

ON p -ADIC ABSOLUTE HODGE COHOMOLOGY AND SYNTOMIC COEFFICIENTS, I.

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ABSTRACT. We interpret syntomic cohomology defined in [48] as a p -adic absolute Hodge cohomology. This is analogous to the interpretation of Deligne-Beilinson cohomology as an absolute Hodge cohomology by Beilinson [8] and generalizes results of Bannai in the good reduction case [6]. This interpretation yields a simple construction of the syntomic descent spectral sequence and its degeneration for proper and smooth varieties. We introduce syntomic coefficients and show that in dimension zero they form a full triangulated subcategory of the derived category of potentially semistable Galois representations.

Along the way, we obtain p -adic realizations of mixed motives including p -adic comparison isomorphisms. We apply this to the motivic fundamental group generalizing results of Olsson and Vologodsky [54], [68].

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1. INTRODUCTION

In [8], Beilinson gave an interpretation of Deligne-Beilinson cohomology as an *absolute Hodge cohomology*, i.e., as derived Hom in the derived category of mixed Hodge structures. This approach is advantageous: absolute Hodge cohomology allows coefficients. It follows that Deligne-Beilinson cohomology can be interpreted as derived Hom between Tate twists in the derived category of Saito's mixed Hodge modules [37, A.2.7].

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Syntomic cohomology is a p -adic analog of Deligne-Beilinson cohomology. The purpose of this paper is to give an analog of the above results for syntomic cohomology. Namely, we will show that the syntomic cohomology introduced in [48] is a p -adic absolute Hodge cohomology, i.e., it can be expressed as derived Hom in the derived category of p -adic Hodge structures, and we will begin the study of syntomic coefficients - an approximation of p -adic Hodge modules. This generalizes the results of Bannai in the good reduction case [6].

Let K be a complete discrete valuation field of mixed characteristic $(0, p)$ with perfect residue field k . Let $G_K = \text{Gal}(\overline{K}/K)$ be the Galois group of K . For the category of p -adic Hodge structures we take the abelian category DF_K of (weakly) admissible filtered (φ, N, G_K) -modules defined by Fontaine. For a variety X over K , we construct a complex $\text{R}\Gamma_{DF_K}(X_{\overline{K}}, r) \in D^b(DF_K)$, $r \in \mathbf{Z}$. The absolute Hodge cohomology of X is then by definition

$$\text{R}\Gamma_{\mathcal{H}}(X, r) := \text{RHom}_{D^b(DF_K)}(K(0), \text{R}\Gamma_{DF_K}(X_{\overline{K}}, r)), \quad r \in \mathbf{Z}.$$

For $r \geq 0$, it coincides with the syntomic cohomology $\text{R}\Gamma_{\text{syn}}(X, r)$ defined in [48]. Recall that the latter was defined as the following mapping fiber

$$\text{R}\Gamma_{\text{syn}}(X, r) = [\text{R}\Gamma_{\text{HK}}^B(X)^{\varphi=p^r, N=0} \xrightarrow{\iota_{\text{dR}}} \text{R}\Gamma_{\text{dR}}(X)/F^r],$$

where $\text{R}\Gamma_{\text{HK}}^B(X)$ is the Beilinson-Hyodo-Kato cohomology from [10], $\text{R}\Gamma_{\text{dR}}(X)$ is the Deligne de Rham cohomology, and the map ι_{dR} is the Beilinson-Hyodo-Kato map.

We present two approaches to the definition of the complex $\text{R}\Gamma_{DF_K}(X_{\overline{K}}, r)$. In the first one, we follow Beilinson's construction of the complex of mixed Hodge structures associated to a variety [8]. Thus, we build the dg category \mathcal{D}_{pH} of p -adic Hodge complexes (an analog of Beilinson's mixed Hodge complexes) which is obtained by gluing two dg categories, one, corresponding morally to the special fiber, whose objects are equipped with an action of a Frobenius and a monodromy operator, and the other one, corresponding to the generic fiber, whose objects are equipped with a filtration thought of as the Hodge filtration on de Rham cohomology. It contains a dg subcategory of *admissible p -adic Hodge complexes* with cohomology groups belonging to DF_K . The category $\mathcal{D}_{pH}^{\text{ad}}$ admits a natural t -structure whose heart is the category DF_K and $\mathcal{D}_{pH}^{\text{ad}}$ is equivalent to the derived category of its heart. That is, we have the following equivalences of categories

$$\theta : DF_K \xrightarrow{\sim} \mathcal{D}_{pH}^{\text{ad}, \heartsuit}, \quad \theta : \mathcal{D}^b(DF_K) \xrightarrow{\sim} \mathcal{D}_{pH}^{\text{ad}}.$$

The interest of the category $\mathcal{D}_{pH}^{\text{ad}}$ lies in the fact that, for $r \in \mathbf{Z}$, a variety X over K gives rise to the admissible p -adic Hodge complex

$$\text{R}\Gamma_{pH}(X_{\overline{K}}, r) := (\text{R}\Gamma_{\text{HK}}^B(X_{\overline{K}}, r), (\text{R}\Gamma_{\text{dR}}(X), F^{\bullet+r}), \iota_{\text{dR}}) \in \mathcal{D}_{pH}^{\text{ad}}$$

We define $\text{R}\Gamma_{DF_K}(X_{\overline{K}}, r) := \theta^{-1}\text{R}\Gamma_{pH}(X_{\overline{K}}, r)$.

Since the category DF_K is equivalent to that of potentially semistable representations [19], i.e., we have a functor $V_{\text{pst}} : DF_K \xrightarrow{\sim} \text{Rep}_{\text{pst}}(G_K)$, we can also write

$$\text{R}\Gamma_{\mathcal{H}}(X, r) = \text{Hom}_{\mathcal{D}^b(\text{Rep}_{\text{pst}}(G_K))}(\mathbf{Q}_p, \text{R}\Gamma_{\text{pst}}(X_{\overline{K}}, r)),$$

for $\text{R}\Gamma_{\text{pst}}(X_{\overline{K}}, r) := V_{\text{pst}}\text{R}\Gamma_{DF_K}(X_{\overline{K}}, r)$. Using Beilinson's comparison theorems [10] we prove that $\text{R}\Gamma_{\text{pst}}(X_{\overline{K}}, r) \simeq \text{R}\Gamma_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r))$ as Galois modules. It follows that there is a functorial syntomic descent spectral sequence (constructed originally by a different, more complicated, method in [48])

$${}^{\mathcal{H}}E_2^{i,j} := H_{\text{st}}^i(G_K, H_{\text{ét}}^j(X_{\overline{K}}, \mathbf{Q}_p(r))) \Rightarrow H_{\mathcal{H}}^{i+j}(X, r),$$

where $H_{\text{st}}^i(G_K, \cdot) := \text{Ext}_{\text{Rep}_{\text{pst}}(G_K)}^i(\mathbf{Q}_p, \cdot)$. By a classical argument of Deligne [24], it follows from Hard Lefschetz Theorem, that it degenerates at E_2 for X proper and smooth.

A more direct definition of the complex $\text{R}\Gamma_{DF_K}(X_{\overline{K}}, r)$, or, equivalently, of the complex $\text{R}\Gamma_{\text{pst}}(X_{\overline{K}}, r)$ of potentially semistable representations associated to a variety was proposed by Beilinson [11] using Beilinson's Basic Lemma. This lemma allows one to associate a potentially semistable analog of a cellular complex (of a CW-complex) to an affine variety X over K : one stratifies the variety by closed subvarieties such that consecutive relative geometric étale cohomology is concentrated in the top degree

(and is a potentially semistable representation). For a general X one obtains Beilinson's potentially semistable complex by a Čech gluing argument.

All the p -adic cohomologies mentioned above (de Rham, étale, Hyodo-Kato, and syntomic) behave well hence they lift to realizations of both Nori's abelian and Voevodsky's triangulated category of mixed motives. We also lift the comparison maps between them, thus obtaining comparison theorems for mixed motives. We illustrate this construction by two applications. The first one is a p -adic realization of the motivic fundamental group including a potentially semistable comparison theorem. We rely on Cushman's motivic (in the sense of Nori) theory of the fundamental group [21]. This generalizes results obtained earlier for curves and proper varieties with good reduction [36], [1], [68], [54]. The second is a compatibility result. We show that Beilinson's p -adic comparison theorems (with compact support or not) are compatible with Gysin morphisms and (possibly mixed) products.

To define a well-behaved notion of syntomic coefficients (i.e., coefficients for syntomic cohomology) we use Morel-Voevodsky *motivic homotopy theory*, and more precisely the concept of modules over (motivic) ring spectra. Recall that objects of motivic stable homotopy theory, called spectra, represent cohomology theories with suitable properties. A multiplicative structure on the cohomology theory corresponds to a monoid structure on the representing spectrum, which is then called a *ring spectrum*. These objects have to be thought of as a generalization of (h -sheaves of) differential graded algebras. In fact, as we will only consider ordinary cohomology theories (as opposed to K-theory or algebraic cobordism with integral coefficients), we will always restrict to this later concept. Therefore modules over ring spectra have to be understood as the more familiar concept of modules over differential graded algebras.

One of the basic examples of a representable cohomology theory is de Rham cohomology in characteristic 0. Denote the corresponding motivic ring spectrum by $\mathcal{E}_{\mathrm{dR}}$. By [17], [27], working relatively to a fix complex variety X , modules over $\mathcal{E}_{\mathrm{dR},X}$ satisfying a suitable finiteness condition correspond naturally to (regular holonomic) \mathcal{D}_X -modules of geometric origin.

In [48] it is shown that syntomic cohomology can be represented by a motivic dg algebra $\mathcal{E}_{\mathrm{syn}}$, i.e., we have

$$\boxed{\text{form}} \quad (1.1) \quad \mathrm{R}\Gamma_{\mathrm{syn}}(X, r) = \mathrm{R}\mathrm{Hom}_{DM_h(K, \mathbf{Q}_p)}(M(X), \mathcal{E}_{\mathrm{syn}}(r)),$$

where $M(X)$ is the Voevodsky's motive associated to X and $DM_h(K, \mathbf{Q}_p)$ is the category of h -motives. So we have the companion notion of *syntomic modules*, that is, modules over the motivic dg-algebra $\mathcal{E}_{\mathrm{syn}}$. The main advantage of this definition is that the link with mixed motives is rightly given by the construction and, most of all, the 6 functors formalism follows easily from the motivic one.

Now the crucial question is to understand how the category of syntomic modules is related to the category of filtered (φ, N, G_K) -modules, the existing candidates for syntomic smooth sheaves [29], [30], [64], [60], and the category of syntomic coefficients introduced in [23] by a method analogous to the one we use but based on Gros-Besser's version of syntomic cohomology. In this paper we study this question only in dimension zero, i.e., for syntomic modules over the base field. With a suitable notion of finiteness for syntomic modules, called *constructibility*, we prove the following theorem.

Theorem (Theorem 5.13). The triangulated monoidal category of constructible syntomic modules over a p -adic field K is equivalent to a full subcategory of the derived category of admissible filtered (φ, N, G_K) -modules.

It implies, by adjunction from (1.1), that p -adic absolute Hodge cohomology coincides with derived Hom in the (homotopy) category of syntomic modules, i.e., we have

$$\mathrm{R}\Gamma_{\mathcal{H}}(X, r) = \mathrm{R}\mathrm{Hom}_{\mathcal{E}_{\mathrm{syn}}\text{-mod}_X}(\mathcal{E}_{\mathrm{syn},X}, \mathcal{E}_{\mathrm{syn},X}(r)).$$

In the conclusion of the paper, we use syntomic modules to introduce new notions of p -adic Galois representations (Definition 5.20). We define *geometric representations* which correspond to the common intuition of representations associated to (mixed) motives, and *constructible representations*, corresponding to cohomology groups of Galois realizations of syntomic modules.

We expect that the categories of geometric, constructible, and potentially semi-stable representations are not the same. This is at least what is predicted by the current general conjectures. Note that this

is in contrast to the case of number fields where the analogs of these notions are conjectured to coincide with the known definition of “representations coming from geometry” [33].

1.0.1. *Notation.* Let \mathcal{O}_K be a complete discrete valuation ring with fraction field K of characteristic 0, with perfect residue field k of characteristic p . Let \overline{K} be an algebraic closure of K . Let $W(k)$ be the ring of Witt vectors of k with fraction field K_0 and denote by K_0^{nr} the maximal unramified extension of K_0 . Set $G_K = \text{Gal}(\overline{K}/K)$ and let I_K denote its inertia subgroup. Let φ be the absolute Frobenius on K_0^{nr} . We will denote by \mathcal{O}_K , \mathcal{O}_K^\times , and \mathcal{O}_K^0 the scheme $\text{Spec}(\mathcal{O}_K)$ with the trivial, canonical (i.e., associated to the closed point), and $(\mathbf{N} \rightarrow \mathcal{O}_K, 1 \mapsto 0)$ log-structure respectively. For a scheme X over $W(k)$, X_n will denote its reduction mod p^n , X_0 will denote its special fiber.

Let $\mathcal{V}ar_K$ denote the category of varieties over K . For a dg category \mathcal{C} with a t -structure, we will denote by \mathcal{C}^\heartsuit the heart of the t -structure.

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2. A p -ADIC ABSOLUTE HODGE COHOMOLOGY, I

2.1. The derived category of admissible filtered (φ, N, G_K) -modules.

2.1. For a field K , let V_K denote the category of K -vector spaces. It is an abelian category. We will denote by $\mathcal{D}^b(V_K)$ its bounded derived dg category and by $D^b(V_K)$ – its bounded derived category. Let V_{dR}^K denote the category of K -vector spaces with a descending exhaustive separated filtration F^\bullet . The category V_{dR}^K (and the category of bounded complexes $C^b(V_{\text{dR}}^K)$) is additive but not abelian. It is an exact category in the sense of Quillen [55], where short exact sequences are exact sequences of K -vector spaces with *strict morphisms* (recall that a morphism $f : M \rightarrow N$ is *strict* if $f(F^i M) = F^i N \cap \text{Im}(f)$). It is also a quasi-abelian category in the sense of [59] (see [58, 2] for a quick review). Thus its derived category can be studied as usual (see [12]).

An object $M \in C^b(V_{\text{dR}}^K)$ is called a *strict complex* if its differentials are strict. There are canonical truncation functors on $C^b(V_{\text{dR}}^K)$:

$$\begin{aligned} \tau_{\leq n} M &:= \cdots \rightarrow M^{n-2} \rightarrow M^{n-1} \rightarrow \ker(d^n) \rightarrow 0 \rightarrow \cdots \\ \tau_{\geq n} M &:= \cdots \rightarrow 0 \cdots \rightarrow \text{coim}(d^{n-1}) \rightarrow M^n \rightarrow M^{n+1} \rightarrow \cdots \end{aligned}$$

The cohomology object

$$\tau_{\leq n} \tau_{\geq n}(M) = \cdots \rightarrow 0 \rightarrow \text{coim}(d^{n-1}) \rightarrow \ker(d^n) \rightarrow 0 \rightarrow \cdots$$

We will denote the bounded derived dg category of V_{dR}^K by $\mathcal{D}^b(V_{\text{dR}}^K)$. It is defined as the dg quotient [28] of the dg category $C^b(V_{\text{dR}}^K)$ by the full dg subcategory of strictly exact complexes [47]. A map of complexes is a quasi-isomorphism if and only if it is a quasi-isomorphism on the grading. The homotopy category of $\mathcal{D}^b(V_{\text{dR}}^K)$ is the bounded filtered derived category $D^b(V_{\text{dR}}^K)$.

For $n \in \mathbf{Z}$, let $D_{\leq n}^b(V_{\text{dR}}^K)$ (resp. $D_{\geq n}^b(V_{\text{dR}}^K)$) denote the full subcategory of $D^b(V_{\text{dR}}^K)$ of complexes that are strictly exact in degrees $k > n$ (resp. $k < n$). The above truncation maps extend to truncations functors $\tau_{\leq n} : D^b(V_{\text{dR}}^K) \rightarrow D_{\leq n}^b(V_{\text{dR}}^K)$ and $\tau_{\geq n} : D^b(V_{\text{dR}}^K) \rightarrow D_{\geq n}^b(V_{\text{dR}}^K)$. The pair $(D_{\leq n}^b(V_{\text{dR}}^K), D_{\geq n}^b(V_{\text{dR}}^K))$ defines a t -structure on $D^b(V_{\text{dR}}^K)$ by [59]. The heart $D_{\leq 0}^b(V_{\text{dR}}^K)^\heartsuit$ is an abelian category $LH(V_{\text{dR}}^K)$. We have an embedding $V_{\text{dR}}^K \hookrightarrow LH(V_{\text{dR}}^K)$ that induces an equivalence $D^b(V_{\text{dR}}^K) \xrightarrow{\sim} D^b(LH(V_{\text{dR}}^K))$. This t -structure pulls back to a t -structure on the derived dg category $\mathcal{D}^b(V_{\text{dR}}^K)$.

2.2. Let the field K be again as at the beginning of this article. A φ -module over K_0 is a pair (D, φ) , where D is a K_0 -vector space and the Frobenius $\varphi = \varphi_D$ is a φ -semilinear endomorphism of D . We will usually write D for (D, φ) . The category $M_{K_0}(\varphi)$ of φ -modules over K_0 is abelian and we will denote by $\mathcal{D}_{K_0}^b(\varphi)$ its bounded derived dg category.

For $D_1, D_2 \in M_{K_0}(\varphi)$, let $\text{Hom}_{K_0, \varphi}(D_1, D_2)$ denote the group of Frobenius morphisms. We have the exact sequence

$$(2.1) \quad 0 \rightarrow \text{Hom}_{K_0, \varphi}(D_1, D_2) \rightarrow \text{Hom}_{K_0}(D_1, D_2) \rightarrow \text{Hom}_{K_0}(D_1, \varphi_* D_2),$$

where the last map is $\delta : x \mapsto \varphi_{D_2} x - \varphi_*(x)\varphi_{D_1}$. Set $\text{Hom}_{K_0, \varphi}^\sharp(D_1, D_2) := \text{Cone}(\delta)[-1]$. Beilinson proves the following lemma.

Lemma 2.3. ([10, 1.13, 1.14])

For $D_1, D_2 \in \mathcal{D}_{K_0}^b(\varphi)$, the map $R\text{Hom}_{K_0, \varphi}(D_1, D_2) \rightarrow \text{Hom}_{K_0, \varphi}^\sharp(D_1, D_2)$ is a quasi-isomorphism, i.e.,

$$R\text{Hom}_{K_0, \varphi}(D_1, D_2) = \text{Cone}(\text{Hom}_{K_0}(D_1, D_2) \xrightarrow{\delta} \text{Hom}_{K_0}(D_1, \varphi_* D_2))[-1]$$

Proof. Note that, for $D_1, D_2 \in \mathcal{D}_{K_0}^b(\varphi)$, from the exact sequence (2.1), we get a map

$$\alpha : R\text{Hom}_{K_0, \varphi}(D_1, D_2) \rightarrow \text{Cone}(R\text{Hom}_{K_0}(D_1, D_2) \xrightarrow{\delta} R\text{Hom}_{K_0}(D_1, R\varphi_* D_2))[-1]$$

We will show that it is a quasi-isomorphism.

The forgetful functor $M_{K_0}(\varphi) \rightarrow V_{K_0}$ has a right adjoint $M \rightarrow M_\varphi$, where the φ -module $M_\varphi := \prod_{n \geq 0} \varphi_*^n M$ with Frobenius $\varphi_{M_\varphi} : (x_0, x_1, \dots) \rightarrow (x_1, x_2, \dots)$. The functor $M \rightarrow M_\varphi$ is left exact and preserves injectives. Since all K_0 -modules are injective, the map $M \rightarrow M_\varphi$, $m \mapsto (m, \varphi(m), \varphi^2(m), \dots)$, embeds M into an injective φ -module. It suffices thus to check that the map α is a quasi-isomorphism for D_1 any φ -module and $D_2 = G_\varphi$. We calculate

$$\begin{aligned} R\text{Hom}_{K_0, \varphi}(D_1, G_\varphi) &\xrightarrow{\sim} \text{Hom}_{K_0, \varphi}(D_1, G_\varphi) \xrightarrow{\sim} \text{Cone}(\text{Hom}_{K_0}(D_1, G_\varphi) \xrightarrow{\delta} \text{Hom}_{K_0}(D_1, \varphi_* G_\varphi))[-1] \\ &\xrightarrow{\sim} \text{Cone}(R\text{Hom}_{K_0}(D_1, G_\varphi) \xrightarrow{\delta} R\text{Hom}_{K_0}(D_1, R\varphi_* G_\varphi))[-1] \end{aligned}$$

This proves the lemma. \square

2.4. A (φ, N) -module is a triple (D, φ_D, N) (abbreviated often to D), where (D, φ_D) is a finite rank φ -module over K_0 and φ_D is an automorphism, and N is a K_0 -linear endomorphism of D such that $N\varphi_D = p\varphi_D N$ (hence N is nilpotent). The category $M_{K_0}(\varphi, N)$ of (φ, N) -modules is naturally a Tannakian tensor \mathbf{Q}_p -category and $(M, \varphi_M, N) \mapsto M$ is a fiber functor over K_0 . Denote by $\mathcal{D}_{\varphi, N}^b(K_0)$ and $D_{\varphi, N}^b(K_0)$ the corresponding bounded derived dg category and bounded derived category, respectively.

For (φ, N) -modules M, T , let $\text{Hom}_{\varphi, N}(M, T)$ be the group of (φ, N) -module morphisms. Let $\text{Hom}^\sharp(M, T)$ be the complex [10, 1.15]

$$\text{Hom}_{K_0}(M, T) \rightarrow \text{Hom}_{K_0}(M, \varphi_* T) \oplus \text{Hom}_{K_0}(M, T) \rightarrow \text{Hom}(M, \varphi_* T)$$

beginning in degree 0 and with the following differentials

$$\begin{aligned} d_0 : x &\mapsto (\varphi_2 x - x\varphi_1, N_2 x - xN_1); \\ d_1 : (x, y) &\mapsto (N_2 x - pxN_1 - p\varphi_2 y + y\varphi_1) \end{aligned}$$

Clearly, we have $\text{Hom}_{\varphi, N}(M, T) = H^0 \text{Hom}_{\varphi, N}^\sharp(M, T)$. Complexes $\text{Hom}_{\varphi, N}^\sharp$ compose naturally and supply a dg category structure on the category of bounded complexes of (φ, N) -modules.

Beilinson states the following fact.

diagram1

Lemma 2.5. ([10, 1.15]) For $D_1, D_2 \in \mathcal{D}_{K_0}^b(\varphi, N)$, the map $R\text{Hom}_{\varphi, N}(D_1, D_2) \rightarrow \text{Hom}_{\varphi, N}^\sharp(D_1, D_2)$ is a quasi-isomorphism, i.e.,

$$R\text{Hom}_{\varphi, N}(D_1, D_2) = \left[\begin{array}{ccc} \text{Hom}_{K_0}(D_1, D_2) & \xrightarrow{\delta_1} & \text{Hom}_{K_0}(D_1, \varphi_* D_2) \\ \downarrow \delta_2 & & \downarrow \delta'_2 \\ \text{Hom}_{K_0}(D_1, D_2) & \xrightarrow{\delta'_1} & \text{Hom}_{K_0}(D_1, \varphi_* D_2) \end{array} \right]$$

Here

$$\begin{aligned}\delta_1 : x &\mapsto \varphi_2 x - x\varphi_1, & \delta'_1 : x &\mapsto p\varphi_2 x - x\varphi_1; \\ \delta_2 : x &\mapsto N_2 x - xN_1, & \delta'_2 : x &\mapsto N_2 x - pxN_1\end{aligned}$$

The outside bracket denotes the homotopy limit (the total complex of the double complex in this case).

Proof. We pass first to the category of (φ, N) -modules defined as above but with modules of any rank and with the Frobenius being just an endomorphism. It suffices to prove our lemma in this (larger) category. The forgetful functor $M_{K_0}(\varphi, N) \rightarrow V_{K_0}$ has a right adjoint $M \mapsto M_{\varphi, N}$, where $M_{\varphi, N}$ is the product $\prod_{i \geq 0} M_{\varphi}^i$, $M_{\varphi}^i = M_{\varphi}$. We will often write $M_{\varphi, N} = \prod_{i, j \geq 0} M_{ij}$. The K_0 -action on $M_{\varphi, N}$ is twisted by Frobenius: $am_{ij} = \varphi^i(a)m_{ij}$, $a \in K_0$; the Frobenius is defined by $\varphi = \text{Id} : M_{ij} \rightarrow \varphi_* M_{i, j-1}$ and the monodromy – as $N = p^{-j} : M_{ij} \rightarrow M_{i-1, j}$. The functor $M \mapsto M_{\varphi, N}$ is exact and preserves injectives. We have the adjunction map $M \rightarrow M_{\varphi, N}$, $m \mapsto (m_{ij} = N^i \varphi^j(m))$ that induces the isomorphism $\text{Hom}_{K_0}(T, M) \xrightarrow{\sim} \text{Hom}_{\varphi, N}(T, M_{\varphi, N})$, $T \in V_{K_0}$, $M \in M_{K_0}(\varphi, N)$.

Consider the following injective resolution of a (φ, N) -module M

$$0 \rightarrow M \rightarrow M_{\varphi, N} \xrightarrow{\alpha} \varphi_* M_{\varphi, N} \oplus M_{\varphi, N} \xrightarrow{\beta} \varphi_* M \rightarrow 0$$

with

$$\begin{aligned}\alpha(m_{ij}) &= (\varphi(m_{ij}) - m_{i, j+1}, Nm_{ij} - p^{-j}m_{i+1, j}), \\ \beta(x, y) &= ((i, j) \mapsto Nx_{ij} - p^{1-j}x_{i+1, j} - p\varphi y_{i, j} + y_{i, j+1})\end{aligned}$$

Hence $R\text{Hom}_{\varphi, N}(T, M)$ for $M, T \in M_{K_0}(\varphi, N)$, can be computed by the complex

$$\text{Hom}_{\varphi, N}(T, M_{\varphi, N}) \xrightarrow{\alpha} \text{Hom}_{\varphi, N}(T, \varphi_* M_{\varphi, N}) \oplus \text{Hom}_{\varphi, N}(T, M_{\varphi, N}) \xrightarrow{\beta} \text{Hom}_{\varphi, N}(T, \varphi_* M_{\varphi, N})$$

Passing to the K_0 -linear morphisms we get the complex $\text{Hom}^{\sharp}(T, M)$. This suffices to prove the lemma. \square

2.6. A filtered (φ, N) -module is a tuple $(D_0, \varphi, N, F^{\bullet})$, where (D_0, φ, N) is a (φ, N) -module and F^{\bullet} is a decreasing finite filtration of $D_K := D_0 \otimes_{K_0} K$ by K -vector spaces. There is a notion of a (weakly) admissible filtered (φ, N) -module [19]. Denote by

$$MF_K^{\text{ad}}(\varphi, N) \subset MF_K(\varphi, N) \subset M_{K_0}(\varphi, N)$$

the categories of admissible filtered (φ, N) -modules, filtered (φ, N) -modules, and (φ, N) -modules, respectively. We know [19] that the pair of functors

$$\begin{aligned}D_{\text{st}}(V) &= (\mathbf{B}_{\text{st}} \otimes_{\mathbf{Q}_p} V)^{G_K}, & D_K(V) &= (\mathbf{B}_{\text{dR}} \otimes_{\mathbf{Q}_p} V)^{G_K}; \\ V_{\text{st}}(D) &= (\mathbf{B}_{\text{st}} \otimes_{K_0} D_0)^{\varphi=\text{Id}, N=0} \cap F^0(\mathbf{B}_{\text{dR}} \otimes_K D_K)\end{aligned}$$

defines an equivalence of categories $MF_K^{\text{ad}}(\varphi, N) \simeq \text{Rep}_{\text{st}}(G_K) \subset \text{Rep}(G_K)$, where the last two categories denote the subcategory of semistable Galois representations [31] of the category of finite dimensional \mathbf{Q}_p -linear representations of the Galois group G_K . The rings \mathbf{B}_{st} and \mathbf{B}_{dR} are the semistable and de Rham period rings of Fontaine [31]. The category $MF_K^{\text{ad}}(\varphi, N)$ is naturally a Tannakian tensor \mathbf{Q}_p -category and $(D_0, \varphi, N, F^{\bullet}) \mapsto D_0$ is a fiber functor over K_0 .

A filtered (φ, N, G_K) -module is a tuple $(D_0, \varphi, N, \rho, F^{\bullet})$, where

- (1) D_0 is a finite dimensional K_0^{nr} -vector space;
- (2) $\varphi : D_0 \rightarrow D_0$ is a Frobenius map;
- (3) $N : D_0 \rightarrow D_0$ is a K_0^{nr} -linear monodromy map such that $N\varphi = p\varphi N$;
- (4) ρ is a K_0^{nr} -semilinear G_K -action on D_0 (hence $\rho|_{I_K}$ is linear) that factors through a finite quotient of the inertia I_K and that commutes with φ and N ;
- (5) F^{\bullet} is a decreasing finite filtration of $D_K := (D \otimes_{K_0^{\text{nr}}} \overline{K})^{G_K}$ by K -vector spaces.

Morphisms between filtered (φ, N, G_K) -modules are K_0^{nr} -linear maps preserving all structures. There is a notion of a (weakly) admissible filtered (φ, N, G_K) -module [19], [32]. Denote by

$$DF_K := MF_K^{\text{ad}}(\varphi, N, G_K) \subset MF_K(\varphi, N, G_K) \subset M_K(\varphi, N, G_K)$$

the categories of admissible filtered (φ, N, G_K) -modules (DF stands for Dieudonné-Fontaine), filtered (φ, N, G_K) -modules, and (φ, N, G_K) -modules, respectively. The last category is built from tuples (D_0, φ, N, ρ) having properties 1, 2, 3, 4 above. We know [19] that the pair of functors

$$D_{\text{pst}}(V) = \text{inj} \lim_H (\mathbf{B}_{\text{st}} \otimes_{\mathbf{Q}_p} V)^H, \quad H \subset G_K \text{ -- an open subgroup,} \quad D_K(V) := (V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{dR}})^{G_K};$$

$$V_{\text{pst}}(D) = (\mathbf{B}_{\text{st}} \otimes_{K_0^{\text{nr}}} D_0)^{\varphi=\text{Id}, N=0} \cap F^0(\mathbf{B}_{\text{dR}} \otimes_K D_K)$$

define an equivalence of categories $MF_K^{\text{ad}}(\varphi, N, G_K) \simeq \text{Rep}_{\text{pst}}(G_K)$, where the last category denotes the category of potentially semistable Galois representations [31]. We have the abstract period isomorphisms

$$(2.2) \quad \rho_{\text{pst}} : D_{\text{pst}}(V) \otimes_{K_0^{\text{nr}}} \mathbf{B}_{\text{st}} \simeq V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{st}}, \quad \rho_{\text{dR}} : D_K(V) \otimes_K \mathbf{B}_{\text{dR}} \simeq V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{dR}},$$

where the first one is compatible with the action of φ, N , and G_K , and the second one is compatible with filtration. The category MF_K^{pst} is naturally a Tannakian tensor \mathbf{Q}_p -category and $(D_0, \varphi, N, \rho, F^\bullet) \mapsto D_0$ is a fiber functor over K_0^{nr} . We will denote by $\mathcal{D}^b(DF_K)$ and $D^b(DF_K)$ its bounded derived dg category and bounded derived category, respectively.

The category $M_K(\varphi, N, G_K)$ is abelian. We will denote by $\mathcal{D}_K^b(\varphi, N, G_K)$ and $D_K^b(\varphi, N, G_K)$ its bounded derived dg category and bounded derived category, respectively. For (φ, N, G_K) -modules M, T , let $\text{Hom}_{\varphi, N, G_K}(M, T)$ be the group of (φ, N, G_K) -module morphisms and let $\text{Hom}_{G_K}(M, T)$ be the group of K_0^{nr} -linear and G_K -equivariant morphisms. Let $\text{Hom}_{\varphi, N, G_K}^\sharp(M, T)$ be the complex

$$\text{Hom}_{G_K}(M, T) \rightarrow \text{Hom}_{G_K}(M, \varphi_* T) \oplus \text{Hom}_{G_K}(M, T) \rightarrow \text{Hom}_{G_K}(M, \varphi_* T).$$

This complex is supported in degrees 0, 1, 2 and the differentials are as above for (φ, N) -modules. Clearly, we have $\text{Hom}_{\varphi, N, G_K}(M, T) = H^0 \text{Hom}_{\varphi, N, G_K}^\sharp(M, T)$. Complexes $\text{Hom}_{\varphi, N, G_K}^\sharp$ compose naturally. Arguing as in the proof of Lemma 2.5, we can show that, for $M, T \in \mathcal{D}_K^b(\varphi, N, G_K)$,

$$(2.3) \quad \text{R Hom}_{\varphi, N, G_K}(M, T) \simeq \text{Hom}_{\varphi, N, G_K}^\sharp(M, T).$$

Let M, T be two complexes in $C^b(MF_K(\varphi, N, G_K))$. Define the complex $\text{Hom}^b(M, T)$ as the following homotopy fiber

$$\text{Hom}^b(M, T) := \text{Cone}(\text{Hom}_{\varphi, N, G_K}^\sharp(M_0, T_0) \oplus \text{Hom}_{\text{dR}}(M_K, T_K) \xrightarrow{\text{can} - \text{can}} \text{Hom}_{G_K}(M_{\overline{K}}, T_{\overline{K}}))[-1]$$

Complexes Hom^b compose naturally.

isom1 **Proposition 2.7.** *We have $\text{R Hom}_{DF_K}(M, T) \simeq \text{Hom}^b(M, T)$.*

Proof. We follow the method of proof of Beilinson and Bannai [8, Lemma 1.7], [6, Prop. 1.7]. Denote by $f_{M, T}$ the morphism in the cone defining $\text{Hom}^b(M, T)$. We have the distinguished triangle

$$\ker(f_{M, T}) \rightarrow \text{Hom}^b(M, T) \rightarrow \text{coker}(f_{M, T})[-1]$$

We also have the functorial isomorphism

$$\text{Hom}_{K^b(DF_K)}(M, T[i]) \xrightarrow{\sim} H^i(\ker(f_{M, T}))$$

Hence a long exact sequence

$$\rightarrow H^{i-2}(\text{coker}(f_{M, T})) \rightarrow \text{Hom}_{K^b(DF_K)}(M, T[i]) \rightarrow H^i(\text{Hom}^b(M, T)) \rightarrow H^{i-1}(\text{coker}(f_{M, T})) \rightarrow$$

Let I_T be the category whose objects are quasi-isomorphisms $s : T \rightarrow L$ in $K^b(DF_K)$ and whose morphisms are morphisms $L \rightarrow L'$ in $K^b(DF_K)$ compatible with s . Since $\text{inj} \lim_{I_T} \text{Hom}_{K^b(DF_K)}(M, L[i]) = \text{Hom}_{D(DF_K)}(M, T[i])$, it suffices to show that $\text{inj} \lim_{I_T} H^i(\text{Hom}^b(M, L)) = H^i(\text{Hom}^b(M, T))$ and that $\text{inj} \lim_{I_T} H^i(\text{coker}(f_{M, L})) = 0$. The first fact follows from Lemma 2.5 and the second one from the Lemma 2.8 below. \square

help

Lemma 2.8. *Let $u \in \text{Hom}_{G_K}^j(M_{\bar{K}}, T_{\bar{K}})$. There exists a complex $E \in C^b(DF_K)$ and a quasi-isomorphism $T \rightarrow E$ such that the image of u in the cokernel of the map f is zero.*

Proof. We will construct an extension

$$0 \rightarrow T \rightarrow E \rightarrow \text{Cone}(M \xrightarrow{\text{Id}} M)[-j-1] \rightarrow 0$$

in the category of filtered (φ, N, G_K) -modules. Since the category of admissible modules is closed under extension, E will be admissible. The underlying complex of K_0^{nr} -vector spaces is

$$E_0 := \text{Cone}(M_0[-j-1] \xrightarrow{(0, \text{Id})} T_0 \oplus M_0[-j-1])$$

The Frobenius, monodromy operator, and Galois action are defined on $E_0^{i+j} := T_0^{i+j} \oplus M_0^{i-1} \oplus M_0^i$ coordinatewise. The filtration on $E_K^{i+j} := E_0^{i+j} \otimes_{K_0^{\text{nr}}} \bar{K}$ is defined as

$$\begin{aligned} F^n E_K^{i+j} &= F^n T_K^j \oplus \{(u^i(x), 0, x) | x \in F^n M_K^i\} \\ &\quad \oplus \{(d_T(u^{i-1}(x)), -x, -d_M(x)) | x \in F^n M_K^{i-1}\} \end{aligned}$$

Now take $\xi = (0, 0, \text{Id}) + (u^i, 0, \text{Id}) \in \text{Hom}_{\varphi, N, G_K}^{\sharp}(M_0^i, E_0^{i+j}) \oplus \text{Hom}_{\text{dR}}(M_K^i, E_K^{i+j})$. We have $f(\xi) = (u^i, 0, 0)$, as wanted. \square

2.2. The category of p -adic Hodge complexes.

2.9. Let $V_{\bar{K}}^G$ be the category of \bar{K} -vector spaces with a smooth \bar{K} -linear action of G_K . It is a Grothendieck abelian category. We will consider the following functors:

- $F_{\text{dR}} : V_{\text{dR}}^K \rightarrow V_{\bar{K}}^G$, which to a filtered K -vector space (E, F^\bullet) associates the \bar{K} -vector space $E \otimes_K \bar{K}$ with its natural action of G_K .
- $F_0 : M_K(\varphi, N, G_K) \rightarrow V_{\bar{K}}^G$, which to a (φ, N, G_K) -module M associates the \bar{K} -vector space $M \otimes_{K_0^{\text{nr}}} \bar{K}$ whose G_K -action is induced by the given G_K -action on M .

Both functors are exact and monoidal. Note in particular that they induces functors on the respective categories of complexes which are dg-functors.

df:DpH

2.10. Let $\mathcal{D}^b(V_{\bar{K}}^G)$ and $D^b(V_{\bar{K}}^G)$ denote the bounded derived dg category and the bounded derived category of $V_{\bar{K}}^G$, respectively. We define the dg category \mathcal{D}_{pH} of p -adic Hodge complexes as the homotopy limit

$$\mathcal{D}_{pH} := \text{holim}(\mathcal{D}^b(M_K(\varphi, N, G_K)) \xrightarrow{F_0} \mathcal{D}^b(V_{\bar{K}}^G) \xleftarrow{F_{\text{dR}}} \mathcal{D}^b(V_{\text{dR}}^K))$$

We denote by D_{pH} the homotopy category of \mathcal{D}_{pH} . By [61, Def. 3.1], [13, 4.1], an object of \mathcal{D}_{pH} consists of objects $M_0 \in \mathcal{D}^b(M_K(\varphi, N, G_K))$, $M_K \in \mathcal{D}^b(V_{\text{dR}}^K)$, and a quasi-isomorphism

$$F_0(M_0) \xrightarrow{\alpha_M} F_{\text{dR}}(M_K)$$

in $\mathcal{D}(V_{\bar{K}}^G)$. We will denote the object above by $M = (M_0, M_K, a_M)$. The morphisms are given by the complex $\text{Hom}_{\mathcal{D}_{pH}}((M_0, M_K, a_M), (N_0, N_K, a_N))$:

$$\begin{aligned} (2.4) \quad & \text{Hom}_{\mathcal{D}_{pH}}^i((M_0, M_K, a_M), (N_0, N_K, a_N)) \\ &= \text{Hom}_{\mathcal{D}^b(M_K(\varphi, N, G_K))}^i(M_0, N_0) \oplus \text{Hom}_{\mathcal{D}^b(V_{\text{dR}}^K)}^i(M_K, N_K) \oplus \text{Hom}_{\mathcal{D}^b(V_{\bar{K}}^G)}^{i-1}(F_0(M_0), F_{\text{dR}}(N_K)) \end{aligned}$$

The differential is given by

$$d(a, b, c) = (da, db, dc + a_N F_0(a) - (-1)^i F_{\text{dR}}(b) a_M)$$

and the composition

hom1

$$(2.5) \quad \begin{aligned} & \text{Hom}_{\mathcal{D}_{pH}}((N_0, N_K, a_N), (T_0, T_K, a_T)) \otimes \text{Hom}_{\mathcal{D}_{pH}}((M_0, M_K, a_M), (N_0, N_K, a_N)) \\ & \rightarrow \text{Hom}_{\mathcal{D}_{pH}}((M_0, M_K, a_M), (T_0, T_K, a_T)) \end{aligned}$$

is given by

$$(a', b', c')(a, b, c) = (a'a, b'b, c'F_0(a) + F_{\text{dR}}(b')c)$$

It now follows easily that a (closed) morphism $(a, b, c) \in \text{Hom}_{\mathcal{D}_{pH}}((M_0, M_K, a_M), (N_0, N_K, a_N))$ is a quasi-isomorphism if and only so are the morphisms a and b (see [13, Lemma 4.2]).

By definition, we get a commutative square of dg categories over \mathbf{Q}_p :

$$(2.6) \quad \begin{array}{ccc} \mathcal{D}_{pH} & \xrightarrow{T_{\text{dR}}} & \mathcal{D}^b(V_{\text{dR}}^K) \\ T_0 \downarrow & & \downarrow F_{\text{dR}} \\ \mathcal{D}^b(M_K(\varphi, N, G_K)) & \xrightarrow{F_0} & \mathcal{D}^b(V_{\overline{K}}^G). \end{array}$$

Given a p -adic Hodge complex M , we will call $T_{\text{dR}}(M)$ (resp. $T_0(M)$) the *generic fiber* (resp. *special fiber*) of M . As pointed out above, a morphism f of p -adic Hodge complexes is a quasi-isomorphism if and only if $T_{\text{dR}}(f)$ and $T_0(f)$ are quasi-isomorphisms.

2.11. Let us recall that, since the category \mathcal{D}_{pH} is obtained by gluing, it has a canonical t -structure [35, Prop. 4.1.12]. We will denote by $\mathcal{D}_{pH, \leq 0}$ (resp. $\mathcal{D}_{pH, \geq 0}$) the full dg subcategory of \mathcal{D}_{pH} made of non-positive (resp. non negative) p -adic Hodge complexes. Let M be a p -adic Hodge complex. We define its non positive truncation $\tau_{\leq 0}(M)$ according to the following formula:

$$\tau_{\leq 0}(M) := (\tau_{\leq 0}M_0, \tau_{\leq 0}M_K, \tau_{\leq 0}a_M).$$

The functors F_{dR} and F_0 being exact, this is indeed a p -adic Hodge module. The non negative truncation is obtained using the same formula. According to this definition, we get a canonical morphism of p -adic Hodge complexes:

$$\tau_{\leq 0}(M) \rightarrow M$$

whose cone is positive. This is all we need to get that the pair $(\mathcal{D}_{pH, \leq 0}, \mathcal{D}_{pH, \geq 0})$ forms a t -structure on \mathcal{D}_{pH} .

Definition 2.12. The t -structure $(\mathcal{D}_{pH, \leq 0}, \mathcal{D}_{pH, \geq 0})$ defined above will be called the canonical t -structure on \mathcal{D}_{pH} .

2.13. Let $M \in C^b(MF_K(\varphi, N, G_K))$. Define $\theta(M) \in \mathcal{D}_{pH}$ to be the object

$$\theta(M) := (M_0, M_K, \text{Id}_M : M_{\overline{K}} \simeq M_{\overline{K}})$$

Since θ preserves quasi-isomorphisms, it induces a functor

$$\theta : \mathcal{D}^b(DF_K) \rightarrow \mathcal{D}_{pH}$$

This is a functor between dg categories compatible with the t -structures. Through this functor we can regard $MF_K(\varphi, N, G_K)$ as a subcategory of the heart of the t -structure on \mathcal{D}_{pH} .

heart1 **Lemma 2.14.** *The natural functor*

$$\theta : MF_K(\varphi, N, G_K) \rightarrow \mathcal{D}_{pH}^{\heartsuit}$$

is fully faithful.

Proof. Analogous to [35, Prop. 4.1.12], [59, 1.2.27]. \square

Definition 2.15. We will say that a strict p -adic Hodge complex M is *admissible* if its cohomology filtered (φ, N, G_K) -modules $H^n(M)$ are (weakly) admissible. Denote by $\mathcal{D}_{pH}^{\text{ad}}$ the full dg subcategory of \mathcal{D}_{pH} of admissible p -adic Hodge complexes.

The functor θ induces a canonical functor:

$$\theta : \mathcal{D}^b(DF_K) \rightarrow \mathcal{D}_{pH}^{\text{ad}, \heartsuit}.$$

heart2 **Lemma 2.16.** *The natural functor*

$$\theta : DF_K \xrightarrow{\sim} \mathcal{D}_{pH}^{\text{ad}, \heartsuit}$$

is an equivalence of abelian categories.

Proof. By definition, a strict p -adic Hodge complex M is in the heart of the t -structure if and only if M is isomorphic to $\tau_{\leq 0}\tau_{\geq 0}(M)$. According to the formula for this truncation, we get that M is isomorphic to an object \widetilde{M} such that \widetilde{M}_0 is a (φ, N, G_K) -module, \widetilde{M}_K is a filtered K -vector space, and one has a G_K -equivariant isomorphism

$$\widetilde{M}_0 \otimes_{K_0^{\text{nr}}} \overline{K} \simeq \widetilde{M}_K \otimes_K \overline{K}.$$

In particular, \widetilde{M}_0 has the structure of a filtered (φ, N, G_K) -module. Conversely, it is clear that a filtered (φ, N, G_K) -module induces a p -adic Hodge complex concentrated in degree 0. This proves our lemma. \square

thm1 **Theorem 2.17.** *The functor θ induces an equivalence of dg categories*

$$\theta : \mathcal{D}^b(DF_K) \xrightarrow{\sim} \mathcal{D}_{pH}^{\text{ad}}$$

Proof. Since, by Lemma 2.16, we have the equivalence of abelian categories

$$\theta : DF_K = \mathcal{D}^b(DF_K)^\heartsuit \xrightarrow{\sim} \mathcal{D}_{pH}^{\text{ad}, \heartsuit}$$

it suffices to show that, given two complexes M, M' of $C^b(MF_K^{\text{pst}})$, the functor θ induces a quasi-isomorphism:

$$\theta : \text{Hom}_{\mathcal{D}^b(DF_K)}(M, M') \rightarrow \text{Hom}_{\mathcal{D}_{pH}}(\theta(M), \theta(M'))$$

By (2.3) and Proposition 2.7, since $F_0(M_0) = F_{\text{dR}}(M_K) = M_{\overline{K}}, F_0(M'_0) = F_{\text{dR}}(M'_K) = M'_{\overline{K}}$, we have the following sequence of quasi-isomorphisms

$$\begin{aligned} \text{Hom}_{\mathcal{D}_{pH}}(\theta(M), \theta(M')) &= \text{Hom}_{\mathcal{D}_{pH}}((M_0, M_K, \text{Id}_M), (M'_0, M'_K, \text{Id}_{M'})) \\ &\simeq (\text{Hom}_{\mathcal{D}^b(M_K(\varphi, N, G_K))}(M_0, M'_0) \xrightarrow{F_0} \text{Hom}_{\mathcal{D}^b(V_{\overline{K}}^G)}(M_{\overline{K}}, M'_{\overline{K}}) \xleftarrow{F_{\text{dR}}} \text{Hom}_{\mathcal{D}^b(V_{\overline{K}}^K)}(M_K, M'_K)) \\ &\simeq (\text{Hom}_{\varphi, N, G_K}^\sharp(M_0, M'_0) \xrightarrow{F_0} \text{Hom}_{G_K}(M_{\overline{K}}, M'_{\overline{K}}) \xleftarrow{F_{\text{dR}}} \text{Hom}_{\text{dR}}(M_K, M'_K)) \\ &\simeq \text{Hom}_{\mathcal{D}^b(DF_K)}(M, M'). \end{aligned}$$

This concludes our proof. \square

2.3. The absolute p -adic Hodge cohomology.

2.18. Any potentially semi-stable p -adic representation is a p -adic Hodge complex. Therefore, we can define the Tate twist in \mathcal{D}_{pH} as follows: given any integer $r \in \mathbf{Z}$, we let $K(r)$ be the p -adic Hodge complex

$$K(-r) = (K_0^{\text{nr}}, K, \text{Id}_K : \overline{K} \xrightarrow{\sim} \overline{K})$$

that is equal to K_0^{nr} and K concentrated in degree 0; the Frobenius is $\varphi_{K(-r)}(a) = p^r \varphi(a)$, the Galois action is canonical and the monodromy operator is zero; the filtration is $F^i = K$ for $i \leq r$ and zero otherwise.

As usual, given any p -adic Hodge complex M , we put $M(r) := M \otimes K(r)$. In other words, twisting a p -adic Hodge complex r -times divides the Frobenius by p^r , leaves unchanged the monodromy operator, and shifts the filtration r -times.

ex1 **Example 2.19.** Given any p -adic Hodge complex M , by formula (2.5) and by (2.3), we have the quasi-isomorphism of complexes of \mathbf{Q}_p -vector spaces

$$\text{Hom}_{\mathcal{D}_{pH}}(K(0), M(r)) \simeq \text{Cone}(M_0^\sharp \oplus F^r M_K \xrightarrow{a_M - \text{can}} F_{\text{dR}}(M_K)^{G_K})[-1],$$

where M_0^\sharp is defined as the homotopy limit of the following diagram (we set $\varphi_i := \varphi/p^i$)

$$M_0^\sharp := \begin{array}{ccc} M_0^{G_K} & \xrightarrow{1-\varphi_r} & M_0^{G_K} \\ N \downarrow & & \downarrow N \\ M_0^{G_K} & \xrightarrow{1-\varphi_{r-1}} & M_0^{G_K} \end{array}$$

2.20. Let X be a variety over K . Consider the following complex in $\mathcal{D}_{pH}^{\text{ad}}$

$$\text{R}\Gamma_{pH}(X_{\overline{K}}, 0) := (\text{R}\Gamma_{\text{HK}}^B(X_{\overline{K}}), (\text{R}\Gamma_{\text{dR}}(X), F^\bullet), \text{R}\Gamma_{\text{HK}}^B(X_{\overline{K}}) \xrightarrow{\iota_{\text{dR}}} \text{R}\Gamma_{\text{dR}}(X_{\overline{K}}))$$

Here $\text{R}\Gamma_{\text{HK}}^B(X_{\overline{K}})$ is the (geometric) Beilinson-Hyodo-Kato cohomology [10], [48, 3.3]; by definition it is a bounded complex of (φ, N, G_K) -modules. The filtered complex $\text{R}\Gamma_{\text{dR}}(X)$ is the Deligne de Rham cohomology. The map ι_{dR} is the Beilinson-Hyodo-Kato map [10] that induces a quasi-isomorphism

$$\iota_{\text{dR}} : \text{R}\Gamma_{\text{HK}}^B(X_{\overline{K}}) \otimes_{K_0^{\text{nr}}} \overline{K} \xrightarrow{\sim} \text{R}\Gamma_{\text{dR}}(X_{\overline{K}}).$$

The comparison theorems of p -adic Hodge theory (proved in [30], [63], [51], [10], [14]) imply that the p -adic Hodge complex $\text{R}\Gamma_{pH}(X_{\overline{K}}, 0)$ is admissible.

We will denote by

$$\text{R}\Gamma_{pH}(X_{\overline{K}}, r) := \text{R}\Gamma_{pH}(X_{\overline{K}}, 0)(r) \in \mathcal{D}_{pH}^{\text{ad}}$$

the r 'th Tate twist of $\text{R}\Gamma_{pH}(X_{\overline{K}}, 0)$. We will call it *the geometric p -adic Hodge cohomology of X* . It is a dg \mathbf{Q}_p -algebra. Since the Beilinson-Hyodo-Kato map is a map of dg K_0^{nr} -algebras, the assignment $X \mapsto \text{R}\Gamma_{pH}(X_{\overline{K}}, r)$ is a presheaf of dg \mathbf{Q}_p -algebras on $\mathcal{V}ar_K$ and we also have the external product $\text{R}\Gamma_{pH}(X_{\overline{K}}, r) \otimes \text{R}\Gamma_{pH}(Y_{\overline{K}}, s)$ in $\mathcal{D}_{pH}^{\text{ad}}$.

Lemma 2.21 (Künneth formula). *The natural map*

$$\text{R}\Gamma_{pH}(X_{\overline{K}}, r) \otimes \text{R}\Gamma_{pH}(Y_{\overline{K}}, s) \xrightarrow{\sim} \text{R}\Gamma_{pH}(X_{\overline{K}} \times Y_{\overline{K}}, r + s)$$

is a quasi-isomorphism.

Proof. This follows easily from the Künneth formulas in the filtered de Rham cohomology and the Hyodo-Kato cohomology (use the Hyodo-Kato map to pass to de Rham cohomology). \square

Set

$$\begin{aligned} \text{R}\Gamma_{DF_K}(X_{\overline{K}}, r) &:= \theta^{-1} \text{R}\Gamma_{pH}(X_{\overline{K}}, r) \in \mathcal{D}^b(DF_K), \\ \text{R}\Gamma_{\text{pst}}(X_{\overline{K}}, r) &:= V_{\text{pst}} \theta^{-1} \text{R}\Gamma_{pH}(X_{\overline{K}}, r) \in \mathcal{D}^b(\text{Rep}_{\text{pst}}(G_K)). \end{aligned}$$

Lemma 2.22. *There exists a canonical quasi-isomorphism in $\mathcal{D}^b(\text{Rep}(G_K))$*

$$\text{R}\Gamma_{\text{pst}}(X_{\overline{K}}, r) \simeq \text{R}\Gamma_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r)).$$

Proof. To start, we note that we have the following commutative diagram of dg categories.

$$\begin{array}{ccc} \mathcal{D}^b(\text{Rep}_{\text{pst}}(G_K)) & \xrightarrow{\text{can}} & \mathcal{D}^b(\text{Rep}(G_K)) \\ \uparrow V_{\text{pst}} \quad \downarrow D_{\text{pst}} & & \downarrow \text{can} \\ \mathcal{D}^b(DF_K) & \xrightarrow{\theta} \mathcal{D}_{pH}^{\text{ad}} \xrightarrow{r_{\text{ét}}} & \mathcal{D}(\text{Spec}(K)_{\text{proét}}) \end{array}$$

Here the functor

$$r_{\text{ét}} : \mathcal{D}_{pH}^{\text{ad}} \rightarrow \mathcal{D}(\text{Spec}(K)_{\text{proét}})$$

associates to a p -adic Hodge complex $(M_0, M_K, a_M : F_0(M_0) \rightarrow F_{\text{dR}}(M_K))$ the complex

$$\begin{aligned} & [[M_0 \otimes_{K_0^{\text{nr}}} \mathbf{B}_{\text{st}}]^{\varphi=\text{Id}, N=0} \oplus F^0(M_K \otimes_K \mathbf{B}_{\text{dR}}) \xrightarrow{a_M \otimes \iota - \text{can} \otimes \iota} F_{\text{dR}}(M_K) \otimes_{\overline{K}} \mathbf{B}_{\text{dR}}] \\ & = [[M_0 \otimes_{K_0^{\text{nr}}} \mathbf{B}_{\text{st}}]^{\varphi=\text{Id}, N=0} \xrightarrow{a_M \otimes \iota} (F_{\text{dR}}(M_K) \otimes_{\overline{K}} \mathbf{B}_{\text{dR}}) / F^0] \end{aligned}$$

where $\iota : \mathbf{B}_{\text{st}} \hookrightarrow \mathbf{B}_{\text{dR}}$ is the canonical map of period rings. To see that the diagram commutes, recall that we have the fundamental exact sequence

$$(2.7) \quad 0 \rightarrow \mathbf{Q}_p \rightarrow \mathbf{B}_{\text{st}}^{\varphi=\text{Id}, N=0} \oplus F^r \mathbf{B}_{\text{dR}} \xrightarrow{\iota} \mathbf{B}_{\text{dR}} \rightarrow 0$$

It follows that, for $V \in \mathcal{D}^b(\mathrm{Rep}_{\mathrm{pst}}(G_K))$, we have a canonical morphism

$$\begin{aligned} V &\simeq [V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{st}}^{\varphi=\mathrm{Id}, N=0} \oplus V \otimes_{\mathbf{Q}_p} F^0 \mathbf{B}_{\mathrm{dR}} \xrightarrow{\mathrm{Id} \otimes \iota - \mathrm{can} \otimes \iota} V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}] \\ &\simeq [[V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{st}}]^{\varphi=\mathrm{Id}, N=0} \oplus V \otimes_{\mathbf{Q}_p} F^0 \mathbf{B}_{\mathrm{dR}} \xrightarrow{\mathrm{Id} \otimes \iota - \mathrm{can} \otimes \iota} V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}] \\ &\xrightarrow{(\rho_{\mathrm{pst}} \oplus \rho_{\mathrm{dR}}, \rho_{\mathrm{dR}})} [[D_{\mathrm{pst}}(V) \otimes_{K_0^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}}]^{\varphi=\mathrm{Id}, N=0} \oplus F^0(D_K(V) \otimes_K \mathbf{B}_{\mathrm{dR}}) \xrightarrow{a_M \otimes \iota - \mathrm{can} \otimes \iota} F_{\mathrm{dR}}(D_K(V)) \otimes_{\overline{K}} \mathbf{B}_{\mathrm{dR}}] \\ &\simeq r_{\mathrm{ét}} \theta D_{\mathrm{pst}}(V) \end{aligned}$$

Since the abstract period morphisms $\rho_{\mathrm{pst}}, \rho_{\mathrm{dR}}$ from (2.2) are isomorphisms, the above morphism is a quasi-isomorphism and we have the commutativity we wanted.

The above diagram gives us the first quasi-isomorphism in the following formula.

$$\mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, r) \simeq r_{\mathrm{ét}} \mathrm{R}\Gamma_{pH}(X_{\overline{K}}, r) \simeq \mathrm{R}\Gamma_{\mathrm{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r)).$$

It suffices now to prove the second quasi-isomorphism. But, we have

$$\mathrm{R}\Gamma_{pH}(X_{\overline{K}}, r) = (\mathrm{R}\Gamma_{\mathrm{HK}}^B(X_{\overline{K}}, r), (\mathrm{R}\Gamma_{\mathrm{dR}}(X), F^{\bullet+r}), \mathrm{R}\Gamma_{\mathrm{HK}}^B(X_{\overline{K}}, r) \otimes_{K_0^{\mathrm{nr}}} \overline{K} \xrightarrow{\iota_{\mathrm{dR}}} \mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K}})),$$

where we twisted the Beilinson-Hyodo-Kato cohomology to remember the Frobenius twist. Recall that Beilinson has constructed period morphisms (of dg-algebras) [9, 3.6], [10, 3.2]¹

$$\begin{aligned} \rho_{\mathrm{pst}} : \quad &\mathrm{R}\Gamma_{\mathrm{HK}}^B(X_{\overline{K}}) \otimes_{K_0^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}} \simeq \mathrm{R}\Gamma_{\mathrm{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{st}}, \\ \rho_{\mathrm{dR}} : \quad &\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K}}) \otimes_{\overline{K}} \mathbf{B}_{\mathrm{dR}} \simeq \mathrm{R}\Gamma_{\mathrm{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}, \end{aligned}$$

The first morphism is compatible with Frobenius, monodromy, and G_K -action; the second one - with filtration. These morphisms allow us to define a quasi-isomorphism

$$\beta : \quad r_{\mathrm{ét}} \mathrm{R}\Gamma_{pH}(X_{\overline{K}}, r) \simeq \mathrm{R}\Gamma_{\mathrm{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r))$$

in $\mathcal{D}(\mathrm{Spec}(K)_{\mathrm{proét}})$ as the composition

$$\begin{aligned} \beta : \quad &r_{\mathrm{ét}} \mathrm{R}\Gamma_{pH}(X_{\overline{K}}, r) = [[\mathrm{R}\Gamma_{\mathrm{HK}}^B(X_{\overline{K}}, r) \otimes_{K_0^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}}]^{\varphi=\mathrm{Id}, N=0} \xrightarrow{\iota_{\mathrm{dR}}} (\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K}}) \otimes_{\overline{K}} \mathbf{B}_{\mathrm{dR}}) / F^r] \\ &\xrightarrow{(\rho_{\mathrm{HK}}, \rho_{\mathrm{dR}})} [\mathrm{R}\Gamma_{\mathrm{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r)) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{st}}^{\varphi=\mathrm{Id}, N=0} \xrightarrow{\iota} (\mathrm{R}\Gamma_{\mathrm{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}) / F^r] \\ &\xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r)) \end{aligned}$$

Here the last quasi-isomorphism follows from the fundamental exact sequence (2.7). We are done. \square

geomsc

Remark 2.23. The geometric p -adic Hodge cohomology $\mathrm{R}\Gamma_{pH}(X_{\overline{K}}, r)$ we work with here is not the same as the geometric syntomic cohomology $\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\overline{K}, h}, r)$ defined in [48]. While the first one, by the above corollary, represents the étale cohomology $\mathrm{R}\Gamma_{\mathrm{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r))$, the second one represents only its piece, i.e., we have $\tau_{\leq r} \mathrm{R}\Gamma_{\mathrm{syn}}(X_{\overline{K}, h}, r) \simeq \tau_{\leq r} \mathrm{R}\Gamma_{\mathrm{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r))$.

construction

2.24. The p -adic absolute Hodge cohomology of X (also called *syntomic cohomology* of X if this does not cause confusion) is defined as

def1

$$(2.8) \quad \mathrm{R}\Gamma_{\mathcal{H}}(X, r) = \mathrm{R}\Gamma_{\mathrm{syn}}(X, r) := \mathrm{Hom}_{\mathcal{D}_{pH}}(K(0), \mathrm{R}\Gamma_{pH}(X_{\overline{K}}, r)).$$

By Theorem 2.17, we have

$$\begin{aligned} \mathrm{R}\Gamma_{\mathcal{H}}(X, r) &\simeq \mathrm{Hom}_{\mathcal{D}^b(D_{F_K})}(K(0), \mathrm{R}\Gamma_{D_{F_K}}(X_{\overline{K}}, r)) \\ &\simeq \mathrm{Hom}_{\mathcal{D}^b(\mathrm{Rep}_{\mathrm{pst}}(G_K))}(\mathbf{Q}_p, \mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, r)). \end{aligned}$$

The assignment $X \mapsto \mathrm{R}\Gamma_{\mathcal{H}}(X, r) = \mathrm{R}\Gamma_{\mathrm{syn}}(X, r)$ is a presheaf of dg \mathbf{Q}_p -algebras on $\mathcal{V}ar_K$.

compsynpH

Theorem 2.25. *Let $\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)$ be the syntomic cohomology defined in [48, 3.3]. There exists a natural quasi-isomorphism (in the classical derived category)*

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{syn}}(X, r), \quad r \geq 0.$$

¹We will be using consistently Beilinson's definition of the period maps. It is likely that the uniqueness criterium stated in [52] can be used to show that these maps coincide with the other existing ones.

Proof. Let $r \geq 0$. Recall that we have a natural quasi-isomorphism [48, Prop. 3.18]

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) \simeq \mathrm{Cone}(\mathrm{R}\Gamma_{\mathrm{HK}}^B(X)^{\varphi, N} \oplus F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X) \xrightarrow{\iota_{\mathrm{dR}} - \mathrm{can}} \mathrm{R}\Gamma_{\mathrm{dR}}(X))[-1],$$

where

$$\mathrm{R}\Gamma_{\mathrm{HK}}^B(X_h)^{\varphi, N} := \begin{bmatrix} \mathrm{R}\Gamma_{\mathrm{HK}}^B(X) \xrightarrow{1-\varphi_r} \mathrm{R}\Gamma_{\mathrm{HK}}^B(X) \\ \downarrow N \qquad \qquad \downarrow N \\ \mathrm{R}\Gamma_{\mathrm{HK}}^B(X) \xrightarrow{1-\varphi_{r-1}} \mathrm{R}\Gamma_{\mathrm{HK}}^B(X) \end{bmatrix}$$

and the complex $\mathrm{R}\Gamma_{\mathrm{HK}}^B(X)$ is the (arithmetic) Beilinson-Hyodo-Kato cohomology [10] that comes equipped with the Beilinson-Hyodo-Kato map $\iota_{\mathrm{dR}} : \mathrm{R}\Gamma_{\mathrm{HK}}^B(X) \rightarrow \mathrm{R}\Gamma_{\mathrm{dR}}(X)$ [48, 3.3].

Since $\mathrm{R}\Gamma_{\mathrm{HK}}^B(X) \simeq \mathrm{R}\Gamma_{\mathrm{HK}}^B(X_{\overline{K}})^{G_K}$ and $\mathrm{R}\Gamma_{\mathrm{dR}}(X) \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K}})^{G_K}$ by [48, Prop. 3.20], Example 2.19 and Theorem 2.17 yield

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) \simeq \mathrm{Hom}_{\mathcal{D}_{pH}}(K(0), \mathrm{R}\Gamma_{pH}(X_{\overline{K}}, r)) \simeq \mathrm{Hom}_{\mathcal{D}^b(D_{F_K})}(K(0), \mathrm{R}\Gamma_{D_{F_K}}(X_{\overline{K}}, r)) \simeq \mathrm{R}\Gamma_{\mathrm{syn}}(X, r),$$

as wanted. \square

Set $H_{\mathrm{syn}}^i(X, r) := H^i \mathrm{R}\Gamma_{\mathrm{syn}}(X, r)$.

Theorem 2.26. (1) *There is a functorial syntomic descent spectral sequence*

$$(2.9) \quad {}^{\mathrm{syn}}E^{i,j} := H_{\mathrm{st}}^i(G_K, H_{\mathrm{ét}}^j(X_{\overline{K}}, \mathbf{Q}_p(r))) \Rightarrow H_{\mathrm{syn}}^{i+j}(X, r),$$

where $H_{\mathrm{st}}^i(G_K, \cdot)$ is the group of (potentially) semistable extensions $\mathrm{Ext}_{\mathrm{Rep}_{\mathrm{pst}}(G_K)}^i(\mathbf{Q}_p, \cdot)$ as defined in [34, 1.19].

(2) *There is a functorial syntomic period morphism*

$$\rho_{\mathrm{syn}} : \mathrm{R}\Gamma_{\mathrm{syn}}(X, r) \rightarrow \mathrm{R}\Gamma_{\mathrm{ét}}(X, \mathbf{Q}_p(r)).$$

(3) *The syntomic descent spectral sequence is compatible with the Hochschild-Serre spectral sequence*

$$(2.10) \quad {}^{\mathrm{ét}}E_2^{i,j} = H^i(G_K, H_{\mathrm{ét}}^j(X_{\overline{K}}, \mathbf{Q}_p(r))) \Rightarrow H_{\mathrm{ét}}^{i+j}(X, \mathbf{Q}_p(r)).$$

More specifically, there is a natural map ${}^{\mathrm{syn}}E_2^{i,j} \rightarrow {}^{\mathrm{ét}}E_2^{i,j}$ that is compatible with the syntomic period map ρ_{syn} .

Proof. From the definition (2.8) of $\mathrm{R}\Gamma_{pH}(X_{\overline{K}}, r)$ we obtain the following spectral sequence

$$E_2^{i,j} = \mathrm{Ext}_{\mathrm{Rep}_{\mathrm{pst}}(G_K)}^i(\mathbf{Q}_p, H^j \mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, r)) \Rightarrow H^{i+j} \mathrm{R}\Gamma_{\mathrm{syn}}(X, r)$$

Since, by Lemma 2.22, we have $\mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, r) \simeq \mathrm{R}\Gamma_{\mathrm{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r))$, the first statement of our theorem follows.

We define the syntomic period map $\rho_{\mathrm{syn}} : \mathrm{R}\Gamma_{\mathrm{syn}}(X, r) \rightarrow \mathrm{R}\Gamma_{\mathrm{ét}}(X, \mathbf{Q}_p(r))$ as the composition

$$\begin{aligned} \rho_{\mathrm{syn}} : \mathrm{R}\Gamma_{\mathrm{syn}}(X, r) &= \mathrm{Hom}_{\mathcal{D}_{pH}}(K(0), \mathrm{R}\Gamma_{pH}(X_{\overline{K}}, r)) \xrightarrow{r_{\mathrm{ét}}} \mathrm{Hom}_{\mathcal{D}(\mathrm{Spec}(K)_{\mathrm{proét}})}(\mathbf{Q}_p, r_{\mathrm{ét}} \mathrm{R}\Gamma_{pH}(X_{\overline{K}}, r)) \\ &\xrightarrow{\beta} \mathrm{Hom}_{\mathcal{D}(\mathrm{Spec}(K)_{\mathrm{proét}})}(\mathbf{Q}_p, \mathrm{R}\Gamma_{\mathrm{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r))) = \mathrm{R}\Gamma_{\mathrm{ét}}(X, \mathbf{Q}_p(r)). \end{aligned}$$

The second statement of the theorem follows.

Finally, since the Hochschild-Serre spectral sequence

$${}^{\mathrm{ét}}E_2^{i,j} := H^i(G_K, H^j(X_{\overline{K}}, \mathbf{Q}_p(r))) \Rightarrow H^{i+j}(X, \mathbf{Q}_p(r))$$

can be identified with the spectral sequence

$${}^{\mathrm{ét}}E_2^{i,j} := H^i(\mathrm{Spec}(K)_{\mathrm{proét}}, H^j(X_{\overline{K}}, \mathbf{Q}_p(r))) \Rightarrow H^{i+j}(X, \mathbf{Q}_p(r))$$

we get that the syntomic descent spectral sequence is compatible with the Hochschild-Serre spectral sequence via the map ρ_{syn} , as wanted. \square

Remark 2.27. The above theorem gives an alternative construction of the syntomic descent spectral sequence from [48, 4.1] (that construction used the geometric syntomic cohomology mentioned in Remark 2.23) and an alternative proof of its compatibility with the Hochschild-Serre spectral sequence [48, Theorem 4.8]. In the present approach the syntomic descent spectral sequence is a genuine descent spectral sequence: from geometric étale cohomology to syntomic cohomology. In the approach of [48] this sequence appears as a piece of a larger descent spectral sequence that remains to be understood.

compact

Remark 2.28. In everything above, the variety X can be replaced by a finite simplicial scheme or a finite diagram of schemes. In particular, we obtain statements about cohomology with compact support: use resolutions of singularities to get a compactification of the variety with a divisor with normal crossing at infinity and then represent cohomology with compact support as a cohomology of a finite simplicial scheme built from the closed strata. In particular, we get the syntomic descent spectral sequence with compact support:

$$\mathrm{syn},c E_2^{i,j} := H_{\mathrm{st}}^i(G_K, H_{\mathrm{ét},c}^j(X_{\overline{K}}, \mathbf{Q}_p(r))) \Rightarrow H_{\mathrm{syn},c}^{i+j}(X, r)$$

that is compatible with the Hochschild-Serre spectral sequence for étale cohomology with compact support.

Corollary 2.29. *For X smooth and proper over K , the syntomic descent spectral sequence (2.9) degenerates at E_2 .*

Proof. The argument proceeds along standard lines [24, Thm 1.5]. First, we treat the case of X smooth and projective, of equal dimension d . Recall that we have the Hard Lefschetz Theorem [25, Thm 4.1.1]: If $L \in H^2(X_{\overline{K}}, \mathbf{Q}_p(1))$ is the class of a hyperplane, then for $i \leq d$, the map $L^i : H^{d-i}(X_{\overline{K}}, \mathbf{Q}_p) \rightarrow H^{d+i}(X_{\overline{K}}, \mathbf{Q}_p(i))$, $a \mapsto a \cup L^i$, is an isomorphism. This gives us the Lefschetz primitive decomposition

LPD

$$(2.11) \quad H^i(X_{\overline{K}}, \mathbf{Q}_p(r)) = \bigoplus_{k \geq 0} L^k H_{\mathrm{prim}}^{i-2k}(X_{\overline{K}}, \mathbf{Q}_p(r-k)).$$

Take $s \geq 2$. Assume that the differentials of our spectral sequence $d_2 = \dots = d_{s-1} = 0$. We want to show that $d_s = 0$. Note that Hard Lefschetz gives us that the differentials

$$d_s : H_{\mathrm{st}}^j(G_K, H_{\mathrm{prim}}^{i-2k}(X_{\overline{K}}, \mathbf{Q}_p(r-k))) \rightarrow H_{\mathrm{st}}^j(G_K, H^{i-2k}(X_{\overline{K}}, \mathbf{Q}_p(r-k)))$$

are trivial and hence that so are the differentials

$$d_s : H_{\mathrm{st}}^j(G_K, L^k H_{\mathrm{prim}}^i(X_{\overline{K}}, \mathbf{Q}_p(r))) \rightarrow H_{\mathrm{st}}^j(G_K, H^i(X_{\overline{K}}, \mathbf{Q}_p(r))).$$

By (2.11), this gives that $d_s = 0$, as wanted.

Remark 2.30. In fact, we have the Decomposition Theorem, i.e., there is a natural quasi-isomorphism in $D^b(\mathrm{Rep}_{\mathrm{pst}}(G_K))$

$$\bigoplus_i H^i(X_{\overline{K}}, \mathbf{Q}_p)[-i] \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, \mathbf{Q}_p)$$

For a general smooth and proper variety X , we first use Chow Lemma and resolution of singularities to find a birational proper map $f : Y \rightarrow X$ from a smooth and projective variety Y over K . It suffices now to note that the maps $f^* : H^i(X_{\overline{K}}, \mathbf{Q}_p) \hookrightarrow H^i(Y_{\overline{K}}, \mathbf{Q}_p)$, $i \geq 0$, are injective to get the degeneration we wanted. \square

3. A p -ADIC ABSOLUTE HODGE COHOMOLOGY, II: BEILINSON'S DEFINITION

In this section we will describe the definition of p -adic absolute Hodge cohomology due to Beilinson [11]. Beilinson associates to any variety over K a canonical complex of potentially semistable representations of G_K representing the geometric étale cohomology of the variety as a Galois module. Then he defines p -adic absolute Hodge cohomology of this variety as the derived Hom in the category of potentially semistable representations from the trivial representation to this complex.

3.1. Potentially semistable complex of a variety.

3.1.1. *Potentially semistable cellular complexes.* The Basic Lemma of Beilinson [7, Lemma 3.3] allows one, in analogy with the cellular complex for CW -complexes, to associate a canonical complex of potentially semistable representations of G_K to any affine variety over K . Recall that the cellular complex associated to a CW -complex X is a complex of singular homology groups

$$(3.1) \quad \dots \rightarrow H_2^B(X^2, X^1) \xrightarrow{d_2} H_1^B(X^1, X^0) \xrightarrow{d_1} H_0^B(X^0, \emptyset) \xrightarrow{d_0} 0$$

where X^j denotes the j -skeleton of X . The homology of the above complex computes the singular homology of X : we have $H_j^B(X^j/X^{j-1}) \simeq H_j^B(\vee_{|I|=j} S^j) \simeq \sum_{i \in I} e_i \mathbf{Z}$, I being the index set of j -cells in X .

We will briefly sketch the construction of potentially semistable (cohomological) cellular complexes and we refer interested reader for details to [42], [53], [38].

Definition 3.1. (1) A *pair* is a triple (X, Y, n) , for a closed K -subvariety $Y \subset X$ of a K -variety X and an integer n .

(2) Pair (X, Y, n) is called a *good pair* if the relative geometric étale cohomology

$$H^j(X_{\overline{K}}, Y_{\overline{K}}, \mathbf{Q}_p) = 0, \quad \text{unless } j \neq n.$$

(3) A good pair is called *very good* if X is affine and $X \setminus Y$ is smooth and either X is of dimension n and Y of dimension $n - 1$ or $X = Y$ is of dimension less than n .

Lemma 3.2. (*Basic Lemma*) *Let X be an affine variety over K and let $Z \subset X$ be a closed subvariety such that $\dim(Z) < \dim(X)$. Then there is a closed subvariety $Y \supset Z$ such that $\dim(Y) < \dim(X)$ and (X, Y, n) , $n := \dim(X)$, is a good pair, i.e.,*

$$H^j(X_{\overline{K}}, Y_{\overline{K}}, \mathbf{Q}_p) = 0, \quad j \neq n.$$

Moreover, $X \setminus Y$ can be chosen to be smooth.

Proof. See [7, Lemma 3.3] (a result in any characteristic), [53], [39, 7]. \square

Corollary 3.3. (1) *Every affine variety X over K has a cellular stratification*

$$F_{\bullet} X : \quad \emptyset = F_{-1} X \subset F_0 X \subset \dots \subset F_{d-1} X \subset F_d X = X$$

That is, a stratification by closed subvarieties such that the triple $(F_j X, F_{j-1} X, j)$ is very good.

(2) *Celullar stratifications of X form a filtered system.*

(3) *Let $f : X \rightarrow Y$ be a morphism of affine varieties over K . Let $F_{\bullet} X$ be a cellular stratification on X . Then there exists a cellular stratification $F_{\bullet} Y$ such that $f(F_i X) \subset F_i Y$.*

Proof. See Corollary D.11, Corollary D.12 in [38]. \square

Having the above facts it is easy to associate a potentially semistable analog of the cellular complex (3.1) to an affine variety X over K [38, Appendix D]. We just pick a cellular stratification

$$F_{\bullet} X : \quad \emptyset = F_{-1} X \subset F_0 X \subset \dots \subset F_{d-1} X \subset F_d X = X$$

and take the complex

$$\mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, F_{\bullet} X) := 0 \rightarrow H^0(F_0 X_{\overline{K}}, \mathbf{Q}_p) \rightarrow \dots \rightarrow H^j(F_j X_{\overline{K}}, F_{j-1} X_{\overline{K}}, \mathbf{Q}_p) \xrightarrow{d_j} H^{j+1}(F_{j+1} X_{\overline{K}}, F_j X_{\overline{K}}, \mathbf{Q}_p) \xrightarrow{d_{j+1}} \dots \rightarrow H^d(X_{\overline{K}}, F_{d-1} X_{\overline{K}}, \mathbf{Q}_p) \rightarrow 0$$

This is a complex of Galois modules that, by p -adic comparison theorems, are potentially semistable. To get rid of the choice we take the homotopy colimit over all cellular stratifications, i.e., we set

$$\mathrm{R}\Gamma_{\mathrm{pst}}^T(X_{\overline{K}}) := \mathrm{hocolim}_{F_{\bullet} X} \mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, F_{\bullet} X)$$

It is a complex in $\mathcal{D}(\mathrm{Ind} - \mathrm{Rep}_{\mathrm{pst}}(G_K))$ whose cohomology groups are in $\mathrm{Rep}_{\mathrm{pst}}(G_K)$ hence we can think of it as being in $\mathcal{D}(\mathrm{Rep}_{\mathrm{pst}}(G_K))$.

The complex $\mathrm{R}\Gamma_{\mathrm{pst}}^T(X_{\overline{K}})$ computes the étale cohomology groups $H^*(X_{\overline{K}}, \mathbf{Q}_p)$ as Galois modules. More precisely, we have the following proposition.

prop1

Proposition 3.4. ([42, Prop. 2.1])

- (1) Let $F_\bullet X$ be a cellular stratification of X . There is a natural quasi-isomorphism

$$\kappa_{(X, F_\bullet X)} : \mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, F_\bullet X) \simeq \mathrm{R}\Gamma_{\mathrm{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$$

that is compatible with the action of G_K .

- (2) Let $f : Y \rightarrow X$ be a map of affine schemes and let $F_\bullet Y$ be a cellular stratification of Y such that, for all i , $F_i Y \subset F_i X$. Then the following diagram commutes (in the dg derived category)

$$\begin{array}{ccc} \mathrm{R}\Gamma_{\mathrm{pst}}(Y_{\overline{K}}, F_\bullet Y) & \xrightarrow[\sim]{\kappa_{(Y, F_\bullet Y)}} & \mathrm{R}\Gamma_{\mathrm{ét}}(Y_{\overline{K}}, \mathbf{Q}_p) \\ f^* \uparrow & & \uparrow f^* \\ \mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, F_\bullet X) & \xrightarrow[\sim]{\kappa_{(X, F_\bullet X)}} & \mathrm{R}\Gamma_{\mathrm{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \end{array}$$

- (3) There exists a natural quasi-isomorphism

$$\kappa_X : \mathrm{R}\Gamma_{\mathrm{pst}}^T(X_{\overline{K}}) \simeq \mathrm{R}\Gamma_{\mathrm{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$$

that is compatible with the action of G_K .

Proof. We have the following commutative diagram of Galois equivariant morphisms

$$\begin{array}{ccccccc} H^0(F_0 X_{\overline{K}}, \mathbf{Q}_p) & \rightarrow & \cdots & \longrightarrow & H^k(F_k X_{\overline{K}}, F_{k-1} X_{\overline{K}}, \mathbf{Q}_p) & \longrightarrow & \cdots & \longrightarrow & H^d(X_{\overline{K}}, F_{d-1} X_{\overline{K}}, \mathbf{Q}_p) \\ \downarrow \wr & & & & \downarrow \wr & & & & \downarrow \wr \\ \mathrm{R}\Gamma_{\mathrm{ét}}(F_0 X_{\overline{K}}, \mathbf{Q}_p) & \rightarrow & \cdots & \rightarrow & [\mathrm{R}\Gamma_{\mathrm{ét}}(F_k X_{\overline{K}}, \mathbf{Q}_p) \rightarrow \mathrm{R}\Gamma_{\mathrm{ét}}(F_{k-1} X_{\overline{K}}, \mathbf{Q}_p)][k] & \rightarrow & \cdots & \rightarrow & [\mathrm{R}\Gamma_{\mathrm{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \rightarrow \mathrm{R}\Gamma_{\mathrm{ét}}(F_{d-1} X_{\overline{K}}, \mathbf{Q}_p)][d] \\ \downarrow & & & & \downarrow & & & & \downarrow \\ 0 & \rightarrow & \cdots & \rightarrow & 0 & \rightarrow & \cdots & \rightarrow & \mathrm{R}\Gamma_{\mathrm{ét}}(X_{\overline{K}}, \mathbf{Q}_p)[d] \end{array}$$

Here the square brackets denote homotopy limits. The first vertical maps are the truncations $\tau_{\leq d} \tau_{\geq d}$. We obtain the map $\kappa_{(X, F_\bullet X)}$ from the first statement of the proposition by taking homotopy limits of the rows of the diagram. Second statement is now clear. The third one is an immediate corollary of Proposition 3.4 and Corollary 3.3. \square

pstB

3.1.2. *Potentially semistable complex of a variety.* For a general variety X over K , one (Zariski) covers it with (rigidified) affine varieties defined over K , takes the associated Čech covering, and applies the above construction to each level of the covering [38, D.5-D.10]. Then, to make everything canonical, one goes to limit over such coverings.

Proposition 3.4 implies now the following result [38, Prop. D.3].

Theorem 3.5. *Let X be a variety over K . There is a canonical complex $\mathrm{R}\Gamma_{\mathrm{pst}}^B(X_{\overline{K}}) \in \mathcal{D}^b(\mathrm{Rep}_{\mathrm{pst}})$ which represents the étale cohomology $\mathrm{R}\Gamma_{\mathrm{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ of $X_{\overline{K}}$ together with the action of G_K , i.e., there is a natural quasi-isomorphism*

$$\kappa_X : \mathrm{R}\Gamma_{\mathrm{pst}}^B(X_{\overline{K}}) \simeq \mathrm{R}\Gamma_{\mathrm{ét}}(X_{\overline{K}}, \mathbf{Q}_p),$$

that is compatible with the action of G_K .

3.2. **Beilinson's p -adic absolute Hodge cohomology.** Beilinson [11] uses the above construction of the potentially semistable complexes to define his syntomic complexes.

Definition 3.6. ([11]) Let X be a variety over K , $r \in \mathbf{Z}$. Set $\mathrm{R}\Gamma_{\mathrm{pst}}^B(X_{\overline{K}}, \mathbf{Q}_p(r)) := \mathrm{R}\Gamma_{\mathrm{pst}}^B(X_{\overline{K}})(r)$ and $\mathrm{R}\Gamma_{\mathcal{H}}^B(X, r) = \mathrm{R}\Gamma_{\mathrm{syn}}^B(X, r) := \mathrm{Hom}_{\mathcal{D}^b(\mathrm{Rep}_{\mathrm{pst}}(G_K))}(\mathbf{Q}_p, \mathrm{R}\Gamma_{\mathrm{pst}}^B(X_{\overline{K}}, \mathbf{Q}_p(r)))$, $H_{\mathrm{syn}}^i(X, r) := H^i \mathrm{R}\Gamma_{\mathrm{syn}}^B(X, r)$.

Immediately from this definition we obtain that

- (1) For $X = \mathrm{Spec}(K)$, we have $\mathrm{R}\Gamma_{\mathrm{syn}}^B(X, r) = \mathrm{Hom}_{\mathcal{D}^b(\mathrm{Rep}_{\mathrm{pst}}(G_K))}(\mathbf{Q}_p, \mathbf{Q}_p(r))$.

(2) There is a natural syntomic descent spectral sequence

$$(3.2) \quad \text{syn} E_2^{i,j} := H_{\text{st}}^i(G_K, H^j(X_{\overline{K}}, \mathbf{Q}_p(r))) \Rightarrow H_{\text{syn}}^{i+j}(X, r)$$

(3) We have a natural map

$$\kappa_X : \text{R}\Gamma_{\text{syn}}^B(X, r) \rightarrow \text{R}\Gamma_{\text{ét}}(X, \mathbf{Q}_p(r))$$

defined as the composition

$$\begin{aligned} \text{R}\Gamma_{\text{syn}}^B(X, r) &= \text{Hom}_{\mathcal{D}^b(\text{Rep}_{\text{pst}}(G_K))}(\mathbf{Q}_p, \text{R}\Gamma_{\text{pst}}^B(X_{\overline{K}}, \mathbf{Q}_p(r))) \rightarrow \text{Hom}_{\mathcal{D}^b(\text{Spec}(K)_{\text{proét}})}(\mathbf{Q}_p, \text{R}\Gamma_{\text{pst}}^B(X_{\overline{K}}, \mathbf{Q}_p(r))) \\ &\xrightarrow{\kappa_X} \text{Hom}_{\mathcal{D}^b(\text{Spec}(K)_{\text{proét}})}(\mathbf{Q}_p, \text{R}f_*^X \mathbf{Q}_p(r)) = \text{R}\Gamma_{\text{ét}}(X, \mathbf{Q}_p(r)) \end{aligned}$$

It follows that the syntomic descent spectral sequence is compatible with the Hochschild-Serre spectral sequence.

3.3. Comparison of the two constructions of syntomic cohomology. We will show now that the syntomic complexes defined in [48] and by Beilinson are naturally quasi-isomorphic.

Corollary 3.7. (1) *There is a canonical quasi-isomorphism in $\mathcal{D}^b(\text{Rep}_{\text{pst}}(G_K))$*

$$\text{R}\Gamma_{\text{pst}}(X_{\overline{K}}, r) \xrightarrow{\sim} \text{R}\Gamma_{\text{pst}}^B(X_{\overline{K}}, \mathbf{Q}_p(r)).$$

(2) *There is a canonical quasi-isomorphism*

$$\text{R}\Gamma_{\text{syn}}^B(X, r) \simeq \text{R}\Gamma_{\text{syn}}(X, r), \quad r \in \mathbf{Z}.$$

It is compatible with the syntomic descent spectral sequences (2.10) and (3.2).

Proof. The second statement follows immediately from the first one. To prove the first statement, consider the complex $\text{R}\Gamma_{pH}^B(X_{\overline{K}}, r)$ in $\mathcal{D}^b(DF_K)$ defined as in Proposition 3.4 but starting with $\text{R}\Gamma_{pH}(X_{\overline{K}}, r)$ instead of $\text{R}\Gamma_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r))$. This is possible since, for a good pair (X, Y, j) , we have

$$\text{R}\Gamma_{pH}^B(X_{\overline{K}}, Y_{\overline{K}}, r) \simeq (H_{\text{HK}}^j(X_{\overline{K}}, Y_{\overline{K}}, r), (H_{\text{dR}}^j(X, Y), F^{\bullet+r}), H_{\text{HK}}^j(X_{\overline{K}}, Y_{\overline{K}}) \xrightarrow{\iota_{\text{dR}}} H_{\text{dR}}^j(X_{\overline{K}}, Y_{\overline{K}})),$$

and, by p -adic comparison theorems, this is an element of DF_K . We get a functorial quasi-isomorphism in $\mathcal{D}^b(DF_K)$:

$$\kappa_X : \text{R}\Gamma_{pH}^B(X_{\overline{K}}, r) \simeq \text{R}\Gamma_{DF_K}(X_{\overline{K}}, r).$$

For good pairs (X, Y, j) , the Beilinson period maps $\rho_{\text{HK}}, \rho_{\text{dR}}$ [9, 3.6], [10, 3.2] induce the period isomorphism $V_{\text{pst}} \text{R}\Gamma_{pH}^B(X_{\overline{K}}, Y_{\overline{K}}, r) \xrightarrow{\sim} H^j(X_{\overline{K}}, Y_{\overline{K}}, \mathbf{Q}_p(r))$. This period map lifts to a period map

$$V_{\text{pst}} \text{R}\Gamma_{pH}^B(X_{\overline{K}}, r) \xrightarrow{\sim} \text{R}\Gamma_{\text{pst}}^B(X_{\overline{K}}, \mathbf{Q}_p(r)).$$

We define the map $\text{R}\Gamma_{\text{pst}}(X_{\overline{K}}, r) \xrightarrow{\sim} \text{R}\Gamma_{\text{pst}}^B(X_{\overline{K}}, \mathbf{Q}_p(r))$ as the following composition

$$\text{R}\Gamma_{\text{pst}}(X_{\overline{K}}, r) \xrightarrow{\kappa_X^{-1}} V_{\text{pst}} \text{R}\Gamma_{pH}^B(X_{\overline{K}}, r) \simeq \text{R}\Gamma_{\text{pst}}^B(X_{\overline{K}}, \mathbf{Q}_p(r)).$$

□

4. p -ADIC REALIZATIONS OF MOTIVES

4.1. p -adic realizatons of Nori's motives. We start with a quick review of Nori's motives. We follow [38], [43], [5], and [2, 2].

Take an embedding $K \hookrightarrow \mathbf{C}$ and a field $F \supset \mathbf{Q}$. A diagram Δ is a directed graph. A representation $T : \Delta \rightarrow V_F$ assigns to every vertex in Δ an object in V_F and to every edge e from v to v' a homomorphism $T(e) : T(v) \rightarrow T(v')$. Let $\mathcal{C}(\Delta, T)$ be its associated diagram category [43, Thm 41], [2, 2.1]: the category of finite dimensional right $\text{End}^{\vee}(T)$ -comodules. It is the universal abelian category together with a unique representation $\tilde{T} : \Delta \rightarrow \mathcal{C}(\Delta, T)$ and a faithful, exact, F -linear functor $T : \mathcal{C}(\Delta, T) \rightarrow V_F$ extending the original representation T . If Δ is an abelian category then we have an equivalence $\Delta \simeq \mathcal{C}(\Delta, T)$.

More specifically we have the following result of Nori.

Nori1 **Proposition 4.1.** (Nori, [2, Cor. 2.2.10], [2, Cor. 2.2.11])

- (1) Let \mathcal{R} be an F -linear abelian category with a faithful exact functor $\rho : \mathcal{R} \rightarrow V_F$. Assume that the representation $T : \Delta \rightarrow V_F$ factors, up to natural equivalence, as $T_1\rho$. Let \mathcal{A} be an F -linear abelian category equipped with a faithful exact functor $U : \mathcal{A} \rightarrow \mathcal{R}$. If $G : \Delta \rightarrow \mathcal{A}$ is a morphism of directed graphs such that T_1 is equivalent to UG , then there exist functors $\mathcal{C}(\Delta, T) \rightarrow \mathcal{R}$, $\tilde{G} : \mathcal{C}(\Delta, T) \rightarrow \mathcal{A}$ such that the following diagram

$$\begin{array}{ccc}
 \Delta & \xrightarrow{G} & \mathcal{A} \\
 \downarrow \tilde{T} & \searrow \tilde{G} & \downarrow U \\
 \mathcal{C}(\Delta, T) & \xrightarrow{\tilde{G}} & \mathcal{R} \\
 & \searrow T & \downarrow \rho \\
 & & V_F
 \end{array}$$

commutes up to natural equivalence.

- (2) For a commutative (up to natural equivalence) diagram

$$\begin{array}{ccc}
 \Delta & \xrightarrow{G} & \mathcal{A} \\
 \downarrow \pi & \searrow T & \downarrow U \\
 \Delta' & \xrightarrow{G'} & \mathcal{A}' \\
 & \searrow T' & \downarrow U' \\
 & & V_F
 \end{array}$$

we have a commutative (up to natural equivalence) diagram

$$\begin{array}{ccc}
 \mathcal{C}(\Delta, T) & \xrightarrow{\tilde{G}} & \mathcal{A} \\
 \downarrow \pi & & \downarrow \\
 \mathcal{C}(\Delta', T') & \xrightarrow{\tilde{G}'} & \mathcal{A}'
 \end{array}$$

Example 4.2. The following diagrams appear in the construction of Nori's motives.

- (1) The diagram Δ^{eff} of effective pairs consists of pairs (X, Y, i) and two types of edges:
 - (a) (functoriality) for every morphism $f : X \rightarrow X'$, with $f(Y) \subset Y'$, an edge $f^* : (X', Y', i) \rightarrow (X, Y, i)$.
 - (b) (coboundary) for every chain $X \supset Y \supset Z$ of closed K -subvarieties of X , an edge $\partial : (Y, Z, i) \rightarrow (X, Y, i + 1)$.
- (2) The diagram Δ_g^{eff} (resp. Δ_{vg}^{eff}) of effective good (resp. of effective very good) pairs is the full subdiagram of Δ^{eff} with vertices good (resp. very good) pairs (X, Y, i) .
- (3) The diagrams Δ of pairs, Δ_g of good pairs, and Δ_{vg} of very good pairs are obtained by localization with respect to the pair $(\mathbb{G}_m, \{1\}, 1)$ [38, B.18].

Let $H^* : \Delta_g \rightarrow V_F$ be the representation which assigns to (X, Y, i) the relative singular cohomology $H^i(X(\mathbf{C}), Y(\mathbf{C}), F)$.

Definition 4.3. The category of (reps. effective) Nori motives $\text{MM}(K)_F$ (resp. $\text{EMM}(K)_F$) is defined as the diagram category $\mathcal{C}(\Delta_g, H^*)$ (resp. $\mathcal{C}(\Delta_g^{\text{eff}}, H^*)$). For a good pair (X, Y, i) , we denote by $H_{\text{mot}}^i(X, Y)$ the object of $\text{EMM}(K)_F$ (resp. $\text{MM}(K)_F$) corresponding to it and we define the Tate object as

$$1(-1) := H_{\text{mot}}^1(\mathbb{G}_{m, K}, \{1\}) \in \text{EMM}(K)_F, \quad 1(-n) := 1(-1)^{\otimes n}.$$

We have [38, Thm 1.6, Cor. 1.7]

- $\text{EMM}(K)_F \simeq \text{EMM}(K)_{\mathbf{Q}} \otimes_{\mathbf{Q}} F$ and $\text{MM}(K)_F \simeq \text{MM}(K)_{\mathbf{Q}} \otimes_{\mathbf{Q}} F$.

- As an abelian category $\text{EMM}(K)_F$ is generated by Nori motives of the form $H_{\text{mot}}^i(X, Y)$ for good pairs (X, Y, i) ; every object of $\text{EMM}(K)_F$ is a subquotient of a finite direct sum of objects of the form $H_{\text{mot}}^i(X, Y)$.
- $\text{EMM}(K)_F \subset \text{MM}(K)_F$ are commutative tensor categories.
- $\text{MM}(K)_F$ is obtained from $\text{EMM}(K)_F$ by \otimes -inverting $1(-1)$.
- The diagram categories of Δ^{eff} and of Δ_{vg}^{eff} with respect to singular cohomology with coefficients in F are equivalent to $\text{EMM}(K)_F$ as abelian categories. The diagram categories of Δ and of Δ_{vg} are equivalent to $\text{MM}(K)_F$.² In particular, any pair (X, Y, i) defines a Nori motive $H_{\text{mot}}^i(X, Y)$.
- Nori shows that these categories are independent of the embedding $K \hookrightarrow \mathbf{C}$.

From the universal property of the category $\text{EMM}(K)_F$ it is easy to construct realizations. We will describe the ones coming from p -adic Hodge Theory.

constr1 **Construction 4.4.** (Galois realization) Consider the map $\Delta^{\text{eff}} \rightarrow \text{Rep}(G_K)$:

$$(X, Y, i) \mapsto H^i(X_{\overline{K}}, Y_{\overline{K}}, \mathbf{Q}_p).$$

We have $H^i(X_{\overline{K}}, Y_{\overline{K}}, \mathbf{Q}_p) \simeq H^i(X(\mathbf{C}), Y(\mathbf{C}), \mathbf{Q}_p)$. Thus, by Theorem 4.1, we obtain an extension which is the exact étale realization functor

$$\text{R}_{\text{ét}} : \text{EMM}(K)_{\mathbf{Q}_p} \rightarrow \text{Rep}(G_K).$$

Note that $\text{R}_{\text{ét}}(1(-1)) = H^1(\mathbb{G}_{m, \overline{K}}, \{1\}, \mathbf{Q}_p) = \mathbf{Q}_p(-1)$. Hence the functor $\text{R}_{\text{ét}}$ lifts to $\text{MM}(K)_{\mathbf{Q}_p}$.

In analogous way we obtain the exact potentially semistable realization

$$\text{R}_{\text{pst}} : \text{MM}(K)_{\mathbf{Q}_p} \rightarrow \text{Rep}_{\text{pst}}(G_K).$$

It factors $\text{R}_{\text{ét}}$ via the natural functor $\text{Rep}_{\text{pst}}(G_K) \rightarrow \text{Rep}(G_K)$.

constr1 **Construction 4.5.** (Filtered (φ, N, G_K) realization) Consider the map $\Delta^{\text{eff}} \rightarrow DF_K$:

$$(X, Y, i) \mapsto H_{DF}^i(X, Y) := (H_{\text{HK}}^i(X_{\overline{K}}, Y_{\overline{K}}), (H_{\text{dR}}^i(X, Y), F^\bullet), \iota_{\text{dR}} : H_{\text{HK}}^i(X_{\overline{K}}, Y_{\overline{K}}) \otimes_{K_0^{\text{nr}}} \overline{K} \xrightarrow{\sim} H_{\text{dR}}^i(X_{\overline{K}}, Y_{\overline{K}})).$$

By p -adic comparison theorems, we have

$$D_{\text{pst}}(H_{DF}^i(X, Y)) \simeq H^i(X_{\overline{K}}, Y_{\overline{K}}, \mathbf{Q}_p) \simeq H^i(X(\mathbf{C}), Y(\mathbf{C}), \mathbf{Q}_p).$$

Thus, by Theorem 4.1, we obtain an extension which is the exact filtered (φ, N, G_K) realization functor

$$\text{R}_{DF_K} : \text{EMM}(K)_{\mathbf{Q}_p} \rightarrow DF_K.$$

Since $\text{R}_{DF}(1(-1)) = K(-1)$, the functor R_{DF} lifts to $\text{MM}(K)_{\mathbf{Q}_p}$.

Projections yield faithful exact functors from DF_K to the categories $M_K(\varphi, N, G_K)$ and V_{dR}^K . Composing them with the realization R_{DF} we get

- the exact Hyodo-Kato realization

$$\text{R}_{\text{HK}} : \text{MM}(K)_{\mathbf{Q}_p} \rightarrow M_K(\varphi, N, G_K),$$

- the exact de Rham realization

$$\text{R}_{\text{dR}} : \text{MM}(K)_{\mathbf{Q}_p} \rightarrow V_{\text{dR}}^K.$$

Composing R_{DF_K} with the projection on the third factor of the filtered (φ, N, G_K) -module, we obtain the Hyodo-Kato natural equivalence

$$(4.1) \quad \iota_{\text{dR}} : \text{R}_{\text{HK}} \otimes_{K_0^{\text{nr}}} \overline{K} \simeq \text{R}_{\text{dR}} \otimes_K \overline{K} : \text{MM}(K)_{\mathbf{Q}_p} \rightarrow V_{\overline{K}},$$

where the tensor product is taken pointwise.

²This is shown by an argument analogous to the one we have used in the construction of Beilinson's potentially semistable complex of a variety in Section 3.1.2 : via cellular complexes and Čech coverings one lifts the representation H^* from very good pairs to all pairs to a representation that canonically computes relative singular cohomology.

4.2. p -adic realizations of Voevodsky's motives.

Recall 4.8. The category of Voevodsky's motives $DM(K, \mathbf{Q}_p)$ with rational coefficients admits several equivalent constructions, each interesting in its own. In this section, we will be using the one of Morel (see [46]) for a review of which we refer the reader to [23, §1].

By construction, the triangulated category $DM(K, \mathbf{Q}_p)$ is stable under taking arbitrary coproducts. In this category, each smooth K -scheme X admits a homological motive $M(X)$, covariant with respect to morphism of K -schemes (and even finite correspondences). Each motive can be twisted by an arbitrary integer power of the Tate object $\mathbf{Q}_p(1)$, and as a triangulated category stable under taking coproducts, $DM(K, \mathbf{Q}_p)$ is generated by motives of the form $M(X)(n)$, X/K smooth, and $n \in \mathbf{Z}$.

The category of *constructible* motives (see also 5.4) is the thick³ triangulated subcategory of $DM(K, \mathbf{Q}_p)$ generated by the motives $M(X)(n)$, X/K smooth, and $n \in \mathbf{Z}$, without requiring stability by infinite coproducts. It is equivalent to Voevodsky's category of geometric motives $DM_{gm}(K, \mathbf{Q}_p)$ ([67, chap. 5]) and can also be described in an elementary way as follows. Let $\mathbf{Q}_p[\mathrm{Sm}_K^{aff}]$ be the \mathbf{Q}_p -linearization of the category of smooth affine K -varieties, $K^b(\mathbf{Q}_p[\mathrm{Sm}_K^{aff}])$ its bounded homotopy category. This is a triangulated monoidal category, the tensor structure being induced by cartesian products of K -schemes. First we get the geometric \mathbb{A}^1 -derived category $D_{\mathbb{A}^1, gm}(K, \mathbf{Q}_p)$ out of $K^b(\mathbf{Q}_p[\mathrm{Sm}_K^{aff}])$ by the following operations:

- (1) Take the Verdier quotient with respect to the triangulated subcategory generated by complexes of the form:
 - (*homotopy*) $\dots \rightarrow 0 \rightarrow \mathbb{A}_X^1 \xrightarrow{p} X \rightarrow 0 \dots$, for $X \in \mathrm{Sm}_K^{aff}$, p canonical projection;
 - (*excision*) $\dots \rightarrow 0 \rightarrow W \xrightarrow{q-k} U \oplus V \xrightarrow{j+p} X \dots$, for any cartesian square
$$\begin{array}{ccc} W & \xrightarrow{k} & V \\ q \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X \end{array}$$

such that j is an open immersion, p is étale and an isomorphism above the complement of j .

- (2) Formally invert the Tate object $\mathbf{Q}_p(1)$, which is the cokernel of $\{1\} \rightarrow \mathbb{G}_m$ placed in cohomological degree $+1$.
- (3) Take the pseudo-abelian envelope.

Let τ be the automorphism of $\mathbf{Q}_p(1)[1] \otimes \mathbf{Q}_p(1)[1]$ in $D_{\mathbb{A}^1, gm}(K, \mathbf{Q}_p)$ which permutes the factors. Because $\mathbf{Q}_p(1)$ is invertible, it induces an automorphism ε of \mathbf{Q}_p in $D_{\mathbb{A}^1, gm}(K, \mathbf{Q}_p)$ such that $\varepsilon^2 = 1$. Then we can define complementary projectors: $p_+ = (1 - \varepsilon)/2$, $p_- = (\varepsilon - 1)/2$, which cut the objects, and therefore the category, into two pieces:

$$D_{\mathbb{A}^1, gm}(K, \mathbf{Q}_p)_+ = \mathrm{Im}(p_+), \quad D_{\mathbb{A}^1, gm}(K, \mathbf{Q}_p)_- = \mathrm{Im}(p_-).$$

Then, according to a theorem of Morel (cf. [17, 16.2.13]), $DM_{gm}(K, \mathbf{Q}_p) \simeq D_{\mathbb{A}^1, gm}(K, \mathbf{Q}_p)_+$.

realization

Example 4.9. Let F be an extension field of \mathbf{Q}_p and \mathcal{A} be a Tannakian F -linear category with a fiber functor $\omega : \mathcal{A} \rightarrow V_F$. Consider a contravariant functor:

$$R : (\mathrm{Sm}_K^{aff})^{op} \rightarrow C^b(\mathcal{A}).$$

It automatically extends to a contravariant functor $R' : K^b(\mathbf{Q}_p[\mathrm{Sm}_K^{aff}])^{op} \rightarrow D^b(\mathcal{A})$. The conditions for R' to induce a contravariant functor defined on $DM_{gm}(K, \mathbf{Q}_p)$ are easy to state given the description of DM_{gm} given above. We will use the following simpler criterion:

We now suppose that the functor R takes its values in the bigger category $C^b(\mathrm{Ind} - \mathcal{A})$ but we assume that there exists a functorial isomorphism

$$H^i \omega R(X) \simeq H^i(X(\mathbf{C}), F)$$

and that the product map $H^i(X(\mathbf{C}), F) \otimes H^i(Y(\mathbf{C}), F) \rightarrow H^i(X(\mathbf{C}) \times Y(\mathbf{C}), F)$ can be lifted to a map $R(X) \otimes R(Y) \rightarrow R(X \times_k Y)$ in $C^b(\mathcal{A})$.

Then the functor R' uniquely extends to a realization functor

$$\tilde{R}^\vee : DM_{gm}(K, \mathbf{Q}_p)^{op} \rightarrow D^b(\mathcal{A})$$

³*i.e.* stable by direct factors

which is monoidal and such that $H^i(\tilde{R}^\vee(M(X))) = H^i(R(X))$.⁴ After composing this functor with the canonical duality endofunctor of the (rigid) triangulated monoidal category $D^b(\mathcal{A})$, we get a covariant realization:

$$\tilde{R} : DM_{gm}(K, \mathbf{Q}_p) \rightarrow D^b(\mathcal{A})$$

such that $H^i \tilde{R}(M(X)) = H^i(R(X))^\vee$. Note also that, by construction, the preceding identification can be extended to closed pairs. Also, because $DM_{gm}(K, \mathbf{Q}_p)$ satisfies h-descent (see section 5.5), it can be extended to singular K -varieties and pairs of such.

Using this example we can easily build realizations:

Proposition 4.10. *Let F be an extension field of \mathbf{Q}_p and \mathcal{A} be a Tannakian F -linear category with a fiber functor $\omega : \mathcal{A} \rightarrow V_F$. Consider a representation $A^* : \Delta_g \rightarrow \mathcal{A}$ such that ωA^* is isomorphic to the singular representation (see Definition 4.3).*

Then there exists a canonical covariant monoidal realization:

$$R_A : DM_{gm}(K, \mathbf{Q}_p) \rightarrow D^b(\mathcal{A})$$

such that for any good pair (X, Y, i) , $H^i R_A(M(X, Y)) = A^i(X, Y)^\vee$ and this identification is functorial in (X, Y, i) – including with respect to boundaries.

Moreover, this construction is functorial with respect to exact morphisms of representations.

Proof. Let X be a smooth affine K -scheme. To any cellular stratification of X (cf. Corollary 3.3) $F_\bullet X$, we can associate the complex

$$R'_A(F_\bullet X) := 0 \rightarrow A^0(F_0 X) \rightarrow A^1(F_1 X, F_0 X) \rightarrow \dots \rightarrow A^d(X, F_{d-1} X) \rightarrow 0.$$

We put: $R'_A(X) := \text{colim}_{F_\bullet X} R'_A(F_\bullet X)$. This defines a contravariant functor:

$$R'_A : (\text{Sm}_K^{aff})^{op} \rightarrow C^b(\text{Ind} - \mathcal{A})$$

which satisfies the assumptions of the previous example. Hence we get the proposition by applying the construction of this example. \square

Remark 4.11. Consider again a fiber functor $\omega : \mathcal{A} \rightarrow V_F$ and a contravariant functor

$$R : (\text{Sch}_K)^{op} \rightarrow C^b(\text{Ind} - \mathcal{A})$$

such that for any K -variety X , one has a functorial isomorphism $H^i \omega R(X) \simeq H^i(X(\mathbf{C}), F)$. Then we can apply the preceding example to $R|_{\text{Sm}_K^{aff}}$ and also the preceding proposition to the unique representation A^* induced by R such that $A^i(X, Y) = H^i(\text{Cone}(R(X) \rightarrow R(Y)[-1]))$. By applying the construction of the preceding proof, we get for any smooth affine K -scheme a canonical map of complexes

$$R(X) \rightarrow R_A(X)$$

which is a quasi-isomorphism. For the functoriality of the construction of the previous example, we thus get a canonical isomorphism between the two realizations of any Voevodsky's motive M :

$$\tilde{R}(M) \xrightarrow{\sim} \tilde{R}_A(M)$$

Remark 4.12. Voevodsky's motives $M(X)$ are homological: they are covariant in X . In fact, the monoidal category $DM_{gm}(K, \mathbf{Q}_p)$ is rigid: any object has a strong dual – cf. [56]. Then for any smooth K -variety X , $M(X)^\vee$ is the cohomological motive of X/K . Using the notations of the previous proposition, because R_A is monoidal and therefore commutes with strong duals, we get: $H^i R_A(M(X)^\vee) = A^i(X)$.

Recall that the category $DM_{gm}(K, \mathbf{Q}_p)$ can be extended to any base and satisfies the 6 functors formalism (cf. [17], in particular 16.1.6). According to *loc. cit.*, 15.2.4, $M(X)^\vee = f_*(1_X)$ where $f : X \rightarrow \text{Spec}(K)$ is the structural morphism. The preceding relation can be rewritten:

$$H^i R_A(f_*(1_X)) = A^i(X).$$

⁴Note in particular that the permutation ε acts by -1 on singular cohomology.

Note finally that f_* exists for any K -variety X . One can extend the above identification to this general case using De Jong resolution of singularities and h -descent, which is true for Voevodsky's rational motives ([17, 14.3.4]) and for Betti cohomology.

There is fully faithful monoidal functor

$$\mathrm{CHM}(K)_{\mathbf{Q}_p}^{op} \rightarrow \mathrm{DM}_{gm}(K, \mathbf{Q}_p), \quad h(X) \mapsto M(X)$$

from the category of Chow motives (X is smooth projective over K). Applying duality on the right hand side, we get a covariant fully faithful monoidal functor:

$$\mathrm{CHM}(K)_{\mathbf{Q}_p} \rightarrow \mathrm{DM}_{gm}(K, \mathbf{Q}_p), \quad h(X) \mapsto M(X)^\vee = f_*(1_X).$$

In view of this embedding, it is convenient to identify the Chow motive $h(X)$ with the Voevodsky's (cohomological) motive $M(X)^\vee$.

Let us also state the following corollary which follows from the preceding proposition and [26]:

Corollary 4.13. *In the assumptions of the previous proposition, for any smooth projective K -scheme X of dimension d , the complex $R_A(h(X)) = R_A(M(X)^\vee)$ is split: there exists a canonical isomorphism:*

$$R_A(h(X)) = \bigoplus_{i=0}^{2d} H^i(R_A(h(X)))[-i] = \bigoplus_{i=0}^{2d} A^i(X)[-i].$$

This decomposition statement follows simply from *loc. cit.* as the derived category $D^b(\mathcal{A})$ satisfies the assumptions of *loc. cit.* and the object $R_A(h(X))$ satisfies the assumption (L.V.) for the map $h(X) \rightarrow h(X)(1)[2]$ given by multiplication by the (motivic) first Chern class of an ample invertible bundle on X .

Example 4.14. In particular, applying the preceding proposition to the singular representation, we get the classical realization,⁵ due to Nori, of (cohomological) Nori's motives:

$$\Gamma : \mathrm{DM}_{gm}(K, \mathbf{Q}_p) \rightarrow D^b(\mathrm{MM}(K)_{\mathbf{Q}_p}).$$

By definition, and applying the preceding remark, we get for any smooth projective (resp. smooth, any) K -variety $f : X \rightarrow \mathrm{Spec}(K)$:

$$H^i\Gamma(h(X)) = H_{mot}^i(X), \text{ resp. } H^i\Gamma(M(X)) = H_{mot}^i(X)^\vee, \quad H^i\Gamma(f_*(1_X)) = H_{mot}^i(X).$$

When X is smooth projective of dimension d , we also get by the above corollary the decomposition:

$$\Gamma(h(X)) = \bigoplus_{i=0}^{2d} H_{mot}^i(X)[-i]$$

Moreover, because of the functoriality statement of the proposition, this realization of Voevodsky's motives is the universal (initial) one.

4.15. More interestingly, using either Example 4.9 or Proposition 4.10, we can get various p -adic realizations of Voevodsky's motives, and extend the de Rham p -adic comparison theorem to the derived situation as summarized in the following essentially commutative diagram of triangulated monoidal functors:

$$\begin{array}{ccccc}
 & & \mathrm{R}\Gamma_{\acute{e}t} & \longrightarrow & D^b(\mathrm{Rep}(G_K)) \\
 & & & & \otimes_{\mathbf{Q}_p} \mathbf{B}_{dR} \\
 & & & \nearrow \iota & \longrightarrow & D^b(MF_{\mathbf{B}_{dR}}) \\
 \mathrm{DM}_{gm}(K, \mathbf{Q}_p) & \xrightarrow{\mathrm{R}\Gamma_{\mathrm{pst}}} & D^b(\mathrm{Rep}_{\mathrm{pst}}(G_K)) & & & \\
 & & \uparrow V_{\mathrm{pst}} & & & \\
 & & D^b(DF_K) & \longrightarrow & D^b(V_{dR}^K) & \\
 & & & & \otimes_K \mathbf{B}_{dR} \\
 & & \mathrm{R}\Gamma_{dR} & \longrightarrow & &
 \end{array}$$

⁵Conjecturally, this is more than a realization: it is thought to be an equivalence of categories!

where ι is the canonical functor.⁶ The functors $\mathrm{R}\Gamma_{\acute{\mathrm{e}}\mathrm{t}}$, $\mathrm{R}\Gamma_{\mathrm{pst}}$ and $\mathrm{R}\Gamma_{DF_K}$ are obtained either from 4.9 or equivalently from 4.10 (according to Remark 4.11) by considering respectively the following functors:

- $X \in \mathrm{Sm}_K^{aff}, f : X \rightarrow \mathrm{Spec}(K) \mapsto \mathrm{R}f_*(\mathbf{Q}_p)$ and $(X, Y, i) \mapsto H_{\acute{\mathrm{e}}\mathrm{t}}^i(X_{\bar{K}}, Y_{\bar{K}}, \mathbf{Q}_p)$;
- $X \in \mathrm{Sm}_K^{aff} \mapsto \mathrm{R}\Gamma_{\mathrm{pst}}(X_{\bar{K}}, r) \simeq \mathrm{R}\Gamma_{\mathrm{pst}}^B(X_{\bar{K}}, \mathbf{Q}_p(r))$
and $(X, Y, i) \mapsto H_{\acute{\mathrm{e}}\mathrm{t}}^i(X_{\bar{K}}, Y_{\bar{K}}, \mathbf{Q}_p) \in \mathrm{Rep}_{\mathrm{pst}}(G_K)$;
- $(X, Y, i) \mapsto H_{DF}^i(X, Y)$ (see Construction 4.5).

The functor $\mathrm{R}\Gamma_{\mathrm{dR}}$ is obtained by composing $\mathrm{R}\Gamma_{DF_K}$ with the canonical functor $DF_K \rightarrow V_{\mathrm{dR}}^K$.

For $\varepsilon = \acute{\mathrm{e}}\mathrm{t}, \mathrm{pst}, DF_K$, one has defined in the preceding section an analogous exact monoidal realization functor R_ε from the category of Nori's motives $\mathrm{MM}(K)_{\mathbf{Q}_p}$. This functor being exact induces a functor on the (bounded) derived categories and according to the functoriality in Proposition 4.10, one gets for any Voevodsky motive $M \in \mathrm{DM}_{\mathrm{gm}}(K, \mathbf{Q}_p)$:

$$(4.2) \quad \mathrm{R}\Gamma_\varepsilon(M) = R_\varepsilon(\Gamma(M)).$$

Same for the de Rham realizations: we have $\mathrm{R}\Gamma_{\mathrm{dR}}(M) = \mathrm{R}_{\mathrm{dR}}(\Gamma(M))$. Therefore, the essential commutativity of the previous diagram simply follows from the de Rham comparison theorem for Nori's motives. More precisely, it yields, for any Voevodsky's motive M , the de Rham comparison isomorphism:

$$\rho_{\mathrm{dR}} : \quad \mathrm{R}\Gamma_{\mathrm{dR}}(M) \otimes_K \mathbf{B}_{\mathrm{dR}} \simeq \mathrm{R}\Gamma_{\acute{\mathrm{e}}\mathrm{t}}(M) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}$$

which is a quasi-isomorphism of complexes of filtered finite rank \mathbf{B}_{dR} -modules.

This comparison can be made more precise through the Hyodo-Kato realization, as illustrated in the essentially commutative diagram:

$$\begin{array}{ccccc}
 & & \mathrm{R}\Gamma_{\acute{\mathrm{e}}\mathrm{t}} & \xrightarrow{\quad} & D^b(\mathrm{Rep}(G_K)) \\
 & & \nearrow & \nearrow \iota & \searrow \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{st}} \\
 & & D^b(\mathrm{Rep}_{\mathrm{pst}}(G_K)) & & (1) \quad D^b(M_{\mathbf{B}_{\mathrm{st}}}(\varphi, N, G_K)) \\
 & \mathrm{R}\Gamma_{\mathrm{pst}} & \nearrow & \nearrow \otimes_{K_0^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}} & \\
 \mathrm{DM}_{\mathrm{gm}}(K, \mathbf{Q}_p) & \xrightarrow{\quad} & D^b(M_K(\varphi, N, G_K)) & & \\
 & \mathrm{R}\Gamma_{\mathrm{HK}} \downarrow V_{\mathrm{pst}} & \nearrow & \nearrow F_0 & \\
 & & D^b(DF_K) & & (2) \quad D^b(V_{\bar{K}}^G) \\
 & \mathrm{R}\Gamma_{DF_K} & \nearrow & \nearrow F_{\mathrm{dR}} & \\
 & & D^b(V_{\mathrm{dR}}^K) & & \\
 & \mathrm{R}\Gamma_{\mathrm{dR}} & \searrow & \searrow &
 \end{array}$$

The Hyodo-Kato realization $\mathrm{R}\Gamma_{\mathrm{HK}}$ is obtained by composing $\mathrm{R}\Gamma_{\mathrm{dR}}$ with the projection $DK_F \rightarrow M_K(\varphi, N, G_K)$. Then the essential commutativity of the part (1) and (2) of the above diagram corresponds, respectively, for any Voevodsky's motive M , to the potentially semistable comparison theorem and to the Hyodo-Kato quasi-isomorphism:

$$\begin{aligned}
 \rho_{\mathrm{pst}} : \quad & \mathrm{R}\Gamma_{\mathrm{HK}}(M) \otimes_{K_0^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}} \simeq \mathrm{R}\Gamma_{\acute{\mathrm{e}}\mathrm{t}}(M) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{st}}, \\
 \iota_{\mathrm{dR}} : \quad & \mathrm{R}\Gamma_{\mathrm{HK}}(M) \otimes_{K_0^{\mathrm{nr}}} \bar{K} \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(M) \otimes_K \bar{K}.
 \end{aligned}$$

Again, the identification (4.2) holds when $\varepsilon = \mathrm{HK}$ and the above canonical comparison quasi-isomorphisms correspond to the comparison isomorphisms obtained in the previous section.

Remark 4.16. By construction, for any (smooth) K -variety $f : X \rightarrow \mathrm{Spec}(K)$, one has a canonical identification: $\mathrm{R}\Gamma_{\acute{\mathrm{e}}\mathrm{t}}(f_*(1_X)) = \mathrm{R}f_*(\mathbf{Q}_p)$ where the right hand side denotes the right derived functor of the direct image for étale p -adic sheaves.

This implies that the realization functor $\mathrm{R}\Gamma_{\acute{\mathrm{e}}\mathrm{t}}$ constructed above coincides with that of [18, 7.2.24], denoted by ρ_p^* , and equivalently to the one defined in [4]. In particular, it can be extended to any base

⁶One should be careful that though ι is induced by a fully faithful functor on the corresponding abelian categories, it is a *non full* faithful functor.

and commutes with the six functors formalism. This explains the preceding relation and why we have preferred the covariant realization rather than the contravariant one (see the end of Example 4.9).⁷

Example 4.17. The above realizations allow us to define syntomic cohomology of a motive M in $DM_{gm}(K, \mathbf{Q}_p)$ as

$$\mathrm{R}\Gamma_{\mathrm{syn}}(M) := \mathrm{R}\mathrm{Hom}_{D(\mathrm{Rep}_{\mathrm{pst}})}(\mathbf{Q}_p, \mathrm{R}\Gamma_{\mathrm{pst}}(M)) = \mathrm{R}\mathrm{Hom}_{D(DF_K)}(K(0), \mathrm{R}\Gamma_{DF_K}(M)).$$

In particular, we have the syntomic descent spectral sequence

$${}^{\mathrm{syn}}E_2^{i,j} := H_{\mathrm{st}}^i(G_K, H^j \mathrm{R}\Gamma_{\mathrm{ét}}(M)) \Rightarrow H^{i+j} \mathrm{R}\Gamma_{\mathrm{syn}}(M).$$

If we apply it to the cohomological Voevodsky's motive $M(X)^\vee = f_*(1_X)$ of any K -variety X with structural morphism f , we get back the results of Theorem 2.26.

An interesting case is obtained by using the (homological) motive with compact support $M^c(X)$ in $DM_{gm}(K, \mathbf{Q}_p)$ of Voevodsky for any K -variety X , and its dual $M^c(X)^\vee = \underline{\mathrm{Hom}}(M(X), \mathbf{Q}_p)$ which belongs to $DM_{gm}(K, \mathbf{Q}_p)$. Then $\mathrm{R}\Gamma_{\mathrm{syn}}(M^c(X)^\vee(r))$ is the n -th twisted syntomic complex with compact support and we recover the syntomic descent spectral sequence with compact support from Remark 2.28:

$${}^{\mathrm{syn},c}E_2^{i,j} := H_{\mathrm{st}}^i(G_K, H_{\mathrm{ét},c}^j(X_{\overline{K}}, \mathbf{Q}_p(r))) \Rightarrow H_{\mathrm{syn},c}^{i+j}(X, r).$$

Indeed, in terms of the 6 functors formalism, $M^c(X)^\vee(r) = f_!(1_X)(r)$ and the identification relevant to compute the above E_2 -term follows from the previous remark.

4.3. Example I: p -adic realizations of the motivic fundamental group. Let $\mathrm{EHM}(K)_{\mathbf{Q}_p}$ denote the category of effective homological Nori's motives, i.e., the diagram category $\mathcal{C}(\tilde{\Delta}_g^{\mathrm{eff}}, H_*)$, $H_* := (H^*)^* := \mathrm{Hom}(H^*, \mathbf{Q}_p)$, where the diagram $\tilde{\Delta}_g^{\mathrm{eff}}$ is obtained from the diagram Δ_g^{eff} by reversing the edge f^* to $f_* : (X, Y, i) \rightarrow (X', Y', i)$ and changing ∂ to $\partial : (X, Y, i) \rightarrow (Y, Z, i - 1)$. There is a duality functor $\vee : \mathrm{EHM}(K)_{\mathbf{Q}_p} \rightarrow \mathrm{MM}(K)_{\mathbf{Q}_p}^{\mathrm{op}}$ respecting the representations H_* and H^* via the usual duality that sends a good pair (X, Y, i) to (X, Y, i) . This induces an equivalence on the derived categories $\vee : D^b(\mathrm{EHM}(K)_{\mathbf{Q}_p}) \xrightarrow{\sim} D^b(\mathrm{MM}(K)_{\mathbf{Q}_p}^{\mathrm{op}})$.

In [20] Cushman developed a motivic theory of the fundamental group, i.e., he showed that the unipotent completion of the fundamental group of varieties over complex numbers carries a motivic structure in the sense of Nori. We will recall his main theorem.

- Let $\mathcal{V}ar_K^*$ be the category of pairs (X, x) , where X is a variety defined over K and x is a K -rational base point; morphism between such pairs are morphisms between the corresponding varieties defined over K that are compatible with the base points.

- Let $\mathcal{V}ar_K^{**}$ be the category of triples $(X; x_1, x_2)$, where X is defined over K and x_1, x_2 are K -rational base points.

For a variety X over \mathbf{C} , let $\pi_1(X, x)$ be the fundamental group of X with base point x and let $\pi_1(X; x_1, x_2)$ be the space of based paths up to homotopy from x_1 to x_2 . Denote by I_{x_2} – the augmentation ideal in $\mathbf{Q}_p[\pi_1(X, x_2)]$ (i.e., the kernel of the augmentation map $\mathbf{Q}_p[\pi_1(X, x_2)] \rightarrow \mathbf{Q}_p$) which acts on the right on $\pi_1(X; x_1, x_2)$. The following theorem [21, Thm 3.1] shows that the quotient $\mathbf{Q}_p[\pi_1(X; x_1, x_2)]/I_{x_2}^n$, $n \in \mathbf{N}$, has motivic version $\Pi^n(X; x_1, x_2)$ (in the sense of Nori).

Theorem 4.18. *For every $n \in \mathbf{N}$, there are functors*

$$\Pi^n : \mathcal{V}ar_K^{**} \rightarrow \mathrm{EHM}(K)_{\mathbf{Q}_p}, \quad \Pi^n : \mathcal{V}ar_K^* \rightarrow \mathrm{EHM}(K)_{\mathbf{Q}_p}.$$

These functors have the following properties.

(1) *There is a natural transformation*

$$\Pi^{n+1}(X; x_1, x_2) \rightarrow \Pi^n(X; x_1, x_2).$$

(2) *We have a natural isomorphism of \mathbf{Q}_p -vector spaces*

$$\tilde{H}(\Pi^n(X(\mathbf{C}); x_1, x_2)) \simeq \mathbf{Q}_p[\pi_1(X(\mathbf{C}); x_1, x_2)]/I_{x_2}^n.$$

⁷In Section 5, we will similarly extend the realization functor $\mathrm{R}\Gamma_{\mathrm{pst}}$ to arbitrary K -bases (see more precisely Rem. 5.16).

(3) *There are natural transformations*

$$\begin{aligned}\Pi^n(X; x_1, x_2) \otimes \Pi^n(X; x_2, x_3) &\rightarrow \Pi^n(X; x_1, x_3) \\ \Pi^{n+m+1}(X, x_2) &\rightarrow \Pi^{m+1}(X, x_2) \otimes \Pi^{n+1}(X, x_2)\end{aligned}$$

Via the natural isomorphisms in (2), these transformations are compatible with the product and coproduct structures as well as with the inversion in the path space.

This data is equivalent to giving a pro-EHM structure on the inverse limit $\mathbf{Q}_p[\pi_1(X(\mathbf{C}); x_1, x_2)]/I_{x_2}^n$ such that all the obvious maps are motivic, and the completed ideal $I_{x_2}^\wedge$ is a sub-motive.

Dualizing the realization functors of Nori's motives used in Constructions 4.4, 4.5 we obtain the following functors

$$\begin{aligned}\Pi_{\text{ét}}^n &: \mathcal{V}ar_K^{**} \rightarrow \text{Rep}(G_K), \quad \Pi_{\text{ét}}^n := R_{\text{ét}}\Pi^n; \\ \Pi_{\text{HK}}^n &: \mathcal{V}ar_K^{**} \rightarrow M_K(\varphi, N, G_K), \quad \Pi_{\text{HK}}^n := R_{\text{HK}}\Pi^n; \\ \Pi_{\text{dR}}^n &: \mathcal{V}ar_K^{**} \rightarrow V_{\text{dR}}^K, \quad \Pi_{\text{dR}}^n := R_{\text{dR}}\Pi^n.\end{aligned}$$

These realizations are compatible with change of the index n and with the structure maps that endow these realizations with Hopf algebra structures.

From Constructions 4.5, 4.6 (again dualizing) we obtain also the following comparison isomorphisms.

Corollary 4.19. (1) *There exists the Hyodo-Kato natural equivalence*

$$\iota_{\text{dR}} : \Pi_{\text{HK}}^n(X; x_1, x_2) \otimes_{K_0^{\text{nr}}} \overline{K} \simeq \Pi_{\text{dR}}^n(X; x_1, x_2) \otimes_K \overline{K}.$$

(2) *There exists a natural equivalence (potentially semistable period isomorphism)*

$$\rho_{\text{pst}} : \Pi_{\text{HK}}^n(X; x_1, x_2) \otimes_{K_0^{\text{nr}}} \mathbf{B}_{\text{st}} \simeq \Pi_{\text{ét}}^n(X; x_1, x_2) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{st}}$$

that is compatible with Galois action, Frobenius, the monodromy operator. Extending to \mathbf{B}_{dR} and using the Hyodo-Kato equivalence, we get the de Rham period isomorphism

$$\rho_{\text{dR}} : \Pi_{\text{dR}}^n(X; x_1, x_2) \otimes_K \mathbf{B}_{\text{dR}} \simeq \Pi_{\text{ét}}^n(X; x_1, x_2) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{dR}}$$

that is compatible with filtrations.

These comparison isomorphisms are compatible with change of the index n and with Hopf algebra structures.

The above comparison statements were proved before in the case of curves in [36], [1], for varieties with good reduction over slightly ramified base in [68], and for varieties with good reduction over an unramified base in [54]. The various realizations appearing in these constructions should be naturally isomorphic with ours but we did not check it.

4.4. Example II: p -adic comparison maps with compact support and compatibilities.

4.20. When $\varepsilon = \text{HK}, \text{ét}, \text{dR}, DF_K, \text{pst}$, we get from the preceding section, for any K -variety, a complex

$$\text{R}\Gamma_\varepsilon(X) := \text{R}\Gamma_\varepsilon(M(X)^\vee) = \text{R}\Gamma_\varepsilon(M(X))^*$$

which computes the ε -cohomology with enriched coefficients. When $\varepsilon = \text{ét}, \text{HK}, \text{dR}$ this is the usual complex, respectively, of Galois representations, (φ, N, G_K) -modules, filtered K -vector spaces which computes, respectively, geometric étale cohomology, Hyodo-Kato cohomology and De Rham cohomology with their natural algebraic structures. These complexes are related by the comparison isomorphisms $\rho_{\text{dR}}, \rho_{\text{pst}}$, and ι_{dR} .

An interesting point is that these complexes, as well as the comparison isomorphisms are contravariantly functorial in the homological motive $M(X)$. Recall Voevodsky's motives are equipped with special covariant functorialities.

Let X and Y be K -varieties. A *finite correspondence* α from X to Y is an algebraic cycle in $X \times_K Y$ whose support is finite equidimensional over X and which is *special* over X in the sense of [17, 8.1.28].⁸ Then by definition, α induces a map $\alpha_* : M(X) \rightarrow M(Y)$.

Assume now that X and Y are smooth. Let $f : X \rightarrow Y$ be any morphism of schemes of constant relative dimension d . Then we have the Gysin maps $f^* : M(Y) \rightarrow M(X)(d)[2d]$ (cf. [22]).

Corollary 4.21. *Consider the notations above.*

Then $\mathrm{R}\Gamma_\varepsilon(X)$ is contravariant with respect to finite correspondences and covariant with respect to morphisms of smooth K -varieties.

Moreover, the comparison isomorphisms $\rho_{\mathrm{dR}}, \rho_{\mathrm{pst}}, \iota_{\mathrm{dR}}$ are natural with respect to these functorialities.

Remark 4.22. (1) Note in particular that covariance with respect to finite correspondences implies the existence of transfer maps f_* for any finite equidimensional morphism $f : X \rightarrow Y$ which is special (eg. flat, or X is geometrically unibranch).

(2) The syntomic descent spectral sequence and the syntomic period map of Example 4.17 are natural with respect to the functorialities of the corollary.

(3) We can deduce from [22] the usual good properties of covariant functoriality (compatibility with composition, projection formulas, excess of intersection formulas,...)

4.23. Products. Consider again the notations of the Paragraph 4.20. As said previously, from the Künneth formula, $\mathrm{R}\Gamma_\varepsilon$ is a monoidal functor and the comparison isomorphisms are isomorphisms of monoidal functors.

Consider a K -variety X with structural morphism f . Recall from Remark 4.12 that $M(X)^\vee = f_*(1_X)$. The functor f_* is left adjoint to a monoidal functor. Therefore it is weakly monoidal and we get a pairing:

$$\mu : M(X)^\vee \otimes M(X)^\vee = f_*(1_X) \otimes f_*(1_X) \rightarrow f_*(1_X) = M(X)^\vee$$

in $DM_{gm}(K, \mathbf{Q}_p)$. This induces a cup-product on the ε -complexes:

$$(4.3) \quad \mathrm{R}\Gamma_\varepsilon(X) \otimes \mathrm{R}\Gamma_\varepsilon(X) = \mathrm{R}\Gamma_\varepsilon(f_*(1_X)) \otimes \mathrm{R}\Gamma_\varepsilon(f_*(1_X)) \xrightarrow{K} \mathrm{R}\Gamma_\varepsilon(f_*(1_X) \otimes f_*(1_X)) \xrightarrow{\mu_*} \mathrm{R}\Gamma_\varepsilon(f_*(1_X)) = \mathrm{R}\Gamma_\varepsilon(X),$$

where the isomorphism labelled K stands for the structural morphism of the monoidal functor $\mathrm{R}\Gamma_\varepsilon$ – and corresponds to the Künneth formula in ε -cohomology. When $\varepsilon = \text{ét}, \mathrm{HK}, \mathrm{dR}$, we deduce from the definition of this structural isomorphism that these products correspond to the natural products on the respective cohomology. As the comparison isomorphisms are isomorphisms of monoidal functors, we deduce that they are compatible with the above cup-products.

From the end of Example 4.17, we can also define the ε -complex of X with compact support:

$$\mathrm{R}\Gamma_{\varepsilon,c}(X) = \mathrm{R}\Gamma_\varepsilon(f_!(1_X)).$$

Because we have a natural map $f_* \rightarrow f_!$ of functors ([17, 2.4.50(2)]), we also deduce, as usual, a natural map:

$$\mathrm{R}\Gamma_{\varepsilon,c}(X) \rightarrow \mathrm{R}\Gamma_\varepsilon(X).$$

From the 6 functors formalism, we get a pairing in $DM_{gm}(K, \mathbf{Q}_p)$:

$$\mu_c : f_*(1_X) \otimes f_!(1_X) \xrightarrow{(1)} f_!(f^* f_*(1_X) \otimes 1_X) = f_!(f^* f_*(1_X)) \xrightarrow{(2)} f_!(1_X)$$

where the isomorphism (1) stands for the projection formula ([17, 2.4.50(5)]) and the map (2) is the unit map of the adjunction (f^*, f_*) . Then, using μ_c instead of μ in formula (4.3), we get the pairing between cohomology and cohomology with compact support:

$$(4.4) \quad \mathrm{R}\Gamma_\varepsilon(X) \otimes \mathrm{R}\Gamma_{\varepsilon,c}(X) \rightarrow \mathrm{R}\Gamma_{\varepsilon,c}(X).$$

Using again the fact that the comparison isomorphisms $\rho_{\mathrm{dR}}, \rho_{\mathrm{pst}}, \iota_{\mathrm{dR}}$ are isomorphisms of monoidal functor, we deduce that they are compatible with this pairing. Let us summarize:

⁸If X is geometrically unibranch, every α whose support is finite equidimensional over X is special (cf. [17, 8.3.27]). If Z is a closed subset of $X \times_K Y$ which is flat and finite over X , the cycle associated with Z is a finite correspondence (cf. [17, 8.1.31]).

Proposition 4.24. *For $* = \emptyset, c$, we have comparison isomorphisms*

$$\begin{aligned} \iota_{\mathrm{HK},*} &: \mathrm{R}\Gamma_{\mathrm{HK},*}(X) \otimes_{K_0^{\mathrm{nr}}} \overline{K} \simeq \mathrm{R}\Gamma_{\mathrm{dR},*}(X) \otimes_K \overline{K}, \\ \rho_{\mathrm{pst},*} &: \mathrm{R}\Gamma_{\mathrm{HK},*}(X) \otimes_{K_0^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}} \simeq \mathrm{R}\Gamma_{\mathrm{ét},*}(X_{\overline{K}}) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{st}}, \\ \rho_{\mathrm{dR},*} &: \mathrm{R}\Gamma_{\mathrm{dR},*}(X) \otimes_K \mathbf{B}_{\mathrm{dR}} \simeq \mathrm{R}\Gamma_{\mathrm{ét},*}(X_{\overline{K}}) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}, \end{aligned}$$

that are compatible with cup-products (4.3) and with the pairing (4.4).

5. SYNTOMIC MODULES

5.1. Definition. In this section we use the dg-algebra $\mathcal{E}_{\mathrm{syn},K}$, which represents syntomic cohomology of varieties over K [48, Appendix] to define a category of syntomic modules over any such variety. This is our candidate for coefficients systems (of geometric origin) for syntomic cohomology. We prove that in the case of $\mathrm{Spec} K$ itself the category of syntomic coefficients is (via the period map) a subcategory of potentially semistable representations that is closed under extensions. We call such representations *constructible representations*.

Let us first recall the setting of Voevodsky's h -motives, with coefficients in a given ring R and over any noetherian base scheme S . We let $\mathrm{Sh}(S, R)$ be the category of h -sheaves of R -modules on Sch_S – the category of separated schemes of finite type over S . This is a monoidal Grothendieck abelian category with generators the free R -linear h -sheaves represented by any X in Sch_S ; we denote them by $R_S^h(X)$. In particular, its derived category $\mathcal{D}(\mathrm{Sh}(S, R))$ has a canonical structure of a stable monoidal ∞ -category in the sense of [57, Def. 3.5] (see also [45]).⁹ Moreover, it admits infinite direct sums. Let us define the Tate object as the following complex of R -sheaves: $R_S(1) := R_S^h(\mathbb{P}_S^1)/R_S^h(\{\infty\})[-2]$.

The following theorem is an ∞ -categorical summary of a classical construction phrased in terms of model categories in [18]:

Theorem 5.1. *There exists a universal monoidal ∞ -category $\underline{\mathcal{D}M}_h(S, R)$ which admits infinite direct sums and is equipped with a monoidal ∞ -functor*

$$\Sigma^\infty : \mathcal{D}(\mathrm{Sh}(S, R)) \rightarrow \underline{\mathcal{D}M}_h(S, R)$$

such that:

- \mathbb{A}^1 -Homotopy: for any scheme X in Sch_S , the induced map $\Sigma^\infty R_S^h(\mathbb{A}_X^1) \rightarrow \Sigma^\infty R_S^h(X)$ is an isomorphism;
- \mathbb{P}^1 -stability: the object $\Sigma^\infty R_S(1)$ is \otimes -invertible.

Moreover, the monoidal ∞ -category $\underline{\mathcal{D}M}_h(S, R)$ is stable and presentable.

Concerning the first point, the statement follows from the existence of localization for monoidal ∞ -categories. The statement for the second point follows from [57, 4.16] and the fact that, up to \mathbb{A}^1 -homotopy, the cyclic permutation on $R_S(1)^{\otimes 3}$ is the identity.

Remark 5.2. According to [18] and [57, 4.29], the ∞ -category $\underline{\mathcal{D}M}_h(S, R)$ is associated with an underlying symmetric monoidal model category – this also implies it can be described by a canonical R -linear dg-category. According to the description of this model category, up to quasi-isomorphism, the objects of $\underline{\mathcal{D}M}_h(S, R)$ can be understood as \mathbb{N} -graded complexes of R -linear h -sheaves $(\mathcal{E}_r)_{r \in \mathbb{N}}$ which satisfy the following properties:

- (Homotopy invariance) for any integer r , the h -cohomology presheaves $H_h^*(-, \mathcal{E}_r)$ are \mathbb{A}^1 -invariant;
- (Tate twist) there exists a (structural) quasi-isomorphism $\mathcal{E}_r \rightarrow \underline{\mathrm{Hom}}(R_S(1), \mathcal{E}_{r+1})$.

One should be careful however that, in order to get the right *symmetric* monoidal structure on the underlying model category, one has to consider in addition an action of the symmetric group of order r on \mathcal{E}_r , in a way compatible with the structural isomorphism associated with Tate twists. The corresponding objects are called *symmetric Tate spectra*.¹⁰

⁹Actually, this follows from the existence of a closed monoidal category structure on the category of complexes of $\mathrm{Sh}(S, R)$ (cf. [15] or [18]) and from [57, Sec. 3.9.1].

¹⁰See [17, Sec. 5.3] for the construction in motivic homotopy theory.

sec:modules

m:exists_DM

d_presheaves

Example 5.3. Let $S = \text{Spec}(K)$ and $R = \mathbf{Q}_p$. Consider the h -sheaf associated with the presheaf of dg- \mathbf{Q}_p -algebras

$$X \mapsto (\mathbf{R}\Gamma_{\text{syn}}(X_h, r) \simeq \mathbf{R}\Gamma_{\text{syn}}(X, r))$$

defined in 2.8 (see Theorem 2.25 for the isomorphism). Because of [48], it satisfies the homotopy invariance and Tate twist properties stated above; thus as explained in Appendix B of [48], it canonically defines an object \mathcal{E}_{syn} of $\mathcal{DM}_h(K, \mathbf{Q}_p)$. Moreover, the dg-structure allows to get a canonical ring structure on this object, which corresponds to a strict structure (the diagrams encoding commutativity and associativity are commutative not only up to homotopy).

For any scheme X in Sch_S , we put $M_S(X) := \Sigma^\infty R_S^h(X)$, called the (homological) h -motive associated with X/S .

Definition 5.4. We define the stable monoidal ∞ -category of h -motives $\mathcal{DM}_h(S, R)$ (resp. constructible h -motives $\mathcal{DM}_{h,c}(S, R)$) over S with coefficients in R as the sub- ∞ -category of $\mathcal{DM}_h(S, R)$, spanned by infinite direct sums of objects of the form $M_S(X)(n)[i]$ (resp. objects of the form $M_S(X)(n)[i]$) for a smooth S -scheme X and integers $(n, i) \in \mathbf{Z}^2$.

We let $DM_h(S, R)$ (resp. $DM_{h,c}(S, R)$) be the associated homotopy category, as a triangulated monoidal category.

Example 5.5. When R is a \mathbf{Q} -algebra (resp. R is a \mathbf{Z}/n -algebra where n is invertible on S), $DM_h(S, R)$ is equivalent to the triangulated monoidal category of rational mixed motives (resp. derived category of R -sheaves on the small site étale of S): see [18], Th. 5.2.2 (resp. Cor. 5.4.4). In particular, $\mathcal{DM}_h(S, R)$ is presentable by a monoidal model category.

The justification of the axioms of \mathbb{A}^1 -homotopy and \mathbb{P}^1 -stability added to the derived category of h -sheaves comes from the following theorem:

Theorem 5.6 ([18]). *The triangulated categories $DM_h(S, R)$ for various schemes S are equipped with Grothendieck 6 functors formalism and satisfy the absolute purity property. If one restricts to quasi-excellent schemes S and morphisms of finite type, the subcategories $DM_{h,c}(S, R)$ are stable under the 6 operations, and satisfy Grothendieck-Verdier duality.*

We refer the reader to [17, A.5] of [18, Appendix A] for a summary of Grothendieck 6 functors formalism and Grothendieck-Verdier duality.

Let us now take the notations of Example 5.3. We view \mathcal{E}_{syn} in the model category underlying $\mathcal{DM}_h(K, \mathbf{Q}_p)$, equipped with its structure of (commutative) dg-algebra. According to [17, 7.1.11(d)], one can assume that \mathcal{E}_{syn} is cofibrant (by taking a cofibrant resolution in the category of dg-algebras according to *loc. cit.*). Given any morphism $f : S \rightarrow \text{Spec}(K)$, we put

$$\mathcal{E}_{\text{syn},S} := \mathbf{L}f^*(\mathcal{E}_{\text{syn}})$$

which is again a dg-algebra because f^* is monoidal. According to the construction of [17, Sec. 7.2], the category $\mathcal{E}_{\text{syn}}\text{-Mod}_S$ of modules over this dg-algebra is endowed with a monoidal model structure, and therefore with a structure of monoidal ∞ -category. The free \mathcal{E}_{syn} -module functor induces an adjunction of ∞ -categories:

$$R_{\text{syn}} : \mathcal{DM}_h(S, \mathbf{Q}_p) \rightleftarrows \mathcal{E}_{\text{syn}}\text{-Mod}_S : \mathcal{O}_{\text{syn}}.$$

Given any S -scheme X , and any integer $n \in \mathbf{Z}$, we put $\mathcal{E}_{\text{syn},S}(X)(n) := R_{\text{syn}}(M_S(X)(n))$.

Definition 5.7. Using the above notations, we define the ∞ -category of *syntomic modules* (resp. *constructible syntomic modules*) over S as the ∞ -subcategory of $\mathcal{E}_{\text{syn}}\text{-Mod}_S$ stable under taking infinite direct sums (resp. finite direct sums) and generated by $\mathcal{E}_{\text{syn},S}(X)(n)[i]$ for a smooth S -scheme X and integers $(n, i) \in \mathbf{Z}^2$.

We denote it by $\mathcal{E}_{\text{syn}}\text{-Mod}_S$ (resp. $\mathcal{E}_{\text{syn}}\text{-Mod}_{c,S}$) and let $\mathcal{E}_{\text{syn}}\text{-mod}_S$ (resp. $\mathcal{E}_{\text{syn}}\text{-mod}_{c,S}$) be its associated homotopy category. This is a monoidal triangulated category.

In particular, we get an adjunction of triangulated categories:

$$R_{\text{syn}} : DM_h(S, \mathbf{Q}_p) \rightleftarrows \mathcal{E}_{\text{syn}}\text{-mod}_S : \mathcal{O}_{\text{syn}},$$

such that R_{syn} , called the realization functor, is monoidal and sends constructible motives to constructible syntomic modules.

mod&compact

Remark 5.8. By definition, the triangulated category $\mathcal{E}_{\text{syn}}\text{-mod}_S$ (resp. $DM_h(S, \mathbf{Q}_p)$) is generated by the objects of the form $\mathcal{E}_{\text{syn},S}(X)(n)$ (resp. $M_S(X)(n)$) for a smooth S -scheme X and an integer $n \in \mathbf{Z}$. By construction, the functor \mathcal{O}_{syn} commutes with arbitrary direct sums. Thus, because $M_S(X)(n)$ is compact¹¹ in $DM_h(S, \mathbf{Q}_p)$ (see [17, 15.1.4]), we deduce that $\mathcal{E}_{\text{syn},S}(X)(n)$ is compact. This implies that a syntomic module is constructible if and only if it is compact.¹²

Note also that $\mathcal{E}_{\text{syn}}\text{-mod}_S$ is a *compactly generated* triangulated category.

Essentially using the previous theorem and the good properties of the forgetful functor \mathcal{O}_{syn} – we get the following result:

realization

Theorem 5.9. *The triangulated categories $\mathcal{E}_{\text{syn}}\text{-mod}_S$ for various schemes S are equipped with Grothendieck 6 functors formalism and satisfy the absolute purity property. If one restricts to quasi-excellent K -schemes S and morphisms of finite type, the subcategories $\mathcal{E}_{\text{syn}}\text{-mod}_{c,S}$ are stable under the 6 operations, and satisfy Grothendieck-Verdier duality.*

If one restricts to K -varieties S , the syntomic (pre-)realization functors:

$$R'_{\text{syn}} : DM_{h,c}(S, \mathbf{Q}_p) \rightarrow \mathcal{E}_{\text{syn}}\text{-mod}_{c,S},$$

for various S , commute with the 6 operations and in particular with duality.

See Corollary 5.15 for the computation of this functor over the base field K .

Proof. The first assertion comes from [17, Prop. 7.2.18], which also implies that the motivic category $\mathcal{E}_{\text{syn}}\text{-mod}$ is separated.¹³ Thus the second assertion comes from [17, Th. 4.2.29].¹⁴ The last assertion is [17, Th. 4.4.25]. \square

Remark 5.10. To get a feeling for the category $\mathcal{E}_{\text{syn}}\text{-mod}_{c,S}$ the reader might want to recall a more classical case of coefficients defined by de Rham cohomology. Let $K = \mathbf{C}$ be the field of complex numbers; let \mathcal{E}_{dR} be the commutative ring spectrum representing de Rham cohomology $X \mapsto R\Gamma_{\text{dR}}(X)$, for varieties X over K . We have

$$H_{\text{dR}}^n(X) = R\text{Hom}_{DM_h(K, \mathbf{C})}(M(X), \mathcal{E}_{\text{dR}}[n]).$$

We can define, in a way analogous to what we have done above, the category of constructible de Rham coefficients $\mathcal{E}_{\text{dR}}\text{-mod}_{c,S}$, for varieties S that are smooth over K . By [17, Example 17.2.22] (using the Riemann-Hilbert correspondence) or by [27, Theorem 3.3.20] (more directly, using the isomorphism between Betti and de Rham cohomologies) this category is equivalent to the bounded derived category of analytic regular holonomic \mathcal{D} -modules on S that are constructible, of geometric origin.

num:duality

5.11. Recall the Grothendieck-Verdier duality property means that for any regular K -scheme S and any separated morphism of finite type $f : X \rightarrow S$, the syntomic module $M_X = f^!(\mathcal{E}_{\text{syn},S})$ is dualizing for the category of constructible syntomic modules over X . In other words, the functor

synt_module

$$(5.1) \quad D_X := \underline{\text{Hom}}(-, M_X) : (\mathcal{E}_{\text{syn}}\text{-mod}_{c,X})^{op} \rightarrow \mathcal{E}_{\text{syn}}\text{-mod}_{c,X}$$

is an anti-equivalence of monoidal triangulated categories. Moreover, it exchanges usual functors with exceptional functors: given any separated morphism of finite type $p : Y \rightarrow X$, one has: $D_Y p^* = p^! D_X$ and $D_X p_* = p_! D_Y$.

¹¹Recall an object M of a triangulated category \mathcal{T} is compact when the functor $\text{Hom}_{\mathcal{T}}(M, -)$ commutes with arbitrary direct sums.

¹²This corresponds to the description of perfect complexes of a ring as compact objects of the derived category.

¹³Recall from [17, 2.1.7] that this means the following: for any K -schemes X, Y and surjective morphism $f : Y \rightarrow X$ of finite type, the base change functor $f^* : \mathcal{E}_{\text{syn}}\text{-mod}_X \rightarrow \mathcal{E}_{\text{syn}}\text{-mod}_Y$ is conservative.

¹⁴As we work over a field of characteristic 0, the absolute purity property is easy to get. Thus the pre-motivic triangulated category $\mathcal{E}_{\text{syn}}\text{-mod}$ is compatible with Tate twist in the sense of Def. 4.2.20 of [17].

5.2. Comparison theorem.

5.12. Consider the abelian category $\text{Rep}_{\text{pst}}(G_K)$ of potentially semi-stable representations and the coinvariants functor

$$\omega_! : \text{Rep}_{\text{pst}}(G_K) \rightarrow V_{\mathbf{Q}_p}^f$$

where the right hand side is the category of finite dimensional \mathbf{Q}_p -vector spaces. It admits a right adjoint denoted by $\omega^!$ which to a finite dimensional \mathbf{Q}_p -vector space V associates the representation V with trivial action of G_K . It is obviously exact and monoidal. One could also put $\omega^* = \omega^!$ because it also admits a right adjoint ω_* which to a potentially semi-stable representation V associates the \mathbf{Q}_p -vector V^{G_K} of G_K -invariants. The situation can be pictured as follows:

$$\text{Rep}_{\text{pst}}(G_K) \begin{array}{c} \xrightarrow{\omega_!} \\ \xleftarrow{\omega^! = \omega^*} \\ \xrightarrow{\omega_*} \end{array} V_{\mathbf{Q}_p}^f.$$

It will be convenient for what follows to enlarge the category $\text{Rep}_{\text{pst}}(G_K)$. Consider the category

$$\text{Rep}_{\text{pst}}^\infty(G_K) := \text{Ind} - \text{Rep}_{\text{pst}}(G_K)$$

of ind-objects. Thus, for us, an infinite potentially semi-stable representation V will be a \mathbf{Q}_p -vector space V with an action of G_K which is a filtering union of sub- \mathbf{Q}_p -vector spaces stable under the action of G_K which are potentially semi-stable representations of G_K . The category $\text{Rep}_{\text{pst}}^\infty(G_K)$ is an abelian (symmetric closed) monoidal category which contains $\text{Rep}_{\text{pst}}(G_K)$ as a full abelian thick subcategory. Moreover, it is a Grothendieck abelian category – it admits infinite direct sums and filtering colimits are exact. The above diagram of functors extends to this larger category. Note in particular that according to this definition, Formula (2.8) can be rewritten:

$$(5.2) \quad V_{\text{pst}}\theta^{-1} : \text{R}\Gamma_{\text{syn}}(X, r) \xrightarrow{\sim} \text{R}\omega_*\text{R}\Gamma_{\text{pst}}(X_{\overline{K}}, r).$$

Due to the Drew's thesis [27] together with our main construction (§2.24), we get the following computation of syntomic modules over K :

Theorem 5.13. *There exist a canonical pair of adjoints of triangulated categories:*

$$\rho^* : \mathcal{E}_{\text{syn}} - \text{mod}_K \rightleftarrows \text{D}(\text{Rep}_{\text{pst}}^\infty(G_K)) : \rho_*$$

such that ρ^* is monoidal and which can be promoted to an adjunction of stable ∞ -categories. Moreover, the functor ρ^* is fully faithful and induces by restriction a monoidal fully faithful triangulated functor:

$$\rho^* : \mathcal{E}_{\text{syn}} - \text{mod}_{c,K} \rightarrow \text{D}^b(\text{Rep}_{\text{pst}}(G_K))$$

such that for any K -variety X with structural morphism f , there exists a canonical quasi-isomorphism of complexes of G_K -representations:

$$(5.3) \quad \rho^*(f_*\mathcal{E}_{\text{syn},X}(r)) \simeq \text{R}\Gamma_{\text{pst}}(X_{\overline{K}}, r).$$

Proof. We will apply Theorem 2.2.7 and Proposition 2.2.21 of [27]. To be consistent with the notations of *loc. cit.*, we take $B = \text{Spec}(K)$ and put $\mathcal{T}_0 = \text{Rep}_{\text{pst}}(G_K)$, $\mathcal{T} = \text{Rep}_{\text{pst}}^\infty(G_K)$.

Consider the functor $\tilde{\mathcal{E}}_{\text{syn}} : X \mapsto \text{R}\Gamma_{\text{pst}}(X_{\overline{K}}, 0)$ (recall that $\text{R}\Gamma_{\text{pst}}(X_{\overline{K}}, 0) \simeq \text{R}\Gamma_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(0))$ as Galois representations). This is a presheaf of dg- \mathbf{Q}_p -algebras on K -varieties with values in \mathcal{T}_0 . Then $\tilde{\mathcal{E}}_{\text{syn}}$ satisfies the axioms of a mixed Weil \mathcal{T}_0 -theory in the sense of [27, 2.1.1]: the axiom (W1) comes from the fact $\tilde{\mathcal{E}}_{\text{syn}}$ satisfies h -descent which is stronger than Nisnevich descent, (W2), (W3) comes from homotopy invariance of geometric p -adic Hodge cohomology and the computation of the syntomic cohomology of K , (W4) comes from the projective bundle formula for geometric p -adic Hodge cohomology, and (W5) was proved in Lemma 2.21. Then we can apply 2.2.7 and 2.2.21 of *loc. cit.* to $\tilde{\mathcal{E}}_{\text{syn}}$ and this gives the theorem.

Let us explain this in more detail. First, Drew generalizes Theorem 5.1, to the category $SH_{\text{Rep}_{\text{pst}}(G_K)}(S)$ of Nisnevich sheaves with values in the category of ind-representations \mathcal{T} , seen as an enriched category over \mathcal{T} – morphisms are not simply sets but ind-representations. This defines the $\text{Rep}_{\text{pst}}(G_K)$ -enriched stable homotopy category over any base scheme S . Drew proves that this category is a stable monoidal

∞ -category – actually it is defined by a monoidal model category – that we will denote here by $\mathcal{D}_{\mathbb{A}^1}(K, \mathcal{T})$. We will denote by $\mathcal{D}_{\mathbb{A}^1}(K, \mathbf{Q}_p)$ the usual monoidal ∞ -category of \mathbb{A}^1 -homology, obtained by replacing \mathcal{T} with the category of \mathbf{Q}_p -vector spaces – and the associated homotopy category still satisfies the 6 functors formalism (cf. *loc. cit.*, Prop. 1.6.7).¹⁵

Then applying Theorem 2.1.4 of *loc. cit.* to the presheaf $\tilde{\mathcal{E}}_{syn}$ we get that the geometric p -adic Hodge cohomology is representable in $SH_{\text{Rep}_{\text{pst}}(G_K)}(S)$ by a commutative monoid $\tilde{\mathcal{E}}_{syn}$ in the underlying model category – in our case the corresponding object is simply the collections of presheaves $X \mapsto \text{R}\Gamma_{\text{pst}}(X_{\bar{K}}, r)$, as a \mathbf{N} -graded dg-algebra indexed by r , seen as presheaves on Sm_K (the category of smooth K -varieties) with values in \mathcal{T} .

Then Drew shows that one can define a monoidal ∞ -category of modules over the dg-algebra $\tilde{\mathcal{E}}_{syn}$ which is enriched over \mathcal{T} , that we will denote here by $\tilde{\mathcal{E}}_{syn}\text{-mod}_K$. It follows that we have the following interpretation of the Künneth formula: by Theorem 2.2.7 of *loc. cit.* the functor

$$\tilde{\rho}: \tilde{\mathcal{E}}_{syn}\text{-mod}_K \xrightarrow{\sim} D(\mathcal{T}), \quad M \mapsto \text{RHom}_{\tilde{\mathcal{E}}_{syn}}^{\mathcal{T}}(\tilde{\mathcal{E}}_{syn}, M),$$

where $\text{Hom}_{\mathcal{T}}$ indicates the enriched Hom (with values in complexes of \mathcal{T}), is an equivalence of monoidal triangulated categories. Recall that any smooth K -variety X defines a canonical $\tilde{\mathcal{E}}_{syn}$ -module $\tilde{\mathcal{E}}_{syn}(X)$. It follows from the construction that, for any smooth K -variety X and any integer $r \in \mathbf{Z}$, there exists a canonical quasi-isomorphism:

$$(5.4) \quad \text{RHom}_{\tilde{\mathcal{E}}_{syn}}^{\mathcal{T}}(\tilde{\mathcal{E}}_{syn}(X), \tilde{\mathcal{E}}_{syn}(r)) \simeq \text{R}\Gamma_{\text{pst}}(X_{\bar{K}}, r)$$

functorial in X .

Now we descend. According to *loc. cit.*, 1.6.8, the pair of adjoint functors (ω^*, ω_*) induces an adjunction of stable ∞ -categories:

$$\text{L}\omega^*: \mathcal{D}_{\mathbb{A}^1}(K, \mathbf{Q}_p) \rightleftarrows \mathcal{D}_{\mathbb{A}^1}(K, \mathcal{T}) : \text{R}\omega_*$$

such that $\text{L}\omega^*$ is monoidal. Then Drew defines (*loc. cit.*, 2.2.13) the absolute cohomology associated with the enriched mixed Weil cohomology $\tilde{\mathcal{E}}_{syn}$ as $\text{R}\omega_*(\tilde{\mathcal{E}}_{syn})$, seen as a monoid in $\mathcal{D}_{\mathbb{A}^1}(K, \mathbf{Q}_p)$ – recall $\text{R}\omega_*$ is weakly monoidal. According to this definition, Formula (5.2), and the definition recalled in Example 5.3, we get:

$$\mathcal{E}_{syn} \simeq \text{R}\omega_*(\tilde{\mathcal{E}}_{syn}),$$

the absolute cohomology associated with $\tilde{\mathcal{E}}_{syn}$. According to this definition, we deduce from the adjunction $(\text{L}\omega^*, \text{R}\omega_*)$ an adjunction of stable ∞ -categories:

$$\text{L}\tilde{\omega}^*: \mathcal{E}_{syn}\text{-mod}_K \rightleftarrows \tilde{\mathcal{E}}_{syn}\text{-mod}_K : \text{R}\tilde{\omega}_*$$

whose left adjoint, $\text{L}\tilde{\omega}^*$, is monoidal. Therefore, one gets the first two statements of the Theorem by putting:

$$\rho^* = \tilde{\rho} \circ \text{L}\tilde{\omega}^*, \quad \rho_* = \tilde{\omega}_* \circ \text{R}\tilde{\rho}^{-1}.$$

Moreover, Prop. 2.2.21 of *loc. cit.* tells us that $\text{L}\tilde{\omega}^*$ is an equivalence of categories if one restricts to constructible objects on both sides (*i.e.*, generated by, respectively, the objects of the form $\mathcal{E}_{syn}(X)(r)$ and $\tilde{\mathcal{E}}_{syn}(X)(r)$ for a smooth K -scheme X and an integer $r \in \mathbf{Z}$). The fact that ρ^* is fully faithful is a formal consequence of this result together with the fact that $\mathcal{E}_{syn}\text{-mod}_K$ is compactly generated (cf. Rem. 5.8).

Recall that, for any smooth K -variety X with structural morphism $f: X \rightarrow \text{Spec}(K)$, one gets:

$$\tilde{\mathcal{E}}_{syn}(X) = \text{L}\tilde{\omega}^*(\mathcal{E}_{syn}(X)) = \text{L}\tilde{\omega}^*(f_! f^! \mathcal{E}_{syn,K}) = \text{L}\tilde{\omega}^* D_K(f_* f^* \mathcal{E}_{syn,K}) = \text{L}\tilde{\omega}^* D_K(f_* \mathcal{E}_{syn,X}),$$

where D_K is the Grothendieck-Verdier duality operator on constructible syntomic modules over K defined in Paragraph 5.11. Thus, in the case when X is a smooth K -variety, Formula (5.3) follows from this identification, the definition of ρ^* , and (5.4). One removes the assumption that X is smooth using the fact that the quasi-isomorphism (5.3) can be extended to diagrams of smooth K -varieties and that both the left and the right hand side satisfies (by definition) cohomological descent for the h-topology.

¹⁵Essentially, its object are graded presheaves on the category of smooth S -scheme with values in \mathcal{T} satisfying homotopy invariance, Tate twist, as in Remark 5.2, but we have to add the Nisnevich descent property.

□

syntomic_modl/ K

Remark 5.14. As a consequence, the category of constructible syntomic modules over K can be identified with a full triangulated subcategory \mathcal{D} of the derived category $D^b(\mathrm{Rep}_{\mathrm{pst}}(G_K))$.

It is easy to describe this subcategory: using resolution of singularities, all objects of $\mathcal{E}_{\mathrm{syn}} - \mathrm{mod}_{c,K}$ are obtained by taking iterated extensions¹⁶ or retracts of syntomic modules of the form $f_*(\mathcal{E}_{\mathrm{syn},X})(r)$ for a smooth projective morphism $f : X \rightarrow \mathrm{Spec}(K)$ and an integer $r \in \mathbf{Z}$ (this is an easy case of the general result [17, 4.4.3]). So \mathcal{D} is the full subcategory of $D^b(\mathrm{Rep}_{\mathrm{pst}}(G_K))$ whose objects are obtained by taking retract of iterated extensions of complexes of the form $\mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, r)$ for X/K smooth projective and $r \in \mathbf{Z}$.

Similarly, the (essential image of the) category of (not necessarily constructible) syntomic modules over K can be identified with the smallest full triangulated subcategory of $D(\mathrm{Rep}_{\mathrm{pst}}^\infty(G_K))$ stable under taking (infinite) direct sums and which contains complexes of the form $\mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, r)$ with the same assumptions as above.

Composing the syntomic realization functor over K with the fully faithful functor ρ^* above, we get:

real&modules

Corollary 5.15. *The syntomic (pre-)realization functor of Theorem 5.9 in the case $S = \mathrm{Spec}(K)$ defines a triangulated monoidal realization functor:*

$$R_{\mathrm{syn}} : DM_{\mathrm{gm}}(K, \mathbf{Q}_p) \simeq DM_{h,c}(K, \mathbf{Q}_p) \xrightarrow{R'_{\mathrm{syn}}} \mathcal{E}_{\mathrm{syn}} - \mathrm{mod}_{c,K} \xrightarrow{\rho^*} D^b(\mathrm{Rep}_{\mathrm{pst}}(G_K)).$$

It coincides with the functor $\mathrm{R}\Gamma_{\mathrm{pst}}$ defined in Paragraph 4.15.

Proof. Only the last statement requires a proof. By definition, $\mathrm{R}\Gamma_{\mathrm{pst}}$ is the functor defined on $DM_{\mathrm{gm}}(K, \mathbf{Q}_p)$ applying Example 4.9 to the functor which to a smooth affine K -variety X associates the complex $\mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, r)$. Thus the statement follows from the description of the functor ρ^* in the above proof and the identification (5.4). □

synt_extends

Remark 5.16. The corollary means in particular that the realization R'_{syn} of Theorem 5.9 does indeed extend the realization $\mathrm{R}\Gamma_{\mathrm{pst}}$ to arbitrary K -bases in a way compatible with the 6 operations.

Corollary 5.17. *For a variety $f : X \rightarrow \mathrm{Spec}(K)$, we have a natural quasi-isomorphism*

$$\mathrm{R}\Gamma_{\mathcal{H}}(X, r) = \mathrm{R}\mathrm{Hom}_{\mathcal{E}_{\mathrm{syn}} - \mathrm{mod}_X}(\mathcal{E}_{\mathrm{syn},X}, \mathcal{E}_{\mathrm{syn},X}(r)).$$

Proof. Since, by the above theorem, $\rho^*(f_*\mathcal{E}_{\mathrm{syn},X}(r)) \simeq \mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, r)$, we have

$$\begin{aligned} \mathrm{R}\mathrm{Hom}_{\mathcal{E}_{\mathrm{syn}} - \mathrm{mod}_X}(\mathcal{E}_{\mathrm{syn},X}, \mathcal{E}_{\mathrm{syn},X}(r)) &= \mathrm{R}\mathrm{Hom}_{\mathcal{E}_{\mathrm{syn}} - \mathrm{mod}_X}(f^*\mathcal{E}_{\mathrm{syn},K}, \mathcal{E}_{\mathrm{syn},X}(r)) \\ &= \mathrm{R}\mathrm{Hom}_{\mathcal{E}_{\mathrm{syn}} - \mathrm{mod}_K}(\mathcal{E}_{\mathrm{syn},K}, f_*\mathcal{E}_{\mathrm{syn},X}(r)) = \mathrm{R}\mathrm{Hom}_D(\mathrm{Rep}_{\mathrm{pst}}(G_K))(\mathbf{Q}_p, \mathrm{R}\Gamma_{\mathrm{pst}}(X_{\overline{K}}, r)) \\ &\simeq \mathrm{R}\Gamma_{\mathcal{H}}(X, r), \end{aligned}$$

as wanted. □

This means that we can define syntomic cohomology of a syntomic module in the following way.

Definition 5.18. Let X be a variety over K and $\mathcal{M} \in \mathcal{E}_{\mathrm{syn}} - \mathrm{mod}_X$. *Syntomic cohomology of \mathcal{M} is the complex*

$$\mathrm{R}\Gamma_{\mathcal{H}}(X, \mathcal{M}) = \mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathcal{M}) := \mathrm{R}\mathrm{Hom}_{\mathcal{E}_{\mathrm{syn}} - \mathrm{mod}_X}(\mathcal{E}_{\mathrm{syn},X}, \mathcal{M}).$$

This definition is compatible with the definition of syntomic cohomology of Voevodsky's motives from Example 4.17. That is, for $M \in DM_{\mathrm{gm}}(K, \mathbf{Q}_p)$, we have a canonical quasi-isomorphism

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathrm{Spec}(K), R'_{\mathrm{syn}}(M)) \simeq \mathrm{R}\Gamma_{\mathrm{syn}}(M).$$

This follows easily from Theorem 5.13 and Corollary 5.15.

¹⁶Recall: in a triangulated category \mathcal{T} , an object M is an extension of M'' by M' if there exists a distinguished triangle $M' \rightarrow M \rightarrow M'' \rightarrow M'[1]$ in \mathcal{T} .

Remark 5.19. Syntomic cohomology with coefficients was studied before in [53], [50], [62], [6]. The coefficients used there could be called "syntomic local systems". They are variants of the crystalline and semistable local systems introduced by Faltings [29], [30]. There exists also a notion of "de Rham local systems". Those were introduced by Tsuzuki in his (unpublished) thesis [64] and later by Scholze [60] in the rigid analytic setting.

In all these cases, syntomic local systems have a de Rham avatar and an étale one. These two avatars are related by relative Fontaine theory and their cohomologies (de Rham, étale, and syntomic) satisfy p -adic comparison isomorphisms. We hope that this is also the case for the syntomic coefficients introduced here and we will discuss it in a forthcoming paper.

5.3. Geometric and constructible representations.

Definition 5.20. Keep the notations of the previous section. We define the category $\text{Rep}_{gm}(G_K)$ (resp. $\text{Rep}_{Ngm}(G_K)$, resp. $\text{Rep}_c(G_K)$) of *geometric* (resp. *Nori's geometric*, resp. *constructible*) p -adic representations of G_K as the essential image of the following (composite) functor:

$$\begin{aligned} DM_{gm}(K, \mathbf{Q}_p) &\xrightarrow{R_{\text{syn}}} D^b(\text{Rep}_{\text{pst}}(G_K)) \xrightarrow{H^0} \text{Rep}_{\text{pst}}(G_K), \\ \text{resp. } R_{\text{pst}} : \quad \text{MM}(K)_{\mathbf{Q}_p} &\rightarrow \text{Rep}_{\text{pst}}(G_K), \\ \text{resp. } \mathcal{E}_{\text{syn}}\text{-mod}_{c,K} &\xrightarrow{\rho^*} D^b(\text{Rep}_{\text{pst}}(G_K)) \xrightarrow{H^0} \text{Rep}_{\text{pst}}(G_K). \end{aligned}$$

Thus a geometric G_K -representation can be described as the geometric étale p -adic cohomology of a Voevodsky's motive over K with its natural Galois action and Nori's geometric G_K -representation - as the geometric étale p -adic cohomology of a Nori's motive. By Corollary 5.15, a geometric G_K -representation is constructible and by the compatibility of realizations of Nori's and Voevodsky's motives (4.2) geometric representation is Nori's geometric. So we have the following inclusions of categories

$$(5.5) \quad \text{Rep}_{gm}(G_K) \subset \text{Rep}_{Ngm}(G_K) \subset \text{Rep}_c(G_K) \subset \text{Rep}_{\text{pst}}(G_K).$$

We do not know much about these subcategories. For example we even do not have a conjectural description of them in purely algebraic terms (for example in terms of (φ, N, G_K) -modules) – this contrasts very much with the case of number fields, see [33].

Here are few trivial facts:

- All three subcategories are stable under taking tensor products and twists.
- All three categories contain representations of the form $H_{\text{ét}}^i(X_{\overline{K}}, \mathbf{Q}_p(r))$ for any integers $i, r \in \mathbf{N} \times \mathbf{Z}$ and any K -variety X (possibly singular). They also contain kernel of projectors of these particular representations when the projector is induced by an algebraic correspondence modulo rational equivalence for X/K projective smooth, and any finite correspondence for an arbitrary X/K .

We do not know if any of these subcategories are stable under taking sub-objects, quotients, or even direct factors.

The following fact is the only nontrivial result about stability.

Proposition 5.21. *The category $\text{Rep}_c(G_K)$ contains all potentially semistable extensions of representations of the form $H_{\text{ét}}^i(X_{\overline{K}}, \mathbf{Q}_p(r))$ for X/K smooth and projective, $i \in \mathbf{N}$, $r \in \mathbf{Z}$.*

Proof. Let \mathcal{D} be the essential image of the functor $\rho^* : \mathcal{E}_{\text{syn}}\text{-mod}_{c,K} \rightarrow D^b(\text{Rep}_{\text{pst}}(G_K))$. Note that \mathcal{D} is stable under taking retracts, suspensions, and extensions (see Remark 5.14). We first prove that for any smooth projective morphism $f : X \rightarrow \text{Spec}(K)$ and any integer $r \in \mathbf{Z}$, the representation $H_{\text{ét}}^i(X_{\overline{K}}, \mathbf{Q}_p(r))$ belongs to \mathcal{D} .

The complex or representations $R\Gamma_{\text{pst}}(X_{\overline{K}}, r) \simeq Rf_*(\mathbf{Q}_p)(r)$ belongs to \mathcal{D} (according to the end of Theorem 5.13). Moreover, using [25, 4.1.1] and [24], there exists an isomorphism in $D^b(\text{Rep}_{\text{pst}}(G_K))$:

$$Rf_*(\mathbf{Q}_p)(r) \simeq \bigoplus_{i \in \mathbf{Z}} R^i f_*(\mathbf{Q}_p)(r)[-i].$$

This means that $R^i f_*(\mathbf{Q}_p)(r)$ is the kernel of a projector of $R f_*(\mathbf{Q}_p)(r)$, thus belongs to \mathcal{D} because the later is stable under taking retracts.

Thus the result follows, using the fact that \mathcal{D} is stable under taking extensions in $D^b(\text{Rep}_{\text{pst}}(G_K))$. \square

Remark 5.22. The preceding proof shows that the essential image \mathcal{D} of constructible syntomic modules in complexes of pst-representations contains arbitrary truncations of the complexes $R\Gamma_{\text{pst}}(X_{\overline{K}}, r)$. A natural question would be to determine if, more generally, \mathcal{D} is stable under taking truncation. This would immediately imply that $\text{Rep}_c(G_K)$ is a thick abelian subcategory of $\text{Rep}(G_K)$ (i.e. it is stable under taking sub-objects and quotients) and that \mathcal{D} is the category of bounded complexes of pst-representations whose cohomology groups are constructible in the above sense.

Remark 5.23. In the sequence of inclusions

$$\text{Rep}_{gm}(G_K) \subset \text{Rep}_{Ngm}(G_K) \subset \text{Rep}_c(G_K) \subset \text{Rep}_{\text{pst}}(G_K)$$

we believe that the first one is an equality and the following two are strict. We can support this belief with the following observations. The first inclusion should be an equality since the category of Nori's motives is expected to be the heart of a motivic t -structure on $DM_{gm}(K, \mathbf{Q}_p)$ (see [41, p. 374]). The second inclusion should be strict by the philosophy of weights: by Proposition 5.21, we allow all potentially semistable extensions as extensions of certain geometric representations in the constructible category but in the geometric category such extensions should satisfy a weight filtration condition. For properties of geometric representations coming from abelian varieties over \mathbf{Q}_p see the work of Volkov [65], [66].

For the third inclusion, take $k = F_q$, the finite field with $q = p^s$ elements. Let $V \in \text{Rep}_c(G_K)$ be a constructible representation. Then, by the Conjecture of purity of the weight filtration, the φ -module $D_{\text{pst}}(V)$ is an extension of "pure" φ -modules, i.e., φ -modules such that, for a number $a \geq s$, φ^a has eigenvalues that are p^a -Weil numbers¹⁷ (c.f., [40, Conjecture 2.6.5]). But there are crystalline representations that do not have this property. For example, any unramified character $\chi : G_{K_0} \rightarrow \mathbf{Q}_p^*$, $Fr \mapsto \mu \in \mathbf{Q}_p^*$, such that μ is not a p^a -Weil number for any $a \geq 0$ (such a μ exists by a uncountability of \mathbf{Q}_p).

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