# BIVARIANT THEORIES IN MOTIVIC STABLE HOMOTOPY
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## Introduction

The purpose of this work is to study the notion of bivariant cohomology of [FM81] in the context of motivic stable homotopy theory, and more generaly in the broader framework of Grothendieck six functors formalism specialized in the axiomatic of motivic triangulated categories.

## Contents



## Notations and conventions

All schemes in this paper are assumed to be noetherian of finite dimension. We will say that an $S$-scheme $X$, or equivalently its structure morphism, is *projective* if it admits an $S$-embedding into $\mathbb{P}^n_S$ for a suitable integer $n$.[1]

In the whole text, unless stated otherwise, $\mathscr{S}$ stands for a sub-category of the category of such schemes. We will assume that $\mathscr{S}$ is stable by blow-up and contains any open subscheme of (resp. projective bundle over) a scheme in $\mathscr{S}$. In all the examples of the text, the category $\mathscr{S}$ is always the category of all schemes (noetherian of finite dimension).

### 0.1. **Borel-Moore homology.**

**0.1.1.** *Exceptional functors.*– In this part, we use the more involved functoriality of the stable homotopy category of schemes established by Ayoub in [Ayo07] following the theory and strategy of cross functors introduced by Voevodsky. More precisely, we use the slightly more general construction explained in [CD09, sec. 2] which allows to consider separated morphisms of finite type instead of quasi-projective ones. We will abbreviate separated morphisms of finite type by *s-morphisms*.

The underlying axiomatic is well known as the *Grothendieck six functors formalism*. Let us state the properties that we need, completing the axiomatic of [Dég14], Par. 1.1.1: for any s-morphism $f : Y \to X$, there exists a pair of adjoint functors:

$$f_! : S\mathscr{H}(Y) \to S\mathscr{H}(X) : f^!$$

satisfying the following properties:

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[1] For example, if one works with quasi-projective schemes over a noetherian affine scheme (or more generally a noetherian scheme which admits an ample line bundle), then a morphism is proper if and only if it is projective with our convention – use [Gro61, Cor. 5.3.3].

(B0) given another s-morphisms of schemes $g$, composable on the right with $f$, there exists an isomorphism:

$$f_! g_! \to (fg)_!$$

which satisfies the usual cocycle condition.

(B1) There exists a natural transformation:

$$\alpha_f : f_! \to f_*$$

which is an isomorphism whenever $f$ is proper.

(B1') Assume $f$ is a smooth s-morphism with tangent vector bundle $T_f/Y$, there exists an isomorphism:

$$\mathfrak{p}_f : f_\sharp(. \otimes \mathrm{Th}(-T_f)) \to f_!$$

compatible with composition. *i.e.* Given composable smooth s-morphisms $f$ and $g$, one has a commutative diagram of isomorphisms:

$$\begin{array}{ccc}
(fg)_\sharp\big(\mathbb{E} \wedge \mathrm{Th}(-T_{fg})\big) & \longrightarrow & f_\sharp g_\sharp\big(\mathbb{E} \wedge \mathrm{Th}(-T_{fg})\big) \\
 & & \downarrow{\scriptstyle \epsilon_\sigma} \\
 & & f_\sharp g_\sharp\big(\mathbb{E} \wedge \mathrm{Th}(-T_g) \wedge g^* \mathrm{Th}(-T_f)\big) \\
{\scriptstyle \mathfrak{p}_{fg}}\downarrow & & \downarrow{\scriptstyle (A3)} \\
 & & f_\sharp\big((g_\sharp(\mathbb{E} \wedge \mathrm{Th}(-T_g)) \wedge \mathrm{Th}(-T_f)\big) \\
 & & \downarrow{\scriptstyle \mathfrak{p}_f . \mathfrak{p}_g} \\
(fg)_!(\mathbb{E}) & \longrightarrow & f_! g_!(\mathbb{E})
\end{array}$$

where the vertical maps are the respective isomorphisms of the 2-functors $f_\sharp$ and $f_!$, the map labeled (A3) stands for the isomorphism of Axiom (A3) – see [Dég14] paragraph 1.1.1 – and the isomorphism $\epsilon_\sigma$ is the one associated with the generalized Thom space functor defined in [Dég14], Remark 2.4.8, with respect to the exact sequence of vector bundles over $Y$:

$$(\sigma) \qquad 0 \to g^{-1}(T_f) \to T_{fg} \to T_g \to 0.$$

(B2) For any cartesian square:

$$\begin{array}{ccc}
Y & \xrightarrow{q} & X \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\
T & \xrightarrow{p} & S
\end{array}$$

such that $f$ is a s-morphism, there exists an isomorphism:

$$p^* f_! \to g_! q^*$$

compatible with composition with respect to $?^*$ and $?_!$.

(B3) For spectrum $\mathbb{E}$ over $Y$ and $\mathbb{F}$ over $X$, there exists an isomorphism:

$$f_!(f^*(\mathbb{E}) \wedge \mathbb{F}) \to \mathbb{E} \wedge f_!(\mathbb{F})$$

compatible with composition.

For the case of quasi-projective morphisms, properties (B0), (B1) and (B2) are established in [Ayo07, 1.4.2, §2.3] (cf. section 1.7.1 for the construction of $\alpha_f$). Property (B1') follows from *loc. cit.* together with [Ayo07, §2.3.3] while (B3) is proved in [Ayo07, 2.3.40]. For the case of s-morphisms, we refer the reader to [CD09, Th. 2.4.50].

*Remark* 0.1.2. Let us recall that property (B2) comes with the following *exchange transformation*:

$$(0.1.2.a) \qquad q^* f^! \xrightarrow{ad'} g^! g_! q^* f^! \xrightarrow{(B3)} g^! p^* f_! f^! \xrightarrow{ad} g^! p^*$$

where $ad'$ (resp. $ad$) stands for the counit (resp. unit) of the adjunction $(f_!, f^!)$.

*Remark* 0.1.3. The generalization of the work of Ayoub to s-morphisms is obtained by using the construction of Deligne of exceptional functors (see [AGV73, XVII, sec. 5, Th. 5.1.8]) together with the Chow lemma of Grothendieck to get some of its properties. The construction of Ayoub used the factorisation of a quasi-projective morphisms in a smooth morphism followed by a closed immersion (more generally called a "lissifiable" morphism). It is interesting to point out the remark of Deligne in [AGV73, XVIII, 0.4]: this method was originally used by Grothendieck to get the exceptional functors in étale setting.

**0.1.4.** Using the exceptional functors of the preceding paragraph, it is possible to introduce the following well-behaved homologies of *Borel-Moore* type:

**Definition 0.1.5.** Let $\mathbb{E}$ be an absolute ring spectrum. For any s-morphism $p : X \to S$, and any couple of integers $(n, m)$, we define the Borel-Moore homology of $f$ with coefficients in $\mathbb{E}$ as:

$$\mathbb{E}^{BM}_{n,m}(X \xrightarrow{p} S) := \mathrm{Hom}_X\left(\mathbb{1}_X(m)[n], p^!(\mathbb{E}_S)\right) = \mathrm{Hom}_X\left(p_!(\mathbb{1}_X)(m)[n], \mathbb{E}_S\right)$$

Usually, we simply denote by $\mathbb{E}^{BM}_{n,m}(X/S)$ the previous groups. We use the more precise notation of the previous definition when the structural morphism of $X/S$ is not clear.

*Remark* 0.1.6. As we will se below, Borel-Moore homology is an instance of what Fulton and MacPherson have called a *bivariant theory* in [FM81, §2].

Note also that it is a generalization of the cohomology with support. Indeed, given a closed immersion $i : Z \to X$, property (B1) gives an isomorphism:

$$\mathbb{E}^{BM}_{n,m}(Z \xrightarrow{i} X) = \mathrm{Hom}_X\left(i_!(\mathbb{1}_X)(m)[n], \mathbb{E}_S\right) \xrightarrow{\alpha_i^*} \mathrm{Hom}_X\left(i_*(\mathbb{1}_X)(m)[n], \mathbb{E}_S\right) = \mathbb{E}_Z^{-n,-m}(X).$$

**Example 0.1.7.** According to a work in progress by R. Casalis [?], if $f : X \to S$ is a quasi-projective morphism (or more generally a separated morphism of finite type) and $S$ is regular, there exists a canonical isomorphism:

(0.1.7.a) $$\mathbf{KGL}^{BM}_{n,m}(X/S) \simeq K'_{n-2m}(X).$$

Recall also from [Rio10, 5.3.2] that there exists for any non zero integer $k$ an endomorphism $\Psi^k$ on the absolute ring spectrum $\mathbf{KGL}_\mathbb{Q}$ which induces the usual Adams operations on $\mathbf{KGL}^{**}_\mathbb{Q}(S) \simeq K_*(S) \otimes \mathbb{Q}$ when $S$ is regular. This $\Psi^k$ induces an endomorphism on $\mathbf{KGL}^{BM}_{**}(X/S)$ which coincides with the operation $\phi^k$ on $K'_*(X) \otimes \mathbb{Q}$ introduced by Soul in [Sou85, Th. 7].[2]

Therefore, by definition, the Borel-Moore homology associated with the absolute ring spectrum $\mathbf{H}_Б$ coincide with Soul motivic homology defined in [Sou85, 7.4]:

(0.1.7.b) $$\mathbf{H}^{Б,BM}_{n,m}(X/S) \simeq H_n(X, m) := Gr_m K'_{n-2m}(X)_\mathbb{Q}.$$

where the two groups on the right hand side are defind in *loc. cit.* Note as a by-product that $\mathbf{H}^{Б,BM}_{2*,*}(X/S)$ coincide with Grothendieck Chow ring $Gr_* K'_0(X)$ associated with the topological filtration (see [BGI71, X, 1.1.1] for the definition). Equivalently, it coincides with Chow homology groups $A_*(X)$ defined by Fulton (see [Sou85, Th. 8]).

**0.1.8.** According to the previous remark, Borel-Moore homology shares the same properties as the ones described earlier for cohomology with support:

- *Base change*: Given any quasi-projective scheme $X/S$ and any morphism $f : T \to S$, one has a pullback map:
$$f^* : \mathbb{E}^{BM}_{n,i}(X/S) \to \mathbb{E}^{BM}_{n,i}(X \times_S T/T).$$
It is simply induced by the functor $f^*$ using property (B2).
- *Covariant functoriality*: let $p : X \to S$ and $q : Y \to X$ be quasi-projective morphisms. If we assume in addition that $q : Y \to X$ is proper, then one defines a pushout morphism:
$$q_! : \mathbb{E}^{BM}_{n,i}(Y/S) \to \mathbb{E}^{BM}_{n,i}(X/S)$$

[2]From the proof of Th.7; in *loc. cit.* it suffices to note that $\Psi^k$ induces the Adams operation used by Soul on K-thory with suppport and to invoke the duality Riemann-Roch formula of 0.1.19.

as follow: given a map $y : (pq)_!(\mathbb{1}_Y) \to \mathbb{E}_S$, we define $q_!(y)$ as the composite:

$$p_!(\mathbb{1}_X) \to p_!q_*q^*(\mathbb{1}_X) \xrightarrow{\alpha_q^{-1}} p_!q_!q^*(\mathbb{1}_X) \simeq (pq)_!(\mathbb{1}_Y) \xrightarrow{y} \mathbb{E}_S$$

where the first map is given by the adjunction $(q^*, q_*)$, the second and third isomorphisms uses respectively properties (B1) and (B0).

- *Products*: Given quasi-projective morphisms $p : X \to S$ and $q : Y \to X$, we define a pairing of the form:

$$\mathbb{E}^{BM}_{n,i}(Y/X) \otimes_{\mathbb{Z}} \mathbb{E}^{BM}_{m,j}(X/S) \to \mathbb{E}^{BM}_{n+m,i+j}(Y/S)$$

as follows: given $y : q_!(\mathbb{1}_Y)(i)[n] \to \mathbb{E}_X$ and $x : p_!(\mathbb{1}_X)(j)[m] \to \mathbb{E}_S$, we define the product $x \cdot y$ as the following composite:

$$\begin{aligned} p_!q_!(\mathbb{1}_Y)(i+j)[n+m] &\xrightarrow{p_!(y)} p_!(\mathbb{E}_X)(j)[m] \simeq p_!(p^*(\mathbb{E}_S) \otimes \mathbb{1}_X)(j)[m] \\ &\xrightarrow{Ex} \mathbb{E}_S \wedge p_!(\mathbb{1}_X)(j)[m] \xrightarrow{Id \wedge x} E_S \wedge \mathbb{E}_S \xrightarrow{\mu} \mathbb{E}_S. \end{aligned}$$

where the identification on the first line uses the stuctural isomorphism of the absolute ring spectrum $\mathbb{E}$ and the map labelled $Ex$ stands for the isomorphisms of Property (B3).

The reader will remark that these definition are exactly the same than the one given in [Dég14], respective paragraphs 1.2.5, 1.2.7 and 1.2.8. In particular, we get the analog of [Dég14], Proposition 1.2.9 as follows:

(F1) $f^*g^* = (gf)^*$, $q'_!q_! = (q'q)_!$ whenever defined.

(F3) Consider the following cartesian squares:

$$\begin{array}{ccccc} Y' & \xrightarrow{q'} & X' & \longrightarrow & S' \\ \downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{q} & X & \longrightarrow & S \end{array}$$

where horizontal maps are quasi-projective and $q$ is in addition proper. Then: $f^*q_! = q'_!f^*$.

(F4) Consider quasi-projective morphisms: $W \xrightarrow{r} Y \xrightarrow{q} X \xrightarrow{p} S$.
Then for any triple $(\lambda, \alpha, \beta) \in \mathbb{E}^{BM}_{**}(W/Y) \times \mathbb{E}^{BM}_{**}(Y/X) \times \mathbb{E}^{BM}_{**}(X/S)$, one has:
$\lambda \cdot (\alpha \cdot \beta) = (\lambda \cdot \alpha) \cdot \beta$.

(F5) Under the assumption of (F3), for any couple $(\lambda, \rho) \in \mathbb{E}^{BM}_{**}(Y/X) \times \mathbb{E}^{BM}_{**}(X/S)$, one has:
$f^*(\lambda \cdot \rho) = g^*(\lambda) \cdot f^*(\rho)$.

(F6) Under the assumption of (F4), for any couple $(\lambda, \rho) \in \mathbb{E}^{BM}_{**}(Y/X) \times \mathbb{E}^{BM}_{**}(X/S)$, one has:
$q_!(\lambda \cdot \rho) = q_!(\lambda) \cdot \rho$.

(F7) Consider the following diagram:

$$\begin{array}{ccccc} Y' & \xrightarrow{q'} & X' & & \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} & & \\ Y & \xrightarrow{q} & X & \xrightarrow{p} & S \end{array}$$

made of quasi-projective morphisms, with $g$ being proper and the square assumed to be cartesian. Then for any couple $(\lambda, \rho) \in \mathbb{E}^{BM}_{**}(Y/X) \times \mathbb{E}^{BM}_{**}(X'/S)$, one has:
$h_!(g^*(\lambda) \cdot \rho) = \lambda \cdot g_!(\rho)$.

The proof is exactly the same as in [Dég14], Prop. 1.2.9, using the properties (B-) stated in 0.1.1 instead of (A-) stated *loc. cit.*

*Remark* 0.1.9. The above properties correspond to that of Fulton and MacPherson in [FM81, 2.2]. In other words, $\mathbb{E}^{BM}_{**}$ is a bivariant theory defined on the category $\mathscr{S}$ whose morphisms are restricted to $s$-morphisms, and such that confined maps are proper morphisms. In fact, it is a $\mathbb{Z}$-graded such theory, with respect to twists and our convention for degree is homological. It is also skew commutative with respect to the first grading.

**Example 0.1.10.** The isomorphism (0.1.7.a) is compatible with covariant functoriality of $S$-morphisms and with base change in $S$. The product when $X$ and $S$ are regular amounts to action of $K$-theory on $K'$-theory.

**Example 0.1.11.** Let $k$ be a perfect field and consider the $k$-absolute ring spectrum $\mathbf{H}_\Lambda$ representing motivic cohomology with coefficients in $\Lambda$ - [Dég14], Example 1.2.2(4)). For a separated $k$-scheme $X$ of finite type, we let $H^{\mathcal{M},BM}_{n,m}(X/k,\Lambda)$ be the Borel-Moore homology of $X/k$ associated with $\mathbf{H}_\Lambda$.

Using the niveau spectral sequence as in [BO74], one deduces an isomorphism:

$$H^{\mathcal{M},BM}_{2n,n}(X/k,\mathbb{Z}) \simeq CH_n(X).$$

The functorial properties of this isomorphism are studied in detail in the forthcoming work [**?**].

If $p$ is the exponential characteristic of $k$, it follows from [CD14] that there is a canonical isomorphism:

$$H^{\mathcal{M},BM}_{n,m}(X/k,\mathbb{Z}[1/p]) \simeq CH_n(X,n+2m)[1/p]$$

where the left hand side denotes Bloch's higher Chow groups.

**0.1.12.** Using this more general homology, understood as an extension of cohomology with support, one can easily extend the previously done work on fundamental classes as follows.

**Theorem 0.1.13.** *Assume $\mathbb{E}$ is an absolutely pure oriented ring spectrum.*

*There exists a family of classes $\bar\eta_f \in \mathbb{E}^{BM}_{**}(X/S)$ indexed by quasi-projective lci morphisms $f$ uniquely characterized by the following properties:*

(1) *If $i : Z \to X$ is a regular closed immersion in $\mathscr{S}$, $\bar\eta_i$ coincides with the fundamental class in $\mathbb{E}^{**}_Z(X)$ defined in* [Dég14]*, 2.3.1.*
(2) *If $f$ is a smooth quasi-projective morphism, $\bar\eta_f$ is the pre-image of the Thom class $\bar{\mathfrak{t}}(-T_f)$ defined in* [Dég14]*, Remark 2.4.8 by the sequence of isomorphisms:*

$$\begin{aligned}\mathbb{E}^{BM}_{**}(X/S) = \mathrm{Hom}_S(f_!(\mathbb{1}_X),\mathbb{E}_S)_{**} &\xrightarrow{\mathfrak{p}_f^*} \mathrm{Hom}_S(f_\sharp(\mathrm{Th}(-T_f)),\mathbb{E}_S)_{**} \\ &= \mathrm{Hom}_S(\mathrm{Th}(-T_f), f^*\mathbb{E}_S)_{**} \\ &\simeq \mathrm{Hom}_S(\mathrm{Th}(-T_f),\mathbb{E}_X)_{**} = \mathbb{E}^{**}(\mathrm{Th}(-T_f)),\end{aligned} \tag{0.1.13.a}$$

*where the last isomorphism exchanges the homological bigraduation to the cohomological bigraduation by the usual change of sign.*

(3) *for composable quasi-projective lci morphisms $Y \xrightarrow{g} X \xrightarrow{f} S$, one has:*

$$\bar\eta_g \cdot \bar\eta_f = \bar\eta_{fg} \in \mathbb{E}^{BM}_{**}(Y/S).$$

*Proof.* Because any quasi-projective morphism $p : X \to S$ admits a factorization $X \xrightarrow{i} X' \xrightarrow{f} S$ where $f$ is smooth and $i$ is a regular closed immersion, we have to prove that the class $\bar\eta_i \cdot \bar\eta_f$ is independant of the factorization.

The proof is formally the same as for [Dég14], Lemma 3.2.4. Thus, we reduce to prove (3) in the following cases:

(a) $f$ and $g$ are regular closed immersions.
(b) $f$ and $g$ are smooth $\mathcal{P}$-morphisms.
(c) $g$ is a smooth quasi-projective morphism and $f$ is a section of $g$.

Case (a) is [Dég14], Theorem 2.4.9. Case (b) follows from property (B1') of paragraph 0.1.1 together with the additivity of the Thom class with respect to the exact sequence of vector bundle ($\sigma$) (cf. [Dég14], Rem. 2.4.8).

Let us consider case (c). We have to prove: $\bar\eta_g \cdot \bar\eta_f = 1$. Let $V = N_Y(X)$ be the normal bundle of $Y$ in $X$. By construction of the deformation space, we get a commutative diagram made of cartesian squares:

$$\begin{array}{ccccc}
Y & \xrightarrow{g} & X & \xrightarrow{f} & Y \\
{\scriptstyle s_1}\downarrow & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle s_1} \\
\mathbb{A}^1_Y & \longrightarrow & D_Y(X) & \xrightarrow{\tilde{f}} & \mathbb{A}^1_Y \\
{\scriptstyle s_1}\uparrow & & \uparrow{\scriptstyle d_0} & & \uparrow{\scriptstyle s_0} \\
Y & \xrightarrow{\sigma} & V & \xrightarrow{p} & Y
\end{array}$$

where the two left column are made by the deformation diagram [Dég14], (1.3.1.a), associated with $g$, the morphism $p$ is the canonical projection of $V/Y$ and $\tilde{f}$ is induced by $D_Y(X) \to D_Y(Y) \simeq \mathbb{A}^1_Y$. An easy computation shows that $\tilde{f}$ is smooth.

Because Thom classes are obviously stable by pullback, we deduce get the same result for fundamental classes associated with smooth quasi-projective morphisms. Therefore, using formula (F5) and the fact $s_0^*, s_1^* : \mathbb{E}^{**}(S) \to \mathbb{E}^{**}(S)$ are isomorphisms, we are reduce to the prove $\bar{\eta}_\sigma \,.\, \bar{\eta}_p = 1$, *i.e.* we can assume $X = V$ is a vector bundle, $g = \sigma$ is its zero section and $f = p$ its canonical projection. By definition, we obviously have: $\mathfrak{t}(V) \cdot \mathfrak{t}(-V) = 1$. Thus, we are reduced to the following formal lemma:

**Lemma 0.1.14.** *Given a vector bundle $V/Y$ with zero section $\sigma$ and canonical projection $p$, the following diagram is commutative:*

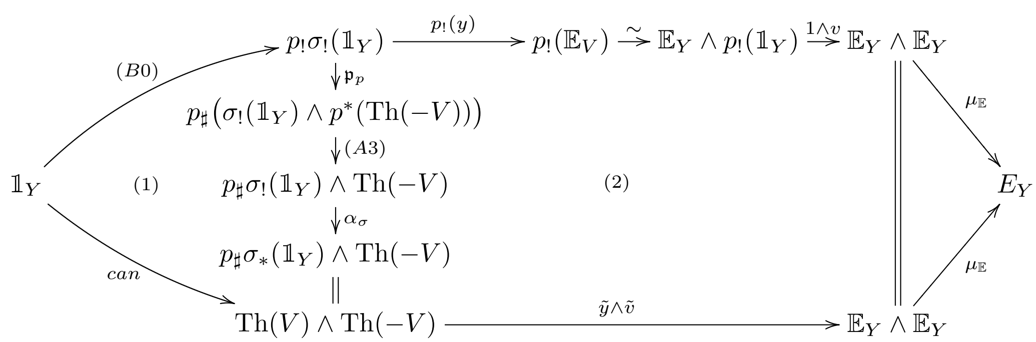
$$\begin{array}{ccc}
\mathbb{E}^{BM}_{**}(Y/V) \otimes \mathbb{E}^{BM}_{**}(V/Y) & \xrightarrow{\bar{\mu}} & \\
{\scriptstyle \alpha^*_\sigma \otimes \mathfrak{p}^*_p}\downarrow & & \mathbb{E}^{**}(Y) \\
\mathbb{E}^{**}(\mathrm{Th}(V)) \otimes \mathbb{E}^{**}(\mathrm{Th}(-V)) & \xrightarrow{\mu} &
\end{array}$$

*where $\mu$ is the product on cohomology, $\bar{\mu}$ the (refined) product on Borel-Moore homology, while $\alpha^*_\sigma$ and $\mathfrak{p}^*_p$ are induced repsectively by the isomorphisms of (B1) and of (B1') through obvious identifications.*

To give the proof of the lemma, it is sufficient to consider classes classes of degree $(0,0)$:

$$y : \sigma_!(\mathbb{1}_Y) \to \mathbb{E}_V, v : p_!(\mathbb{1}_V) \to \mathbb{E}_Y.$$

Let $(\tilde{y}, \tilde{v}) \in \mathbb{E}^{0,0}(\mathrm{Th}(V)) \otimes \mathbb{E}^{0,0}(\mathrm{Th}(-V))$ be there image by the above vertical map. Then the lemma can be reduced to the commutativity of the following diagram:




where the map labelled $(B0)$ stands for the inverse of the isomorphism $Id = (p \circ \sigma)_! \xrightarrow{p} _! \sigma_!$ stated in (B0) and *can* by definition of the inverse Thom space. The commutativity of part (2) is obvious (by definition of $\tilde{y}$ and $\tilde{v}$), so that only the commutativity of part (1) require an argument: it follows by definition of the isomorphism (B0): [Ayo07, 1.6.37]. □

**Definition 0.1.15.** Use the notations and assumptions of the previous theorem.
We call $\bar{\eta}_f \in \mathbb{E}^{BM}_{**}(X/S)$ the *fundamental class* associated with the quasi-projective lci morphism $f : X \to S$.

Note that when $f$ has pure relative dimension $d$, $\bar{\eta}_f$ has (homological) bidegree $(2d, d)$.

*Remark* 0.1.16. According to the current progress, one can generalize the preceding definition to the following cases:

(1) Using the construction of fundamental classes in the singular case by A. Navarro [**?**], one can avoid the assumption of absolute purity on the absolute spectrum $\mathbb{E}$ and define $\bar{\eta}_f$ for any quasi-projective lci maps between arbitrary (Noetherian finite dimensional) schemes – *i.e.* we can drop the assumptions we are working either with regular schemes or smooth scheme over some base.

(2) Using the construction of $(f_!, f^!)$ for separated morphisms of finite type of [CD09] and the property of cohomological (Zariski) descent for $S\mathscr{H}$ as formulated following Deligne in [CD09, §3], one should be able to extend the definition of $\bar{\eta}_f$ to any lci morphism separated of finite type, between eventually singular schemes if we use the first point.[3]

Using the previous definition, we can easily extend the results obtained so far in the case of fundamental classes of closed immersion to the case of these more general fundamental classes:

**Theorem 0.1.17.** *Let $\mathbb{E}$ be an absolutely pure oriented ring spectrum.*

(1) *Consider a cartesian square in $\mathscr{S}$:*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & S' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & S \end{array}$$

*such that $f$ is quasi-projective and lci. Let $\tau_f \in K_0(X)$ (resp. $\tau_{f'} \in K_0(X')$) be the virtual tangent bundle of $f$ (resp. $f'$). We put $\xi = f^*(\tau_f) - \tau_{f'}$ and let $e(\xi)$ be the Top Chern class of $\xi$ in $\mathbb{E}^{**}(X')$. Then the following* excess intersection formula *holds:*

$$p^*(\bar{\eta}_f) = e(\xi) \cdot \bar{\eta}_{f'} .$$

(2) *Given any quasi-projective lci morphism $f : X \to S$, the following map is an isomorphism:*

$$\delta_f : \mathbb{E}^{**}(X) \to \mathbb{E}^{BM}_{**}(X/S), x \mapsto x \cdot \bar{\eta}_f$$

*called the* duality isomorphism *associated with $f$. When $f$ has pure relative dimension $d$, it gives for any couple of integer $(n, i)$ an isomorphism:*

$$\mathbb{E}^{n,i}(X) \xrightarrow{\sim} \mathbb{E}^{BM}_{2d-n,d-i}(X/S)$$

(3) *Let $\mathbb{F}$ be an absolutely pure oriented ring spectrum and $\varphi : \mathbb{E} \to \mathbb{F}$ be a morphism of absolute ring spectra. Let $\mathrm{Td}_\varphi : K_0 \to \mathbb{F}^{0,0}$ be the Todd class associated with $\varphi$ (cf.* [Dég14] *4.1.4). Then given any quasi-projective lci morphism $f$ with virtual tangent bundle $\tau_f$, the following* generalized Riemann-Roch formula *holds:*

$$\varphi_{X/S}(\bar{\eta}^{\mathbb{E}}_f) = \mathrm{Td}_\varphi(-\tau_f).\,\bar{\eta}^{\mathbb{F}}_f$$

*where $\varphi_{X/S} : \mathbb{E}^{BM}_{**}(X/S) \to \mathbb{F}^{BM}_{**}(X/S)$ is the map induced by $\varphi$.*

*Proof.* Note first that each of these assertions can be reduced to either the case where $f$ is a regular closed immersion or a quasi-projective smooth morphism.

Point (1) easily follows from [Dég14], Th. 2.4.2 as the case of a smooth morphism is formal.

Point (2) follows because, by definition, the map $\delta_f$ coincides with $\mathfrak{p}_{()}X, Z$ of [Dég14], (2.3.1.b) when $f = i : Z \to X$ is a closed immersion and with the map induced by the following composite of isomorphisms:

$$\mathbb{E}^{**}(X) \xrightarrow{\bar{\mathfrak{t}}(-T_f)} \mathbb{E}^{**}(\mathrm{Th}(-T_f)) \to \mathbb{E}^{BM}_{**}(X/S)$$

where the first map is given by product with the refined (inverse) Thom class and the second map is the inverse of (0.1.13.a).

[3] In fact, one should even be able to drop the separatedness assumption on $f$ for the definition of $(f_!, f^!)$ and $\bar{\eta}_f$ using cohomological descent.

Point (3) follows from the Riemann-Roch formula in the case of a closed immersion (more precisely, [Dég14], (4.2.1.a) and [Dég14], Lemma 4.2.2) and the case of a smooth morphism follows from the same fact applied to the (inverse) refined Thom class of the tangent bundle of $f$. □

*Remark* 0.1.18. While points (1) and (3) of the previous Theorem can easily be generalized to the cases pointed out in Remark 0.1.16, without requiring absolute purity of $\mathbb{E}$, it is not the case of point (2) – tautologycally.

**0.1.19.** The preceding formulation of the Riemann-Roch formula is the more general we know. It can be reformulated as a comparison of duality isomorphisms. Using the notations of point (2) and (3) above, one gets a commutative diagram:

$$\begin{array}{ccc} \mathbb{E}^{**}(X) & \xrightarrow[\sim]{\delta_f^{\mathbb{E}}} & \mathbb{E}^{BM}_{**}(X/S) \\ {\scriptstyle\varphi_X}\downarrow & & \downarrow{\scriptstyle\varphi^{BM}_{X/S}} \\ \mathbb{F}^{**}(X) & \xrightarrow[\sim]{\mathrm{Td}(\tau_f).\delta_f^{\mathbb{F}}} & \mathbb{F}^{BM}_{**}(X/S). \end{array}$$

Thus the invertible class $\mathrm{Td}(T_f)$ appears as the correction between the dualities with respect to $\mathbb{E}$ and $\mathbb{F}$.

Then, using this duality, one recovers the classical formulation of the Grothendieck-Riemann-Roch formula. Assume $f : Y \to X$ is a projective lci morphism, one defines the Gysin morphism in cohomology as follows:

$$f_* : \mathbb{E}^{**}(Y) \xrightarrow{\delta_f} \mathbb{E}^{BM}_{**}(Y/X) \xrightarrow{f_!} \mathbb{E}^{BM}_{**}(X/X) = \mathbb{E}^{**}(X),$$

where the last map used the natural covariant functoriality of Borel-Moore homology (cf. 0.1.8).[4] It follows easily from the preceding formula that this Gysin morphism satisfies the properties (1)-(4) of [Dég14], Theorem 3.3.1. As it obviously coincides with the Gysin morphism defined in [Dég14], Def. 3.2.6, for closed immersions, it agrees with the one defined previously by the Theorem mentioned. Then the preceding Riemann-Roch formula implies the one formulated in [Dég14], Th. 4.3.2.

Moreover, we can formulate another kind of Riemann-Roch formula. Assume now $f : Y \to X$ is a morphism of quasi-projective lci $S$-schemes. Then it defines a pullback morphism, that we will simply call the Gysin morphism, in Borel-Moore homology as follows:

$$f^* : \mathbb{E}^{BM}_{**}(X/S) \xrightarrow{\delta_{X/S}^{-1}} \mathbb{E}^{**}(X) \xrightarrow{f^*} \mathbb{E}^{**}(Y) \xrightarrow{\delta_{Y/S}} \mathbb{E}^{BM}_{**}(Y/S).$$

If $f$ has pure dimension $d$, then $f^*$ is homogenous of degree $(2d, d)$. Given a morphism $\varphi : \mathbb{E} \to \mathbb{F}$, the Riemann-Roch formula for this Gysin morphism reads as the following commutative diagram:

$$\begin{array}{ccc} \mathbb{E}^{BM}_{**}(X/S) & \xrightarrow{f^*} & \mathbb{E}^{BM}_{**}(Y/S) \\ {\scriptstyle\varphi^{BM}_{X/S}}\downarrow & & \downarrow{\scriptstyle\varphi^{BM}_{Y/S}} \\ \mathbb{F}^{BM}_{**}(X/S) & \xrightarrow{\mathrm{Td}(\tau_f).f^*} & \mathbb{F}^{BM}_{**}(Y/S). \end{array}$$

*Remark* 0.1.20. In fact, all the formulas of the preceding paragraph have to be compared with the Baum-Fulton-MacPherson formulas for Chow homology (see [Ful98, 18.2]).

**Example 0.1.21.** We can the results explained above to the absolute spectrum **KGL**, its rational version and the Chern character $\mathrm{ch}_t$ of [Dég14], paragraph 5.3.3.

Let $S$ be a regular scheme and $f : Y \to X$ be a quasi-projective lci $S$-morphism such that $S$ one of the following assumptions hold:

(1) $X$ and $Y$ are regular.
(2) $X$, $Y$ and $S$ are smooth over some common base scheme.

[4] If $f$ has pure dimension $d$, this map is homogenous of degree $(-2d, -d)$.

Then, taking into account the isomorphism (0.1.7.a), we get a pullback morphism in K'-theory:

$$f^* : K'_r(X) \to K'_r(Y)$$

which can be identified with the pullback defined by Quillen on $K'$-theory (see [Qui69, §2.5]).

Moreover, using ismorphism (0.1.7.b), we get moreover, when $f$ is of pure dimension $d$, a pullback

$$f^* : Gr_n K'_r(X)_{\mathbb{Q}} \to Gr_{n+d} K'_r(Y)_{\mathbb{Q}}$$

which is induced by the previous pullback.

Finally, the Riemann-Roch formula for Borel-Moore **KGL**-homology is given by the following diagram:

$$\begin{array}{ccc} K'_r(X)_{\mathbb{Q}} & \xrightarrow{\quad f^* \quad} & K'_r(Y)_{\mathbb{Q}} \\ {\scriptstyle \mathrm{ch}_t}\downarrow & & \downarrow{\scriptstyle \mathrm{ch}_t} \\ \oplus_{n\geq 0} Gr_n K'_r(X)_{\mathbb{Q}} & \xrightarrow{\mathrm{Td}(\tau_f).f^*} & \oplus_{n\geq 0} Gr_{n+d} K'_r(Y)_{\mathbb{Q}}. \end{array}$$

0.2. **The four theories.** In the preceding paragraph, we have seen that to an absolute ring spectrum is associated a pair of cohomology/Borel-Moore homology satisfying an extension of the axioms of Bloch-Ogus whenever it is absolutely pure and oriented. In fact, one can do better:

**Definition 0.2.1.** Let $\mathbb{E}$ be an absolute spectrum. Then, to any s-morphism $p : X \to S$ and any integers $(n, m)$ we define the following theories:

| | |
|---|---|
| Cohomology | $\mathbb{E}^{n,m}(X) = \mathrm{Hom}_X(S^0, \mathbb{E}_X(m)[n])$ |
| Borel-Moore homology | $\mathbb{E}^{BM}_{n,m}(X/S) = \mathrm{Hom}_X(S^0, p^!\mathbb{E}_S(m)[n])$ |
| Cohomology with compact support | $\mathbb{E}^{n,m}_c(X/S) = \mathrm{Hom}_S(S^0, p_!\mathbb{E}_X(m)[n])$ |
| Homology | $\mathbb{E}_{n,m}(X/S) = \mathrm{Hom}_S(S^0, p_!p^!\mathbb{E}(-m)[-n])$ |

We will say *BM-homology* (resp. *c-cohomology*) for Borel-Moore homology (resp. cohomology with compact support).

Note that the four theories corresponds to the four possible ways to combine $(p_*, p_!)$ with $(p^*, p^!)$.

**0.2.2.** These theories enjoy the following elementary properties:

- *Natural functoriality*:

| | |
|---|---|
| $\mathbb{E}^{n,m}(X)$ | contravariant in $X$ |
| $\mathbb{E}^{BM}_{n,m}(X/S)$ | covariant in $X/S$ wrt proper morphisms,<br>contravariant in $X/S$ wrt étale morphisms,<br>contravariant in $S$ (base change) |
| $\mathbb{E}^{n,m}_c(X/S)$ | contravariant in $X/S$ wrt proper morphisms,<br>covariant in $X/S$ wrt étale morphisms,<br>contravariant in $S$ (base change) |
| $\mathbb{E}_{n,m}(X/S)$ | covariant in $X/S$<br>contravariant in $S$ (base change) |

The case of cohomology and homology is obvious (use adjunction map). The case of BM-homology was zexplained in Paragraph 0.1.8 except for its étale contravariance: given $u : V \to X$ an étale morphism, the pullback $u^*$ on BM-homology is given by applying the functor $u^*$ and using the fact it is isomorphic to $u^!$ according to (B1'). The case of c-cohomology is completely analogous to the one of BM-homology: it follows respectively from properties (B2), (B1') and (B1).

Given a pullback square of $S$-morphisms,

$$\begin{array}{ccc} W & \xrightarrow{g} & V \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle u} \\ Y & \xrightarrow{f} & X \end{array}$$

such that $f$ is proper and $u$ is étale, we get:
– $u^*f_* = g_*v^*$ in BM-homology,
– $f^*u_* = v_*g^*$ in c-cohomology.
These formulas are lengthy exercices going back to definitions and using the elementary property of adjoint functors (the so called "zig-zag equations").

Note finally that base change maps are compatible with the other two functorialities. Moreover, BM-homology and c-cohomology are homotopy invariant with respect to base change: give any vector bundle $f : E \to S$, the maps

$$f^* : \mathbb{E}^{BM}_{n,m}(X/S) \to \mathbb{E}^{BM}_{n,m}(X \times_S E/E),$$
$$f^* : \mathbb{E}^{n,m}_c(X/S) \to \mathbb{E}^{n,m}_c(X \times_S E/E),$$

are isomorphisms (this follows easily from the fact the functor $p^* : S\mathscr{H}(S) \to S\mathscr{H}(E)$ is fully faithful).

- *Comparison*: there exists canonical morphisms:

$$\nu_{X/S} : \mathbb{E}^{n,m}_c(X/S) \to \mathbb{E}^{n,m}(X),$$
$$\nu^{BM}_{X/S} : \mathbb{E}_{n,m}(X/S) \to \mathbb{E}^{BM}_{n,m}(X/S)$$

which are isomorphisms whenever $X/S$ is proper. This is a direct traduction of (B1).

Note that $\nu_{X/S}$ (resp. $\nu^{BM}_{X/S}$) is natural with respect to contravariance (resp. covariance) in $X/S$ with respect to proper morphisms. Moreover, $\nu_{X/S}$ is contravariantly functorial in $S$.

- *Localization*: Let $i : Z \to X$ be an open immersion of quasi-projective $S$-schemes with complementary open immersion $j : U \to X$. Then there exists *localization long exact sequences*[5]:

$$\mathbb{E}^{BM}_{n,m}(Z/S) \xrightarrow{i_*} \mathbb{E}^{BM}_{n,m}(X/S) \xrightarrow{j^*} \mathbb{E}^{BM}_{n,m}(U/S) \xrightarrow{\partial_i^{BM}} \mathbb{E}^{BM}_{n-1,m}(Z/S),$$
$$\mathbb{E}^{n,m}_c(U/S) \xrightarrow{j_*} \mathbb{E}^{n,m}_c(X/S) \xrightarrow{i^*} \mathbb{E}^{n,m}_c(Z/S) \xrightarrow{\partial_i^c} \mathbb{E}^{n+1,m}_c(U/S)$$

which are contravariantly natural in $S$ (resp. in $X/S$ with respect to proper morphisms). This follows directly from (A4) – given that $j_\sharp = j_!$. Note that the statement about naturality follows in particular from the fact that the boundary morphisms $\partial_i$ in (A4) is unique (see [CD09, 2.3.3]).

Note this property immediately implies that BM-homology and c-cohomology of $X/S$ only depends on the reduced scheme structure of $X$.

*Remark* 0.2.3. As already pointed out, the pair of theories $(\mathbb{E}^{**}, \mathbb{E}^{BM}_{**})$ satisfies an extension of the Bloch-Ogus formalism described in [BO74, (1.2)]. This extension corresponds to the bivariant theory described by Fulton-MacPherson in [FM81] (which was inspired by [BGI71, IV, 3.3]).

The fact this pair is incorporated in a 4 theories formalism seems to have remained unnoticed so far. As well as the fact that cohomology with support is a particular case of Borel-Moore homology, up to the passage from homological to cohomological bigraduation (Remark 0.1.6).

Finally, it is important to note that c-cohomology (resp. BM-homology) is uniquely characterized by the cohomological functor $\mathbb{E}^{**}$ (resp. homological functor $\mathbb{E}_{**}$).

Indeed, a s-morphism $p : X \to S$ can be compactified according to Nagata theorem: $X \xrightarrow{j} \bar{X} \xrightarrow{\pi} S$, $j$ an open immersion, $\pi$ a proper morphism. Put $X_\infty = (\bar{X} - X)$, equipped with its reduced structure of (closed) subscheme of $\bar{X}$. Then, taking care about the Comparison and Localization properties, one gets long exact sequences:

$$\mathbb{E}_{n,m}(X_\infty/S) \xrightarrow{i_*} \mathbb{E}_{n,m}(\bar{X}/S) \to \mathbb{E}^{BM}_{n,m}(X/S) \to \mathbb{E}_{n-1,m}(X_\infty/S),$$
$$\mathbb{E}^{n-1,m}(\bar{X}) \xrightarrow{i^*} \mathbb{E}^{n-1,m}(X_\infty) \to \mathbb{E}^{n,m}_c(X/S) \to \mathbb{E}^{n,m}(\bar{X}/S)$$

[5] We have already used the first exact sequence in the particular case where $S = X$: it was [Dég14], (3.1.1.a).

**0.2.4.** *Descent properties*: These properties are based on the reformulation of basic descent properties in terms of the 6 functors formalism obtained in [CD09]:

Nisnevich (resp. cdh) descent: given any Nisnevich (resp. cdh) distinguished square

$$\begin{array}{ccc} Y' & \xrightarrow{k} & X' \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle u} \\ Y & \xrightarrow{i} & X, \end{array}$$

setting $a = iv = uk$, and given any object $\mathbb{F}$ of $S\mathscr{H}(X)$ one has homotopy pullback squares in $S\mathscr{H}(X)$, made of adjunction maps:

$$\begin{array}{ccc} \mathbb{F} & \longrightarrow & u_*u^*(\mathbb{F}) \\ \downarrow & & \downarrow \\ i_*i^*(\mathbb{F}) & \longrightarrow & a_*a^*(\mathbb{F}). \end{array} \qquad \begin{array}{ccc} a_!a^!(\mathbb{F}) & \longrightarrow & u_!u^!(\mathbb{F}) \\ \downarrow & & \downarrow \\ i_!i^!(\mathbb{F}) & \longrightarrow & \mathbb{F} \end{array}$$

See [CD09, 3.3.3] (resp. [CD09, 3.3.9]).

From these distinguished triangles, we get the following descent long exact sequences:

- *Nisnevich descent.*– given a Nisnevich distinguished square of $S$-schemes as above, one has long exact sequences:

$$\mathbb{E}^{n,m}(X) \xrightarrow{i^*+u^*} \mathbb{E}^{n,m}(Y) \oplus \mathbb{E}^{n,m}(X') \xrightarrow{v^*-k^*} \mathbb{E}^{n,m}(Y') \to \mathbb{E}^{n+1,m}(X)$$
$$\mathbb{E}^{BM}_{n,m}(X/S) \xrightarrow{i^*+u^*} \mathbb{E}^{BM}_{n,m}(Y/S) \oplus \mathbb{E}^{BM}_{n,m}(X'/S) \xrightarrow{v^*-k^*} \mathbb{E}^{BM}_{n,m}(Y'/S) \to \mathbb{E}^{BM}_{n-1,m}(X/S)$$
$$\mathbb{E}^{n,m}_c(Y'/S) \xrightarrow{v_*+k_*} \mathbb{E}^{n,m}_c(Y/S) \oplus \mathbb{E}^{n,m}(X'/S) \xrightarrow{i_*-u_*} \mathbb{E}^{n,m}_c(X/S) \to \mathbb{E}^{n+1,m}_c(Y'/S)$$
$$E_{n,m}(Y'/S) \xrightarrow{v_*+k_*} \mathbb{E}_{n,m}(Y/S) \oplus \mathbb{E}^{n,m}(X'/S) \xrightarrow{i_*-u_*} \mathbb{E}_{n,m}(X/S) \to \mathbb{E}_{n-1,m}(Y'/S)$$

- *cdh-descent.*– given a cdh-distinguished square of separated $S$-schemes of finite type as above, one has long exact sequences:

$$\mathbb{E}^{n,m}(X) \xrightarrow{i^*+u^*} \mathbb{E}^{n,m}(Y) \oplus \mathbb{E}^{n,m}(X') \xrightarrow{v^*-k^*} \mathbb{E}^{n,m}(Y') \to \mathbb{E}^{n+1,m}(X)$$
$$\mathbb{E}^{BM}_{n,m}(Y'/S) \xrightarrow{v_*+k_*} \mathbb{E}^{BM}_{n,m}(Y/S) \oplus \mathbb{E}^{n,m}(X'/S) \xrightarrow{i_*-u_*} \mathbb{E}^{BM}_{n,m}(X/S) \to \mathbb{E}^{BM}_{n-1,m}(Y'/S)$$
$$\mathbb{E}^{n,m}_c(X/S) \xrightarrow{i^*+u^*} \mathbb{E}^{n,m}_c(Y/S) \oplus \mathbb{E}^{n,m}_c(X'/S) \xrightarrow{v^*-k^*} \mathbb{E}^{n,m}_c(Y'/S) \to \mathbb{E}^{n-1,m}_c(X/S)$$
$$E_{n,m}(Y'/S) \xrightarrow{v_*+k_*} \mathbb{E}_{n,m}(Y/S) \oplus \mathbb{E}^{n,m}(X'/S) \xrightarrow{i_*-u_*} \mathbb{E}_{n,m}(X/S) \to \mathbb{E}_{n-1,m}(Y'/S)$$

In each case, one apply one of the preceding homotopy pullback squares for $\mathbb{F} = S^0$ or $\mathbb{F} = \mathbb{E}_S$ and we apply either property (B1') (for $f$ étale) or property (B1).

*Remark* 0.2.5. Stronger descent properties can be obtained when $\mathbb{E}$ satisfies suitable assumptions (see [CD09, 3.3]).

**0.2.6.** *Products.*– We now assume that $\mathbb{E}$ is an absolute ring spectrum.

In the 6 functors formalism, products appears as the following pairing of functors, for a given s-morphism $p$:

$$p^*(\mathbb{E}) \wedge p^!(\mathbb{F}) \xrightarrow{ad} p^!p_!\big(p^*(\mathbb{E}) \wedge p^!(\mathbb{F})\big) \xrightarrow{\sim} p^!\big(\mathbb{E} \wedge p_!p^!(\mathbb{F})\big) \xrightarrow{ad} p^!\big(\mathbb{E} \wedge \mathbb{F}\big), \tag{0.2.6.a}$$

$$p_!(\mathbb{E}) \wedge p_*(\mathbb{F}) \xrightarrow{\sim} p_!\big(\mathbb{E} \wedge p^*p_*(\mathbb{F})\big) \xrightarrow{ad} p_!(\mathbb{E} \wedge \mathbb{F}), \tag{0.2.6.b}$$

where the maps labelled $ad$ are the obvious unit/counit of the adjunctions $(p_!, p^!)$ of $(p^*, p_*)$ and the isomorphisms are given by property (B3).

Using these pairings, one obtains the following products (degrees are added as usual):

| | |
|---|---|
| cup-product | $\mathbb{E}^{**}(X) \otimes \mathbb{E}^{**}(X) \to \mathbb{E}^{**}(X)$ |
| refined product | $\mathbb{E}^{BM}_{**}(Y/X) \otimes \mathbb{E}^{BM}_{**}(X/S) \to \mathbb{E}^{BM}_{**}(Y/S)$ |
| refined product | $\mathbb{E}^{**}_c(Y/X) \otimes \mathbb{E}^{**}_c(X/S) \to \mathbb{E}^{**}_c(Y/S)$ |
| exterior product | $\mathbb{E}_{**}(X/S) \otimes \mathbb{E}_{**}(Y/S) \to \mathbb{E}_{**}(X \times_S Y/S)$ |
| (refined) cap-product | $\mathbb{E}^{**}_c(X/S) \otimes \mathbb{E}^{BM}_{**}(X/S) \to \mathbb{E}_{**}(X/S)$ |

Note that taking $Y = X$ (resp. $Y = S$), we obtain that BM-homology $\mathbb{E}^{BM}_{**}(X/S)$ and c-cohomology $\mathbb{E}^{**}_c(X/S)$ (resp. homology $\mathbb{E}_{**}(X/S)$) have the structure of a bigraded $\mathbb{E}^{**}(X)$-module (resp. $\mathbb{E}^{**}(S)$-module).

The case of cohomology is easy and BM-homology is 0.1.8.

The case of c-cohomology for given s-morphisms $Y \xrightarrow{q} X \xrightarrow{p} S$. We give the formula for classes of bidegree $(0,0)$ for short, $y : S^0 \to q_!(\mathbb{E}_Y)$ and $x : S^0 \to p_!(\mathbb{E}_X)$. We define $y.x$ as the following composite:

$$S^0 \simeq S^0 \wedge S^0 \xrightarrow{y' \wedge x} p_* q_!(\mathbb{E}_Y) \wedge p_!(\mathbb{E}_X) \xrightarrow{(0.2.6.b)} p_!\big(q_!(\mathbb{E}_Y) \wedge \mathbb{E}_X\big) \xrightarrow{(B3)} p_! q_!\big(\mathbb{E}_Y \wedge q^*(\mathbb{E}_X)\big)$$
$$\xrightarrow{\tau} (pq)_!(\mathbb{E}_Y \wedge \mathbb{E}_Y) \xrightarrow{\mu} (pq)_!(\mathbb{E}_Y).$$

The exterior product: we consider a cartesian diagram made of s-morphism:

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{p'} & Y \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{p} & S, \end{array}$$

and put $q = pq' = qp'$. The exterior product is obtained directly from the following pairing:

$$p_! p^!(\mathbb{E}_S) \wedge q_! q^!(\mathbb{E}_S) \xrightarrow{(B3)} p_!\big(p^!(\mathbb{E}_S) \wedge p^* q_! q^!(\mathbb{E}_S)\big) \xrightarrow{(B2)} p_!\big(p^!(\mathbb{E}_S) \wedge q'_! p'^* q^!(\mathbb{E}_S)\big)$$
$$\xrightarrow{(B3)} p_! q'_!\big(q' p^!(\mathbb{E}_S) \wedge p'^* q^!(\mathbb{E}_S)\big) \xrightarrow{(0.1.2.a)} q_!\big(q'^* p^!(\mathbb{E}_S) \wedge q'^! p^*(\mathbb{E}_S)\big)$$
$$\xrightarrow{(0.2.6.a)} a! a^!(\mathbb{E}_S \wedge \mathbb{E}_S) \xrightarrow{\mu} a! a^!(\mathbb{E}_S).$$

Finally, the (refined) cap-product is induced by the following pairing of functors:

$$p_* p^!(\mathbb{E}_S) \wedge p_! p^*(\mathbb{E}_S) \xrightarrow{(0.2.6.b)} p_!\big(p^!(\mathbb{E}_S) \wedge p^*(\mathbb{E}_S)\big) \xrightarrow{(0.2.6.a)} p_! p^!(\mathbb{E}_S \wedge \mathbb{E}_S) \xrightarrow{\mu} p! p^!(\mathbb{E}_S).$$

*Remark* 0.2.7. (1) The refined product (on BM-homology and c-cohomology) is to be compared with the bivariant axioms of Fulton and MacPherson: [FM81, 2.2(1)].

(2) The cap-product with support of [BO74, (1.3.1)] is in fact an instance of the refined product in BM-homology: with the notations of *loc. cit.*, the product appearing *loc. cit.* is:
$$\mathbb{E}^{BM}_{i,m}(X/k) \otimes \mathbb{E}^{BM}_{-j,-n}(Z/X) \to \mathbb{E}^{BM}_{i-j,m-n}(Z/k).$$

(3) What one usually call the cap-product corresponds in fact to the $\mathbb{E}^{**}(X)$-module structure on $\mathbb{E}^{BM}_{**}(X/S)$. To motivate our terminology, note that the following diagram is commutative:
$$\begin{array}{ccc} \mathbb{E}^{**}_c(X/S) \otimes \mathbb{E}^{BM}_{**}(X/S) & \longrightarrow & \mathbb{E}_{**}(X/S) \\ {\scriptstyle \nu_{X/S} \otimes 1}\downarrow & & \downarrow{\scriptstyle \nu^{BM}_{X/S}} \\ \mathbb{E}^{**}(X) \otimes \mathbb{E}^{BM}_{**}(X/S) & \longrightarrow & \mathbb{E}^{BM}_{**}(X/S) \end{array}$$

(4) With stronger assumptions on $X/S$, one has other products: see [Dég08, 2.6].

(5) As for BM-homology (see 0.1.8), it is possible to express the behaviour of all the products with respect to functorialities: compatibilities with certain pullbacks, projection formulas. We left the formulation and proofs to the interested reader.

**0.2.8.** Consider the notations of Property (B3) in Paragraph 0.1.1.

Then we get a canonical pairing:

$$f^*(\mathbb{E}) \wedge f^!(\mathbb{F}) \to f^! f_!\big(f^*(\mathbb{E}) \wedge f^!(\mathbb{F})\big) \xrightarrow{\sim} f^!\big(\mathbb{E} \wedge f_! f^!(\mathbb{F})\big) \to f^!\big(\mathbb{E} \wedge \mathbb{F}\big). \tag{0.2.8.a}$$

where the first and third morphism are given from the adjunction $(f_!, f^!)$ and the middle one from property (B3).

**Corollary 0.2.9.** *Let $\mathbb{E}$ be an absolutely pure oriented ring spectrum.*

(1) *Given a quasi-projective lci morphism $f : X \to S$ of pure relative dimension $d$, the following composite morphism:*

$$\tilde{\eta}_f : \mathbb{E}_X(d)[2d] \xrightarrow{1\wedge\eta_f} \mathbb{E}_X \wedge f^!(\mathbb{E}_S) \xrightarrow{\tau_f\wedge 1} f^*(\mathbb{E}_S) \wedge f^!(\mathbb{E}_S) \xrightarrow{(0.2.8.a)} f^!(\mathbb{E}_S \wedge \mathbb{E}_S) \xrightarrow{\mu_{\mathbb{E}}} f^!(\mathbb{E}_S)$$

*is an isomorphism in $S\mathscr{H}(X)$.*

(2) *Let $\mathbb{F}$ be an absolutely pure oriented ring spectrum and $\varphi : \mathbb{E} \to \mathbb{F}$ be a morphism of absolute ring spectra. Let $\mathrm{Td}_\varphi : K_0 \to \mathbb{F}^{0,0}$ be the Todd class associated with $\varphi$ (cf. 4.1.4). Then given any quasi-projective lci morphism $f$ of pure relative dimension $d$ and with virtual tangent bundle $\tau_f$, the following* categorical Riemann-Roch formula *holds:*

$$\begin{array}{ccc} \mathbb{E}_X(d)[2d] & \xrightarrow[\sim]{\tilde{\eta}_f^{\mathbb{E}}} & f^!(\mathbb{E}_S) \\ {\scriptstyle \varphi_X}\downarrow & & \downarrow{\scriptstyle f^!(\varphi_S)} \\ \mathbb{F}_X(d)[2d] & \xrightarrow[\sim]{\mathrm{Td}(\tau_f).\tilde{\eta}_f^{\mathbb{F}}} & f^!(\mathbb{F}_S), \end{array}$$

*where $\mathrm{Td}(\tau_f)$ is considered as an automorphism of $\mathbb{F}_X$.*

*Proof.* By definition of $\tilde{\eta}_f$, the Riemann-Roch formula of point (2) readily follows from its analog in point (3) of the preceding theorem.

Let us prove point (1). Recall that the family of objects of the form $p_!(S^0)(i)$ for an integer $i \in \mathbb{Z}$ and a smooth morphism $p : W \to X$ is generating in the triangulated category $S\mathscr{H}(X)$.[6]

Therefore, it is sufficient to check that the morphism of bigraded abelian groups

$$[p_!(S^0), \tilde{\eta}_f]_{**} : [p_!(S^0), \mathbb{E}_X((d))]_{**} \to [p_!(S^0), f^!(\mathbb{E}_S)]_{**} \tag{0.2.9.a}$$

is an isomorphism for any smooth morphism $p : W \to X$.

We first consider the case $f = Id_X$. Then by definition of $\tilde{\eta}_f$, we get that the map (0.2.9.a) coincide with the map $\delta_f$ of point (2) of Theorem 0.1.17. It is therefore an isomorphism

We consider the case of a general smooth morphism $p$. We can assume it is of pure dimension $n$. According to property (3) of Theorem 0.1.13, the following diagram is commutative:

$$\begin{array}{ccccc} \mathbb{E}_W((n+d)) & \xrightarrow{\tilde{\eta}_p((d))} & p^!(\mathbb{E}_X)((d)) = p^!\big(\mathbb{E}_X((d))\big) & \xrightarrow{p^!(\tilde{\eta}_f)} & p^! f^!(\mathbb{E}_X) \\ \| & & & & \downarrow\simeq \\ \mathbb{E}_W & & \xrightarrow{\qquad\tilde{\eta}_{fp}\qquad} & & (fp)^!(\mathbb{E}_X). \end{array}$$

Thus, one gets that the following diagram is commutative:

$$\begin{array}{ccccc} \mathbb{E}^{BM}_{**}(X/X) & \xrightarrow{(\tilde{\eta}_p)_*} & \mathbb{E}^{BM}_{**}(W/X) & = & [p_!(S^0), \mathbb{E}_X((d))]_{**} \\ & \searrow^{(\tilde{\eta}_{fp})_*} & \downarrow{\scriptstyle [p^!(\tilde{\eta}_f)]_*} & & \downarrow{\scriptstyle (\tilde{\eta}_f)_*} \\ & & \mathbb{E}^{BM}_{**}(X/S) & = & [p_!(S^0), f^!(\mathbb{E}_S)]_{**} \end{array}$$

The case treated previously shows the map $(\tilde{\eta}_p)_*$ and $(\tilde{\eta}_{fp})_*$ are isomorphisms, which concludes. □

[6] Use the fact $\Sigma^\infty W_+ = p_! p^!(S^0)$ and the dual of property (B1').

*Remark* 0.2.10. As it appears in the preceding proof, Point (1) of the preceding corollary is a functorial formulation of the duality between cohomology and (Borel-Moore) homology. In fact, it is essentially an extension of Grothendieck ideas on duality. Therefore, it has to be compared with the similar formula in étale cohomology as it appeared in the most general form in [**?**, XVI, 3.1.2]. With respect to the absolute oriented ring spectrum **KGL**, the preceding isomorphism first appeared implicitly in [CD09, Rem. 13.7.5] where it was denoted by $\eta'_f$. It was made explicit in [AH10, Ex. 2.4].

Note also that Point (2) when $\mathbb{E} = \mathbf{KGL}$ is a generalization of [AH10, Th. 2.5] as the trace map

$$\mathrm{tr}_f^{\mathbf{KGL}} : f_* f^*(\mathbf{KGL}_S) \to \mathbf{KGL}_S$$

involved in *loc. cit.* when $f$ is projective is obtained by adjunction from $\tilde{\eta}_f^{\mathbf{KGL}}$ when one uses the isomorphism

$$f_!(\mathbf{KGL}_X((d))) \simeq f_!(\mathbf{KGL}_X) = f_* f^*(\mathbf{KGL}_S)$$

where the first map uses the Bott periodicity isomorphism.

**0.2.11.** The preceding result allows to extend again the formulas obtained in 0.1.19. In fact, given an absolutely pure oriented ring spectrum $\mathbb{E}$ and a quasi-projective morphism $f : X \to S$, one can define the cohomology with compact support and homology of $X/S$ with coefficients in $\mathbb{E}$ as follows:

$$\mathbb{E}_c^{n,i}(X/S) = \mathrm{Hom}(\mathbb{1}_S, f_! f^* \mathbb{E}(i)[n])$$
$$\mathbb{E}_{n,i}(X/S) = \mathrm{Hom}(\mathbb{1}_S, f_! f^! \mathbb{E}(-i)[-n]).$$

Then the duality isomorphism $\tilde{\eta}_f$ of the preceding corollary, when $f$ is lci of pure relative dimension $d$, immediately yields a duality isomorphism:

$$\mathbb{E}_c^{n,i}(X/S) \xrightarrow{\delta'_f} \mathbb{E}_{2d-n,d-i}(X/S),$$

dual to the duality isomorphism of Theorem 0.1.17(2).

As in 0.1.19, given a base scheme $S$, this isomorphism to deduce that $\mathbb{E}$-homology is contravariant with respect to projective lci $S$-morphisms while $\mathbb{E}$-cohomology with compact support is covariant with respect to quasi-projective lci morphisms.

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