf{H} \to \mathbb{E}.$$

which gives back the cycle class of point (2) of Proposition 4.4.5.

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LAGA, CNRS (UMR 7539), Université Paris 13, Avenue Jean-Baptiste Clément, 93430 Villetaneuse, France

E-mail address: `deglise@math.univ-paris13.fr`

URL: `http://www.math.univ-paris13.fr/~deglise/`