Cut a cake fairly: not so easy...

7 juin 2007

Plan

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Context

This work has been done with Lionel Eyraud.

- If the cake is homogeneous, the problem is a geometrical one (it may be complicated !)
- We are interested here in cake division as an alternative method for dealing with the fairness concept when the users have their own metrics (participants have their own view on the cake).

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Origine of the problem

Old problem.
First paper in Computer Science in 1948 (Hugo Steinhaus).
Many results in teh decade 60-70.
Regular results (several papers each year).

Division in two pieces

To avoid family quarrels.

There exists a simple and well-known solution : ask one to cut, let the other choose.

In the worst case, each will have at least half of the cake according to his-her own criterion.

Remark : This method guarantees the fairness, but it is not symmetric. The one who cuts will have exactly half of the cake, the other usually get more.

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Notion of "measure"

Goal : to represent the diversity of the criteria.
Each "player" has his-her own measure, i.e. he-she is able to give a mark on each part of the cake.
The cake is modelized by an interval. The measure is defined on this interval.

Property.

A measure is continuous and additive :
For all parts, P and P', $m(P) + m(P') = m(P \cup P')$.

Example

A Christmas cake.

The measures are normalized (i.e. the grad on the whole cake, corresponding to the entire interval, is equal to 1).

Notion of fairness

The cut is fair if and only if each player get a piece of cake whose mark is at least $\frac{1}{n}$ according to his-her own measure.

Variants :

- $\forall i, \quad m_i(P_i) \geq \frac{1}{n}$
- $\forall i, j, \quad m_i(P_i) \geq m_i(P_j)$ (envy-free)
- The existence is not easy to prove
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Moving Knife

Stromquist 1980.

Principle : Consider an external referee.

The referee places the knife on the left side of the interval (cake) and slowly moves it to the right. As soon as a player says "STOP", the referee cuts and gives the left piece.

The game continues until all participants have received their piece.

With n players, this method guarantees the fairness with only $n-1$ cuts (which is of course the minimum).

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Analysis (1)

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It guarantees to obtain a "good" piece, independently of the others.
Bluffer (wait more before saying **stop**) is risky and may not lead to a good solution (think to case where all players have the same measure...).

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Analysis (2)

Property.

moving knife is a fair strategy, but it is not envy-free.

Summarize

- $n - 1$ cuts
- Fair, but not envy-free
- need an infinity of measure evaluations !

Finite protocol and strategies

We define a **cutting protocol** as an interactive procedure, composed of successive steps.
At each step, the protocol can satisfied the requests of some players whose answers can influence further decisions.

Properties.

(1) If everyone follow the protocol, then, he-she will finish with a piece after a finite number of steps.
(2) As soon as someone cut a piece, he-she must do without any interaction with the others.
(3) There is no reliable information about the measures of the other players.

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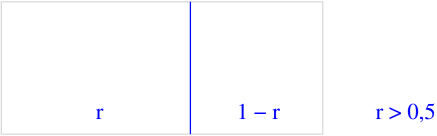
cutting with a minimum number of cuts

For 3 players, every fair solution obtained by a protocol need at least 3 cuts (the trivial lower bound is 2).

Let us prove the result by constructing an adversary :



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r	1 – r

$r > 0{,}5$

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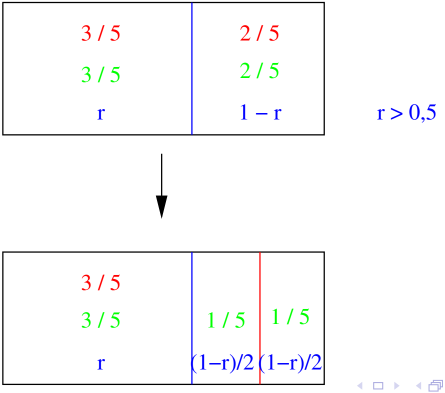

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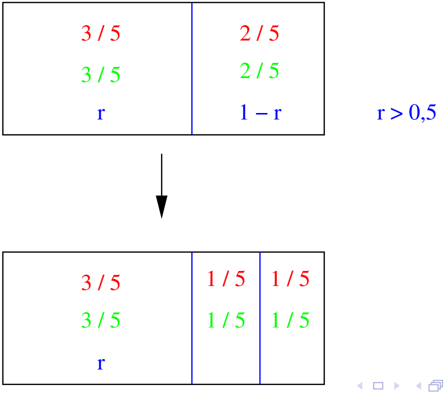

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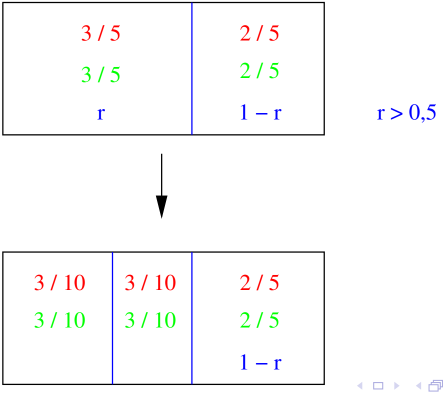

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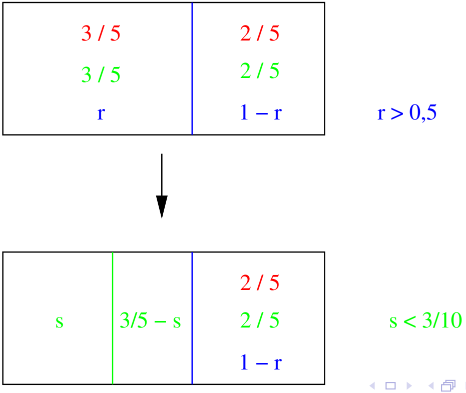

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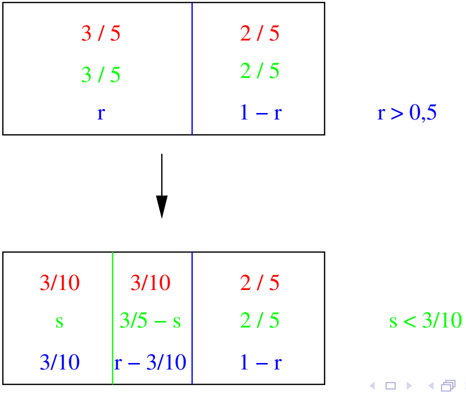
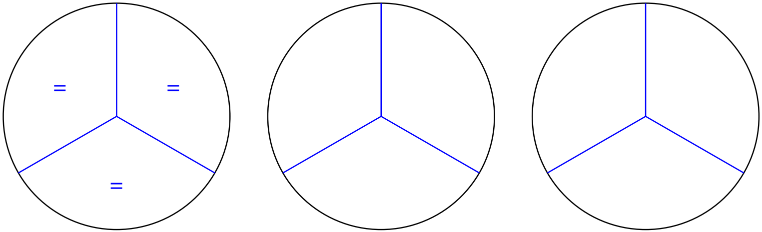


Protocol for 3 players

Let us consider 3 persons : Denis, Fredo and Yves, denoted by A, B et C. The idea here is to extend the protocol for 2 players "**one cuts – the other chooses**".

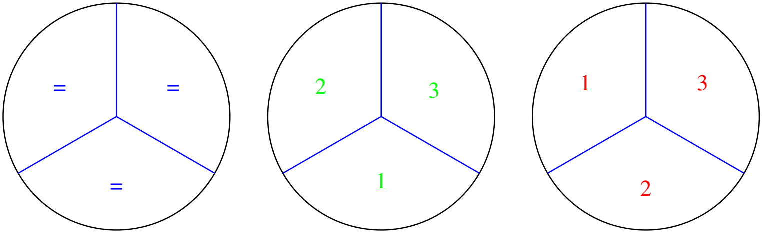
Description

Le joueur Bleu coupe en trois parts qu'il juge égales



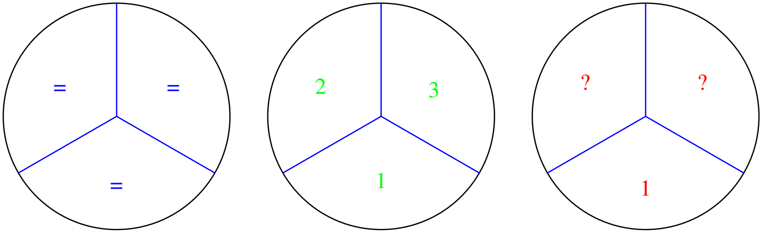
Description

Les deux autres annoncent leur classement.



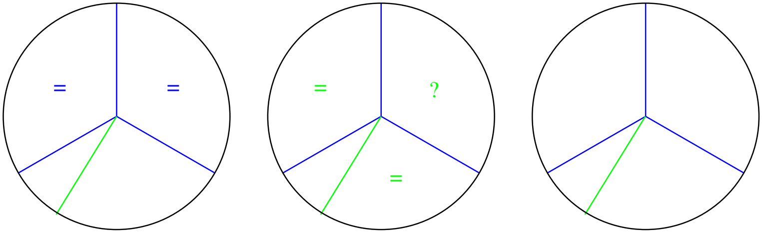
Description

Cas difficile : ils se disputent la même part.



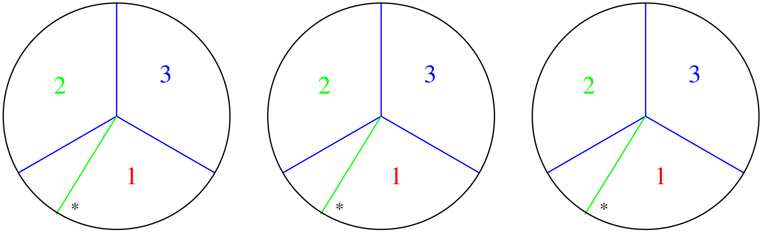
Description

Vert diminue la grosse part pour qu'elle soit égale à la deuxième.



Description

Chacun choisit une part, de droite à gauche.

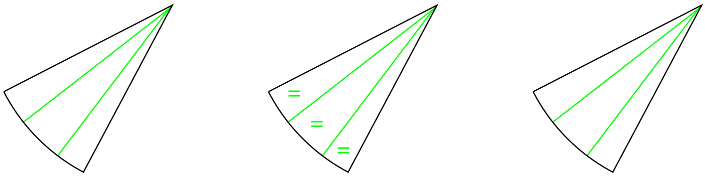


Si Rouge n'a pas pris la part *, Vert la prend forcément.

Description

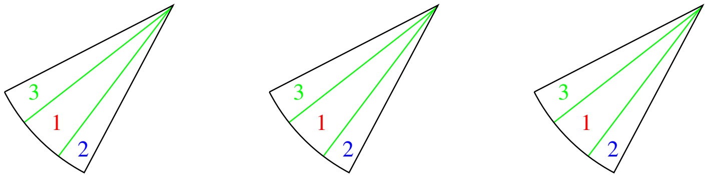
Il reste à partager le petit bout.

Celui qui n'a pas choisi * (ici Vert) le coupe en trois parts égales.



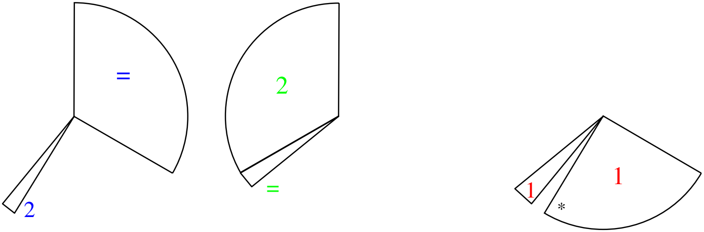
Description

Choix: d'abord celui qui a choisi * (Rouge),
puis Bleu, puis celui qui a coupé (Vert)



Description

Voilà le résultat. Tout le monde pense avoir eu la plus grosse part.



Analysis of fairness (1)

Proof by case analysis.

Let assume that the piece chosen by C is b

then, we have : $mC(b) \geq mC(a1)$ and $mC(a3)$

In this case, B cuts the piece r into three parts such that : $mB(r1) = mB(r2) = mB(r3)$.

C choses first a piece among these pieces, then, A chooses and finally B.

C has piece b plus the better piece of r according to his-her own measure. Let suppose it is r1 ($mC(r1) \geq mC(r2)$ and $mC(r3)$).

A get piece a3 plus one over the remaining pieces of r (say r2).

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Analysis of fairness (2)

$mA(a3 + r2) \geq \frac{1}{3}$ because $mA(a3) = \frac{1}{3}$

$mC(b + r1) \geq \frac{1}{3}$

$mC(b + r1) = mC(b) + mC(r1)$

$mC(b) \geq mC(a2)$ and $mC(a3)$

Thus, $mC(b) \geq \frac{mC(b)+mC(a2)+mC(a3)}{3}$

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Thus, $mC(r1) \geq \frac{mC(r)}{3}$

We get the result because

$mC(b) + mC(r) + mC(a2) + mC(a3) = 1$.

Similarly for B.

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C is not jealous

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Indeed, C prefered b to a2 and he-she has chosen r1 rather than r3.
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Extensions

There exists a not-trivial extension :
a fair envy-free protocol with 11 cuts for 4 players (1997).
Fair algorithm in $O(nlog(n))$ cuts for n players.
Interesting recent paper on divide and conquer (recursive) approaches for cutting with minimum number of cuts (minimizing the envy).