

THE RIGID SYNTOMIC RING SPECTRUM

F. DÉGLISE, N. MAZZARI

ABSTRACT. The aim of this paper is to show that Besser syntomic cohomology is representable by a rational ring spectrum in the motivic homotopical sense. In fact, extending previous constructions, we exhibit a simple representability criterion and we apply it to several cohomologies in order to get our central result. This theorem gives new results for syntomic cohomology such as h-descent and the compatibility of cycle classes with Gysin morphisms. Along the way, we prove that motivic ring spectra induces a complete Bloch-Ogus cohomological formalism and even more. Finally, following a general motivic homotopical philosophy, we exhibit a natural notion of syntomic coefficients.

MSC: 14F42; 14F30.

Key words: Rigid syntomic cohomology, Beilinson motives, Bloch-Ogus.

INTRODUCTION

In the eighties A. Beilinson stated its conjectures relating the special values of L -functions and the regulator map of a variety X defined over a number field [Bei84, Bei86a]. The regulator considered by Beilinson is a map from the K -theory of X with target the Deligne-Beilinson cohomology¹ with real coefficients

$$\text{reg} : K_{2i-n}(X)^{(i)} \otimes \mathbb{Q} \rightarrow H_{DB}^n(X, \mathbb{R}(i)) .$$

One can define $H_{DB}^n(X, A(i))$ for any subring $A \subset \mathbb{R}$ and for $A = \mathbb{Z}$ he proved that the latter is the absolute Hodge cohomology theory: i.e. has a natural interpretation in terms of group homomorphisms in the derived category of mixed Hodge structures:

$$H_{DB}^n(X, i) = \text{Hom}_{D^b(MHS)}(\mathbb{Z}, R\Gamma_{Hdg}(X)(i)[n])$$

where $R\Gamma_{Hdg}(X)$ is the mixed Hodge complex whose cohomology is the Betti cohomology of X endowed with its mixed Hodge structure [Bei86b]. Further he conjectured that the higher K -theory groups form an absolute cohomology theory, in fact the universal one, called motivic cohomology. This vision is now partly accomplished. We do not have the category of mixed motives, but we can construct a triangulated category playing the role of its derived category. More precisely Cisinski and Déglise proved that for any finite dimensional noetherian scheme X there exists a monoidal triangulated category $DM_{\mathbb{B}}(X) = DM_{\mathbb{B}}(X, \mathbb{Q})$ (along with the 6 operations) such that

$$H_{\mathbb{B}}^{n,i}(X) := \text{Hom}_{DM_{\mathbb{B}}(S)}(\mathbb{1}_S, \pi_* \mathbb{1}_X(i)[n]) \simeq K_{2i-n}(X)^{(i)} \otimes \mathbb{Q}$$

when $\pi : X \rightarrow S$ is a smooth morphism and S is regular [CD09b].

Now let K be a p -adic field (i.e. a finite extension of \mathbb{Q}_p) with ring of integers R . Given X a smooth and algebraic R -scheme Besser, extending a construction initiated by Gros, defined the analogue of the Deligne-Beilinson cohomology in order to study the Beilinson conjectures for p -adic L -functions (see [Gro90], [Bes00]). It is the *rigid syntomic*² cohomology, denoted by $H_{\text{syn}}^n(X, i)$. Roughly it is defined as follows: let $R\Gamma_{\text{rig}}(X_s)$ (resp. $R\Gamma_{\text{dR}}(X_\eta)$) be a complex of \mathbb{Q}_p -vector spaces

Date: November 23, 2012.

¹Here we assume that the weight filtration is part of the definition. This is not the case in the original definition by Deligne, where only the Hodge filtration was considered. See [Bei86b] for a complete discussion.

²The word *rigid* is due to the fact that the rigid cohomology of Berthelot plays a role in the definition. The word *syntomic* comes from the work of Fontaine-Messing [FM87] where the syntomic site was used to define a cohomology theory strictly related to the one of Besser in the smooth and projective case.

whose cohomology is the rigid (resp. de Rham) cohomology of the special fiber X_s (resp. generic fiber X_η) of X , then

$$H_{\text{syn}}^n(X, i) = H^{n-1}(\text{Cone}(f : R\Gamma_{\text{rig}}(X_s) \oplus F^i R\Gamma_{\text{dR}}(X_\eta) \rightarrow R\Gamma_{\text{rig}}(X_s) \oplus R\Gamma_{\text{rig}}(X_s)))$$

where $f(x, y) = (\phi(x) - p^i x, sp(y) - x)$, ϕ is the Frobenius map, sp is Berthelot specialization map.

There is a regulator map for this theory and it can be interpreted as an absolute cohomology too [Ban02, CCM12].

The aim of the present paper is to represent rigid syntomic cohomology in the triangulated category of motives by a ring object \mathbb{E}_{syn} . This allows to prove that rigid syntomic cohomology is a Bloch-Ogus theory and satisfies h-descent (*i.e.* proper and fppf descent, see below). In particular we obtain that the Gysin map is compatible with the direct image of cycles as conjectured by Besser. We can say that this paper is the natural pushout of the work of the first author in collaboration with Cisinski [CD12] and that of the second author in collaboration with Chiarellotto and Ciccioni [CCM12].

Let's review in more details the content of this work.

First we recall some results of the motivic homotopy theory. Let S be a base scheme (noetherian and finite dimensional). To any object \mathbb{E} in $DM_{\mathbb{B}}(S)$ we can associate a bi-graded cohomology theory

$$\mathbb{E}^{n,i}(X) := \text{Hom}_{DM_{\mathbb{B}}(S)}(M(X), \mathbb{E}(i)[n])$$

where $M(X) := \pi_! \pi^! \mathbb{1}_S$ is the (covariant) motive of $\pi : X \rightarrow S$. The cohomology defined by the unit object $\mathbb{1}_S$ of the monoidal category $DM_{\mathbb{B}}(S)$ represents rational motivic cohomology denoted by $H_{\mathbb{B}}$. When X is regular, $H_{\mathbb{B}}^{n,i}(X)$ coincides with the original definition of Beilinson using Adams operations on rational Quillen K-theory.

The category of Beilinson motives $DM_{\mathbb{B}}(S)$ can be constructed using some homotopical machinery starting with the category $\mathcal{C}(S, \mathbb{Q})$ of complexes of \mathbb{Q} -linear pre-sheaves on the category of affine and smooth S -schemes (See § 1). An object of $DM_{\mathbb{B}}(S)$ should be thought as a cohomology theory on the category of S -schemes which is \mathbb{A}^1 -homotopy invariant, satisfies the Nisnevich excision and is oriented (in the sense of remark 1.4.11 point (1)).

We are particularly concerned with a certain class of objects in $DM_{\mathbb{B}}(S)$ which we call *motivic ring spectra* (Def. 1.4.13). Given such an object \mathbb{E} the associated cohomology theory $\mathbb{E}^{n,i}(X)$ is naturally a bi-graded \mathbb{Q} -linear algebra and further we have the following properties:

Theorem (See § 3.1). *Let S be a regular scheme and \mathbb{E} a motivic ring spectrum in $DM_{\mathbb{B}}(S)$.*

- (1) Higher cycle class/regulator. – *The unit section of the ring spectrum \mathbb{E} induces a canonical morphism, called regulator:*

$$\sigma : H_{\mathbb{B}}^{n,i}(X) \rightarrow \mathbb{E}^{n,i}(X)$$

which is functorial in X and compatible with products.

- (2) Gysin. – *For any projective morphism $f : Y \rightarrow X$ between smooth S -schemes there is a (functorial) morphism*

$$f_* : \mathbb{E}^{n,i}(Y) \rightarrow \mathbb{E}^{n-2d, i-d}(X).$$

where d is the dimension of f .

- (3) Projection formula. – *For f as above and any pair $(x, y) \in \mathbb{E}^{*,*}(X) \times \mathbb{E}^{*,*}(Y)$, one has:*

$$f_*(f^*(x).y) = x.f_*(y).$$

- (3') Degree formula. – *For any finite morphism $f : Y \rightarrow X$ between smooth connected S -schemes, and any $x \in \mathbb{E}^{n,i}(X)$,*

$$f_* f^*(x) = d.x$$

where d is the degree of the function fields extension associated with f .

- (4) Excess intersection formula. – Consider a cartesian square of smooth R -schemes:

$$\begin{array}{ccc} Y' & \xrightarrow{q} & X' \\ g \downarrow & & \downarrow f \\ Y & \xrightarrow{p} & X \end{array}$$

such that p is projective. Let ξ be the excess intersection bundle associated with that square. Then for any $y \in \mathbb{E}^{*,*}(Y)$, one gets:

$$f^*p_*(y) = q_*(c_e(\xi) \cdot g^*(y)).$$

- (5) The regulator map σ is natural with respect to the Gysin functoriality.
 (5') The regulator map σ induces a Chern character

$$\text{ch}_n : K_n(X)_{\mathbb{Q}} \rightarrow \bigoplus_{i \in \mathbb{Z}} \mathbb{E}^{2i-n, i-n}(X)$$

which satisfies the (higher) Riemann-Roch formula of Gillet (cf [Gil81])

This theorem uses only standard properties of motives. Using Grothendieck six functors formalism for $DM_{\mathbb{F}}$, we get the following refinement of the preceding results:

Theorem (See § 3.2). Consider the assumptions of the previous theorem.

- (6) Descent. – The cohomology $\mathbb{E}^{n,i}$ admits a functorial extension to diagrams of S -schemes and satisfies cohomological descent for the h -topology³: given any hypercover $p : \mathcal{X} \rightarrow X$ for the h -topology, the induced morphism:

$$p^* : \mathbb{E}^{n,i}(X) \rightarrow \mathbb{E}^{n,i}(\mathcal{X})$$

is an isomorphism.⁴

- (7) Bloch-Ogus theory. – One can associate with \mathbb{E} a canonical homology theory, the Borel-Moore \mathbb{E} -homology. For any separated S -scheme X with structural morphism f , and any pair of integers (n, i) , put:

$$\mathbb{E}_{n,i}^{\text{BM}}(X) = \text{Hom}(\mathbb{1}_S, f_* f^! \mathbb{E}(-i)[-n]).$$

Then, the pair $(\mathbb{E}, \mathbb{E}^{\text{BM}})$ is a twisted Poincaré duality theory with support in the sense of Bloch and Ogus (cf [BO74]). Moreover Borel-Moore \mathbb{E} -homology is contravariantly functorial with respect to smooth morphisms.

In fact, we obtain more than just point (7) in section 3.2. We also associate with \mathbb{E} a cohomology with compact support as well as a homology. We explain how to deduce from Grothendieck 6 functors formalism a rich functoriality for these theories, extending for example the existence of Gysin morphisms in cohomology to the case of smooth proper morphisms.⁵

Since our aim is to prove the above properties for the rigid syntomic cohomology of Besser we just need to represent it as a motivic ring spectrum. Thus we prove the following criterion, which is the main result of the first section. Before stating it we introduce the following notation: for any complex $E \in C(S, \mathbb{Q})$ and X/S smooth and affine let

$$H^n(X, E) := H^n(E(X)) .$$

Theorem (cf. Prop. 1.4.10). Let $(E_i)_{i \in \mathbb{N}}$ be a family of complexes in $C(S, \mathbb{Q})$ forming a \mathbb{N} -graded commutative monoid together with a section $c : \mathbb{Q}[0] \rightarrow E_1(\mathbb{G}_m)[1]$ satisfying the following properties:

- (1) Excision. – Let E_i^{Nis} be the associated Nisnevich sheaves. For any integer i and any X/S affine and smooth, $H^n(X, E_i) \simeq H_{\text{Nis}}^n(X, E_i^{\text{Nis}})$.
 (2) Homotopy. – For any integer i and any X/S affine and smooth, $H^n(X, E_i) \simeq H^n(\mathbb{A}_X^1, E_i)$.

³The h -topology was introduced by Voevodsky. Recall that covers for this topology are given by morphisms of schemes which are universal topological epimorphism.

⁴One deduce easily from this isomorphism the usual descent spectral sequence.

⁵By the very construction, the regulator is still compatible with this generalized Gysin morphism.

- (3) Stability. – Let \bar{c} be the image of c in $H^1(\mathbb{G}_m, E_1)$. For any smooth S -scheme X and any pair of integers (n, i) the following map

$$H^n(X, E_i) \rightarrow \frac{H^{n+1}(X \times \mathbb{G}_m, E_{i+1})}{H^{n+1}(X, E_{i+1})}, \quad x \mapsto \pi_X(x \times \bar{c})$$

is an isomorphism.

- (4) Orientation. – Let $u : \mathbb{G}_m \rightarrow \mathbb{G}_m$ be the inverse map of the group scheme \mathbb{G}_m , and denote by \bar{c}' the image of c in the group $H^1(\mathbb{G}_m, E_1)/H^1(S, E_1)$. The following equality holds: $u^*(\bar{c}') = -\bar{c}'$.

Then there exists a motivic ring spectrum \mathbb{E} together with canonical isomorphisms

$$\mathrm{Hom}_{DM_{\mathbb{B}}(S)}(M(X), \mathbb{E}(i)[n]) \simeq H^n(X, E_i)$$

for integers $(n, i) \in \mathbb{Z} \times \mathbb{N}$, functorial in the smooth S -scheme X and compatible with products. Moreover, \mathbb{E} depends functorially on $(E_i)_{i \in \mathbb{N}}$ and c .

The main difficulty of the above result is that the monoid structure on E_i is defined at the level of complexes of pre-sheaves and not just in the homotopy category. Using this result we can prove (in section 2) the existence of motivic ring spectra representing several cohomology theories. First we prove that for any algebraic scheme X defined over a field of characteristic zero there is a motivic ring $\mathbb{E}_{\mathrm{FdR}}$ such that $\mathbb{E}_{\mathrm{FdR}}^{n,i}(X) \simeq F^i H_{\mathrm{dR}}^n(X)$ is the i -th step of the Hodge filtration of the de Rham cohomology of X as defined by Deligne [Del74]. Then we prove that the rigid cohomology of Berthelot is represented by a motivic ring spectrum $\mathbb{E}_{\mathrm{rig}}$ too. As we already mentioned the syntomic cohomology of Besser is defined using a kind of mapping cone complex whose components are differential graded algebras (namely is the homotopy limit of the diagram in 2.5.1). Thus we cannot apply directly the above criterion since we would need to define a multiplication on the cone compatible with that of its components. To go around this problem we prove that a homotopy limit of motivic ring spectra is a motivic ring spectrum. Hence the rigid syntomic cohomology can be represented by a motivic ring spectrum too.

As already mentioned, the existence of $\mathbb{E}_{\mathrm{syn}}$ allows us to naturally extend the rigid syntomic cohomology to singular schemes. By *devissage*, we show how to compute the syntomic cohomology of a semi-stable curve.

In passing we show some results about what we call the *absolute rigid cohomology* given by

$$H_{\phi}^n(X, i) := \mathrm{Hom}_{D^b(F\text{-isoc})}(\mathbb{1}, R\Gamma(X)(i)[n])$$

where $R\Gamma(X)$ is a complex of F -isocrystals such that $H^n(R\Gamma(X)) = H_{\mathrm{rig}}^n(X)$, for X a scheme over a perfect field k .

The last application of the representability theorem of rigid syntomic cohomology is the existence of a natural theory of *rigid syntomic coefficients* for R -schemes (section 3.3). Using the techniques of [CD09b, sec. 16], we set up the theory of *rigid syntomic modules*: over any R -scheme X , they are modules (in a strict homotopical sense) over the ring spectrum $\mathbb{E}_{\mathrm{syn}, X}$ obtained by pullback along the structural morphism of X/R . The corresponding category $\mathbb{E}_{\mathrm{syn}\text{-mod}} X$ for various R -schemes X , shared many of the good properties of the category $DM_{\mathbb{B}}$, such as the complete Grothendieck 6 functors formalism. It receives a natural realization functor from $DM_{\mathbb{B}}$, which is triangulated, monoidal (and commutes with f^* and $f_!$).

This construction might be the main novelty of our representability theorem. However, to be complete we should relate these modules with more concrete categories of coefficients, probably related with F -isocrystals. This relation will be investigated in a future work.

Acknowledgments. The authors are grateful to Joseph Ayoub, Amnon Besser, Bruno Chiarellotto, Denis-Charles Cisinski, Andreas Langer, Victor Rotger and Georg Tamme, Jörg Wildeshaus for our initial motivation, their useful comments or stimulating conversations about their research in connection to this work. The second author is grateful to Michael Harris and the “Fondation Simone et Cino del Duca de l’Institut de France” for the year spent in Jussieu.

1. MOTIVIC HOMOTOPY THEORY

This section is concerned with recall on the motivic homotopy theory initiated by Morel and Voevodsky, more specifically on the part on ring spectra. We will recall the context of [CD12] with some complements coming from [CD09b].

Throughout this section, S will be a base scheme, assumed to be noetherian finite dimensional and Λ will be a ring of coefficients. We will denote by Sm/S either the category of smooth S -schemes of finite type or the category of such schemes which in addition are affine (absolutely). Note that equipped with the Nisnevich topology, the two induced topoi are equivalent.

1.1. The effective \mathbb{A}^1 -derived category.

1.1.1. We let $\text{PSh}(S, \Lambda)$ be the category of presheaves of Λ -modules on Sm/S and $\text{C}(\text{PSh}(S, \Lambda))$ the category of complexes of such presheaves. Given such a complex K , a smooth S -scheme X and an integer $n \in \mathbb{Z}$, we put:

$$H^n(X, K) := H^n(K(X)).$$

This is the cohomology of K computed in the derived category of $\text{PSh}(S, \Lambda)$: if we denote by $\Lambda(X)$ the presheaf of Λ -modules represented by X , we get:

$$H^n(X, K) = \text{Hom}_{\text{D}(\text{PSh}(S, \Lambda))}(\Lambda(X), K[n]).$$

A closed pair will be a couple (X, Z) such that X is a smooth S -scheme and Z is a closed subscheme of X – in fact one requires that X and $(X - Z)$ are in Sm/S . We also define the n -th cohomology group of (X, Z) – equivalently: of X with support in Z – with coefficients in K as:

$$H_Z^n(X, K) := H^{n-1}(\text{Cone}(K(X - Z) \rightarrow K(X))).$$

A morphism of closed pairs $f : (Y, T) \rightarrow (X, Z)$ is a morphism of schemes $f : Y \rightarrow X$ such that $f^{-1}(Z) \subset T$. We say f is *excisive* if it is étale and $f^{-1}(Z)_{red} = T_{red}$. Cohomology with coefficients in K is contravariant with respect to f .

Definition 1.1.2. Let K be a complex of $\text{PSh}(S, \Lambda)$.

- (1) We say that K is *Nis-local* if for any excisive morphism of closed pairs $f : (Y, T) \rightarrow (X, Z)$, the pullback morphism

$$f^* : H_Z^*(X, K) \rightarrow H_T^*(Y, K)$$

is an isomorphism.

- (2) We say that K is \mathbb{A}^1 -*local* if for any smooth S -scheme X , the pullback induced by the canonical projection p of the affine line over X

$$p^* : H^*(X, K) \rightarrow H^*(\mathbb{A}_X^1, K)$$

is an isomorphism.

Following Morel, we define the *effective \mathbb{A}^1 -derived category over S with coefficients in Λ* as the full subcategory of $\text{D}(\text{PSh}(S, \Lambda))$ made by complexes which are Nis-local and \mathbb{A}^1 -local. We will denote it by $\text{D}_{\mathbb{A}^1}^{eff}(S, \Lambda)$.

1.1.3. Let us recall the following facts on the category defined above:

- (1) Let $\text{Sh}(S, \Lambda)$ be the category of sheaves of Λ -modules on Sm/S for the Nisnevich topology. Then $\text{D}_{\mathbb{A}^1}^{eff}(S, \Lambda)$ is equivalent to the \mathbb{A}^1 -localization of the derived category $\text{D}(\text{Sh}(S, \Lambda))$, as defined in [CD12, § 1.1].

This comes from the fact that the pair of adjoint functors, whose left adjoint is the associated Nisnevich sheaf a , induces a derived adjunction

$$a : \text{D}(\text{PSh}(S, \Lambda)) \rightleftarrows \text{D}(\text{Sh}(S, \Lambda)) : \mathcal{O}$$

whose right adjoint \mathcal{O} is fully faithful with essential image the complexes which are Nis-local – this is classical see for example [CD09b, 5.2.10 and 5.2.13]. In particular, Nis-local complexes can be described as those complexes K which satisfy Nisnevich descent: for any Nisnevich hypercover $P_\bullet \rightarrow X$ of any smooth S -scheme X , the induced map:

$$K(X) \rightarrow \text{Tot}(K(P_\bullet))$$

is a quasi-isomorphism – the right hand-side is the total complex associated with the obvious double complex.

- (2) The reliability of the category $D_{\mathbb{A}^1}^{eff}(S, \Lambda)$ comes from the fact it can be described as the homotopy category associated with an explicit model category on the category $C(\text{PSh}(S, \Lambda))$ of complexes on the Grothendieck abelian category $\text{PSh}(S, \Lambda)$:

- *Weak equivalences* (also called weak \mathbb{A}^1 -equivalences) are the morphisms of complexes f such that for any complex K which is \mathbb{A}^1 -local and Nis-local, $\text{Hom}_{D(\text{PSh}(S, \Lambda))}(f, K)$ is an isomorphism.
- Fibrant objects are the complexes which are Nis-local and \mathbb{A}^1 -local. *Fibrations* are the morphisms of complexes which are epimorphisms and whose kernel is fibrant.

For the proof that this defines a model category, we refer the reader to [CD09a]: we first consider the model category structure associated with the Grothendieck abelian category $\text{PSh}(S, \Lambda)$ (see *loc. cit.* Ex. 2.3) and we localize it with respect to Nisnevich hypercovers and \mathbb{A}^1 -homotopy (*loc. cit.* Section 4). Let us recall that a typical example of *cofibrant objects* for this model structure are the presheaves of the form $\Lambda(X)$ for a smooth S -scheme X .

We derive from this model structure the existence of fibrant (resp. cofibrant) resolutions: associated with a complex of presheaves K , we get a map

$$K \rightarrow K_f \text{ (resp. } K_c \rightarrow K),$$

which is a fibration (resp. cofibration) and a weak \mathbb{A}^1 -equivalence. These resolutions can be chosen to be natural in K .

This can be used to derive functors. In particular, the natural tensor product \otimes of $C(\text{PSh}(S, \Lambda))$ as well as its internal complex morphism $\underline{\text{Hom}}$ can be derived using the formulas:

$$K \otimes^{\mathbf{L}} L = K_c \otimes L_c, \quad \mathbf{R}\underline{\text{Hom}}(K, L) = \underline{\text{Hom}}(K_c, L_f);$$

see *loc. cit.* Sections 3 and 4.⁶

1.2. The \mathbb{A}^1 -derived category.

1.2.1. We define the *Tate object* as the following complex of presheaves of Λ -modules:

$$(1.2.1.a) \quad \Lambda(1) := \text{coKer}(\Lambda \xrightarrow{s_1^*} \Lambda(\mathbb{G}_m))[-1]$$

where s_1 is the unit section of the group scheme \mathbb{G}_m , considered as an S -scheme. Given a complex K and an integer $i \geq 0$, we denote by $K(i)$ the tensor product of K with the i -th tensor power of $\Lambda(1)$ (on the right).

As usual in the general theory of motives, one is led to invert the object $\Lambda(1)$ for the tensor product. In the context of motivic homotopy theory, this is done using the construction of spectra, borrowed from algebraic topology.

For any integer $i > 0$, we will denote by Σ_i the group of permutations of the set $\{1, \dots, i\}$, $\Sigma_0 = 1$.

Definition 1.2.2. A *Tate spectrum* (over S with coefficients in Λ), is a sequence $\mathbb{E} = (E_i, \sigma_i)_{i \in \mathbb{N}}$ such that:

- for each $i \in \mathbb{N}$, E_i is a complex of $\text{PSh}(S, \Lambda)$ equipped with an action of Σ_i ,
- for each $i \in \mathbb{N}$, σ_i is a morphism of complexes

$$\sigma_i : E_i(1) \rightarrow E_{i+1},$$

called the *suspension map* (in degree n).

- For any integers $i \geq 0$, $r > 0$, the map induced by the morphisms $\sigma_i, \dots, \sigma_{i+r}$:

$$E_i(r) \rightarrow E_{i+r}$$

⁶ Note in particular that, according to Prop. 4.11, the model category described above is a monoidal model category which satisfies the monoid axiom.

is compatible with the action of $\Sigma_i \times \Sigma_r$, given on the left by the structural Σ_i -action on E_i and the action of Σ_r via the permutation isomorphism of the tensor structure on $C(\text{PSh}(S, \Lambda))$, and on the right via the embedding $\Sigma_i \times \Sigma_r \rightarrow \Sigma_{i+r}$.

A morphism of Tate spectra $f : \mathbb{E} \rightarrow \mathbb{F}$ is a sequence of Σ_i -equivariant maps $(f_i : E_i \rightarrow F_i)_{i \in \mathbb{N}}$ compatible with the suspension maps. The corresponding category will be denoted by $\text{Sp}(S, \Lambda)$.

A morphism f as above is called a *level weak equivalence* if for any integer $i \geq 0$, the morphism of complexes f_i is a quasi-isomorphism. We denote by $D_{\text{Tate}}(S, \Lambda)$ the localization of $\text{Sp}(S, \Lambda)$ with respect to level weak equivalences (See [CD12, Sec. 1.4]).

Complexes and spectra are linked by a pair of adjoint functors $(\Sigma^\infty, \Omega^\infty)$ defined respectively for a complex K and a Tate spectrum \mathbb{E} as follows:

$$(1.2.2.a) \quad \Sigma^\infty K := (K(i))_{i \in \mathbb{N}}, \quad \Omega^\infty(\mathbb{E}) = E_0,$$

where $K(i)$ is equipped with the action of Σ_i by its natural action on $K(i)$ through the symmetry isomorphism of the tensor structure on $C(\text{PSh}(S, \Lambda))$.

1.2.3. This category can be described using the category of symmetric sequences of $C(\text{PSh}(S, \Lambda))$: the objects of this category are the sequences $(E_i)_{i \in \mathbb{N}}$ of complexes on $\text{PSh}(S, \Lambda)$ such that E_i is equipped with an action of Σ_i . This is a Grothendieck abelian category on which one can construct a closed symmetric monoidal structure (see [CD09a, Section 7]). Moreover, the obvious symmetric sequence

$$\text{Sym}(\Lambda(1)) := (\Lambda(i))_{i \in \mathbb{N}}$$

has a canonical structure of a commutative monoid.

The category $\text{Sp}(S, \Lambda)$ is equivalent to the category of modules over $\text{Sym}(\Lambda(1))$ (see again *loc. cit.*). Therefore, it is formally a Grothendieck abelian category equipped with a closed symmetric monoidal structure. Note that the tensor product can be described by the following universal property: to give a morphism of Tate spectra $\mu : \mathbb{E} \otimes \mathbb{F} \rightarrow \mathbb{G}$ is equivalent to give a family of morphisms

$$\mu_{i,j} : E_i \otimes F_j \rightarrow G_{i+j}$$

which is $\Sigma_i \times \Sigma_j$ -equivariant and compatible with the suspension maps (see *loc. cit.* Remark 7.2).

Definition 1.2.4. Let \mathbb{E} be a Tate spectrum over S with coefficients in Λ .

- (1) We say that \mathbb{E} is *Nis-local* (resp. \mathbb{A}^1 -local) if for any integer $i \geq 0$, the complex E_i is *Nis-local* (resp. \mathbb{A}^1 -local).
- (2) We say that \mathbb{E} is a *Tate Ω -spectrum* if the morphism of $D_{\mathbb{A}^1}^{eff}(S, \Lambda)$ induced by adjunction from σ_j :

$$E_i \rightarrow \mathbf{R}\underline{\text{Hom}}(\Lambda(1), E_{i+1})$$

is an isomorphism (i.e. a weak \mathbb{A}^1 -equivalence).

For short, we say that \mathbb{E} is *stably fibrant* if it is an Ω -spectrum which is Nis-local and \mathbb{A}^1 -local.

We define the *\mathbb{A}^1 -derived category over S with coefficients in Λ* , denoted by $D_{\mathbb{A}^1}(S, \Lambda)$, as the full subcategory of $D_{\text{Tate}}(S, \Lambda)$ made by the stably fibrant Tate spectra.

1.2.5. Recall the following facts on the previous construction:

- (1) The construction of $D_{\mathbb{A}^1}(S, \Lambda)$ through spectra is a classical construction derived from algebraic topology (see [Hov01]). In particular, the monoidal model structure on the category $C(\text{PSh}(S, \Lambda))$ induces a canonical monoidal model structure on $\text{Sp}(S, \Lambda)$ whose homotopy category is precisely $D_{\mathbb{A}^1}(S, \Lambda)$. It is called the *stable model category*.

Therefore $D_{\mathbb{A}^1}(S, \Lambda)$ is a symmetric monoidal triangulated category with internal Hom. Moreover, the adjoint functors (1.2.2.a) can be derived:

$$(1.2.5.a) \quad \Sigma^\infty : D_{\mathbb{A}^1}^{eff}(S, \Lambda) \rightleftarrows D_{\mathbb{A}^1}(S, \Lambda) : \Omega^\infty.$$

The functor Σ^∞ is monoidal.⁷ Recall also that given a Tate Ω -spectrum \mathbb{E} as above and an integer $i \geq 0$, we get:

$$(1.2.5.b) \quad \Omega^\infty(\mathbb{E}(i)) = E_i.$$

We will simply denote by Λ or $\mathbb{1}$ the unit of $D_{\mathbb{A}^1}(S, \Lambda)$ – instead of $\Sigma^\infty \Lambda$.

- (2) In fact the triangulated categories of the form $D_{\mathbb{A}^1}(S, \Lambda)$ for various schemes S are not only closed monoidal but they are equipped with the complete formalism of Grothendieck 6 operations

$$(f^*, f_*, f!, f^\dagger, \otimes, \underline{\text{Hom}})$$

as established by Ayoub in [Ayo07].⁸

1.3. Triangulated mixed motives.

1.3.1. In this section, Λ is a \mathbb{Q} -algebra.

We recall the construction of Morel for deriving the triangulated category of mixed motives from the category $D_{\mathbb{A}^1}(S, \Lambda)$ (see [CD09b, 15.2] for details).

Let us consider the inverse map u of the multiplicative group scheme \mathbb{G}_m , corresponding to the map:

$$\mathcal{O}_S[t, t^{-1}] \rightarrow \mathcal{O}_S[t, t^{-1}], t \mapsto t^{-1}.$$

Recall from formula (1.2.1.a) the decomposition $\Lambda(\mathbb{G}_m) = \Lambda \oplus \Lambda(1)[1]$, considered in $D_{\mathbb{A}^1}(S, \Lambda)$. Given this decomposition, the map $u_* : \Lambda(\mathbb{G}_m) \rightarrow \Lambda(\mathbb{G}_m)$ can be written in matrix form as:

$$\begin{pmatrix} 1 & 0 \\ 0 & \epsilon_1 \end{pmatrix}$$

Because $\Lambda(1)[1]$ is \otimes -invertible in $D_{\mathbb{A}^1}(S, \Lambda)$, there exists a unique endomorphism ϵ of Λ in $D_{\mathbb{A}^1}(S, \Lambda)$ such that $\epsilon_1 = \epsilon(1)[1]$.

Because $u^2 = 1$, we get $\epsilon^2 = 1$. Thus we can define two complementary projectors in $\text{End}_{D_{\mathbb{A}^1}(S, \Lambda)}(\Lambda)$:

$$p_+ = \frac{1}{2} \cdot (\epsilon - 1_\Lambda), \quad p_- = \frac{1}{2} \cdot (\epsilon + 1_\Lambda).$$

Given any object \mathbb{E} in $D_{\mathbb{A}^1}(S, \Lambda)$, we deduce projectors $p_+ \otimes \mathbb{E}, p_- \otimes \mathbb{E}$ of \mathbb{E} . Because $D_{\mathbb{A}^1}(S, \Lambda)$ is pseudo-abelian⁹, we deduce a canonical decomposition:

$$\mathbb{E} = \mathbb{E}_+ \oplus \mathbb{E}_-$$

where \mathbb{E}_+ (resp. \mathbb{E}_-) is the image of $p_+ \otimes \mathbb{E}$ (resp. $p_- \otimes \mathbb{E}$). The following triangulated category was introduced by Morel.

Definition 1.3.2. An object \mathbb{E} in $D_{\mathbb{A}^1}(S, \Lambda)$ will be called a Morel motive if $\mathbb{E}_- = 0$. We denote by $D_{\mathbb{A}^1}(S, \Lambda)_+$ the full subcategory of $D_{\mathbb{A}^1}(S, \Lambda)$ made by Morel motives.

Note that according to the above, the fact \mathbb{E} is a Morel motive is equivalent to the property:

$$(1.3.2.a) \quad \epsilon \otimes \mathbb{E} = -1_{\mathbb{E}};$$

in other words, ϵ acts as -1 on \mathbb{E} .

1.3.3. Recall the following facts, which legitimate the terminology of “Morel motives”:

- (1) Obviously, the category $D_{\mathbb{A}^1}(S, \Lambda)_+$ is a triangulated monoidal sub-category of $D_{\mathbb{A}^1}(S, \Lambda)$. Moreover, the 6 operations on $D_{\mathbb{A}^1}(-, \Lambda)$ induce similar operations on $D_{\mathbb{A}^1}(-, \Lambda)_+$ which satisfy all of the 6 functors formalism.

⁷In fact, the homotopy category $D_{\mathbb{A}^1}(S, \Lambda)$ equipped with its left derived functor Σ^∞ , is universal for the property that Σ^∞ is monoidal and $\Sigma^\infty(K(1))$ is \otimes -invertible (see again [Hov01]).

⁸Ayoub treats only the case where f is quasi-projective for the existence of the adjoint pair $(f!, f^\dagger)$. The general case can be easily derived as explained in [CD09b, section 2]. The reader will also find a summary of the 6 operations formalism in *loc. cit.* Theorem 2.4.21.

⁹This is, for example, an application of the fact it is a triangulated category with arbitrary direct sums by a result of Neeman.

(2) According to [CD09b, 15.2.13], there is an equivalence of triangulated monoidal categories:

$$D_{\mathbb{A}^1}(S, \Lambda)_+ \simeq DM_{\mathbb{B}}(S, \Lambda)$$

where $DM_{\mathbb{B}}(S, \Lambda)$ is the triangulated category of Beilinson motives introduced in [CD09b, Def. 13.2.1]. In $DM_{\mathbb{B}}(S, \Lambda)$, given a smooth S -scheme X , we simply denote by $M(X)$ the object corresponding to $\Sigma^\infty \Lambda(X)$ and call it the *motive* of X .

Concretely, the above isomorphism means that when S is regular, for any smooth S -scheme X and any pair $(n, i) \in \mathbb{Z}^2$, one has a canonical isomorphism:

$$(1.3.3.a) \quad \text{Hom}_{D_{\mathbb{A}^1}(S, \Lambda)_+}(\Sigma^\infty \Lambda(X), \Lambda(i)[n]) \simeq K_{2i-n}^{(i)}(X) \otimes_{\mathbb{Q}} \Lambda$$

where $K_{2i-n}^{(i)}(X)$ denotes the i -th Adams subspace of the rational Quillen K-theory of X in homological degree $(2i - n)$.¹⁰

Note in particular that according to the coniveau spectral sequence in K-theory and a computation of Quillen, a particular case of the above isomorphism is the following one:

$$(1.3.3.b) \quad \text{Hom}_{D_{\mathbb{A}^1}(S, \Lambda)_+}(\Sigma^\infty \Lambda(X), \Lambda(n)[2n]) \simeq CH^n(X) \otimes_{\mathbb{Z}} \Lambda$$

where the left hand side is the Chow group of n -codimensional Λ -cycles in X (S is still assumed to be regular).

1.4. Ring spectra.

1.4.1. Recall that a commutative monoid in a symmetric monoidal category $(\mathcal{M}, \otimes, \mathbb{1})$ is an object M , a unit map $\eta : \mathbb{1} \rightarrow M$, a multiplication map $\mu : M \otimes M \rightarrow M$ such that the following diagrams are commutative:

$$\begin{array}{ccc} \text{Unit:} & \text{Associativity:} & \text{Commutativity:} \\ \begin{array}{ccc} M & \xrightarrow{1 \otimes \eta} & M \otimes M \\ & \searrow & \downarrow \mu \\ & & M \end{array} & \begin{array}{ccc} M \otimes M \otimes M & \xrightarrow{1 \otimes \mu} & M \otimes M \\ \mu \otimes 1 \downarrow & & \downarrow \mu \\ M \otimes M & \xrightarrow{\mu} & M \end{array} & \begin{array}{ccc} M \otimes M & & M \\ \downarrow \gamma & \searrow \mu & \uparrow \mu \\ M \otimes M & & M \end{array} \end{array}$$

where γ is the obvious symmetry isomorphism.

Definition 1.4.2. A *weak ring spectrum* (resp. *ring spectrum*) \mathbb{E} over S is a commutative monoid in the symmetric monoidal category $D_{\mathbb{A}^1}(S, \Lambda)$ (resp. $\text{Sp}(S, \Lambda)$).¹¹

1.4.3. A spectrum \mathbb{E} in $D_{\mathbb{A}^1}(S, \Lambda)$ defines a bigraded cohomology theory on smooth S -schemes X by the formula:

$$\mathbb{E}^{n,i}(X) = \text{Hom}_{D_{\mathbb{A}^1}(S, \Lambda)}(\Sigma^\infty \Lambda(X), \mathbb{E}(i)[n]).$$

The structure of a weak ring spectrum on \mathbb{E} corresponds to a product in cohomology, usually called the cup-product and defined as follows: given cohomology classes,

$$\alpha : \Sigma^\infty \Lambda(X) \rightarrow \mathbb{E}(i)[n], \quad \beta : \Sigma^\infty \Lambda(X) \rightarrow \mathbb{E}(j)[m]$$

one defines the class $\alpha \cup \beta$ as the following composite:

$$\Sigma^\infty \Lambda(X) \xrightarrow{\delta_*} \Sigma^\infty \Lambda(X) \otimes \Sigma^\infty \Lambda(X) \xrightarrow{\alpha \otimes \beta} \mathbb{E}(i)[n] \otimes \mathbb{E}(j)[m] \xrightarrow{\mu} \mathbb{E}(i+j)[n+m].$$

Using this definition, one can check easily that the commutativity axiom of \mathbb{E} implies the following formula:

$$\alpha \cup \beta = (-1)^{nm-ij} \cdot \epsilon^{ij} \cdot \beta \cup \alpha$$

where ϵ is the endomorphism of Λ introduced in Paragraph 1.3.1. In particular, if \mathbb{E} is a Morel motive, the product on \mathbb{E}^{**} is anti-commutative with respect to the first index and commutative with respect to the second one. Note also the following result which will be used later.

¹⁰This formula was first obtained by Morel but the proof has not been published. In any case, this is a consequence of *loc. cit.*

¹¹Ring spectra have slowly emerged in homotopy theory and the terminology is not fixed. Usually, our weak ring spectra (resp. ring spectra) are simply called ring spectra (resp. highly structured ring spectra).

Lemma 1.4.4. *Let \mathbb{E} be a weak ring spectrum with unit η and multiplication μ . Then the following conditions are equivalent:*

- (i) \mathbb{E} is a Morel motive.
- (ii) $\eta \circ \epsilon = -\eta$.

Proof. Let us remark that according to the Unit property the following equalities hold:

$$\begin{aligned}\mu \circ (1_{\mathbb{E}} \otimes \eta) &= 1_{\mathbb{E}}, \\ \mu \circ (1_{\mathbb{E}} \otimes (\eta \circ \epsilon)) &= \epsilon \otimes \mathbb{E}.\end{aligned}$$

Thus the equivalence between (i) and (ii) directly follows from relation (1.3.2.a) characterizing Morel motives. \square

Remark 1.4.5. Of course, a ring spectrum induces a weak ring spectrum. Concretely, in the non weak case, one requires that the diagrams of Paragraph 1.4.1 commutes in the mere category of spectra, and not only up to weak homotopy. This makes the construction of ring spectra more difficult than usual weak ring spectra.

1.4.6. Let us denote by $\mathrm{Sp}^{\mathrm{ring}}(S, \Lambda)$ the category of ring spectra. Because the category $\mathrm{Sp}(S, \Lambda)$ is a complete and cocomplete monoidal category, $\mathrm{Sp}^{\mathrm{ring}}(S, \Lambda)$ is complete and cocomplete. Moreover, the forgetful functor:

$$U : \mathrm{Sp}^{\mathrm{ring}}(S, \Lambda) \rightarrow \mathrm{Sp}(S, \Lambda)$$

admits a left adjoint which we denote by F . The following result appears in [CD09b, Th. 4.1.8].

Theorem 1.4.7. *Assume Λ is a \mathbb{Q} -algebra.*

Then the category $\mathrm{Sp}^{\mathrm{ring}}(S, \Lambda)$ is a model category whose weak equivalences (resp. fibrations) are the maps f such that $U(f)$ is a weak equivalence (resp. stable fibration) in the stable model category $\mathrm{Sp}(S, \Lambda)$ (see Par. 1.2.5).

We denote by $\mathcal{H}o(\mathrm{Sp}^{\mathrm{ring}}(S, \Lambda))$ the homotopy category associated with this model category.

1.4.8. For a given \mathbb{Q} -algebra Λ , recall the following consequences of this theorem:

- (1) The pair of adjoint functors (F, U) can be derived and induces adjoint functors:

$$\mathbf{L}F : \mathbf{D}_{\mathbb{A}^1}(S, \Lambda) \rightleftarrows \mathcal{H}o(\mathrm{Sp}^{\mathrm{ring}}(S, \Lambda)) : U$$

The essential image of the functor U lies in the category of weak ring spectra. However, it is not essentially surjective on that category.

- (2) As any homotopy category of a model category, the homotopy category $\mathcal{H}o(\mathrm{Sp}^{\mathrm{ring}}(S, \Lambda))$ admits homotopy limits and colimits (see [Cis03, Intro. Th. 1]). In other words, any diagram of $\mathrm{Sp}^{\mathrm{ring}}(S, \Lambda)$ admits a homotopy limit and a homotopy colimit.

1.4.9. A commutative monoid in the category $\mathbf{C}(\mathrm{PSh}(S, \Lambda))$ is usually called a commutative differential graded Λ -algebra with coefficients in the abelian monoidal category $\mathrm{PSh}(S, \Lambda)$.

A \mathbb{N} -graded commutative monoid in $\mathbf{C}(\mathrm{PSh}(S, \Lambda))$ is a sequence $(E_i)_{i \in \mathbb{N}}$ of complexes of pre-sheaves equipped with a unit map $\eta : \Lambda \rightarrow E_0$ and multiplication maps $\mu_{ij} : E_i \otimes E_j \rightarrow E_{i+j}$ for any pair of integers (i, j) such that the following diagrams commute:

$$\begin{array}{ccc} \text{Unit:} & \text{Associativity:} & \text{Commutativity:} \\ \begin{array}{ccc} E_i & \xrightarrow{1 \otimes \eta} & E_i \otimes E_0 \\ & \searrow & \downarrow \mu_{i,0} \\ & & E_i \end{array} & \begin{array}{ccc} E_i \otimes E_j \otimes E_k & \xrightarrow{1 \otimes \mu_{jk}} & E_i \otimes E_{j+k} \\ \mu_{ij} \otimes 1 \downarrow & & \downarrow \mu_{i,j+k} \\ E_{i+j} \otimes E_k & \xrightarrow{\mu_{i+j,k}} & E_{i+j+k} \end{array} & \begin{array}{ccc} E_i \otimes E_j & & \\ \downarrow \gamma_{ij} & \searrow \mu_{ij} & \\ E_j \otimes E_i & \nearrow \mu_{j,i} & E_{i+j} \end{array} \end{array}$$

where γ_{ij} is the obvious symmetry isomorphism. We then define bigraded cohomology groups for any smooth S -scheme X and any couple of integers (n, i) :

$$H^n(X, E_i) = H^n(E_i(X)).$$

The above monoid structure induces an exterior product on these cohomology groups:

$$H^n(X, E_i) \otimes H^m(Y, E_j) \rightarrow H^{n+m}(X \times_S Y, E_{i+j}), (x, y) \mapsto x \times y.$$

Given any smooth S -scheme X , we let $p : X \times \mathbb{G}_m \rightarrow X$ be the canonical projection and consider for the next statement the following split exact sequence:

$$0 \rightarrow H^n(X, E_i) \xrightarrow{p^*} H^n(X \times \mathbb{G}_m, E_i) \xrightarrow{\pi_X} \tilde{H}^n(X \times \mathbb{G}_m, E_i) \rightarrow 0$$

where $\tilde{H}^n(X \times \mathbb{G}_m, E_i) := \text{Coker}(p^*)$.

Proposition 1.4.10. *Suppose given a \mathbb{N} -graded commutative monoid $(E_i)_{i \in \mathbb{N}}$ in $\mathbf{C}(\text{PSh}(S, \Lambda))$ as above together with a section c of $E_1[1]$ over \mathbb{G}_m satisfying the following properties:*

- (1) Excision. – For any integer i , E_i is Nis-local.
- (2) Homotopy. – For any integer i , E_i is \mathbb{A}^1 -local.
- (3) Stability. – Let \bar{c} be the image of c in $H^1(\mathbb{G}_m, E_1)$. For any smooth S -scheme X and any pair of integers (n, i) the following map

$$H^n(X, E_i) \rightarrow \tilde{H}^{n+1}(X \times \mathbb{G}_m, E_{i+1}), x \mapsto \pi_X(x \times \bar{c})$$

is an isomorphism.

Then there exists a ring spectrum \mathbb{E} which is a stably fibrant Tate spectrum together with canonical isomorphisms

$$(1.4.10.a) \quad \text{Hom}_{\mathbf{D}_{\mathbb{A}^1}(S, \Lambda)}(\Sigma^\infty \Lambda(X), \mathbb{E}(i)[n]) \simeq H^n(X, E_i)$$

for integers $(n, i) \in \mathbb{Z} \times \mathbb{N}$, functorial in the smooth S -scheme X and compatible with products. Moreover, \mathbb{E} depends functorially on $(E_i)_{i \in \mathbb{N}}$ and c .

Assume Λ is a \mathbb{Q} -algebra. Let $u : \mathbb{G}_m \rightarrow \mathbb{G}_m$ be the inverse map of the group scheme \mathbb{G}_m , and denote by \bar{c}' the image of c in the group $\tilde{H}^1(\mathbb{G}_m, E_1)$. Then, under the above assumptions, the following conditions are equivalent:

- (i) The Tate spectrum \mathbb{E} is a Morel motive (i.e. defines an object in $DM_{\mathbb{B}}(S, \Lambda)$, Def. 1.3.2 and Par. 1.3.3).
- (ii) The following equality holds in $\tilde{H}^1(\mathbb{G}_m, E_1)$: $u^*(\bar{c}') = -\bar{c}'$.

Remark 1.4.11. (1) The two last properties should be called the *Orientation* property. In fact, they can be reformulated by saying that \mathbb{E} is an oriented ring spectrum (cf [CD09b, Cor. 13.2.15]). Recall also this is equivalent to the existence of a canonical morphism of groups:

$$\text{Pic}(X) \rightarrow H^2(X, E_1)$$

which is functorial in X (and even uniquely determined by c).

- (2) The Stability axiom can be reformulated by saying that for any $x \in H^{n+1}(X \times \mathbb{G}_m, E_{i+1})$ there exists a unique couple $(x_0, x_1) \in H^{n+1}(X, E_{i+1}) \times H^n(X, E_i)$ such that:

$$x = p^*(x_0) + x_1 \times \bar{c}'.$$

- (3) Though we start with a positively graded complex $(E_i)_{i \in \mathbb{N}}$ we get a cohomology theory which possibly has negative twists. These negative twists are given by the following short exact sequence for $i > 0$:

$$0 \rightarrow \mathbb{E}^{n, -i}(X) \rightarrow H^n(X \times \mathbb{G}_m^i, E_0) \rightarrow H^n(X \times \mathbb{G}_m^{i-1}, E_0) \rightarrow 0$$

where the epimorphism is given by the sums of the maps induced i -th inclusion

$$\mathbb{G}_m^{i-1} \rightarrow \mathbb{G}_m^i$$

where one of the coordinate of the target is set to 1.

Proof. We define the Tate spectrum \mathbb{E} to be the complex of presheaves E_i in degree i with trivial action of Σ_i . The class c defines a map of presheaves:

$$c' : \Lambda(1) \rightarrow \Lambda(\mathbb{G}_m)[-1] \xrightarrow{c[-1]} E_1$$

where the first map is given by the canonical inclusion. We define the suspension map of \mathbb{E} in degree i as the following composite:

$$\sigma_i : E_i(1) = E_i \otimes \Lambda(1) \xrightarrow{1 \otimes c'} E_i \otimes E_1 \xrightarrow{\mu_{i,1}} E_{i+1}.$$

One deduces from the commutative diagram called ‘‘Commutativity’’ of Paragraph 1.2.1 that the induced map $E_i(r) \rightarrow E_{i+r}$ is $\Sigma_i \times \Sigma_r$ -equivariant. So that \mathbb{E} is indeed a Tate spectrum.

By definition, Assumptions (1) and (2) exactly say that \mathbb{E} is Nis-local and \mathbb{A}^1 -local. It remains to check it is an Ω -spectra. In other words, the map obtained by adjunction from σ_i

$$\sigma'_i : E_i \rightarrow \mathbf{R}\underline{\mathrm{Hom}}(\Lambda(1), E_{i+1})$$

is an isomorphism in $D_{\mathbb{A}^1}^{eff}(S, \Lambda)$. It is sufficient to check that $\mathrm{Hom}_{D_{\mathbb{A}^1}^{eff}(S, \Lambda)}(\Lambda(X), \sigma'_i[n])$ is an isomorphism for every smooth S -scheme X and any integer $n \in \mathbb{Z}$. According to the definition, we can compute this map as follows:

$$(1.4.11.a) \quad \mathrm{Hom}(\Lambda(X), E_i[n]) \rightarrow \mathrm{Hom}(\Lambda(X) \otimes \Lambda(1), E_{i+1}[n]), x \mapsto x \times \bar{c}'$$

where \bar{c}' is the class of the map c' in $D_{\mathbb{A}^1}^{eff}(S, \Lambda)$. Using the fact E_i is Nis-local and \mathbb{A}^1 -local, the source of this map is isomorphic to $H^n(X, E_i)$. Similarly, the group of morphisms

$$\mathrm{Hom}(\Lambda(X) \otimes \Lambda(\mathbb{G}_m), E_{i+1}[n+1])$$

is isomorphic to $H^{n+1}(X \times \mathbb{G}_m, E_{i+1})$. Under this isomorphism, the target of the above map corresponds to $\tilde{H}^{n+1}(X \times \mathbb{G}_m, E_{i+1})$. Under these identifications, $\bar{c}' = \pi_X(\bar{c})$. Thus, the fact σ'_i is an isomorphism directly follows from Assumption (3).

According to this construction, the maps η and μ_{ij} induces a structure of a ring spectrum on \mathbb{E} (using in particular the description of the tensor product of spectra recalled in Paragraph 1.2.3).

The isomorphism (1.4.10.a) follows using the adjunction (1.2.5.a) and the relation (1.2.5.b) applied to the Tate Ω -spectrum \mathbb{E} . The fact it is functorial and compatible with products is obvious from the above construction.

Let us finally consider the remaining assertion. Note that according to what was just said, the class \bar{c}' introduced in the beginning of the proof coincides with the class \bar{c}' which appears in the statement of the proposition. Under the isomorphism (1.4.10.a), the canonical isomorphism:

$$\mathrm{Hom}_{D_{\mathbb{A}^1}(S, \Lambda)}(\Lambda, \mathbb{E}) \rightarrow \mathrm{Hom}_{D_{\mathbb{A}^1}(S, \Lambda)}(\Lambda(1), \mathbb{E}(1))$$

corresponds to an isomorphism of the form

$$\mathrm{Hom}_{D_{\mathbb{A}^1}^{eff}(S, \Lambda)}(\Lambda, E_0) \rightarrow \mathrm{Hom}_{D_{\mathbb{A}^1}^{eff}(S, \Lambda)}(\Lambda(1), E_1) = \tilde{H}^1(\mathbb{G}_m, E_1)$$

which is a particular case of the isomorphism (1.4.11.a) considered above. Thus, it sends the unit map η of \mathbb{E} to the class \bar{c}' . Thus the equivalence of conditions (i) and (ii) follows from Lemma 1.4.4. \square

Remark 1.4.12. This proposition is an extension of the construction given in [CD12, sec. 2.1]. The main difference is that we consider here theories in which the different twists are not necessarily isomorphic. By contrast, we require the datum of a stability class here whereas we do not need a particular choice in *op. cit.*

Note also that a similar extension has appeared in [AH10] applied to Deligne cohomology.

Definition 1.4.13. A *motivic ring spectrum* is a stably fibrant ring spectrum \mathbb{E} which is also a Morel motive.

2. SYNTOMIC SPECTRUM

In this section we construct several motivic ring spectra (see Def. 1.4.13): $\mathbb{E}_{\mathrm{FDR}}$, $\mathbb{E}_{\mathrm{rig}}$, \mathbb{E}_{ϕ} , $\mathbb{E}_{\mathrm{syn}}$. First for a field K of characteristic zero we construct $\mathbb{E}_{\mathrm{FDR}}$ representing the filtered part of the de Rham cohomology of a K -scheme. i.e.

$$\mathbb{E}_{\mathrm{FDR}}^{n,i}(X) := \mathrm{Hom}_{D_{\mathbb{A}^1}(\eta, \mathbb{Q})}(\Sigma^\infty \mathbb{Q}(X), \mathbb{E}_{\mathrm{FDR}}(i)[n]) \simeq F^i H_{\mathrm{DR}}^n(X).$$

Then we define \mathbb{E}_{rig} which represents the rigid cohomology of Berthelot. This was already proved in [CD12] in a different way. For both \mathbb{E}_{FdR} and \mathbb{E}_{rig} we use the criteria of Proposition 1.4.10.

Finally we get a motivic ring spectrum \mathbb{E}_{syn} for the rigid syntomic cohomology as a homotopy limit of a diagram of ring spectra.

2.1. Cosimplicial tools.

2.1.1. Let Δ be the category of finite ordered sets $[n] := \{0, \dots, n\}$ as objects and monotone nondecreasing functions as morphisms. Let $\delta_i(n) : [n-1] \rightarrow [n]$ (resp. $\sigma_i(n) : [n] \rightarrow [n-1]$) be the usual¹² (co)face (resp. (co)degeneracy) map, for $0 \leq i \leq n$. In case there is no ambiguity we will simply write δ_i, σ_i . Given a category C , a *simplicial* (resp. *cosimplicial*) object of C is a functor from Δ° (resp. Δ) to C .

For instance let $A_n = \mathbb{Q}[T_0, \dots, T_n]/(\sum T_i - 1)$, then this is a simplicial \mathbb{Q} -algebra in an obvious way. It follows that the associated dga of Kahler differentials

$$(2.1.1.a) \quad \omega_n := \Omega_{A_n/\mathbb{Q}}^\bullet \quad n \geq 0$$

is a simplicial dga over \mathbb{Q} . We will denote by $\delta^i = \delta_i^*$ (resp. $\sigma^i = \sigma_i^*$) the structural morphisms

Now let M be a cosimplicial abelian group and sM the associated simple complex ($sM^i = M[i]$ and the differentials are the alternate sums of the coface morphisms). Its *standard normalization* NM is the subcomplex of sM s.t. $N^q M := \bigcap_i \ker(\sigma_i) \subset M^q$. Then inclusion $NM \rightarrow sM$ is a homotopy equivalence. Now if M is also a cosimplicial commutative monoid the Alexander-Whitney product¹³ gives a (differential graded) monoid structure on sM and NM , but this is not necessarily (graded) commutative. Thus consider the following construction due to Thom and Sullivan. Let

$$\tilde{N}^q M \subset \prod_m \omega_m^q \otimes M^q$$

be the submodule whose elements are sequences $(x_m)_{m \geq 0}$ such that

$$(\text{Id} \oplus \delta_i)x_m = (\delta^i \oplus \text{Id})x_{m+1} \quad , \quad (\sigma^i \oplus \text{Id})x_m = (\text{Id} \oplus \sigma_i)x_{m+1}$$

and define the differentials $D : \tilde{N}^q M \rightarrow \tilde{N}^{q+1} M$ by $D = ((-1)^q \sum_i (-1)^i \text{Id} \otimes \delta_i) + \sum_i (-1)^i (\delta^i \otimes \text{Id})$. With the above notation if M is further a cosimplicial commutative monoid then $\tilde{N}M$ is a commutative monoid too. Namely we can define

$$(2.1.1.b) \quad \tilde{N}M \otimes \tilde{N}M \rightarrow \tilde{N}M$$

induced by $(\alpha \otimes m) \otimes (\alpha' \otimes m') = \alpha \wedge \alpha' \otimes (m \cdot m')$.

Moreover the complex $\tilde{N}M$ is quasi-isomorphic to the standard normalization NM (and then to sM).¹⁴

We can extend the above constructions to the setting of cosimplicial dg abelian groups. Given such an $M = M^{pq}$ (where q is the cosimplicial parameter), then sM (resp. $NM, \tilde{N}M$) is naturally a double complex and we can apply the total complex functor, denoted by tot , to obtain a dg abelian group.

Now we are ready to state a technical result well known to the specialists.

Proposition 2.1.2 (cf. [HS87]). *Let M be a cosimplicial (commutative) dga over \mathbb{Q} then there exists a canonical (commutative) dga $\tilde{N}M$ and a quasi-isomorphism $\int : \tilde{N}M \rightarrow sM$ inducing an isomorphism of (commutative) dg algebras in cohomology $H(\int) : H(\tilde{N}M) \rightarrow H(sM)$.*

¹²i.e. the image of $\delta_i(n)$ (resp kernel of $\sigma_i(n)$) is $[n] \setminus \{i\}$.

¹³This is given as follows. Let $\delta^- : [q] \rightarrow [q+q']$ (resp. $\delta^+ : [q'] \rightarrow [q]$) the map with image $\{0, 1, \dots, q\}$ (resp. $\{q, q+1, \dots, q+q'\}$), then define $a * b := \delta^-(a) \cdot \delta^+(q)$.

¹⁴The isomorphism is induced by the integration map $\int : \omega_n^\bullet \otimes M^n \rightarrow \mathbb{Q}[-n] \otimes M^n$ defined by

$$(dT_1 \wedge \dots \wedge dT_n) \otimes m \mapsto \frac{1}{n!} \otimes m .$$

Note that we need to work with \mathbb{Q} -coefficients to define this map.

2.1.3. (Godement resolutions) Let $u : P \rightarrow X$ be a morphism of Grothendieck sites so that P^\sim (resp. X^\sim) is the category of abelian sheaves on P (resp. X). Then we have a pair of adjoint functors (u^*, u_*) , where $u^* : X^\sim \rightarrow P^\sim$, $u_* : P^\sim \rightarrow X^\sim$. For any object \mathcal{F} of X^\sim we can define a co-simplicial object $B^*(\mathcal{F})$ whose component in degree n is $(u_* u^*)^{n+1}(\mathcal{F})$.¹⁵

Proposition 2.1.4. *Let $u : P \rightarrow X$ a morphism of sites with u^* exact and conservative and \mathcal{F} a complex of sheaves on X . Then*

- (1) *The complex $\text{Gdm}_P(\mathcal{F}) := \text{tot} B^\bullet(\mathcal{F})$ is a functorial flask resolution of \mathcal{F}*
- (2) *If \mathcal{F} is a \mathbb{Q} -linear sheaf, the Thom-Sullivan normalization $\widetilde{\text{Gdm}}_P(\mathcal{F}) := \text{tot} \widetilde{N} B^\bullet(\mathcal{F})$ is a functorial resolution of \mathcal{F} ;*
- (3) *If \mathcal{F} is a sheaf of (commutative) dga over \mathbb{Q} , then the complex $\widetilde{\text{Gdm}}_P(\mathcal{F}) = \text{tot} \widetilde{N} B^\bullet(\mathcal{F})$ is a sheaf of commutative dga and the canonical isomorphism $H^\bullet(X, \mathcal{F}) \cong H^\bullet(\Gamma(X \text{Gdm}(\mathcal{F})))$ is compatible with respect to the multiplicative structure.*

Proof. When u^* is exact and conservative the canonical map $b_{\mathcal{F}} : \mathcal{F} \rightarrow sB^*(\mathcal{F})$ is a quasi-isomorphism. The rest follows directly from Prop. 2.1.2 and the existence of a family of canonical maps

$$\cup_n : B^n(F) \otimes B^n(G) \rightarrow B^n(F \otimes G)$$

compatible with the cosimplicial structure. We leave to the reader to check that if \mathcal{F}^* is further a (commutative) dga on X^\sim then $B^*(\mathcal{F}^*)$ is a cosimplicial (commutative) dga.¹⁶ \square

2.1.5 (Enough points). We will use the above construction in the case X is the site associated to a scheme or a dagger space (in the case of a dagger space we take the site associated to its G -topology). In both cases we let P be the category $Pt(X)$ of site-theoretical points of X . For a general X the canonical map $u : Pt(X) \rightarrow X$ is not conservative. The latter property is guaranteed in the two cases we are interested in. It suffices to exhibit a subcategory C of $Pt(X)$ (with the discrete topology) such that u restricted to C is conservative. When X is associated to a scheme (resp. a dagger space) we let C be the category of its Zariski points (resp. its Berkovich or adic points). This is enough as explained in [CCM12, § 3] or [Tam11, § 3].

From now on we will simply write $\widetilde{\text{Gdm}}$ instead of $\widetilde{\text{Gdm}}_{Pt(X)}$ with X as above.

2.2. De Rham cohomology.

2.2.1 (The Hodge Filtration). We recall some well known facts about algebraic de Rham cohomology (see for instance [Jan90]). Let K be a field of characteristic zero and X be a smooth and algebraic K -scheme. Fix a compactification $g : X \rightarrow \bar{X}$ such that the complement $D = \bar{X} \setminus X$ is a normal crossing divisor¹⁷. Then consider the complex $\Omega_{\bar{X}/K}^\bullet \langle D \rangle$ of differential forms on \bar{X} with logarithmic differential poles along D . The natural inclusion $\Omega_{\bar{X}/K}^\bullet \langle D \rangle \subset g_* \Omega_{X/K}^\bullet$ is a quasi-isomorphism and we define the Hodge filtration on the de Rham cohomology of X by

$$F^i H_{dR}^n(X/K) := H^n(\bar{X}, F^i \Omega_{\bar{X}/K}^\bullet \langle D \rangle)$$

where $F^i \Omega_{\bar{X}/K}^\bullet \langle D \rangle$ is the stupid filtration.

¹⁵The cosimplicial structure is defined as follows. First let $\eta : \text{Id}_{X^\sim} \rightarrow u_* u^*$ and $\epsilon : u^* u_* \rightarrow \text{Id}_{P^\sim}$ be the natural transformations induced by adjunction.

Endow $B^n(\mathcal{F}) := (u_* u^*)^{n+1}(\mathcal{F})$ with co-degeneracy maps

$$\sigma_i^n := (u_* u^*)^i u_* \epsilon u^* (u_* u^*)^{n-1-i} : B^n(\mathcal{F}) \rightarrow B^{n-1}(\mathcal{F}) \quad i = 0, \dots, n-1$$

and co-faces

$$\delta_i^{n-1} := (u_* u^*)^i \eta (u_* u^*)^{n-i} : B^{n-1}(\mathcal{F}) \rightarrow B^n(\mathcal{F}) \quad i = 0, \dots, n.$$

¹⁶In fact one needs to take care of the signs:

$$\cup_n^{ab} : B^n(\mathcal{F}^a) \otimes B^n(\mathcal{F}^b) \rightarrow B^n(\mathcal{F}^a \otimes B^n(\mathcal{F}^b)), \quad \cup_n^{ab} = (-1)^{na} \cup_n.$$

¹⁷Such a compactification exists by the Nagata's compactification theorem and the result of Hironaka on the resolution of singularities.

A remarkable result of Deligne says that (for $K = \mathbb{C}$) the Hodge filtration does not depend on the chosen compactification. Moreover given a morphism $f : X \rightarrow Y$ of smooth algebraic schemes over \mathbb{C} the induced morphism on de Rham cohomology is strictly compatible w.r.t the Hodge filtrations¹⁸. Then the same holds for $H_{dR}^n(X/K)$ where $K \subset \mathbb{C}$ is a field of characteristic zero.

Proposition 2.2.2. *Let X be a smooth K -scheme.*

- (1) *For any normal crossing compactification \bar{X} of X the resolution $\widetilde{\text{Gdm}}(\Omega_{\bar{X}/K}^\bullet \langle D \rangle)$ (notation as in § 2.1.5) gives a sheaf of filtered commutative dga¹⁹ and $F^i H_{dR}^n(X/K) \cong H^n(\Gamma(\bar{X}, \widetilde{\text{Gdm}}(F^i \Omega_{\bar{X}/K}^\bullet \langle D \rangle)))$.*
- (2) *The following complexes*

$$E_{\text{FdR},i}(X) := \text{colim}_{\bar{X}} \Gamma(\bar{X}, \widetilde{\text{Gdm}}(F^i \Omega_{\bar{X}/K}^\bullet \langle D \rangle)), \quad E'_{dR}(X) := \text{colim}_{\bar{X}} \Gamma(\bar{X}, \widetilde{\text{Gdm}}(g_* \Omega_X^\bullet))$$

are functorial in X and there are functorial quasi-isomorphisms

$$E_{\text{FdR},0}(X) \rightarrow E'_{dR}(X) \leftarrow E_{dR}(X)$$

Proof. By definition $\Omega_{\bar{X}/K}^\bullet \langle D \rangle$ is a commutative (filtered) dga. Let

$$F^i \widetilde{\text{Gdm}}(\Omega_{\bar{X}/K}^\bullet \langle D \rangle) = \widetilde{\text{Gdm}}(F^i \Omega_{\bar{X}/K}^\bullet \langle D \rangle).$$

Then $\widetilde{\text{Gdm}}(\Omega_{\bar{X}/K}^\bullet \langle D \rangle)$ is a (sheaf of) filtered commutative dga by Proposition 2.1.4. This concludes the proof of point (1).

As the complex of sheaves $\Omega_{\bar{X}/K}^\bullet \langle D \rangle$ is functorial²⁰ with respect to the pair (X, D) , the same is true for $F^i \widetilde{\text{Gdm}}(\Omega_{\bar{X}/K}^\bullet \langle D \rangle)$. Note that the category of normal crossing compactifications is filtered. Hence the above colimit is quasi-isomorphic to any of its elements. What remains to prove follows directly from the definitions. \square

Example 2.2.3. Let $X = \mathbb{P}_K^1 \setminus \{0, \infty\}$. By construction $E_{\text{FdR},1}(X)$ is a complex starting in degree 1. Let $d\log \in \Gamma(\mathbb{P}_K^1, \Omega_{\mathbb{P}_K^1}^1 \langle 0, \infty \rangle) = H^0(E_{\text{FdR},1}(X)[1]) = H^1(E_{\text{FdR},1}(X))$ be the section defined by dT/T , for a local parameter T at 0. Note that the class of $d\log$ is a generator for $F^1 H_{dR}^1(X) \cong K$. We will denote it by c_1^{FdR} .

Proposition 2.2.4. *There exists a motivic ring spectrum \mathbb{E}_{FdR} whose components are the complexes $E_{\text{FdR},i}$ and such that*

$$F^i H_{dR}^n(X) = \text{Hom}_{\mathbb{D}_{\mathbb{A}^1}(K, \mathbb{Q})}(\mathbb{1}, \mathbb{E}_{\text{FdR}}(i)[n]).$$

Proof. By the previous Lemma the family $E_{\text{FdR},i}$ forms a \mathbb{N} -graded commutative monoid. The dlog of the above example gives a morphism $\mathbb{Q}(\mathbb{G}_m, K) \rightarrow E_{\text{FdR},1}$. According to Proposition 1.4.10 we have to prove the following.

(Excision and homotopy) $E_{\text{FdR},i}$ is both Nis-local and \mathbb{A}^1 -local. We know that E_{dR} is Nis/ \mathbb{A}^1 -local so that the same holds for $E_{\text{FdR},0}$. The same holds for $E_{\text{FdR},i}$ since the canonical maps $E_{\text{FdR},i} \rightarrow E_{\text{FdR},0}$ induce the Hodge filtration on cohomology. Then thanks to the strictness it is easy to conclude (see also the paragraph following this proof).

(Stability) The cup product with $d\log = dT/T$ induces an isomorphism

$$H^n(E_i(X)) \cong H^{n+1}(E_{i+1}(\mathbb{G}_m \times X)) / H^{n+1}(E_{i+1}(X)).$$

Let $g : X \rightarrow \bar{X}$ be a normal crossing compactification with complement D . Then $\mathbb{G}_m \times X \rightarrow \mathbb{P}^1 \times \bar{X}$ is a normal crossing compactification with complement $E = \{0, \infty\} \times \bar{X} \cup \mathbb{P}^1 \times D$. We have to prove that $\Omega_{\mathbb{P}^1 \times \bar{X}} \langle E \rangle = p_1^* \Omega_{\mathbb{P}^1} \langle 0, \infty \rangle \otimes p_2^* \Omega_{\bar{X}} \langle D \rangle$. This can be checked locally by choosing étale coordinates. Then it is easy to prove the filtered Künneth decomposition $F^{i+1} H_{dR}^{n+1}(\mathbb{G}_m \times X) = H_{dR}^0(\mathbb{G}_m) \otimes F^{i+1} H_{dR}^{n+1}(X) \oplus H_{dR}^1(\mathbb{G}_m) \otimes F^i H_{dR}^n(X)$ since $F^j H_{dR}^j(\mathbb{G}_m) = H_{dR}^j(\mathbb{G}_m) \cong K$ for $j = 0, 1$. As $H_{dR}^j(\mathbb{G}_m) = K d\log$ the claim is proved.

¹⁸A morphism $f : A \rightarrow B$ of filtered vector spaces is strict if $f(F^i A) = f(A) \cap F^i B$.

¹⁹Set $F^i \widetilde{\text{Gdm}} = \widetilde{\text{Gdm}} F^i$.

²⁰Morphisms of pairs are morphisms of commutative squares.

(Orientation) This is obvious: the morphism of $\mathbb{A}^1 \setminus \{0\}$ induced by $T \mapsto 1/T$ sends dT/T to $-dT/T$ as an element of $H^0(\mathbb{P}_K^1, \Omega_{\mathbb{P}_K^1}^1(0, \infty)) \subset E_{\text{Fdr},1}(\mathbb{A}^1 \setminus \{0\})$. \square

2.2.5 (Variation on dagger spaces). We define a canonical commutative dga $R\Gamma_{\text{dR}}(X)$ for the de Rham cohomology of a dagger space \mathcal{X} over K . Let K be a p -adic field and $\Gamma^* = |K| \otimes \mathbb{Q}$. Consider the following algebra

$$W_n := \left\{ \sum_{\nu} a_{\nu} T^{\nu} \in K[[T_1, \dots, T_n]] \mid \exists \rho > 1, |a_{\nu}| \rho^{|\nu|} \rightarrow 0 \right\}.$$

According to Grosse-Klönne a K -algebra A is a *dagger algebra* if it is a quotient of W_n for some n . To such an A we can associate the spectrum of maximal points $\text{Spm}(A)$ which is a G -ringed space. One has a universal K -derivation of A into finite A -modules, $d : A \rightarrow \Omega_{A/K}^1$ giving rise to de Rham complex $\Omega_{\mathcal{X}/K}$ on a general dagger space \mathcal{X} . Assuming \mathcal{X} to be smooth we can set

$$H_{\text{dR}}^n(\mathcal{X}) := H^n(\mathcal{X}, \Omega_{\mathcal{X}/K})$$

It follows from Proposition 2.1.2 and § 2.1.5 that the complex $R\Gamma_{\text{dR}}(\mathcal{X}) := \Gamma(\mathcal{X}, \widetilde{\text{Gdm}} \Omega_{\mathcal{X}/K}^{\bullet})$ is a functorial commutative dga.

Now let X be a smooth R -scheme. We can associate to it two different dagger spaces: one is the dagger analytification $(X_K)^{\dagger}$ of its generic fiber; the other is the Raynaud fiber $(X^w)_K$ of the weakly formal scheme X^w associated to X . There is a natural inclusion $(X^w)_K \subset (X_K)^{\dagger}$. Further there is a map of sites $\iota : (X_K)^{\dagger} \rightarrow X_K$ as in the classical analytification case.

2.3. Rigid cohomology. We recall the construction given by Besser as rephrased in [Tam11] since there are some simplification. For the sake of the readers we give all the needed definitions.

2.3.1. We consider the category of affine and algebraic k -schemes Aff/k . Let X be such a scheme, then we can choose a closed embedding $X \rightarrow \mathcal{Y}$ in a weak formal S -scheme²¹ \mathcal{Y} having smooth special fiber \mathcal{Y}_k . We call such an embedding a *rigid pair* and we denote it by (X, \mathcal{Y}) .

A morphism of rigid pairs $(X, \mathcal{Y}), (X', \mathcal{Y}')$ is a commutative diagram

$$\begin{array}{ccc}]X[_{\mathcal{Y}} & \xrightarrow{F} &]X'[_{\mathcal{Y}'} \\ \text{sp} \downarrow & & \downarrow \text{sp} \\ X & \xrightarrow{f} & X' \end{array}$$

We denote by RP the category of rigid pairs.

The datum of a rigid pair (X, \mathcal{Y}) is sufficient to compute the rigid cohomology of X (with K coefficients) as follows

$$H_{\text{rig}}^n(X/K) = H_{\text{dR}}^n(]X[_{\mathcal{Y}}) = H^n(]X[_{\mathcal{Y}}, \Omega_{]X[_{\mathcal{Y}}/K}^{\bullet}).$$

The de Rham complex $\Omega_{]X[_{\mathcal{Y}}/K}^{\bullet}$ is functorial in (X, \mathcal{Y}) and its cohomology is independent up to isomorphism of the choice of \mathcal{Y} . Since the tube of X in \mathcal{Y} is a smooth dagger space we get $H_{\text{rig}}^n(X/K) = H^n(R\Gamma_{\text{dR}}(]X[_{\mathcal{Y}}))$ (see 2.2.5).

Proposition 2.3.2. (1) *Let K/K_0 be a finite extension. There exists a ring object $R\Gamma_{\text{rig},K}$ in the category $\text{D}_{\mathbb{A}^1}^{\text{eff}}(\text{Spec } k, \mathbb{Q})$ that represents rigid cohomology: i.e. for any algebraic scheme X over k there is a canonical rational commutative dga $R\Gamma_{\text{rig},K}(X)$ such that $H^i(R\Gamma_{\text{rig},K}(X)) \cong H_{\text{rig}}^i(X/K)$. (The same holds if we replace the coefficient ring \mathbb{Q} by any field L s.t. $\mathbb{Q} \subset L \subset K$)*

(2) *Let X as above and (X, \mathcal{Y}) be a rigid pair. Then there is a commutative dga $\widetilde{R}\Gamma_{\text{rig}}(X, \mathcal{Y})$ together with a diagram of dga quasi-isomorphisms*

$$R\Gamma_{\text{rig},K}(X) \leftarrow \widetilde{R}\Gamma_{\text{rig}}(X, \mathcal{Y}) \rightarrow R\Gamma_{\text{dR}}(]X[_{\mathcal{Y}})$$

functorial in the pair (X, \mathcal{Y}) .

²¹Note that $S_0 = \text{Spec}(W(k))$ and $S = \text{Spec}(R)$, where R is the valuation ring of a finite extension of $\text{Frac}(W(k))$.

- (3) (*Base change*) Let $\rho : R \rightarrow R'$ be a finite map of complete discrete valuation rings. Let k (resp. k') be the residue field of R (resp. R'). Let X be a k -scheme then there is a canonical (both in X and R) quasi-isomorphism

$$K' \otimes_K R\Gamma_{\text{rig},K}(X) \rightarrow R\Gamma_{\text{rig},K'}(X_{k'}) .$$

The latter induces an isomorphism in $D_{\mathbb{A}^1}^{eff}(\text{Spec } k, \mathbb{Q})$

$$R\Gamma_{\text{rig},K} \otimes_K K' \rightarrow f_*(R\Gamma_{\text{rig},K'})$$

where $f : \text{Spec } k' \rightarrow \text{Spec } k$ is the map induced by ρ and $R\Gamma_{\text{rig},?}$ denotes the object of point (1).

- (4) There exists a canonical σ -linear endomorphism of $R\Gamma_{\text{rig},K_0}(X)$ inducing the Frobenius on cohomology: it is defined as the composition of

$$(2.3.2.a) \quad R\Gamma_{\text{rig},K_0}(X) \xrightarrow{\text{Id} \otimes 1} R\Gamma_{\text{rig},K_0}(X) \otimes_{\sigma} K_0 \xrightarrow{\text{b.c.}} R\Gamma_{\text{rig},K_0}(F^*X) \xrightarrow{\text{rel.frob.}} R\Gamma_{\text{rig},K_0}(X)$$

where b.c. stands for the base change morphism of point (3); F is the Frobenius of $\text{Spec } k$; F^*X is the base change of X via F ; the last map on the right is the relative Frobenius.

Proof. The details are given in [Bes00, 4.9, 4.21, 4.22]. Since we adopt the language of dagger spaces there are some formal differences. For the sake of the readers we give the necessary modifications. To obtain a complex functorial in X we have to take a colimit on some filtered category. The category of pairs (X, \mathcal{Y}) with X fixed is not filtered. Hence we have to introduce the following categories. We define the set RP_X (resp. $RP_{(X,\mathcal{Y})}$) of diagrams $X \xrightarrow{f} X' \rightarrow \mathcal{Y}'$ (resp. $(f, F) : (X, \mathcal{Y}) \rightarrow (X', \mathcal{Y}')$ morphism of rigid pairs) where (X', \mathcal{Y}') is a rigid pair. Let RP_X^0 (resp. $RP_{(X,\mathcal{Y})}^0$) be the subset of RP_X (resp. $RP_{(X,\mathcal{Y})}$) with $f = \text{Id}_X$ (resp. $(f, F) = (\text{Id}, \text{Id})$)

Now we can form the category SET_X^0 (resp. $SET_{(X,\mathcal{Y})}^0$) with objects the finite subsets of RP_X (resp. $RP_{(X,\mathcal{Y})}$) having non-empty intersection with RP_X^0 (resp. $RP_{(X,\mathcal{Y})}^0$); morphisms are inclusions. For instance an element of SET_X^0 is a finite family of diagrams $X \xrightarrow{f_a} X'_a \rightarrow \mathcal{Y}'_a$, $a \in A$ (finite set), such that $f_{a_0} = \text{Id}$ for some $a_0 \in A$. To such an object we can associate the complex $R\Gamma_{\text{dR}}(\text{]X[}_{\mathcal{Y}'_A})$, where $\mathcal{Y}'_A = \prod_a \mathcal{Y}'_a$. The categories SET_X^0 and $SET_{(X,\bar{X},\text{p})}^0$ are filtered.

Having this said we define

$$R\Gamma_{\text{rig},K}(X) := \text{colim}_{A \in SET_X^0} R\Gamma_{\text{dR}}(\text{]X[}_{\mathcal{Y}'_A}) \quad R\Gamma_{\text{rig}}(X, \mathcal{Y}) := \text{colim}_{A \in SET_{(X,\mathcal{Y})}^0} R\Gamma_{\text{dR}}(\text{]X[}_{\mathcal{Y}'_A}) .$$

Now one can follow word by word the proof of Besser. Till this point we assumed X to be affine. By the work of Chiarellotto and Tsuzuki [CT03] rigid cohomology satisfies (étale and so) Nisnevich descent. This allows to extend the definition of $R\Gamma_{\text{rig},K}(X)$ to any algebraic k -scheme. \square

Proposition 2.3.3. *There exists a motivic ring spectrum $\mathbb{E}_{\text{rig},K}$ whose components are all equal to the complex $R\Gamma_{\text{rig},K}$ and whose stability class is induced by dlog such that*

$$H_{\text{rig}}^n(X/K) \cong \mathbb{E}_{\text{rig},K}^{n,i}(X) := \text{Hom}_{D_{\mathbb{A}^1}(k,\mathbb{Q})}(M(X), \mathbb{E}_{\text{rig},K}(i)[n]) .$$

Proof. We have to verify the hypothesis of Proposition 1.4.10 for the family $E_i := R\Gamma_{\text{rig},K}$. First we need to define a morphism of complexes $\mathbb{Q}[0] \rightarrow R\Gamma_{\text{rig},K}(\mathbb{G}_{m,k})(1)[1]$. We argue as in the de Rham case. Let us denote by $X = \mathbb{G}_{m,R}$. Then the de Rham cohomology of the dagger space $(X^w)_K$ computes the (K -linear) rigid cohomology of $X_k = \mathbb{G}_{m,k}$ and there is a canonical map from $R\Gamma_{\text{dR}}((X^w)_K)$ to $R\Gamma_{\text{rig},K}(X_k)$. We can apply the construction of 2.1.4 to the inclusion $\Omega_{(X^w)_K/K}^1[-1] \subset \Omega_{(X^w)_K/K}^1$ and we obtain (as in example 2.2.3) an element dlog of $R\Gamma_{\text{dR}}((X^w)_K)$ of degree 1. \square

Remark 2.3.4. With the notations of point (3) of Proposition 2.3.2, one gets a canonical base change isomorphism in $DM_{\mathbb{B}}(k)$:

$$\mathbb{E}_{\text{rig},K} \otimes_K K' \xrightarrow{\sim} f_*(\mathbb{E}_{\text{rig},K'}) .$$

In the sequel, we will simply denote by \mathbb{E}_{rig} (resp. $R\Gamma_{\text{rig}}$) the ring spectrum $\mathbb{E}_{\text{rig},K_0}$ (resp. the complex $R\Gamma_{\text{rig},K_0}$).

2.4. Absolute rigid cohomology.

2.4.1. Along the lines of [Bei86b] and [Ban02] we are going to define the analogue of absolute Hodge theory in the setting of rigid cohomology. Let k be a perfect field of characteristic p . We denote by $F\text{-isoc}$ the category of F -isocrystals (defined over k): *i.e.* finite dimensional K_0 -vector spaces together with a σ -linear automorphism. This is a tensor category with unit object $\mathbb{1}$ given by K_0 together with σ . For any $I \in F\text{-isoc}$ we denote by $I(n)$ the F -isocrystal having the same vector space I and Frobenius twisted by p^{-n} . We would like to define the absolute rigid cohomology of a k -scheme X as follows

$$H_\phi^n(X, i) := \text{Hom}_{D^b(F\text{-isoc})}(\mathbb{1}, R\Gamma(X)(i)[n])$$

where $R\Gamma(X)$ is a complex of F -isocrystals such that $H^n(R\Gamma(X)) = H_{\text{rig}}^n(X)$ together with its Frobenius endomorphism. Since we do not know how to construct directly $R\Gamma$ we follow the strategy of Beilinson in *loc.cit.* and deduce its existence from proposition 2.3.2.

Let C_{rig}^b be the category of bounded complexes of K_0 -vector spaces M together with a quasi-isomorphism $\phi : M^\sigma = M \otimes_{K_0, \sigma} K_0 \rightarrow M$. We define homotopies (resp. quasi-isomorphisms) between objects in C_{rig}^b to be morphisms in C_{rig}^b such that they are homotopies (resp. quasi-isomorphisms) of the underlying complexes of K_0 -vector spaces. Then we can define the category K_{rig}^b to be the category C_{rig}^b modulo the null-homotopic morphisms.

Lemma 2.4.2. (1) *the category K_{rig}^b is triangulated;*
 (2) *the localization of the category K_{rig}^b by the subcategory of acyclic objects exists and it is a triangulated category too. We denote it by D_{rig}^b ;*
 (3) *there is a canonical functor $\iota : D^b(F\text{-isoc}) \rightarrow D_{\text{rig}}^b$ which is an equivalence of categories*

Proof. We leave to the reader to check that all the arguments given in [Ban02, §§1,2] (or [CCM12, § 2]) can be adapted to our (much simpler) setting. We limit ourselves to make explicit the formulas for the Hom groups in $D^b(F\text{-isoc})$, D_{rig}^b .

Let M, N be two bounded complexes of F -isocrystals. Remind that $F\text{-isoc}$ has internal Hom so that we can form the internal Hom complex $\underline{\text{Hom}}^\bullet(M, N)$ with Frobenius $\phi_{M,N}$. Consider the following morphism of \mathbb{Q}_p -linear²² complexes

$$\xi_{M,N} : \underline{\text{Hom}}^\bullet(M, N) \rightarrow \underline{\text{Hom}}^\bullet(M, N), \quad x \mapsto x - \phi_{M,N}x .$$

Then we can prove as in [Ban02, proposition 1.7] that

$$(2.4.2.a) \quad \text{Hom}_{D^b(F\text{-isoc})}(M, N[i]) \cong H^{i-1}(\text{Cone } \xi_{M,N}) .$$

Similarly given two complexes M, N in C_{rig}^b we define the morphism of complexes

$$\xi'_{M,N} : \underline{\text{Hom}}^\bullet(M, N) \rightarrow \underline{\text{Hom}}^\bullet(M^\sigma, N), \quad x \mapsto x \circ \phi_M - \phi_N \circ (x \otimes_\sigma 1) .$$

²³Then the Hom groups in D_{rig}^b can be computed as follows

$$(2.4.2.b) \quad \text{Hom}_{D_{\text{rig}}^b}(M, N[i]) \cong H^{i-1}(\text{Cone } \xi'_{M,N}) .$$

Now it is easy to check that given to F -isocrystals M, N we have

$$(2.4.2.c) \quad \text{Ext}_{F\text{-isoc}}^i(M, N) \cong \text{Hom}_{D_{\text{rig}}^b}(\iota M, \iota N[i])$$

and the faithfulness of ι follows. □

Definition 2.4.3. Let X be an algebraic k -scheme. We define the absolute rigid cohomology as

$$H_\phi^n(X, i) := \text{Hom}_{D_{\text{rig}}^b}(\mathbb{1}, R\Gamma_{\text{rig}}(X)(i)[n]) .$$

It follows from the equivalence ι of the above lemma that the same formula holds in $D^b(F\text{-isoc})$ for some object $R\Gamma(X)$ corresponding to $R\Gamma_{\text{rig}}(X)$.

²²These are not K_0 -linear since (in general) the Frobenius is not.

²³Mind that we cannot use $\underline{\text{Hom}}$ because there is no internal Hom in C_{rig}^b . This is due to the fact that the Frobenius is only a quasi-isomorphism.

Corollary 2.4.4. *There is a natural spectral sequence*

$$(2.4.4.a) \quad E_2^{pq} = \text{Ext}_{\mathbb{F}\text{-isoc}}^p(\mathbb{1}, H^q(X)(i)) \Rightarrow H_\phi^{p+q}(X, i)$$

degenerating to the following short exact sequence

$$0 \rightarrow H_{\text{rig}}^1(X)/\text{Im}(\text{Id} - \phi/p^i) \rightarrow H_\phi^{n,i}(X) \rightarrow H_{\text{rig}}^n(X)^{\phi=p^i} \rightarrow 0 .$$

Proof. The existence of the spectral sequence follows from the formula (2.4.2.c). By (2.4.2.b) it is concentrated in the columns $p = 0, 1$ so that it gives short exact sequences. \square

Proposition 2.4.5. *There exists a motivic ring spectrum $\mathbb{E}_\phi \in DM_{\mathbb{B}}(k)$ representing the absolute rigid cohomology, i.e.*

$$H_\phi^n(X, i) \cong \mathbb{E}_\phi^{n,i}(X) := \text{Hom}_{DM_{\mathbb{B}}(k)}(M(X), \mathbb{E}_\phi(i)[n]) .$$

Proof. By point (4) of proposition 2.3.2 we can define a family of morphism of presheaves of complexes

$$R\Gamma_{\text{rig}} \xrightarrow{\phi/p^i} R\Gamma_{\text{rig}} .$$

We claim that the latter induces a morphism of ring spectra

$$\mathbb{E}_{\text{rig}} \xrightarrow{\Phi} \mathbb{E}_{\text{rig}} .$$

Indeed it is sufficient to notice that $(\phi \otimes 1) \circ \text{dlog} = p \text{dlog}$, where $\text{dlog} : \mathbb{Q}(\mathbb{G}_m)[-1] \rightarrow \mathbb{E}_{\text{rig}}(1)$ is the stability class of the rigid spectrum.

Now we can define \mathbb{E}_ϕ to be the homotopy limit of the following diagram of ring spectra

$$(2.4.5.a) \quad \mathbb{E}_{\text{rig}} \xrightarrow[\text{Id}]{\Phi} \mathbb{E}_{\text{rig}} .$$

the limit exists by 1.4.8.

To conclude the proof note that $\mathbb{E}_{\phi,i}$ is quasi-isomorphic to the cone $\text{Cone}(\text{Id} - \phi/p^i)$ (up to a shift!). Then it is sufficient to compare (2.4.2.b) and (1.4.10.a). \square

Remark 2.4.6. According to the preceding proof, one gets a canonical distinguished triangle of $DM_{\mathbb{B}}(k)$:

$$(2.4.6.a) \quad \mathbb{E}_\phi \rightarrow \mathbb{E}_{\text{rig}} \xrightarrow{\text{Id} - \Phi} \mathbb{E}_{\text{rig}} \xrightarrow{+1}$$

which induces the short exact sequences of the preceding Corollary. In particular, these exact sequences are functorial with respect to the motive of X .

2.5. Syntomic cohomology.

2.5.1. Let X be a smooth R -scheme. With the notation of § 2.2.5, there is a map of commutative dga

$$\text{sp}_X : E_{\text{dR}}(X_K) \rightarrow R\Gamma_{\text{dR}}((X^w)_K) = R\Gamma_{\text{rig}}(X_k, X^w)$$

inducing the specialization on cohomology and functorial in X . Details can be found in [Tam11, §§ 3.3, 5.3].

Now we can recall the definition of syntomic cohomology $H_{\text{syn}}^n(X, i)$ of X : it is the cohomology of a complex $R\Gamma_{\text{syn}}(X, i)$ defined as the homotopy limit of the following diagram

$$\begin{array}{ccccccc} & R\Gamma_{\text{rig}}(X_k) & & R\Gamma_{\text{rig},K}(X_k) & & R\Gamma_{\text{dR}}(\lfloor X_k[X_w] \rfloor) & & E'_{\text{dR}}(X_K) \\ & \uparrow \text{Id} & & \uparrow & & \uparrow & & \uparrow \\ R\Gamma_{\text{rig}}(X_k) & \xlongequal{\phi/p^i} & R\Gamma_{\text{rig}}(X_k) & & R\Gamma_{\text{rig}}(X_k, X^w) & & E_{\text{dR}}(X_K) & & E_{\text{FdR},i}(X_K) \end{array}$$

(cf. [Bes00], [CCM12])

Proposition 2.5.2. *For any complete discrete valuation ring R of mixed characteristics, there exists a ring spectrum \mathbb{E}_{syn} in $DM_{\mathbb{B}}(R, \mathbb{Q}_p)$ representing the syntomic cohomology defined by Besser, i.e. for any smooth R -scheme X and any integer n , there is a canonical isomorphism*

$$H_{\text{syn}}^n(X, i) \cong \mathbb{E}_{\text{syn}}^{n,i}(X) := \text{Hom}_{DM_{\mathbb{B}}(R, \mathbb{Q}_p)}(M(X), \mathbb{E}_{\text{syn}}(i)[n]) .$$

Proof. By construction the absolute rigid spectrum \mathbb{E}_{ϕ} maps to \mathbb{E}_{rig} and so to the base change $\mathbb{E}_{\text{rig},K}$. Then we define \mathbb{E}_{syn} as the homotopy limit (in the category of ring spectra) of the following diagram

$$i_*\mathbb{E}_{\phi} \rightarrow i_*\mathbb{E}_{\text{rig},K} \leftarrow a \rightarrow b \leftarrow c \rightarrow d \leftarrow j_*\mathbb{E}_{\text{dR}}$$

where a, b, c, d are the ring spectra induced by $E_{\text{rig}}(X_k, X^w)$, $R\Gamma_{\text{dR}}(\cdot|_{X_k|_{X^w}})$, $E_{\text{dR}}(X_K)$, $E'_{\text{dR}}(X_K)$, respectively: we leave to the reader the verification that they are ring spectra following the same proof as the one of 2.3.3.

Note that all the elements of the diagram are Morel motives, then \mathbb{E}_{syn} is a Morel motive too. \square

Remark 2.5.3. Given a complete discrete valuation ring R with residue field k and fraction field K , such that $R/W(k)$ is finite, we get a map of ring spectra in $DM_{\mathbb{B}}(k)$:

$$a_0 : \mathbb{E}_{\phi} \rightarrow \mathbb{E}_{\text{rig},K_0} \rightarrow \mathbb{E}_{\text{rig},K_0} \otimes_{K_0} K \xrightarrow{\sim} E_{\text{rig},K}$$

where the last isomorphism comes from Remark 2.3.4. Let us put $a = i^*(a_0)$.

Secondly, we get a morphism of ring spectra in $DM_{\mathbb{B}}(R)$:

$$b : j_*\mathbb{E}_{\text{FdR}} \rightarrow j_*\mathbb{E}_{\text{dR}} \xrightarrow{\text{sp}} i_*\mathbb{E}_{\text{rig},K} .$$

The first map is the canonical morphism, and the second one is the specialization map induced by the morphism sp_X of Paragraph 2.5.1.

Then the syntomic ring spectrum is characterized up to isomorphism by the following homotopy pullback square (of morphisms of ring spectra):

$$(2.5.3.a) \quad \begin{array}{ccc} \mathbb{E}_{\text{syn}} & \xrightarrow{\alpha} & j_*\mathbb{E}_{\text{FdR}} \\ \beta \downarrow & & \downarrow b \\ i_*\mathbb{E}_{\phi} & \xrightarrow{a} & i_*\mathbb{E}_{\text{rig},K} \end{array}$$

In other words, one can define \mathbb{E}_{syn} as the homotopy limit of the lower corner of the above diagram – but this definition is less precise than the one given in the proof of the previous proposition as (in this way) \mathbb{E}_{syn} is defined only up to non unique isomorphism.

The fact that the preceding square is a homotopy pullback can be translated into the existence of a distinguished triangle in $DM_{\mathbb{B}}(R)$:

$$(2.5.3.b) \quad \mathbb{E}_{\text{syn}} \xrightarrow{\alpha+\beta} i_*\mathbb{E}_{\phi} \oplus j_*\mathbb{E}_{\text{FdR}} \xrightarrow{a-b} i_*E_{\text{rig},K} \xrightarrow{+1}$$

which corresponds to the long exact sequence, for X/R smooth:

$$(2.5.3.c) \quad \dots \rightarrow H_{\text{syn}}^n(X, i) \xrightarrow{\alpha_*+\beta_*} H_{\phi}^n(X_k, i) \oplus F^i H_{\text{dR}}^n(X_K) \xrightarrow{a_*-b_*} H_{\text{rig}}^n(X_k/K) \rightarrow \dots$$

Here, α_* (resp. β_*) is the usual projection map from syntomic cohomology to $\mathbb{E}_{\phi}^{n,i}(X_k) = H_{\phi}^n(X_k, i)$ (resp. $F^i H_{\text{dR}}^n(X_K)$) while a_* is the canonical map and b_* is induced by the specialization map from de Rham cohomology to rigid cohomology.

Note also that \mathbb{E}_{syn} is the homotopy limit of the diagram of ring spectra

$$\begin{array}{ccc} & & j_*\mathbb{E}_{\text{FdR}} \\ & & \downarrow b \\ i_*\mathbb{E}_{\text{rig}} & \xrightarrow[\text{Id}]{\Phi} & i_*\mathbb{E}_{\text{rig}} \longrightarrow i_*E_{\text{rig},K} \end{array}$$

so that we also obtain the following distinguished triangle:

$$\mathbb{E}_{\text{syn}} \rightarrow i_*\mathbb{E}_{\text{rig}} \oplus j_*\mathbb{E}_{\text{FdR}} \rightarrow i_*\mathbb{E}_{\text{rig}} \oplus i_*E_{\text{rig},K} \xrightarrow{+1}$$

which precisely induces the long exact sequence originally considered by Besser.

2.6. Localizing syntomic cohomology.

2.6.1. As the fibred triangulated category $DM_{\mathbb{F}}$ satisfies the “gluing formalism” (this is called the localization property in [CD09b]), we get a canonical distinguished triangle:

$$(2.6.1.a) \quad i_* i^! (\mathbb{E}_{\text{syn}}) \xrightarrow{ad'_i} \mathbb{E}_{\text{syn}} \xrightarrow{ad_j} j_* j^* (\mathbb{E}_{\text{syn}}) \xrightarrow{\partial_i} \mathbb{E}_{\text{syn}}[1]$$

for $i : \text{Spec } k \rightarrow \text{Spec } R$ and $j : \text{Spec } K \rightarrow \text{Spec } R$ the natural immersions. The maps ad'_i and ad_j are the obvious adjunction maps and the map ∂_i is the unique morphism which fits in this distinguished triangle (see [CD09b, 2.3.3]).

Remark 2.6.2. One can be more precise about the gluing formalism: given any object M of $DM_{\mathbb{F}}(R)$, there exists a unique distinguished triangle of the form

$$M_k \rightarrow M \rightarrow M_K \xrightarrow{\partial} M_k[1]$$

such that M_k (resp. M_K) has support in $\text{Spec } k$ ie $j^* M_k = 0$ (resp. in $\text{Spec } K$ ie $i^! (M_K = 0)$). This means that there exists a canonical isomorphism of that triangle with the following one:

$$i_* i^! (M) \xrightarrow{ad'_i} M \xrightarrow{ad_j} j_* j^* (M) \xrightarrow{\partial_i} i_* i^! (M)[1].$$

2.6.3. Let us introduce yet another spectrum: we consider the map

$$a_0 : \mathbb{E}_{\phi} \rightarrow E_{\text{rig}, K}$$

which is defined at the level of the underlying model category, and take its homotopy fiber $\check{\mathbb{E}}_{\phi}$. In particular, we have a canonical morphism: $i_* E_{\text{rig}, K} \xrightarrow{\partial_a} i_* \check{\mathbb{E}}_{\phi}[1]$.

Proposition 2.6.4. *Consider the above notations. Then the syntomic spectrum is equivalent to the homotopy fiber of the morphism*

$$\text{s}\check{\mathbb{P}} : j_* \mathbb{E}_{\text{FdR}} \xrightarrow{b} i_* \mathbb{E}_{\text{rig}, K} \xrightarrow{\partial_a} i_* \check{\mathbb{E}}_{\phi}[1].$$

Moreover, there are canonical identification

$$i^! \mathbb{E}_{\text{syn}} = \check{\mathbb{E}}_{\phi}, \quad j^* \mathbb{E}_{\text{syn}} = \mathbb{E}_{\text{FdR}}$$

through which the localization triangle (2.6.1.a) is identified with

$$i_* \check{\mathbb{E}}_{\phi} \rightarrow \mathbb{E}_{\text{syn}} \rightarrow j_* \mathbb{E}_{\text{dR}} \xrightarrow{\text{s}\check{\mathbb{P}}} i_* \check{\mathbb{E}}_{\phi}[1].$$

Remark 2.6.5. In fancy terms, the generic fiber of \mathbb{E}_{syn} is the ring spectrum \mathbb{E}_{FdR} . While we cannot compute the special fiber of \mathbb{E}_{syn} , its exceptional special fiber is the ring spectrum which is “the image of absolute rigid cohomology in rigid cohomology” and \mathbb{E}_{syn} is obtained by gluing these two ring spectra.

Proof. By definition of $\check{\mathbb{E}}_{\phi}$, there is a canonical distinguished triangle in $DM_{\mathbb{F}}(k)$:

$$\check{\mathbb{E}}_{\phi} \xrightarrow{\nu_0} \mathbb{E}_{\phi} \xrightarrow{a_0} E_{\text{rig}, K} \xrightarrow{\partial_{a_0}} \check{\mathbb{E}}_{\phi}[1]$$

which induces the following triangle after applying i_* :

$$i_* \check{\mathbb{E}}_{\phi} \xrightarrow{\nu} i_* \mathbb{E}_{\phi} \xrightarrow{a} i_* E_{\text{rig}, K} \xrightarrow{\partial_a} i_* \check{\mathbb{E}}_{\phi}[1].$$

Now according to the fact the square (2.5.3.b) is a homotopy pullback, one gets a canonical commutative diagram in $DM_{\mathbb{F}}(R)$:

$$\begin{array}{ccccccc} C(\alpha) & \longrightarrow & \mathbb{E}_{\text{syn}} & \xrightarrow{\alpha} & j_* \mathbb{E}_{\text{FdR}} & \longrightarrow & C(\alpha)[1] \\ \sim \downarrow & & \beta \downarrow & & \downarrow b & & \downarrow \sim \\ i_* \check{\mathbb{E}}_{\phi} & \xrightarrow{\nu} & i_* \mathbb{E}_{\phi} & \xrightarrow{a} & i_* E_{\text{rig}, K} & \xrightarrow{\partial_a} & i_* \check{\mathbb{E}}_{\phi}[1]. \end{array}$$

In other words, we get a distinguished triangle of the form:

$$i_* \check{\mathbb{E}}_{\phi} \rightarrow \mathbb{E}_{\text{syn}} \xrightarrow{\alpha} j_* \mathbb{E}_{\text{FdR}} \xrightarrow{\text{s}\check{\mathbb{P}}} i_* \check{\mathbb{E}}_{\phi}[1].$$

Finally, according to the above remark, one gets a canonical isomorphism of triangles:

$$\begin{array}{ccccccc}
i_* \check{\mathbb{E}}_\phi & \longrightarrow & \mathbb{E}_{\text{syn}} & \xrightarrow{\alpha} & j_* \mathbb{E}_{\text{FdR}} & \xrightarrow{\text{sp}} & i_* \check{\mathbb{E}}_\phi[1] \\
\sim \downarrow & & \sim \downarrow & & \downarrow \sim & & \downarrow \sim \\
i_* i^! \mathbb{E}_{\text{syn}} & \xrightarrow{ad_i} & \mathbb{E}_{\text{syn}} & \xrightarrow{ad_j} & j_* j^*(\mathbb{E}_{\text{syn}}) & \xrightarrow{\partial_i} & i_* i^! \mathbb{E}_{\text{syn}}[1] .
\end{array}$$

□

Remark 2.6.6 (The work of Tamme). The *relative cohomology theory* $H_{\text{rel}}^*(X, *)$ of [Tam11] is represented by the (generalized) cone of the diagram

$$i_* \mathbb{E}_{\text{rig}, K} \leftarrow a \rightarrow b \leftarrow c \rightarrow d \leftarrow j_* \mathbb{E}_{\text{dR}}$$

where we use the notation of the above proof. This is morally a Cone of a morphism of ring spectra $A \rightarrow B$, hence it is not a ring spectrum and in particular there is no unit section.

It follows by the localization sequence that that this cohomology theory is represented by the cone of the canonical adjunction map $\mathbb{E}_{\text{syn}} \rightarrow i_* i^! \mathbb{E}_{\text{syn}} = i_* \mathbb{E}_\phi$.

Example 2.6.7. Let $S = \text{Spec}(W(k))$ (for simplicity) and X be the connected component of the Néron model of an elliptic curve with multiplicative reduction, i.e. X is an S -group scheme such that its generic fiber is an elliptic curve and the special fiber is isomorphic to \mathbb{G}_m . Then X/S is smooth and we can easily compute the long exact sequence for syntomic cohomology. For instance we get

$$0 \rightarrow H_{\text{rig}}^0(X_s) \xrightarrow{a} H_{\text{rig}}^0(X_s) \oplus H_{\text{rig}}^0(X_s) \rightarrow H_{\text{syn}}^{1,1}(X) \rightarrow$$

$$\rightarrow H_{\text{rig}}^1(X_s) \oplus F^1 H_{\text{dR}}^1(X_\eta) \xrightarrow{b} H_{\text{rig}}^1(X_s) \oplus H_{\text{rig}}^1(X_s) \rightarrow H_{\text{syn}}^{2,1}(X) \rightarrow F^1 H_{\text{dR}}^2(X_\eta) \rightarrow 0$$

where $a(x) = (x - \phi(x)/p, -x)$ is injective and $b(x, y) = (0, y - x)$. It follows that $H_{\text{syn}}^{n,1} \cong K^2$ (as \mathbb{Q}_p -vector spaces) for $n = 1, 2$.

The same result can be obtained using the localization triangle: explicitly we get the following exact sequence

$$0 \rightarrow H_{\text{syn},s}^{1,1}(X) \rightarrow H_{\text{syn}}^{1,1}(X) \rightarrow F^1 H_{\text{dR}}^1(X_\eta) \xrightarrow{\delta} H_{\text{syn},s}^{2,1}(X) \rightarrow H_{\text{syn}}^{2,1}(X) \rightarrow F^1 H_{\text{dR}}^2(X_\eta) \rightarrow 0 .$$

Here $H_{\text{syn},s}^{1,1}(X)$ stands for $\text{Hom}(\mathbb{Q}(X), i_* i^! \mathbb{E}_{\text{syn}}(1)[1])$. Using proposition 2.6.4 we get $H_{\text{syn},s}^{n,1}(X) = H_{\text{rig}}^{n-1}(X_s)$ for $n = 1, 2$. We also get that δ is the zero map. For a complete account on the de Rham/rigid cohomology of abelian varieties and their reduction we refer to [LS86].

Example 2.6.8 (Semistable elliptic curve). Let X/S be an elliptic curve such that X_k is a nodal cubic. We assume that the singular point $x_0 \in X_k$ is k -rational. The above remark give a recipe to compute (or approximate) the syntomic cohomology of X

$$\mathbb{E}_{\text{syn}}^{n,i}(X) := \text{Hom}_{\mathbb{D}_{A^1}(S, \mathbb{Q}_p)}(M(X), \mathbb{E}_{\text{syn}}(i)[n])$$

where $M(X) = f_! f^!(\mathbb{Q}_S)$ and $f : X \rightarrow S$ is the structural morphism. Let's compute $i^*(M(X))$. Given the pullback square:

$$\begin{array}{ccc}
X_k & \xrightarrow{l} & X \\
f_0 \downarrow & & \downarrow f \\
\text{Spec } k & \xrightarrow{i} & S
\end{array}$$

one has a canonical exchange map:

$$i^* f_! f^!(\mathbb{Q}_S) \simeq f_0! l^* f^!(\mathbb{Q}_S) \rightarrow f_0! f_0^! i^*(\mathbb{Q}_S) = f_0! f_0^!(\mathbb{Q}_k) = M(X_k)$$

(the first iso is due to the base change theorem of the 6 functors formalism). This map is an isomorphism in the two following cases:

- f is smooth.

- X is regular and f is quasi-projective (and so in our case):
this is due to the absolute purity theorem: as f is quasi-projective, it can be factored $f = pi$ where $p : P \rightarrow S$ is smooth and i is a closed immersion and then one computes:

$$f^!(\mathbb{Q}_S) = i^!p^!(\mathbb{Q}_S) \simeq i^!(\mathbb{Q}_P)(d)[2d] \simeq \mathbb{Q}_X(d-n)[2(d-n)]$$

the first iso follows as p is smooth and the second one because i is a closed immersion between regular schemes. Here d (resp. n) is the relative dimension of P/S (resp. codimension of i), so that $d-n$ is the relative dimension of X/S .

Hence for X semistable we have long exact sequences

$$i^!\mathbb{E}_{\text{syn}}^{n,i}(X_k) \rightarrow \mathbb{E}_{\text{syn}}^{n,i}(X) \rightarrow j_*j^*\mathbb{E}_{\text{syn}}^{n,i}(X) = F^i H_{\text{dR}}(X_\eta) \rightarrow +$$

The term $i^!\mathbb{E}_{\text{syn}}^{n,i}(X_k)$ depends only on the special fiber. In this case it is easy to construct a proper and smooth hypercover Y_* of X_k : let $\pi : \tilde{X}_k \rightarrow X_k$ be the normalization map, then we may take $Y_0 = x_0 \sqcup \tilde{X}_k$, $Y_1 = \pi^{-1}(x_0)$ and $Y_i = \emptyset$ for $i > 1$. Since \tilde{X}_k is isomorphic to the projective line we get that $M(X_k) = \mathbb{Q} \oplus \mathbb{Q}[1] \oplus \mathbb{Q}(1)[2]$ in $DM_{\mathbb{B}}(k, \mathbb{Q})$. This decomposition allows to estimate $i^!\mathbb{E}_{\text{syn}}^{n,i}(X_k)$. For instance we can compute

$$i^!\mathbb{E}_{\text{syn}}^{n,i}(X_k) = H_{\text{rig}}^{n-1}(X_k)_K \simeq K \quad \text{for } n = 1, 2.$$

3. APPLICATIONS

3.1. Gysin morphisms and regulators.

3.1.1. Let R be a complete discrete valuation ring of mixed characteristic.

Recall that Beilinson motivic cohomology for smooth R -schemes is the cohomology represented by the unit object of $DM_{\mathbb{B}}(R)$:

$$H_{\mathbb{B}}^{n,i}(X) := \text{Hom}_{DM_{\mathbb{B}}}(M(X), \mathbb{1}(i)[n]).$$

This group can also be described as the i -graded part for the γ -filtration of algebraic rational K -theory:

$$H_{\mathbb{B}}^{n,i}(X) = \text{gr}_{\gamma}^i K_{2i-n}(X)_{\mathbb{Q}}.$$

See [CD09b, 13.2.14].

By construction, the ring spectrum \mathbb{E}_{syn} is a monoid object of $DM_{\mathbb{B}}(R)$. In particular, the ringed cohomology \mathbb{E}^{**} admits a canonical action of Beilinson motivic cohomology $H_{\mathbb{B}}^{**}$. Concretely, for any smooth R -scheme X and any couple of integers (n, i) , the unit map $\mathbb{1} \rightarrow \mathbb{E}_{\text{syn}}$ of syntomic cohomology induces a canonical morphism

(3.1.1.a)

$$\sigma_{\text{syn}} : H_{\mathbb{B}}^{n,i}(X) = \text{Hom}_{DM_{\mathbb{B}}}(M(X), \mathbb{1}(i)[n]) \rightarrow \text{Hom}_{DM_{\mathbb{B}}}(M(X), \mathbb{E}_{\text{syn}}(i)[n]) = H_{\text{syn}}^n(X, i)$$

which is compatible with pullbacks and products. This is the *higher cycle class map* in syntomic cohomology or equivalently the *syntomic regulator*. Note also that this map can be represented in the category $D_{\mathbb{A}^1}(R, \mathbb{Q})$ as a morphism of ring spectra:

$$\sigma : H_{\mathbb{B}} \rightarrow \mathbb{E}_{\text{syn}}$$

which is unique according to [CD09b, 13.2.15].

When $n = i$, it gives in particular the (usual) cycle class map:

(3.1.1.b)

$$\sigma_{\text{syn}} : \text{CH}^n(X) \rightarrow H_{\text{syn}}^{2n}(X, n)$$

which is compatible with pullbacks and products of cycles as defined in [Ful98].

Moreover, all the maps of the homotopy pullback square (2.5.3.a) are morphisms of monoids in $DM_{\mathbb{B}}(R)$. This can be translated into the following commutative diagram:

$$\begin{array}{ccc}
H_{\text{syn}}^n(X, i) & \xrightarrow{\alpha_*} & F^i H_{\text{dR}}^n(X_K) \\
\downarrow \beta_* & \swarrow \sigma_{\text{syn}} & \nearrow \sigma_{\text{FdR}} j^* \\
& H_{\mathbb{B}}^{n,i}(X) & (b) \\
& \swarrow \sigma_{\phi} i^* & \searrow \sigma_{\text{rig}} i^* \\
H_{\phi}^n(X_k, i) & \xrightarrow{\quad} & H_{\text{rig}}^n(X_k/K)
\end{array}$$

(a)

where $\sigma_?$ stands for the higher cycle classes relevant to the corresponding cohomology, and i^* (resp. j^*) denotes the pullback in motivic cohomology by i (resp. j).

Remark 3.1.2. (1) The part (a) of the above commutative diagram simply express the fact that for any smooth k -scheme X_0 , the higher cycle class map

$$\sigma_{\text{rig}} : H_{\mathbb{B}}^{n,i}(X_0) \rightarrow H_{\text{rig}}^n(X_0/K_0)$$

lands into the part $\phi = p^i$ of rigid cohomology and that it admits a canonical lifting to the absolute rigid cohomology $H_{\phi}^{n,i}(X)$ through the canonical surjection

$$H_{\phi}^{n,i}(X) \rightarrow H_{\text{rig}}^n(X/K_0)^{\phi=p^i}$$

of Corollary 2.4.4.

- (2) One can deduce from the commutativity of the part (b) of the above diagram another proof of the fact, already obtained in [CCM12], that the specialization map sp is compatible with the specialization map sp_{CH} in Chow theory as defined in [Ful98, §20.3]. Indeed, in the case $i = 2n$, (2) can be rewritten as follows:

$$\begin{array}{ccccc}
& & \text{CH}^n(X_K) & \xrightarrow{\sigma_{\text{FdR}}} & F^{2n} H_{\text{dR}}^n(X_K) \\
& \nearrow j^* & \downarrow \text{sp}_{\text{CH}} & & \downarrow \text{sp} \\
\text{CH}^n(X) & & & & \\
& \searrow i^* & \text{CH}^n(X_k) & \xrightarrow{\sigma_{\text{rig}}} & H_{\text{rig}}^n(X_k/K)
\end{array}$$

and the assertion follows as j^* is surjective and sp_{CH} is the unique morphism making the left hand side commutative.

- (3) (Concerning the terminology) The term “higher cycle classes” comes from the theory of higher Chow groups – which, for smooth R -schemes, coincide rationally with Beilinson motivic cohomology according to [Lev04, 14.7].

The term “syntomic regulator” has been introduced by M. Gros in [Gro90]. It comes from the intuition that syntomic cohomology is an analogue of Deligne cohomology and that one can transport the setting of Beilinson’s conjectures from Deligne cohomology to syntomic cohomology. One should be careful however that in the case of Deligne cohomology and if $2i - n = 1$, then the higher cycle class map is only a part of the regulator (see [Sou86, §3.3]).

3.1.3. The ring spectrum \mathbb{E}_{syn} , considered as an object of $D_{\mathbb{A}^1}(R, \mathbb{Q}_p)$, is oriented (see Remark 1.4.11). Thus, one can apply to it the orientation theory of \mathbb{A}^1 -homotopy theory (see [Dég11] in the arithmetic case).

This implies syntomic cohomology admits Chern classes, which are nothing else than the image of the Chern classes in Chow theory through the cycle class map, and satisfies the projective bundle formula (see [Dég11, 2.1.9]). One also gets a chern character map in $D_{\mathbb{A}^1}(R, \mathbb{Q})$:

$$\text{ch}_{\text{syn}} : KGL_{\mathbb{Q}} \xrightarrow{\text{ch}} \bigoplus_{i \in \mathbb{Z}} H_{\mathbb{B}}(i)[2i] \xrightarrow{\sigma} \bigoplus_{i \in \mathbb{Z}} \mathbb{E}_{\text{syn}}(i)[2i]$$

where $KGL_{\mathbb{Q}}$ is the ring spectrum representing rational algebraic K-theory over R and ch is the isomorphism of [CD09b, 13.2.6(3)]. This map induces the usual higher Chern character (see [Gil81]) for any smooth R -scheme X :

$$\text{ch}_n : K_n(X)_{\mathbb{Q}} \rightarrow \prod_{i \in \mathbb{N}} H_{\text{syn}}^{2i-n}(X, i).$$

Remark 3.1.4. The Chern classes constructed above are determined by the first Chern class c_1 , which is determined by its value on the canonical line bundle of \mathbb{P}_R^1 . According to our construction, this is nothing else than the class $dlog$. One deduces that the Chern classes obtained here in syntomic cohomology coincide with the one previously constructed by A. Besser in [Bes00].

3.1.5. The syntomic cohomology ring spectrum allows to extend syntomic cohomology to a (co-homological) realization functor of $DM_{\mathbb{B}}(R)$:

$$\mathbb{E}_{\text{syn}} : DM_{\mathbb{B}}(R)^{op} \rightarrow \mathbb{Q}_p - \text{vs}, M \mapsto \text{Hom}_{DM_{\mathbb{B}}(R)}(M, \mathbb{E}_{\text{syn}}).$$

This shows that the syntomic cohomology of a smooth R -scheme X inherits the functorial structure of the motive of X .

In particular, given a projective morphism of smooth R -schemes $f : Y \rightarrow X$, there exists a Gysin morphism on motives:

$$M(X) \rightarrow M(Y)(-d)[-2d]$$

where d is the dimension of f . This was constructed in [Dég08] and several properties of this Gysin morphism were proved there. Thus, after applying the functor \mathbb{E}_{syn} above, one gets:

Theorem 3.1.6. *Consider the above notations. One can associate to f a Gysin morphism in syntomic cohomology:*

$$f_* = \mathbb{E}_{\text{syn}}(f^*) : H_{\text{syn}}^n(Y, i) \rightarrow H_{\text{syn}}^{n-2d}(X, i-d).$$

Moreover, one gets the following properties:

- (1) ([Dég08, 5.14]) *For any composable projective morphisms f, g , $(fg)_* = f_*g_*$.*
- (2) (projection formula, [Dég08, 5.18]) *For any projective morphism $f : Y \rightarrow X$ and any pair $(x, y) \in H_{\text{syn}}^{*,*}(X) \times H_{\text{syn}}^{*,*}(Y)$, one has:*

$$f_*(f^*(x).y) = x.f_*(y).$$

- (3) (Excess intersection formula, [Dég08, 5.17(ii)]) *Consider a cartesian square of smooth R -schemes:*

$$\begin{array}{ccc} Y' & \xrightarrow{q} & X' \\ g \downarrow & & \downarrow f \\ Y & \xrightarrow{p} & X \end{array}$$

such that p is projective. Let ξ be the excess intersection bundle associated with that square.²⁴

Then for any $y \in H_{\text{syn}}^{*,*}(Y)$, one gets:

$$f^*p_*(y) = q_*(c_e(\xi).g^*(y)).$$

- (4) *For any projective morphism $f : Y \rightarrow X$, the following diagram is commutative:*

$$\begin{array}{ccc} H_{\mathbb{B}}^{n,i}(Y) & \xrightarrow{f_*} & H_{\mathbb{B}}^{n-2d,i-d}(X) \\ \sigma_{\text{syn}} \downarrow & & \downarrow \sigma_{\text{syn}} \\ H_{\text{syn}}^n(Y, i) & \xrightarrow{f_*} & H_{\text{syn}}^{n-2d}(X, i-d) . \end{array}$$

²⁴Recall from *loc. cit.* that one defines ξ as follows: let us choose a closed embedding $i : Y \rightarrow P$ into a projective bundle over X and let $Y' \rightarrow P'$ be its pullback over X' . Let N (resp. N') be the normal vector bundle of Y in P (resp. Y' in P'). Then, as the preceding square is cartesian, there is a monomorphism $N' \rightarrow g^{-1}(N)$ of vector bundles over Y' and one puts: $\xi = g^{-1}(N)/N'$.

Remark 3.1.7. • With the notation of Point (3) recall that ξ has dimension $n - m$ where n (resp. m) is the dimension of p (resp. q). In particular, when the square is *transverse i.e.* $n = m$, one gets the more usual formula: $f^*p_* = q_*g^*$.

- Point (2) can simply be derived from the preceding formula applied to the graph morphism $\gamma : Y \rightarrow Y \times_R X$ given that γ^* is compatible with products.
- Point (4) shows in particular that, when $i : Z \rightarrow X$ is a closed immersion, $i_*(1) = \sigma_{\text{syn}}([Z])$ is the (syntomic) fundamental class of Z in X . If Z is a smooth divisor, corresponding to the line bundle \mathcal{L}/X , one gets in particular:

$$i_*(1) = c_1(\mathcal{L}).$$

This property determines uniquely the Gysin morphism in the case of a closed immersion (see [Dég08] or [Pan09]).

When $p : P \rightarrow X$ is the projection of a projective bundle of rank n and canonical line bundle λ , one gets, again applying Point (4):

$$p_*(c_1(\lambda)^i) = \begin{cases} 1 & \text{if } i=n \\ 0 & \text{otherwise} \end{cases}$$

This fact, together with the projective bundle formula in syntomic cohomology, determines uniquely the morphism p_* .

By construction, the Gysin morphism f_* for any projective morphism f is completely determined by the two above properties.

- Point (4) was conjectured by Besser [Bes12, Conjecture 4.2] in the case of any proper morphism and Theorem 1.1 in *loc. cit.* is conditional to the conjecture. The latter result concerns the regulator of a proper and smooth surface S over R . We also note that Point (4) has already been used (in the projective morphism case, although stated for proper maps) in [Lan11, p. 505] but the references given there is draft of [CCM12] which turns to be different from the published version and does not contain the above statement.

Example 3.1.8. Let $f : Y \rightarrow X$ be a finite morphism between smooth connected R -schemes. Let d be the degree of the extension of the corresponding function fields. Then one gets the *degree formula* in syntomic cohomology: for any $x \in H_{\text{syn}}^*(X, *)$,

$$f_*f^*(x) = d.x.$$

Indeed, according to 3.1.6(1),

$$f_*f^*(x) = f_*(1.f^*(x)) = f_*(1).x.$$

Then one gets $f_*(1) = d$ from 3.1.6(4) and the degree formula in Beilinson motivic cohomology.

As a corollary of Point (4) of the preceding theorem, one obtains the Riemann-Roch formula in syntomic cohomology.

Corollary 3.1.9. *Let $f : Y \rightarrow X$ be a projective morphism between smooth R -schemes. Let τ_f be the virtual tangent bundle of f in $K_0(X)$: $\tau_f = [T_X] - [T_Y]$, the difference of the tangent bundle of X/R with that of Y/R . Then for any element $y \in K_n(Y)_{\mathbb{Q}}$, one gets the following formula:*

$$\text{ch}_{\text{syn}}(f_*(y)) = f_*(\text{td}(\tau_f) \cdot \text{ch}_{\text{syn}}(y))$$

where $\text{td}(\tau_f)$ is the Todd class of the virtual vector bundle τ_f in syntomic cohomology (define for example as the image of the usual Todd class in Chow groups by the cycle class map).

In fact, this corollary is deduced from the Riemann-Roch formula in motivic cohomology after applying to it the syntomic higher cycle class and applying Point (4) of the previous theorem.

3.1.10. Applying the same formalism to the motivic ring spectra \mathbb{E}_{FdR} , $\mathbb{E}_{\text{rig}, K}$, \mathbb{E}_{ϕ} , one obtains Gysin morphisms on their cohomology, satisfying the preceding properties. Moreover, using the distinguish triangle (2.5.3.b) of $DM_{\mathbb{B}}(R)$, one gets the following result:

Proposition 3.1.11. *Let $f : Y \rightarrow X$ be a projective morphism between smooth R -schemes, and denote by f_k (resp. f_K) its special (resp. generic) fiber. Then the following diagram is commutative:*

$$\begin{array}{ccccccc}
 H_{\text{syn}}^n(Y, i) & \xrightarrow{\alpha_* + \beta_*} & H_{\phi}^n(Y_k, i) \oplus F^i H_{\text{dR}}^n(Y_K) & \xrightarrow{a_* - b_*} & H_{\text{rig}}^n(Y_k/K) & \longrightarrow & H_{\text{syn}}^{n+1}(Y, i) \\
 f_* \downarrow & & \downarrow f_{k*} + f_{K*} & & \downarrow f_{k*} & & \downarrow f_* \\
 H_{\text{syn}}^{n-2d}(X, i-d) & \xrightarrow{\alpha_* + \beta_*} & H_{\phi}^{n-2d}(X_k, i-d) \oplus F^{i-d} H_{\text{dR}}^{n-2d}(X_K) & \xrightarrow{a_* - b_*} & H_{\text{rig}}^{n-2d}(X_k/K) & \longrightarrow & H_{\text{syn}}^{n-2d+1}(X, i-d)
 \end{array}$$

where the lines are given by the exact sequences (2.5.3.c).

3.2. The six functors formalism and Bloch-Ogus axioms. In this section, we will recall some consequences of Grothendieck 6 functors formalism established for Beilinson motives (see [CD09b, 2.4.21] for a summary) and apply this theory to the spectra considered in this paper. We will fix a base scheme S and consider only separated S -schemes of finite type over S . We will also consider an abstract object \mathbb{E} of $DM_{\mathbb{B}}(S)$.

3.2.1. We associate with \mathbb{E} four homology/cohomology theories defined for an S -scheme X with structural morphism f and a pair of integers (n, i) as follows:

Cohomology	$\mathbb{E}^{n,i}(X) = \text{Hom}(\mathbb{1}_S, f_* f^* \mathbb{E}(i)[n])$
Homology	$\mathbb{E}_{n,i}(X) = \text{Hom}(\mathbb{1}_S, f_! f^! \mathbb{E}(-i)[-n])$
Cohomology with compact support	$\mathbb{E}_c^{n,i}(X) = \text{Hom}(\mathbb{1}_S, f_! f^* \mathbb{E}(i)[n])$
Borel-Moore homology	$\mathbb{E}_{n,i}^{\text{BM}}(X) = \text{Hom}(\mathbb{1}_S, f_* f^! \mathbb{E}(-i)[-n])$

We will use the terminology *c-cohomology* (resp. *BM-homology*) for cohomology with compact support (resp. Borel-Moore homology).

Note that these definitions, apply to the unit object $\mathbb{1}$ of $DM_{\mathbb{B}}(S)$, yields the four corresponding motivic theories. Also, these definitions are (covariantly) functorial in \mathbb{E} . In particular, if \mathbb{E} admits a structure of a monoid in $DM_{\mathbb{B}}(R)$ (*i.e.* \mathbb{E} is a ring spectrum), the unit map $\eta : \mathbb{1} \rightarrow \mathbb{E}$ yields regulators in all four theories.

When X/S is proper, as $f_* = f_!$, one gets identifications:

$$\mathbb{E}^{n,i}(X) = \mathbb{E}_c^{n,i}(X), \quad \mathbb{E}_{n,i}^{\text{BM}}(X) = \mathbb{E}_{n,i}(X).$$

Example 3.2.2. (1) When $S = \text{Spec } R$, we get syntomic theories for singular R -schemes.

According to the above paragraph, the higher cycle class (3.1.1.a) is defined for singular schemes.²⁵

When focusing attention to Chow theory, one gets in particular:

- X regular: $\sigma_{\text{syn}} : \text{CH}^n(X) \rightarrow H_{\text{syn}}^{2n}(X, n)$.
- X regular quasi-projective: $\sigma_{\text{syn}} : \text{CH}_n(X) \rightarrow H_{2n}^{\text{syn, BM}}(X, n)$.

The second point follows from the fact $H_{\mathbb{B}}^{n,i}(X) \simeq H_{2d-n, d-i}^{\text{B, BM}}(X)$ where d is the (Krull) dimension of X according to the motivic absolute purity theorem ([CD09b, 13.4.1]).

- (2) When $S = \text{Spec } k$, we get syntomic rigid (resp. absolute rigid) theories associated with $\mathbb{E}_{\text{syn}, K}$ (resp. \mathbb{E}_{ϕ}) and regulators for these theories.

In the case $K = K_0$, the Frobenius operator Φ of \mathbb{E}_{rig} induces an action of Frobenius on all four theories, compatible with the regulator. Moreover, the distinguished triangle (2.4.6.a) yields long exact sequences in all four theories.

- (3) When $S = \text{Spec } K$, we get De Rham and absolute Hodge theories from \mathbb{E}_{dR} and \mathbb{E}_{FdR} equipped with regulators. The canonical map $\mathbb{E}_{\text{FdR}} \rightarrow \mathbb{E}_{\text{dR}}$ induces natural maps of these theories, compatible with regulators.

Consider the specialization map:

$$\text{sp} : j_* \mathbb{E}_{\text{FdR}} \rightarrow i_* \mathbb{E}_{\text{rig}, K}.$$

²⁵Recall that for singular schemes, Beilinson motivic cohomology is defined after [CD09b] and [Cis03] as the graded part of homotopy invariant K-theory for the γ -filtration.

Given any R -scheme X with structural morphism f , and applying $f_*f^!$ to this map one obtains:

$$\mathrm{sp}_* : H_n^{\mathrm{FdR}, \mathrm{BM}}(X_K, i) \rightarrow H_n^{\mathrm{rig}, K, \mathrm{BM}}(X_k, i)$$

using the exchange isomorphisms: $f^!i_* = i'_*f_k^!$ and $f^!j_* = j'_*f_K^!$. Similarly, if we apply $f_*f^!$ to the distinguished triangle (2.5.3.a), one gets the following long exact sequence:

(3.2.2.a)

$$\dots \rightarrow H_n^{\mathrm{syn}, \mathrm{BM}}(X, i) \xrightarrow{\alpha_* + \beta_*} H_n^{\phi, \mathrm{BM}}(X_k, i) \oplus H_n^{\mathrm{FdR}, \mathrm{BM}}(X_K, i) \xrightarrow{\alpha_* - \mathrm{sp}_*} H_n^{\mathrm{rig}, K, \mathrm{BM}}(X_k, i) \rightarrow \dots$$

3.2.3. Functoriality properties.– We consider a morphism of S -schemes:

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ & \searrow q & \swarrow p \\ & S & \end{array}$$

Using the adjunction map $ad_f : 1 \rightarrow f_*f^*$ (resp. $ad'_f : f_!f^! \rightarrow 1$), we immediately obtain that cohomology is contravariant (resp. homology is covariant) by – composing on the left by p_* (resp. $p_!$) and on the right by p^* (resp. $p^!$).

When f is proper, $f_! = f_*$. Using again ad_f , ad'_f , one deduces that c-cohomology (resp. BM-homology) is contravariant (resp. contravariant) with respect to proper maps.

When f is smooth of relative dimension d , one has the relative purity isomorphism:

$$f^! \simeq f^*(d)[2d]$$

(see [CD09b, 2.4.21]). In particular, one derive from ad_f and ad'_f the following maps:

$$f^* : \mathbb{E}_c^{n, i}(X) \rightarrow \mathbb{E}_c^{n-2d, i-d}(Y), \quad f_* : \mathbb{E}_{n, i}^{\mathrm{BM}}(X) \rightarrow \mathbb{E}_{n+2d, i+d}^{\mathrm{BM}}(Y).$$

Finally, when f is proper and smooth of relative dimension d one gets in addition:

$$f^* : \mathbb{E}_c^{n, i}(X) \rightarrow \mathbb{E}_c^{n-2d, i-d}(Y), \quad f_* : \mathbb{E}_{n, i}(X) \rightarrow \mathbb{E}_{n+2d, i+d}(Y).$$

Let us summarize the situation:

theory	covariance (degree)	contravariance (degree)
Cohomology	smooth proper, (-2d, -d)	any
Homology	any	smooth proper, (+2d, +d)
Cohomology with compact support	smooth, (-2d, -d)	proper
Borel-Moore homology	proper	smooth, (+2d, +d)

Remark 3.2.4. The fact that the functorialities constructed above are compatible with composition is obvious except when a smooth morphism is involved. This last case follows from the functoriality of the relative purity isomorphism proved by Ayoub in [Ayo07].

When considering one of the four theories associated with \mathbb{E} , one can mix the two kinds of functoriality in a projection formula as usual. In fact, given a cartesian square:

$$\begin{array}{ccc} Y' & \xrightarrow{g} & X' \\ q \downarrow & & \downarrow p \\ Y & \xrightarrow{f} & X \end{array}$$

such that f is proper and smooth (or f smooth and g proper when considering \mathbb{E}_c or \mathbb{E}^{BM}), one obtains respectively:

- $f^*p_* = q_*g^*$ for the two homologies,
- $p^*f_* = g_*q^*$ for the two cohomologies.

This is a lengthy check coming back to the definition of the relative purity isomorphism. The essential fact is that

$$g^{-1}(T_{Y/X}) = T_{Y'/X'}$$

where $T_{Y/X}$ (resp. $T_{Y'/X'}$) is the tangent bundle of f (resp. g).

Example 3.2.5. All the theories considered in the previous example satisfies the functorialities described above. Moreover, regulators are compatible with these functorialities. Similarly, the maps sp_* , α_* , β_* , a_* and b_* considered in part (3) of this example are natural with respect to proper covariant and smooth contravariant functorialities.

Moreover, taking care of the functoriality explained in the previous remark for motivic BM-homology, one can check the following diagram is commutative:

$$\begin{array}{ccccc}
 & & H_{n,i}^{\mathbb{B}}(X_K/K) & \xrightarrow{\sigma_{\mathrm{FdR}}} & H_n^{\mathrm{FdR},\mathrm{BM}}(X_K, i) \\
 & \nearrow^{j^*} & & & \downarrow \mathrm{sp}_* \\
 H_{n,i}^{\mathbb{B}}(X/R) & & & & \\
 & \searrow_{i^*} & H_{n,i}^{\mathbb{B}}(X_k/k) & \xrightarrow{\sigma_{\mathrm{rig}}} & H_n^{\mathrm{rig},K,\mathrm{BM}}(X_k, i)
 \end{array}$$

When X/R is quasi-projective regular with good reduction and $i = 2n$, one obtains in particular a generalization of the second part of Remark 3.1.2 (applying the motivic absolute purity theorem [CD09b, 13.4.1], all the motivic BM-homology in the above diagram can be identified with Chow groups in that case).

This fact can be extended to the exact sequence (3.2.2.a) and to its compatibility with the regulator in syntomic BM-homology.

3.2.6. Products.— Let us now assume that \mathbb{E} is a ring spectrum, with unit map $\eta : \mathbb{1}_S \rightarrow \mathbb{E}$ and product map $\mu : \mathbb{E} \otimes \mathbb{E} \rightarrow \mathbb{E}$.

Of course, for any S -scheme X with structural map f , we can define a product on cohomology, sometimes called the cup-product:

$$\mathbb{E}^{n,i}(X) \otimes \mathbb{E}^{m,j}(X) \rightarrow \mathbb{E}^{n+m,i+j}(X), (x, y) \mapsto xy = x \cup y;$$

given cohomology classes

$$x : \mathbb{1}_X \rightarrow f^*\mathbb{E}(i)[n], y : \mathbb{1}_X \rightarrow f^*\mathbb{E}(j)[m],$$

we define xy as the following composite map:

$$\mathbb{1}_X \xrightarrow{x \otimes y} f^*(\mathbb{E})(i)[n] \otimes f^*(\mathbb{E})(j)[m] = f^*(\mathbb{E} \otimes \mathbb{E})(i+j)[n+m] \xrightarrow{\mu} f^*(\mathbb{E})(i+j)[n+m].$$

This product is obviously commutative and associative. Note one can also define an exterior product on cohomology as follows:

$$\mathbb{E}^{n,i}(X) \otimes \mathbb{E}^{m,j}(Y) \rightarrow \mathbb{E}^{n+m,i+j}(X \times_S Y), (x, y) \mapsto p_1^*(x) \cdot p_2^*(y)$$

where p_1 (resp. p_2) is the projection $X \times_S Y/X$ (resp. $X \times_S Y/Y$).

One can also define *exterior products* on c-cohomology. Consider a cartesian square:

$$\begin{array}{ccc}
 X \times_S Y & \xrightarrow{f'} & Y \\
 g' \downarrow & \searrow h & \downarrow g \\
 X & \xrightarrow{f} & S
 \end{array}$$

of separated morphisms of finite type. We define the following product on c-cohomology:

$$\mathbb{E}_c^{n,i}(X) \otimes \mathbb{E}_c^{m,j}(Y) \rightarrow \mathbb{E}_c^{n+m,i+j}(X \times_S Y), (x, y) \mapsto x \times y$$

which associates to any maps

$$x : \mathbb{1}_S \rightarrow f_! f^* \mathbb{E}(i)[n], y : \mathbb{1}_S \rightarrow g_! g^* \mathbb{E}(j)[m],$$

the following composite map $x \times y$:

$$\begin{aligned}
\mathbb{1}_S &\xrightarrow{x \otimes y} f_! f^*(\mathbb{E})(i)[n] \otimes g_! g^*(\mathbb{E})(j)[m] \\
&\simeq f_!(f^*(\mathbb{E})(i)[n] \otimes f^* g_! g^*(\mathbb{E}))(i+j)[n+m] \\
&\simeq f_!(f^*(\mathbb{E})(i)[n] \otimes g'_! f'^* g^*(\mathbb{E}))(i+j)[n+m] \\
&\simeq f_! g'_!(g'^* f^*(\mathbb{E})(i)[n] \otimes f'^* g^*(\mathbb{E}))(i+j)[n+m] \\
&= h_! h^*(\mathbb{E} \otimes \mathbb{E})(i+j)[n+m] \xrightarrow{\mu} h_! h^*(\mathbb{E})(i+j)[n+m]
\end{aligned}$$

where the second and third isomorphisms follows from the projection formula [CD09b, 2.4.21(v)] and the second one from the exchange isomorphism [CD09b, 2.4.21(iv)].

One can check the following formulas:

$$(x \times y) \times z = x \times (y \times z), x \times y = y \times x$$

through the respective isomorphisms

$$(X \times_S Y) \times_S Z \simeq (X \times_S Y) \times_S Z, X \times_S Y \simeq Y \times_S X.$$

Further, because c-cohomology is contravariant with respect to proper morphism, given any S -schemes X (separated of finite type), the diagonal embedding $\delta : X \rightarrow X \times_S X$ allows to define an inner product on c-cohomology:

$$\mathbb{E}_c^{n,i}(X) \otimes \mathbb{E}_c^{m,j}(X) \rightarrow \mathbb{E}_c^{n+m,i+j}(X \times_S X), (x, x') \mapsto \delta^*(x \times x').$$

When X/S is proper, one can check this product coincides with cup-product on cohomology.

Remark 3.2.7. Let $f : Y \rightarrow X$ be a proper smooth morphism. According to the projection formulas established of Remark 3.2.4, one can check that for any couple (y, x) either in $\mathbb{E}^{n,i}(Y) \times \mathbb{E}^{m,j}(X)$ or in $\mathbb{E}_c^{n,i}(Y) \times \mathbb{E}_c^{m,j}(X)$, one gets the following usual projection formula (for products):

$$f_*(x \cdot f^*(y)) = f_*(x) \cdot y.$$

In fact, in each case, one uses the relevant formula of Remark 3.2.4, the external product and the following formulas:

$$y \times f^*(x) = (1_Y \times_S f)^*(y \times x), f_*(y) \times x = (f \times_S 1_X)_*(y \times x).$$

3.2.8. Cap product.– One can extend the cohomology theory associated with E to a theory with support. Given any closed immersion of S -schemes:

$$\begin{array}{ccc}
Z & \xrightarrow{i} & X, \\
& \searrow g & \swarrow f \\
& & S
\end{array}$$

one puts:

$$\mathbb{E}_Z^{n,i}(X) = \mathrm{Hom}(i_*(\mathbb{1}_X), f^* \mathbb{E}(i)[n]) = \mathrm{Hom}(\mathbb{1}_X, i^! f^* \mathbb{E}(i)[n]).$$

This theory satisfies all the usual properties. We refer the reader to [Dég11, §1.2] for a detailed account.

Assuming again \mathbb{E} is a ring spectra with product map $\mu : \mathbb{E} \otimes \mathbb{E} \rightarrow E$, one defines, following Bloch and Ogus,[BO74], the *cap-product with supports*:

$$\mathbb{E}_{n,i}^{\mathrm{BM}}(X) \otimes \mathbb{E}_Z^{m,j}(X) \rightarrow \mathbb{E}_{n-m,i-j}^{\mathrm{BM}}(Z), (x, z) \mapsto x \cap z.$$

Let us first introduce classical pairing of functors (see [Del77, IV, §1.2]): given any objects A and B of $DM_{\mathbb{E}}(S)$, one considers the following composite map

$$f_!(f^!(A) \otimes f^*(B)) \rightarrow \mathrm{Ex}[f_! f^!(A)] \otimes B \xrightarrow{ad'_f} A \otimes B$$

where the first map is the isomorphism of the projection formula ([CD09b, 2.4.21]) and the second one is the counit of the adjunction $(f_!, f^!)$. One thus deduces by adjunction the following pairing

$$f^!(A) \otimes f^*(B) \xrightarrow{\eta_f} f^!(A \otimes B).$$

Thus, given maps

$$x : \mathbb{1}_X \rightarrow f^!(\mathbb{E}), \quad z : i_*(\mathbb{1}_Z) \rightarrow f^*(\mathbb{E})$$

one defines $x \cap z$ from the following composite map:

$$i_*(\mathbb{1}_Z) \xrightarrow{x \otimes z} f^!(\mathbb{E}) \otimes f^*(\mathbb{E}) \xrightarrow{\eta f} f^!(\mathbb{E} \otimes \mathbb{E}) \xrightarrow{\mu} f^!(\mathbb{E})$$

using $i_* = i_!$, the adjunction $(i_!, i^!)$ and $i^! f^! = g^!$.

Remark 3.2.9. Consider a cartesian square of S -schemes

$$\begin{array}{ccc} T & \xrightarrow{k} & Y \\ g \downarrow & & \downarrow f \\ Z & \xrightarrow{i} & X \end{array}$$

such that i is a closed immersion and f is proper. Then, for any couple $(y, z) \in \mathbb{E}_{n,i}^{\text{BM}}(X) \otimes \mathbb{E}_Z^{m,j}(X)$, one obtains the following formula:

$$f_*(y) \cap z = g_*(y \cap f^*(z)).$$

3.2.10. Suppose again \mathbb{E} is a ring spectrum with unit map $\eta : \mathbb{1}_S \rightarrow \mathbb{E}$.

Let $f : X \rightarrow S$ be a smooth S -scheme of relative dimension d . Then, according to [CD09b, 2.4.21(iii)], one obtains a canonical isomorphism of functors:

$$\mathfrak{p}_f : f^! \rightarrow f^*(d)[2d].$$

In particular, one gets a canonical map

$$\eta_X : \mathbb{1}_X = f^*(\mathbb{1}_S) \xrightarrow{f^*(\eta)} f^*(\mathbb{E}) \xrightarrow{\mathfrak{p}_f^{-1}} f^!(\mathbb{E})(-d)[-2d]$$

which corresponds to a homological class $\eta_X \in \mathbb{E}_{2d,d}^{\text{BM}}(X)$. The following result is now a tautology:

Proposition 3.2.11. *Consider the above assumptions, and let $Z \subset X$ be any closed subset. Then the following map:*

$$\mathbb{E}_Z^{n,i}(X) \rightarrow \mathbb{E}_{2d-n,i-n}^{\text{BM}}(Z), \quad z \mapsto \eta_X \cap z$$

is an isomorphism.

One can now summarize some of the main properties we have proved so far as follows:

Corollary 3.2.12. *The couple of functors $(\mathbb{E}^{**}, \mathbb{E}_{**}^{\text{BM}})$ form a Poincaré duality theory with supports in the sense of Bloch and Ogus ([BO74, Def 1.3]).*

This is the case in particular for syntomic cohomology and syntomic BM-homology.

3.2.13. Descent theory.– Recall (see [CD09b, §3.1]) that a diagram of S -schemes (\mathcal{X}, I) is the data of a small category I and a functor $\mathcal{X} : I \rightarrow \mathcal{S}$. A morphism of diagrams $\varphi = (\alpha, f) : (\mathcal{X}, I) \rightarrow (\mathcal{Y}, J)$ is the data of a functor $f : I \rightarrow J$ and a natural transformation $\alpha : \mathcal{X} \rightarrow f^*(\mathcal{Y})$ where $f^*(\mathcal{Y}) = \mathcal{Y} \circ f$.

According to [CD09b, §3.1], the fibered triangulated category $DM_{\mathbb{B}}$ can be extended to the category of diagrams. Moreover, for any morphism of diagrams $\varphi : (\mathcal{X}, I) \rightarrow (\mathcal{Y}, J)$, one has an adjoint pair of functors:

$$\varphi^* : DM_{\mathbb{B}}(\mathcal{Y}, J) \rightleftarrows DM_{\mathbb{B}}(\mathcal{X}, I) : \varphi_*$$

Consider a diagram of S -schemes (\mathcal{X}, I) and the canonical morphism $\varphi : (\mathcal{X}, I) \rightarrow (S, *)$ where $*$ is the final category. Then one defines the cohomology of (\mathcal{X}, I) as:

$$\mathbb{E}^{n,i}(\mathcal{X}, I) = \text{Hom}(\mathbb{1}, \varphi_* \varphi^*(\mathbb{E})(i)[n]).$$

This is contravariant with respect to morphisms of diagrams.

In particular one has extended the cohomology $\mathbb{E}^{*,*}$ to simplicial S -schemes. The h-topology was introduced by Voevodsky in [Voe96]. Recall that a h-cover $f : Y \rightarrow X$ of S -schemes is a universal topological epimorphism (e.g. faithfully flat maps, proper surjective maps). Then the h-descent theorem for Beilinson motives ([CD09b, 13.3.2]) states the following:

For any S -scheme X and any hypercover $p : \mathcal{X} \rightarrow X$ for the h-topology, the canonical map

$$p^* : \mathbb{E}^{n,i}(X) \rightarrow \mathbb{E}^{n,i}(\mathcal{X})$$

is an isomorphism. In particular, one gets the usual spectral sequence:

$$E_1^{p,q} = \mathbb{E}^{p,i}(\mathcal{X}_q) \Rightarrow \mathbb{E}^{p+q,i}(X).$$

Remark 3.2.14. As already remarked in [CD09a], the preceding descent theory together with De Jong resolution of singularities, shows that in the case where S is the spectrum of a field (non necessarily perfect), the cohomology $\mathbb{E}^{*,*}$ is uniquely determined by its restriction to smooth schemes.

Example 3.2.15. Syntomic cohomology can be functionally extended to diagrams of S -schemes, as well as rigid cohomology, absolute rigid cohomology and absolute Hodge cohomology. One should be careful however that the syntomic long exact sequence (2.5.3.b) can be extended only to the case of diagrams of smooth S -schemes.

3.3. Rigid syntomic modules.

3.3.1. The aim of this last section is to apply the theory developed in [CD09b, sec. 4.2] to the syntomic ring spectrum \mathbb{E}_{syn} .

Put $S = \text{Spec } R$. Recall that by construction, \mathbb{E}_{syn} can be seen as an object of $\text{Sp}^{\text{ring}}(S, \mathbb{Q})$ (Paragraph 1.4.6).

Let $f : X \rightarrow S$ be any morphism of schemes. The pullback functor f^* on the category of Tate spectra is monoidal. Thus, it obviously induces a functor:

$$f^* : \text{Sp}^{\text{ring}}(S, \mathbb{Q}) \rightarrow \text{Sp}^{\text{ring}}(X, \mathbb{Q}).$$

In particular, we can define the rigid syntomic ring spectrum over X as follows:

$$\mathbb{E}_{\text{syn},X} := f^*(\mathbb{E}_{\text{syn}}).$$

The collection of these ring spectra defines a cartesian section of the fibred category $\text{Sp}^{\text{ring}}(-, \mathbb{Q})$ over the category of R -schemes. In particular, one can apply [CD09b, Prop. 4.2.10] to it. In particular, the category of modules over $\mathbb{E}_{\text{syn},X}$ in $\text{Sp}(X, \mathbb{Q})$ admits a model structure.

Definition 3.3.2. Consider the above notations.

We define the category $\mathbb{E}_{\text{syn}}\text{-mod}_X$ of *rigid syntomic modules over X* as the homotopy category of the model category of modules over the ring spectrum $\mathbb{E}_{\text{syn},X}$.

3.3.3. According to [CD09b], Prop. 4.2.11 and 4.2.16, rigid syntomic modules inherit the good functoriality properties of the stable homotopy category (in the terminology of [CD09b, Def. 2.4.2], the category $\mathbb{E}_{\text{syn}}\text{-mod}$, fibered over the category of R -schemes, is motivic). Let us recall briefly the 6 functors formalisms: given a morphism $f : T \rightarrow S$ of R -schemes, one has two pair of adjoint functors:

$$\begin{aligned} f^* : \mathbb{E}_{\text{syn}}\text{-mod}_S &\rightleftarrows \mathbb{E}_{\text{syn}}\text{-mod}_T : f_* , \\ f_! : \mathbb{E}_{\text{syn}}\text{-mod}_T &\rightleftarrows \mathbb{E}_{\text{syn}}\text{-mod}_S : f_! , \text{ for } f \text{ separated of finite type,} \end{aligned}$$

and $\mathbb{E}_{\text{syn}}\text{-mod}_X$ is triangulated closed monoidal. We denote by \otimes (resp. $\underline{\text{Hom}}$) the tensor product (resp. internal Hom).

- $f_* = f_!$ for f proper,
- Relative purity: $f_! = f^*(d)[2d]$ for f smooth of constant relative dimension d ,
- Base change formulas: $f^*g_! = g'_!f'^*$, for f any morphism (resp. g any separated morphism of finite type), f' (resp. g') the base change of f along g (resp. g along f).
- Projection formulas: $f_!(M \otimes f^*(N)) = f_!(M) \otimes N$.
- Localization property: given any closed immersion $i : Z \rightarrow S$ of R -schemes, with complementary open immersion j , there exists a distinguished triangle of natural transformations as follows:

$$j_!j^! \rightarrow 1 \rightarrow i_*i^* \xrightarrow{\partial_i} j_!j^![1]$$

where the first (resp. second) denotes the counit (resp. unit) of the relevant adjunction (as in Paragraph 2.6.1).

Remark 3.3.4. An important set of properties is missing in the theory of rigid syntomic modules.

One will say that a syntomic module over X is *constructible* if and only if it is compact in the triangulated category $\mathbb{E}_{\text{syn-mod}}_X$. The category of constructible modules should enjoy the following properties:

- (1) they are stable by the 6 operations (when restricted to excellent R -schemes),
- (2) they satisfy Grothendieck duality (existence of a dualizing module).

To get these properties, one has only to prove the absolute purity for syntomic modules: given any closed immersion $i : Z \rightarrow X$ of regular R -schemes, of pure codimension c , there exists an isomorphism:

$$i^!(\mathbb{1}_X) = \mathbb{1}_Z(c)[2c].$$

3.3.5. Syntomic triangulated realization.— Applying again [CD09b], Prop. 4.2.11, one gets for any R -scheme X an adjunction of triangulated categories:

$$L_X^{\text{syn}} : DM_{\mathbb{B}}(X) \rightleftarrows \mathbb{E}_{\text{syn-mod}}_X : \mathcal{O}_X^{\text{syn}}$$

such that:

- (1) $\mathcal{O}_X^{\text{syn}}$ is conservative.
- (2) For any Beilinson motive M over X , one has an isomorphism

$$\mathcal{O}_X^{\text{syn}} L_X^{\text{syn}}(M) \simeq M \otimes \mathbb{E}_{\text{syn}}$$

functorial in M .

- (3) The functor L_X^{syn} commutes with the operations f^* , f_* , \otimes .

Let us denote by $\mathbb{1}_X$ the unit object of $\mathbb{E}_{\text{syn-mod}}_X$. According to point (2), one obtains a canonical isomorphism:

$$\text{Hom}_{\mathbb{E}_{\text{syn-mod}}_X}(\mathbb{1}_X, \mathbb{1}_X(i)[n]) \simeq \mathbb{E}_{\text{syn}}^{n,i}(X)$$

which is functorial in X and compatible with products.

Remark 3.3.6. In the preceding section, one has derived Bloch-Ogus axioms, for syntomic cohomology and syntomic BM-homology, from the functoriality of $DM_{\mathbb{B}}$. In fact, as in [BO74, Ex. 2.1], one can also obtain these axioms from the properties of syntomic modules stated above.

3.3.7. Descent properties.— According to [CD09b, sec. 3.1], the 2-functor $X \mapsto \mathbb{E}_{\text{syn-mod}}_X$ can be extended to diagrams of R -schemes (as well as the syntomic triangulated realization). Moreover, the pair of functors (f^*, f_*) can be defined when f is a morphism of diagrams of R -schemes.

From [CD09b, 4.2.11], the motivic category $\mathbb{E}_{\text{syn-mod}}$ is separated. Therefore, according to [CD09b, 3.3.36], it satisfies h-descent (see Paragraph 3.2.13 for the h-topology): for any h-hypercover $p : \mathcal{X} \rightarrow X$ of R -schemes, the functor

$$p^* : \mathbb{E}_{\text{syn-mod}}_X \rightarrow \mathbb{E}_{\text{syn-mod}}_{\mathcal{X}}$$

is fully faithful.

Recall also the following more concrete version of descent: given any pseudo-Galois cover²⁶ $f : Y \rightarrow X$ of group G , any syntomic module M over X , the canonical morphism:

$$M \rightarrow (f_* f^*(M))^G$$

is an isomorphism, where we have denoted by $?^G$ the fixed point for the obvious action of G .

²⁶ f is finite surjective and admits a factorization $f = pf'$ where f' is a Galois cover of group G and p is radicial.

REFERENCES

- [AH10] J. Scholbach A. Holmstrom, *Arakelov motivic cohomology I*, arXiv:1012.2523v2, 2010.
- [Ayo07] J. Ayoub, *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique (I, II)*, Astérisque, vol. 314,315, Soc. Math. France, 2007.
- [Ban02] Kenichi Bannai, *Syntomic cohomology as a p -adic absolute Hodge cohomology*, Math. Z. **242** (2002), no. 3, 443–480. MR MR1985460 (2004e:14037)
- [Beĭ84] A. A. Beĭlinson, *Higher regulators and values of L -functions*, Current problems in mathematics, Vol. 24, Itogi Nauki i Tekhniki, Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984, pp. 181–238. MR 760999 (86h:11103)
- [Beĭ86a] ———, *Higher regulators of modular curves*, Applications of algebraic K -theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), Contemp. Math., vol. 55, Amer. Math. Soc., Providence, RI, 1986, pp. 1–34. MR 862627 (88f:11060)
- [Beĭ86b] ———, *Notes on absolute Hodge cohomology*, Applications of algebraic K -theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), Contemp. Math., vol. 55, Amer. Math. Soc., Providence, RI, 1986, pp. 35–68.
- [Bes00] Amnon Besser, *Syntomic regulators and p -adic integration. I. Rigid syntomic regulators*, Proceedings of the Conference on p -adic Aspects of the Theory of Automorphic Representations (Jerusalem, 1998), vol. 120, 2000, pp. 291–334.
- [Bes12] ———, *On the syntomic regulator for K_1 of a surface*, Israel J. Math. **190** (2012), 29–66. MR 2956231
- [BO74] Spencer Bloch and Arthur Ogus, *Gersten’s conjecture and the homology of schemes*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 181–201 (1975).
- [CCM12] B. Chiarellotto and A. Ciccioni and N. Mazzari, *Cycle classes and the syntomic regulator*, To appear in Algebra and Number Theory arXiv:1006.0132, 2012.
- [CD09a] D.-C. Cisinski and F. Déglise, *Local and stable homological algebra in Grothendieck abelian categories*, Homology, Homotopy and Applications **11** (2009), no. 1, 219–260.
- [CD09b] ———, *Triangulated categories of mixed motives*, arXiv:0912.2110, 2009.
- [CD12] ———, *Mixed weil cohomologies*, Adv. in Math. **230** (2012), 55–130.
- [Cis03] Denis-Charles Cisinski, *Images directes cohomologiques dans les catégories de modèles*, Ann. Math. Blaise Pascal **10** (2003), no. 2, 195–244. MR 2031269 (2004k:18009)
- [CT03] Bruno Chiarellotto and Nobuo Tsuzuki, *Cohomological descent of rigid cohomology for étale coverings*, Rend. Sem. Mat. Univ. Padova **109** (2003), 63–215.
- [Dég08] F. Déglise, *Around the Gysin triangle II*, Doc. Math. **13** (2008), 613–675.
- [Dég11] ———, *Orientation theory in arithmetic geometry*, arXiv:1111.4203, 2011.
- [Del74] Pierre Deligne, *Théorie de Hodge. III*, Inst. Hautes Études Sci. Publ. Math. (1974), no. 44, 5–77. MR MR0498552 (58 #16653b)
- [Del77] ———, *Cohomologie étale*, Lecture Notes in Mathematics, Vol. 569, Springer-Verlag, Berlin, 1977, Séminaire de Géométrie Algébrique du Bois-Marie SGA 4 1/2, Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier. MR 0463174 (57 #3132)
- [FM87] Jean-Marc Fontaine and William Messing, *p -adic periods and p -adic étale cohomology*, Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985), Contemp. Math., vol. 67, Amer. Math. Soc., Providence, RI, 1987, pp. 179–207. MR 902593 (89g:14009)
- [Ful98] William Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998. MR 1644323 (99d:14003)
- [Gil81] Henri Gillet, *Riemann-Roch theorems for higher algebraic K -theory*, Adv. in Math. **40** (1981), no. 3, 203–289. MR 624666 (83m:14013)
- [Gro90] Michel Gros, *Régulateurs syntomiques et valeurs de fonctions L p -adiques. I*, Invent. Math. **99** (1990), no. 2, 293–320, With an appendix by Masato Kurihara. MR 1031903 (91e:11070)
- [Hov01] M. Hovey, *Spectra and symmetric spectra in general model categories*, J. Pure Appl. Algebra **165** (2001), no. 1, 63–127.
- [HS87] V. A. Hinich and V. V. Schechtman, *On homotopy limit of homotopy algebras, K -theory, arithmetic and geometry (Moscow, 1984–1986)*, Lecture Notes in Math., vol. 1289, Springer, Berlin, 1987, pp. 240–264. MR 923138 (89d:55052)
- [Jan90] Uwe Jannsen, *Mixed motives and algebraic K -theory*, Lecture Notes in Mathematics, vol. 1400, Springer-Verlag, Berlin, 1990.
- [Lan11] Andreas Langer, *On the syntomic regulator for products of elliptic curves*, J. Lond. Math. Soc. (2) **84** (2011), no. 2, 495–513. MR 2835341 (2012k:19005)
- [Lev04] M. Levine, *K -theory and motivic cohomology of schemes, I*, <http://www.uni-due.de/%7Ebm0032/publ/KthyMotI12.01.pdf>, 2004.
- [LS86] Bernard Le Stum, *Cohomologie rigide et variétés abéliennes*, C. R. Acad. Sci. Paris Sér. I Math. **303** (1986), no. 20, 989–992.

- [Pan09] Ivan Panin, *Oriented cohomology theories of algebraic varieties. II (After I. Panin and A. Smirnov)*, Homology, Homotopy Appl. **11** (2009), no. 1, 349–405. MR 2529164 (2011c:14063)
- [Sou86] Christophe Soulé, *Régulateurs*, Astérisque (1986), no. 133-134, 237–253, Seminar Bourbaki, Vol. 1984/85. MR 837223 (87g:11158)
- [Tam11] Georg Tamme, *Karoubi's relative Chern character, the rigid syntomic regulator, and the Bloch-Kato exponential map*, <http://arxiv.org/abs/1111.4109v1>, 2011.
- [Voe96] V. Voevodsky, *Homology of schemes*, Selecta Math. (N.S.) **2** (1996), no. 1, 111–153. MR 1403354 (98c:14016)

E-mail address, F. Déglise: frederic.deglise@ens-lyon.fr

E-mail address, N. Mazzari: nicola.mazzari@math.u-bordeaux1.fr