Online scheduling

Frédéric Vivien

Frederic.Vivien@inria.fr

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Outline

1. Introduction and first results
2. Lower bound on the competitive ratio of any algorithm: the clairvoyant max-stretch case
3. The non-clairvoyant case
4. How to derive a lower bound: the max-flow case with communications

Outline

Offline vs. online algorithmics

Nature of the problem

Known

Objective function

Known

Characteristics of the instance

Known beforehand

Offline

Offline vs. online algorithmics

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Characteristics of a job discovered

When the job is released

Offline

(Clairvoyant) Online

Offline vs. online algorithmics

Nature of the problem

Known

Objective function

Known

Characteristics of the instance

Known beforehand	Discovered during execution	
	Characteristics of a job discovered	
	When the job is released	When the job completes
Offline	**(Clairvoyant) Online**	**Non-clairvoyant online**

Notation and hypotheses

Notation

- Jobs J_1, ..., J_n
  Job J_j arrives in the system at the *release* date r_j
  Job J_j has a weight (or a priority) w_j
  Job J_j has an execution time p_j
  Δ is the ratio of the largest to the shortest execution time
- Completion time of job J_j: C_j
  Flow of job J_j: $F_j = C_j - r_j$ (time spent in the system)

Hypotheses

- Jobs may be preempted
- One machine (1 | *pmtn* | ???)

What should we optimize?

- Makespan: $\max_j C_j$

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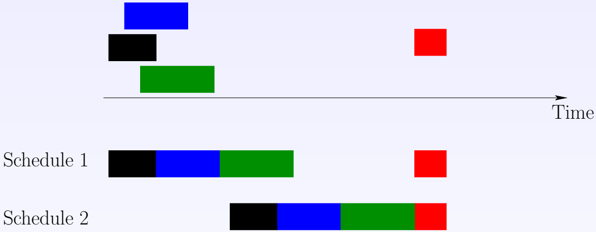
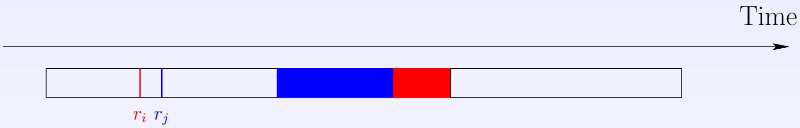


What should we optimize?

- Makespan: $\max_j C_j$
  Release dates are not taken into account
- Average flow or response time: $\sum_j (C_j - r_j)$
  Inconvenient: starvation
- Maximum flow or maximum response time: $\max_j (C_j - r_j)$
  No starvation. Favor long jobs. Worst-case optimization.
- Maximum weighted flow: $\max_j w_j (C_j - r_j)$
  Gives back some importance to short jobs.
  Particular case of the *stretch* or *slowdown*:
  w_j=1/running time of the job on empty platform.

FIFO is optimal for max-flow

Consider any instance and a schedule Θ s.t. there exists two jobs executed consecutively: J_i and J_j with $r_i < r_j$ and $C_i \geq C_j$



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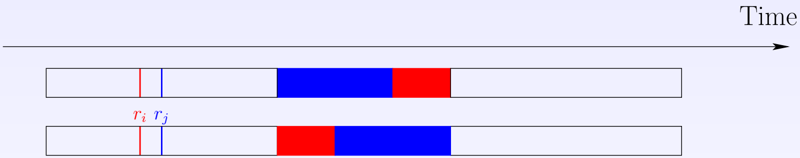


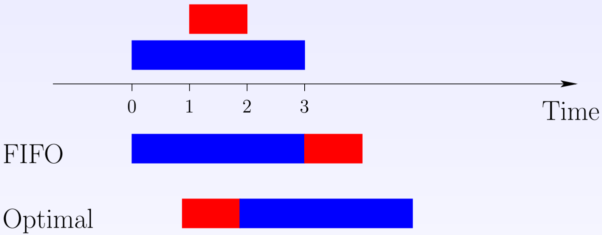
In schedule Θ' we exchange the execution order of J_i and J_j

$$\max_{1 \leq k \leq n} C'_k - r_k = \max\{\max_{\substack{1 \leq k \leq n \\ k \notin \{i,j\}}} C_k - r_k, C'_i - r_i, C'_j - r_j\}$$

$$C'_i - r_i \leq C_i - r_i \quad \text{and} \quad C'_j - r_j = C_i - r_j < C_i - r_i$$

$$\Rightarrow \qquad \max_{1 \leq k \leq n} C'_k - r_k \leq \max_{1 \leq k \leq n} C_k - r_k$$

FIFO is sub-optimal for max-stretch



Max-stretch of FIFO: $\max\{1, \frac{4-1}{1}\} = 3$.

Optimal max-stretch: $\max\{\frac{5-0}{3}, 1\} = \frac{5}{3}$.

Evaluating the quality of an online schedule

An online algorithm has a competitive factor ρ if and only if

Whatever the set of jobs J_1, ..., J_n

$$\text{Online schedule cost}(J_1, ..., J_N) \leq$$
$$\rho \times \text{Optimal off-line schedule cost}(J_1, ..., J_N)$$

The case of list schedules (1/2)

A peculiar framework: tasks are presented one by one to the scheduler that must schedule each task on a processor before seeing the next submitted task (online-list).

Theorem

Any list scheduling algorithm is $2 - \frac{1}{p}$-competitive for the online minimization of the makespan on p processors, and this bound is tight.

The case of list schedules (2/2)

Theorem

If the platform contains 2 or 3 processors (i.e., $p = 2$ or $p = 3$), then any list scheduling algorithm achieves the best possible competitive ratio for the online minimization of the makespan.

$p = 2$. We consider the instances $\mathcal{I}_1 = (1, 1)$ and $\mathcal{I}_2 = (1, 1, 2)$.

$p = 3$. We consider three instances: $\mathcal{I}_1 = (1, 1, 1)$, $\mathcal{I}_2 = (1, 1, 1, 3, 3, 3)$, and $\mathcal{I}_3 = (1, 1, 1, 3, 3, 3, 6)$.

FIFO competitiveness

Theorem

First come, first served is:

- *optimal for the online minimization of max-flow*
- Δ*-competitive for the online minimization of sum-flow*
- Δ*-competitive for the online minimization of max-stretch*
- Δ^2*-competitive for the online minimization of sum-stretch*

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FIFO competitiveness for max-stretch

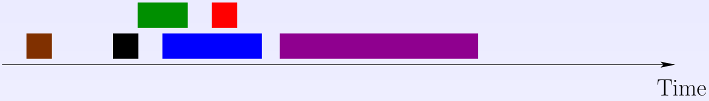
Theorem

FIFO is Δ competitive for maximum stretch minimization

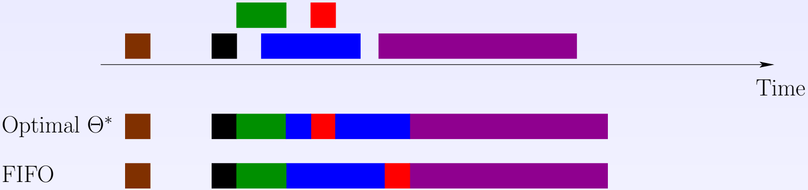
This means that

1. FIFO has a competitive factor of Δ (i.e., on no instance is FIFO's max-stretch more than Δ that of the optimal solution)
2. This bound is tight (=cannot be improved)

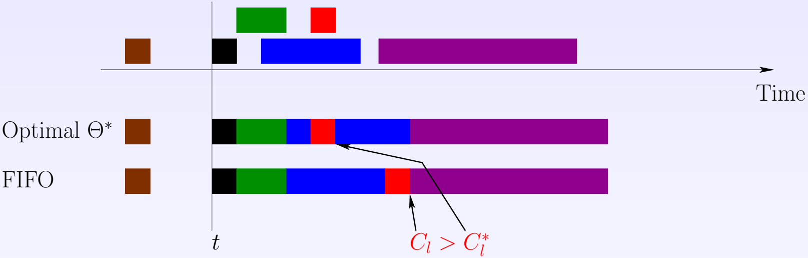
Upper bound for max-stretch



Upper bound for max-stretch



Upper bound for max-stretch

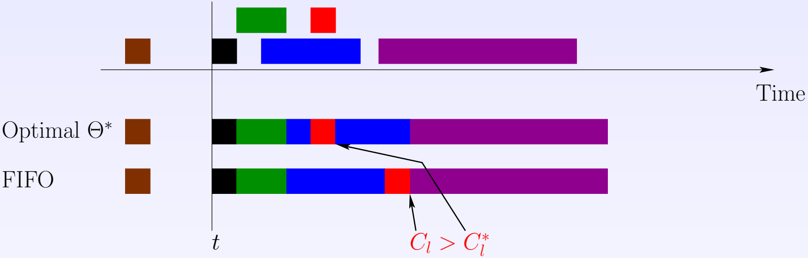


Any job J_l s.t. $\mathcal{S}_l > \mathcal{S}_l^*$ ($\Leftrightarrow C_l > C_l^*$)

t last time before C_l s.t. the processor was idle under FIFO.

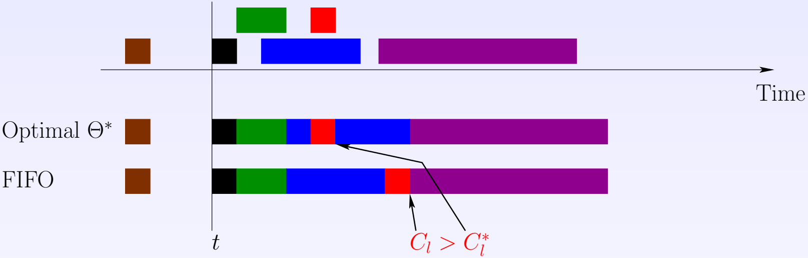
t is the release date r_i of some job J_i.

Upper bound for max-stretch



Any job J_l s.t. $\mathcal{S}_l > \mathcal{S}_l^*$ ($\Leftrightarrow C_l > C_l^*$)
During $[r_i, C_l]$, FIFO exactly executes J_i, J_{i+1}, ..., J_{l-1}, J_l.

Upper bound for max-stretch

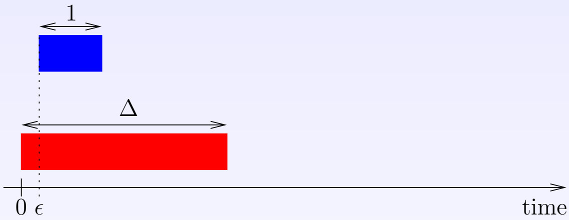


Any job J_l s.t. $\mathcal{S}_l > \mathcal{S}_l^*$ ($\Leftrightarrow C_l > C_l^*$)
During $[r_i, C_l]$, FIFO exactly executes J_i, J_{i+1}, ..., J_{l-1}, J_l.
As $C_l^* < C_l$, there is a job J_k, $i \leq k \leq l-1$ s.t. $C_k^* \geq C_l$. Then:

$$\mathcal{S}^* = \max_j \mathcal{S}_j^* \geq \mathcal{S}_k^* = \frac{C_k^* - r_k}{p_k} \geq \frac{C_l - r_l}{p_k} = \frac{C_l - r_l}{p_l}\frac{p_l}{p_k} \geq \mathcal{S}_l \times \frac{1}{\Delta}$$

$$\forall l, \mathcal{S}_l > \mathcal{S}_l^* \quad \Rightarrow \quad \Delta \times \mathcal{S}^* \geq \mathcal{S}_l$$

The bound is tight



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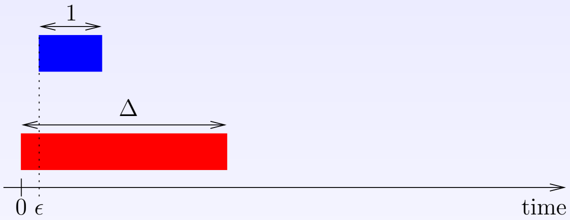
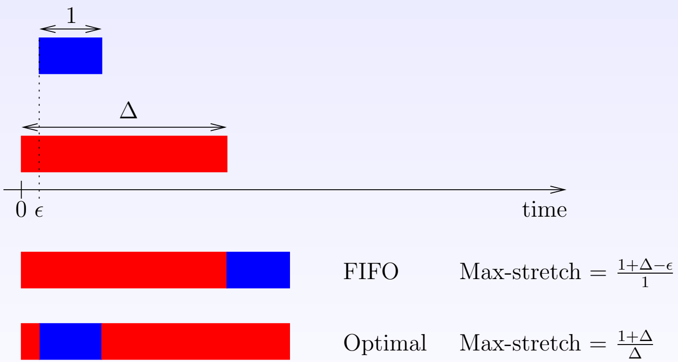


FIFO Max-stretch $= \frac{1+\Delta-\epsilon}{1}$

The bound is tight



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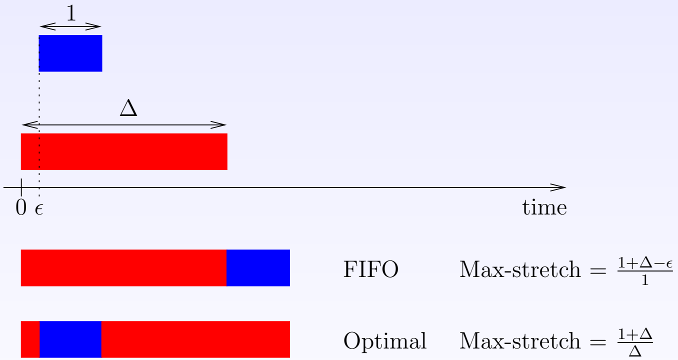

Competitive ratio: $\frac{1+\Delta-\epsilon}{\frac{1+\Delta}{\Delta}} = \Delta\frac{1+\Delta-\epsilon}{1+\Delta} = \Delta - \epsilon\,\frac{\Delta}{1+\Delta} \geq \Delta - \epsilon$

Outline

Bound on the competitive ratio

Theorem

On one processor, any online scheduling algorithm with preemption minimizing the max-stretch has a competitive ratio greater than $\frac{1}{2}\Delta^{\sqrt{2}-1}$, if the system receives at least jobs of three different sizes, and if Δ is the ratio between the size of the largest and the smallest job.

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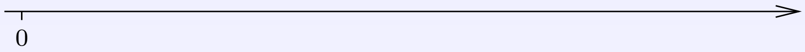
Theorem

On one processor, any online scheduling algorithm with preemption minimizing the max-stretch has a competitive ratio greater than $\frac{1}{2}\Delta^{\sqrt{2}-1}$*, if the system receives at least jobs of three different sizes, and if* Δ *is the ratio between the size of the largest and the smallest job.*

Proof principle: by contradiction we assume that there exists an algorithm and we build a sequence of jobs and a scenario to make the algorithm fail.

The adversary



The adversary

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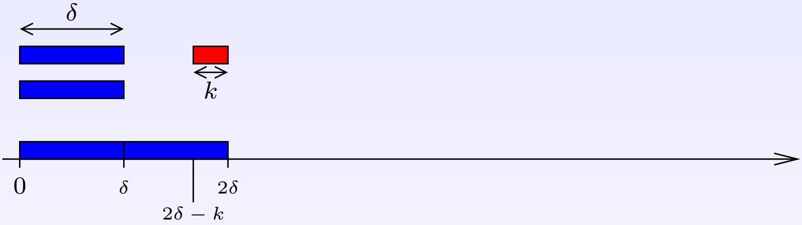
Achievable stretch: $\dfrac{2\delta - 0}{\delta} = 2.$

The adversary



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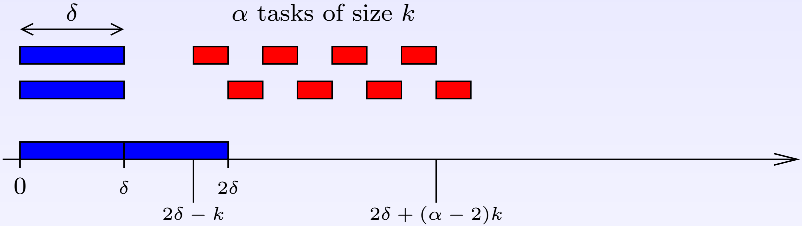
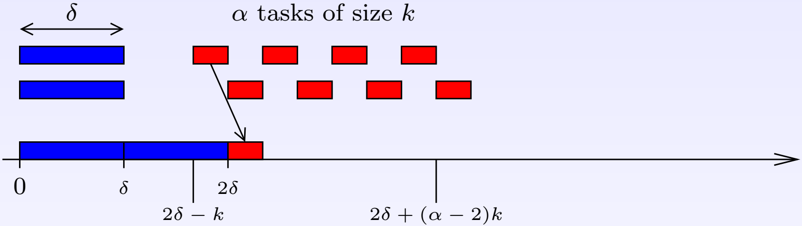


The job J_{2+j} arrives at time $2\delta + (j-2)k$.

The adversary



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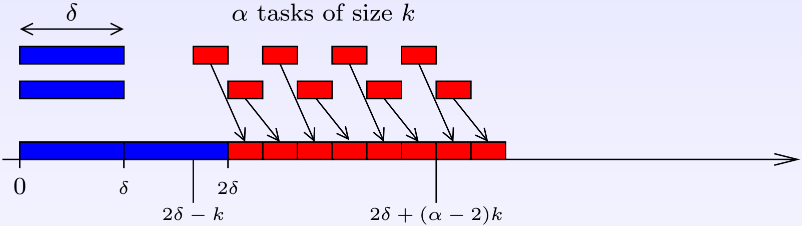
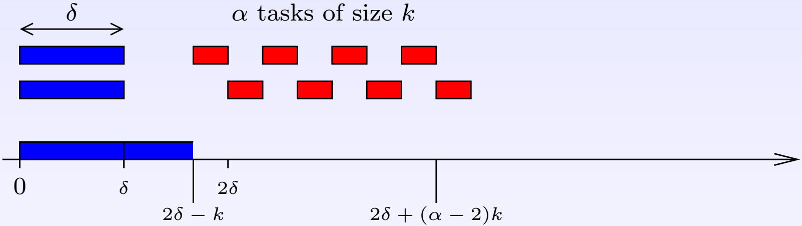


The job J_{2+j} arrives at time $2\delta + (j-2)k$.

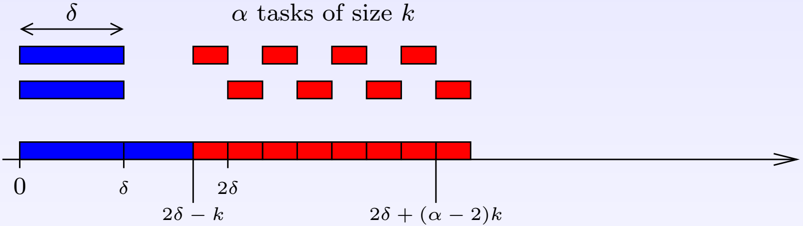
Achievable stretch: $\dfrac{(2\delta + jk) - (2\delta + (j-2)k)}{k} = 2.$

The adversary



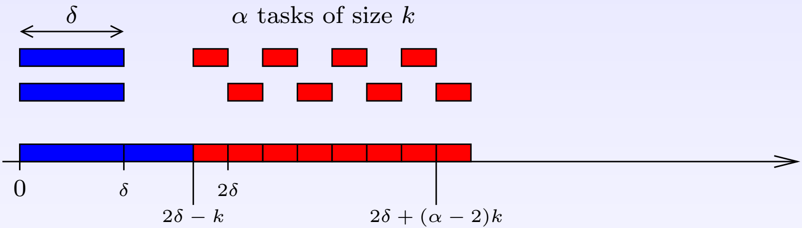
In practice: we do not know what happens after $2\delta - k$.

The adversary



We want to forbid this case (each size-k job being executed at its release date).

The adversary



We want to forbid this case (each size-k job being executed at its release date).

The algorithm being $\frac{1}{2}\Delta^{\sqrt{2}-1}$-competitive, J_1 and J_2 must be completed at the latest at time: $2 \cdot \frac{1}{2}\Delta^{\sqrt{2}-1} \cdot \delta = 2 \cdot \frac{1}{2}\left(\frac{\delta}{k}\right)^{\sqrt{2}-1} \cdot \delta$

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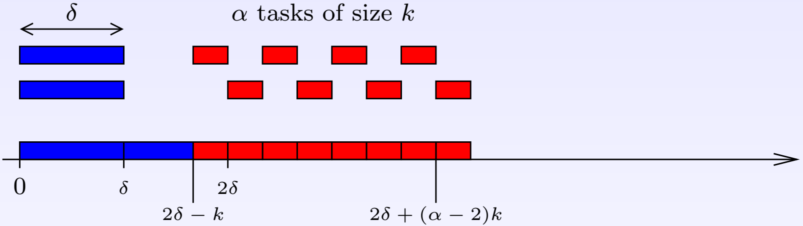

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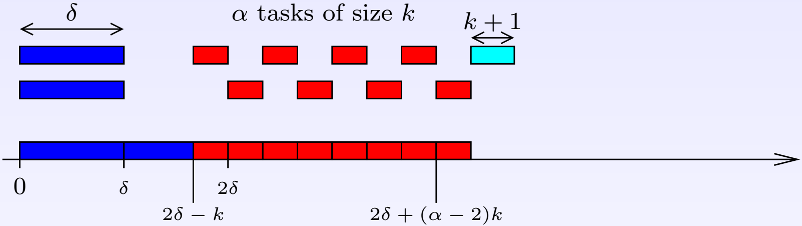
We let $\alpha = \left\lceil 1 + k - \frac{2\delta}{k} \right\rceil$ and then $2\delta + (\alpha - 1)k \geq 2 \cdot \frac{1}{2}\left(\frac{\delta}{k}\right)^{\sqrt{2}-1} \cdot \delta$.

The adversary



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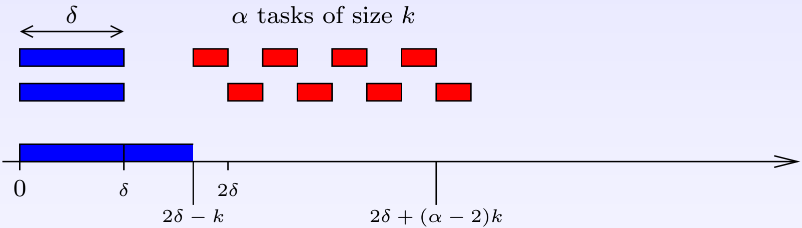

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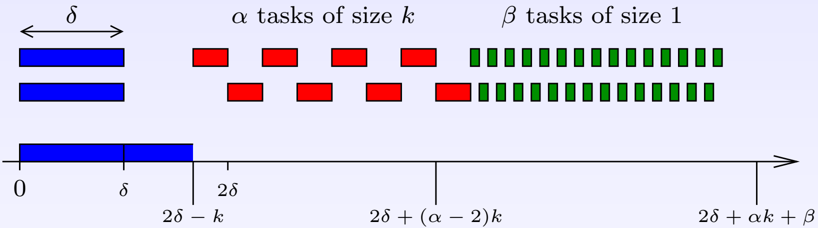
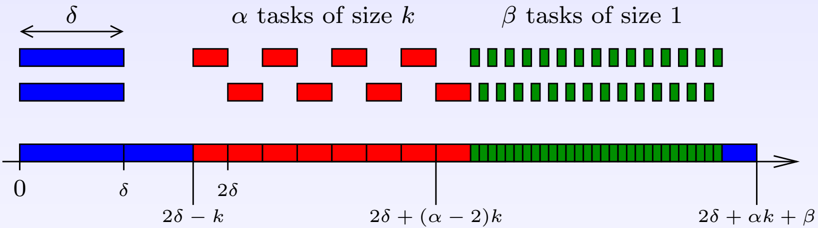


The job $J_{2+\alpha+j}$ arrives at time $2\delta + (\alpha - 1)k + (j - 1)$.

The adversary



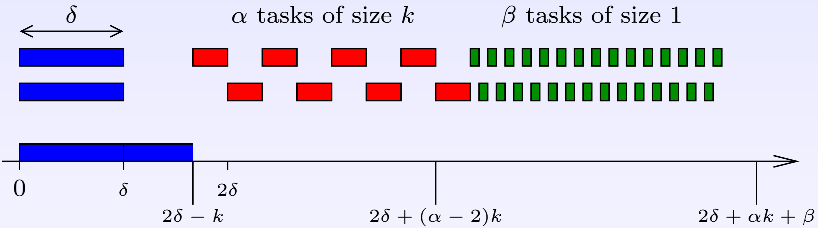
Achievable stretch (off-line)

Stretch of each job of size k or 1: 1.

Stretch of J_1 or J_2: $\dfrac{2\delta + \alpha k + \beta}{\delta}$

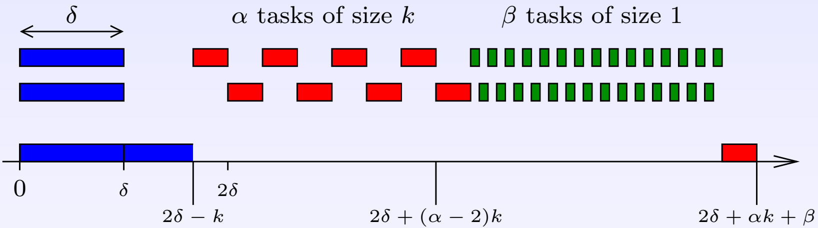
Optimal stretch $\leq \dfrac{2\delta + \alpha k + \beta}{\delta}$

The adversary



Achievable stretch (online)

The adversary

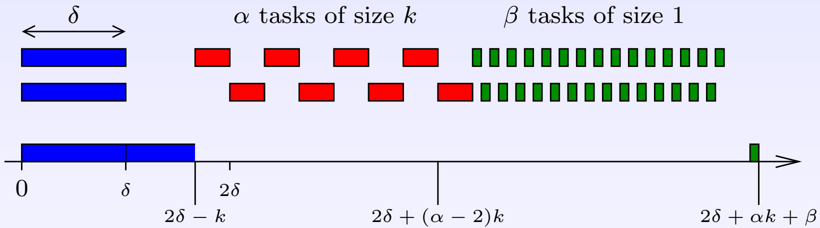


Achievable stretch (online)

The last completed job is of size k.

$$\text{Stretch} \geq \frac{(2\delta + \alpha k + \beta) - (2\delta + (\alpha - 2)k)}{k} = 2 + \frac{\beta}{k}.$$

The adversary

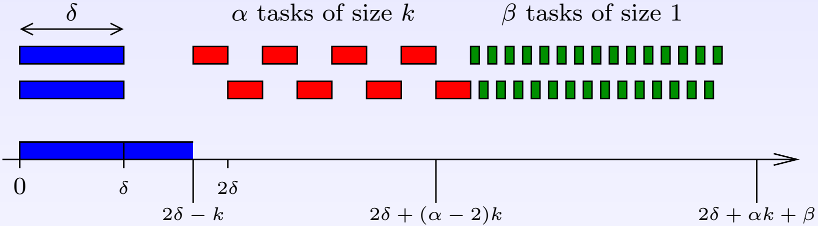


Achievable stretch (online)

The last completed job is of size 1.

$$\text{Stretch} \geq \frac{(2\delta + \alpha k + \beta) - (2\delta + (\alpha - 1)k + (\beta - 1))}{1} = k + 1.$$

The adversary



Achievable stretch (online)

$$\text{Stretch} \geq \min\left\{2 + \frac{\beta}{k}, k + 1\right\}$$

We let: $\beta = \lceil k(k-1) \rceil$

Then: stretch $\geq k + 1$.

The adversary: summing things up

$$\alpha = \left\lceil 1 + k - \frac{2\delta}{k} \right\rceil$$

$$\beta = \lceil k(k-1) \rceil$$

Optimal stretch $\leq \dfrac{2\delta + \alpha k + \beta}{\delta}$

Achieved stretch $\geq k + 1$.

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We let $k = \delta^{2-\sqrt{2}}$

The adversary: summing things up

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Optimal stretch $\leq \dfrac{2\delta + \alpha k + \beta}{\delta}$

Achieved stretch $\geq k + 1$.

We let $k = \delta^{2-\sqrt{2}}$

Therefore $k + 1 > \left(\frac{1}{2}\delta^{\sqrt{2}-1}\right)\left(\frac{2\delta + \alpha k + \beta}{\delta}\right)$

Outline

FIFO competitiveness

Theorem

First come, first served is:

- *optimal for the online minimization of max-flow*
- Δ*-competitive for the online minimization of sum-flow*
- Δ*-competitive for the online minimization of max-stretch*
- Δ^2*-competitive for the online minimization of sum-stretch*

Lower bound as a function of n

Theorem

There is no c-competitive preemptive online algorithm minimizing the maximum stretch with $c < n$

Principle of the proof

- We suppose there exists an algorithm whose ratio $c = n - \epsilon$
- n jobs are released at time 0
- Whatever the scheduler does, no job completes before time n
- Jobs are sorted by non-decreasing cumulative computation time computed at time n: the i-th job is of size λ^{i-1}
- The maximum stretch is at least n (first job has size 1 and is not completed at n)
- Optimal: execute jobs in Shortest Processing Time first order:

$$\frac{\sum_{j=1}^{i} \lambda^{j-1}}{\lambda^{i-1}} = \frac{\lambda^i - 1}{\lambda^{i-1}(\lambda - 1)} \xrightarrow[\lambda \to +\infty]{} 1$$

EquiPartition

Theorem

EquiPartition is n-competitive for the minimization of maximum stretch.

However, EquiPartition is at best $\frac{\Delta+1}{2+\ln(\Delta)}$ competitive (when FIFO is Δ competitive)

Outline

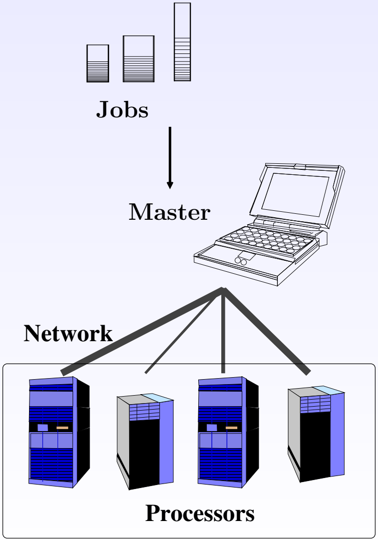
The scheduling problem

The scheduler

- Gather the jobs
- Send them to the processors

The aim

Distribute the *identical* jobs to the processors, for the jobs to be processed in the best possible way

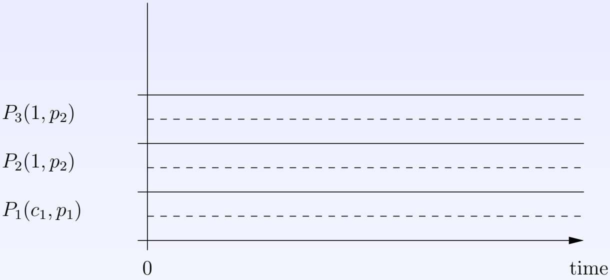


The scheduling problem

Formally

- n jobs, m processors
- p_j: processing time of a job on processor j
- c_j: time to send a job from the master to the worker j
- r_i: release date of job J_i
- C_i: completion time of job J_i
- The objective function:
  - maximal flow: $\max C_i - r_i$

Finding a lower bound on the competitiveness (1)



Idea:

- A fast processor with slow communications ($c_1 > 1$)
- Two identical and slow processors, with fast communications
- If only one job, one must choose the fast processor ($c_1 + p_1 < 1 + p_2$)

Finding a lower bound on the competitiveness (1)

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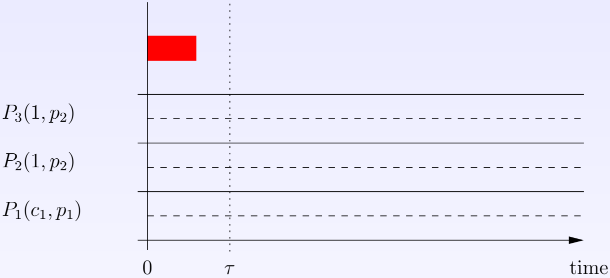
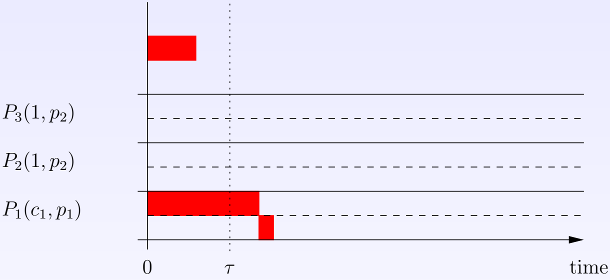


We look at time $\tau \geq 1$ to see what has happened. Three possibilities:

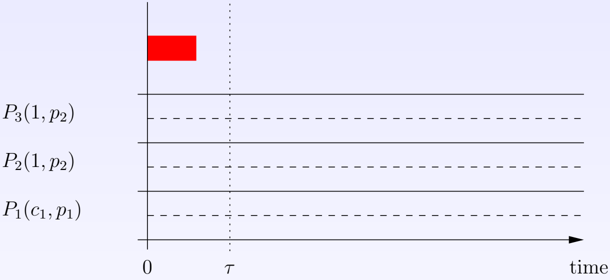
Finding a lower bound on the competitiveness (1)



We look at time $\tau \geq 1$ to see what has happened. Three possibilities:

1. Optimal: job on P_1, max-flow $\geq c_1 + p_1$.

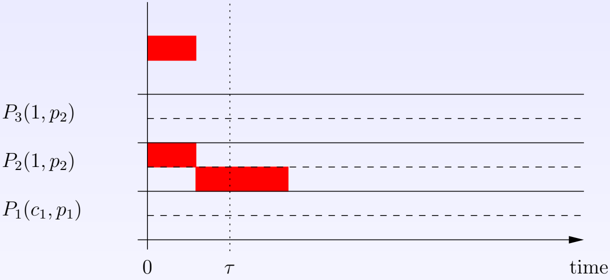
Finding a lower bound on the competitiveness (1)



We look at time $\tau \geq 1$ to see what has happened. Three possibilities:

1. Optimal: job on P_1, max-flow $\geq c_1 + p_1$.
2. Nothing done: max-flow $\geq \tau + c_1 + p_1$, ratio $\geq \frac{\tau+c_1+p_1}{c_1+p_1}$.

Finding a lower bound on the competitiveness (1)

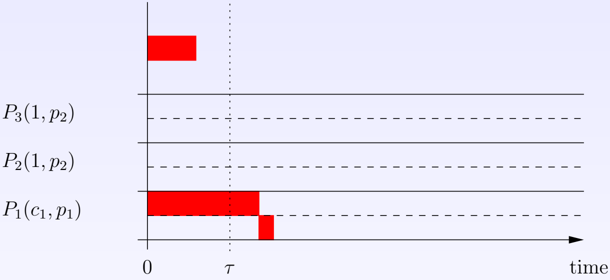


We look at time $\tau \geq 1$ to see what has happened. Three possibilities:

1. Optimal: job on P_1, max-flow $\geq c_1 + p_1$.
2. Nothing done: max-flow $\geq \tau + c_1 + p_1$, ratio $\geq \frac{\tau+c_1+p_1}{c_1+p_1}$.
3. Job sent to P_2, max-flow $\geq 1 + p_2$. Ratio $\geq \frac{1+p_2}{c_1+p_1}$.

We want to force the algorithm to process the first job on P_1.

Finding a lower bound on the competitiveness (1)

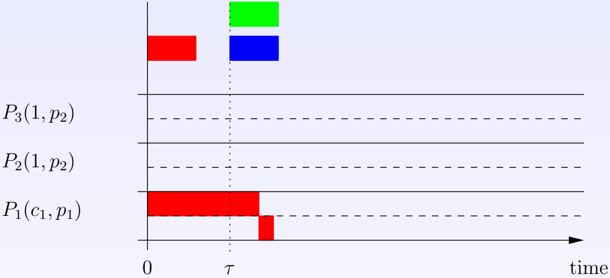


We look at time $\tau \geq 1$ to see what has happened. If the scheduler did not pick the first possibility, the adversary sends no more jobs. Later we will choose τ, c_1, p_1 and p_2 such that the ratio achieved,

$$\min \left\{ \frac{1 + p_2}{c_1 + p_1}, \frac{\tau + c_1 + p_1}{c_1 + p_1} \right\},$$

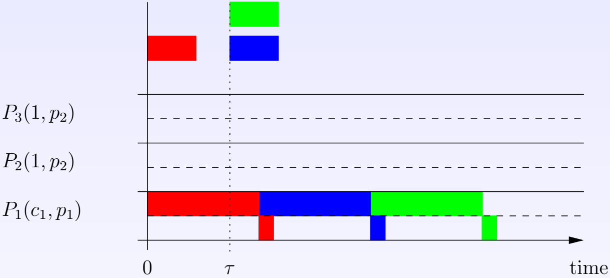
is as large as possible.

Finding a lower bound on the competitiveness (1)



At time τ we send two new jobs.
We consider all the possible cases.

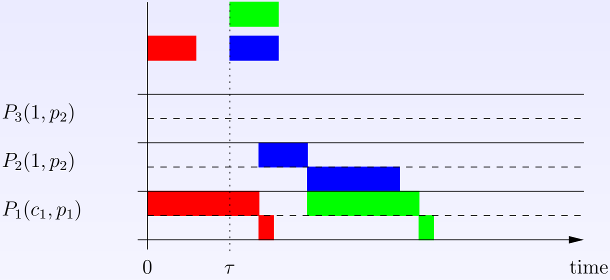
Finding a lower bound on the competitiveness (1)



At time τ we send two new jobs.
The two jobs are executed on P_1:

$$\max\{c_1 + p_1,$$
$$\max\{\max\{c_1, \tau\} + c_1 + p_1, c_1 + 2p_1\} - \tau,$$
$$\max\{\max\{c_1, \tau\} + c_1 + p_1 + \max\{c_1, p_1\}, c_1 + 3p_1\} - \tau\}$$

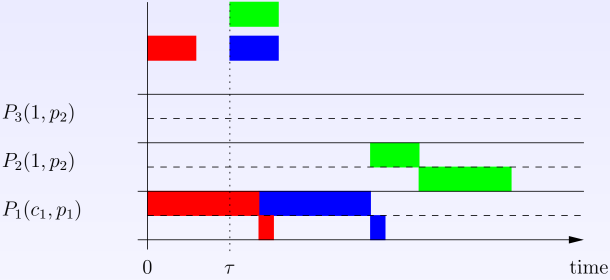
Finding a lower bound on the competitiveness (1)



At time τ we send two new jobs.
The first of the two jobs is executed on P_2 (or P_3), and the other one on P_1.

$$\max\{c_1 + p_1,\\ (\max\{c_1, \tau\} + c_2 + p_2) - \tau,\\ \max\{\max\{c_1, \tau\} + c_2 + c_1 + p_1, c_1 + 2p_1\} - \tau\}$$

Finding a lower bound on the competitiveness (1)

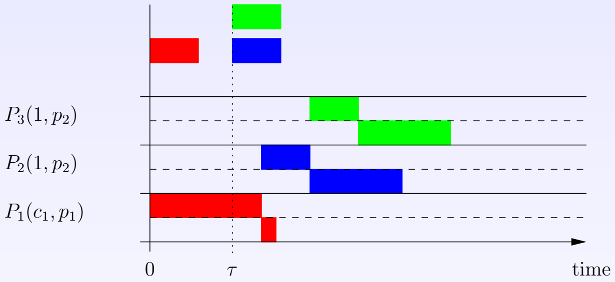


At time τ we send two new jobs.

The first of the two jobs is executed on P_1, and the other one on P_2 (or P_3).

$$\max\{c_1 + p_1,\\ \max\{\max\{c_1, \tau\} + c_1 + p_1, c_1 + 2p_1\} - \tau,\\ (\max\{c_1, \tau\} + c_1 + c_2 + p_2) - \tau\}$$

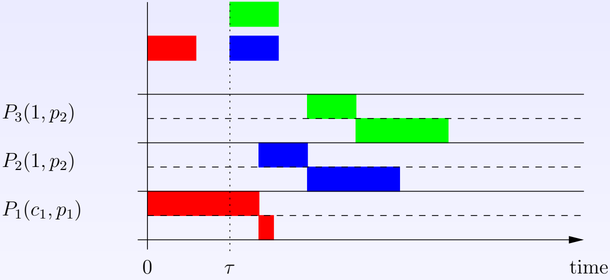
Finding a lower bound on the competitiveness (1)



At time τ we send two new jobs.
One of the two jobs is executed on P_2 and the other one on P_3.

$$\max\{c_1+p_1, (\max\{c_1, \tau\}+c_2+p_2)-\tau, (\max\{c_1, \tau\}+c_2+c_2+p_2)-\tau\}$$

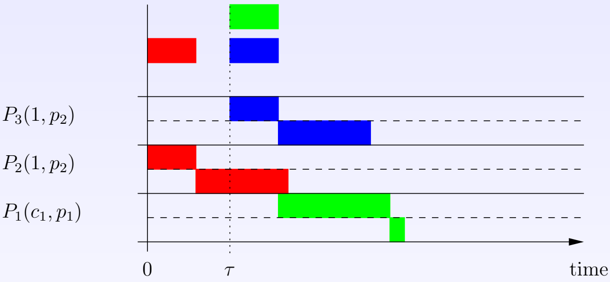
Finding a lower bound on the competitiveness (1)



At time τ we send two new jobs.

The case where both jobs are executed on P_2 (or both on P_3) is worse than the previous one, therefore, we do not need to study it.

Finding a lower bound on the competitiveness (1)



At time τ we send two new jobs.
The (desired) optimal: the first job on P_2, the second on P_3, and the third on P_1.

$$\max\{c_2+p_2, (\max\{c_2,\tau\}+c_2+p_2)-\tau, (\max\{c_2,\tau\}+c_2+c_1+p_1)-\tau\}$$

Finding a lower bound on the competitiveness (2)

Lower bound on the competitiveness of any online algorithm:

$$\min\begin{cases} \frac{\tau+c_1+p_1}{c_1+p_1}, \\ \\ \frac{1+p_2}{c_1+p_1}, \\ \\ \dfrac{\min\begin{cases} \max\{c_1+p_1, \max\{\max\{c_1,\tau\}+c_1+p_1, c_1+2p_1\}-\tau, \\ \qquad \max\{\max\{c_1,\tau\}+c_1+p_1+\max\{c_1,p_1\}, c_1+3p_1\}-\tau\} \\ \max\{c_1+p_1, (\max\{c_1,\tau\}+c_2+p_2)-\tau, \max\{\max\{c_1,\tau\}+c_2+c_1+p_1, c_1+2p_1\}-\tau\} \\ \max\{c_1+p_1, \max\{\max\{c_1,\tau\}+c_1+p_1, c_1+2p_1\}-\tau, (\max\{c_1,\tau\}+c_1+c_2+p_2)-\tau\} \\ \max\{c_1+p_1, (\max\{c_1,\tau\}+c_2+p_2)-\tau, (\max\{c_1,\tau\}+c_2+c_2+p_2)-\tau\} \end{cases}}{\max\{c_2+p_2, (\max\{c_2,\tau\}+c_2+p_2)-\tau, (\max\{c_2,\tau\}+c_2+c_1+p_1)-\tau\}} \end{cases}$$

Problem: to find τ, c_1, p_1, and p_2 (as $c_2=1$) which maximizes this lower bound.
Constraints: $c_1+p_1 < 1+p_2$.

Finding a lower bound on the competitiveness (3)

1. Numeric resolution

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2. Characterization of the shape of the optimal: $\tau < c_1$, $p_1 = 0$, etc.

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1. Numeric resolution
2. Characterization of the shape of the optimal: $\tau < c_1$, $p_1 = 0$, etc.
3. New system:

$$\min \begin{cases} \frac{\tau+c_1}{c_1} \\ \frac{1+p_2}{c_1} \\ \frac{\min\begin{cases} 3c_1 - \tau \\ c_1 + 1 - \tau + p_2 \\ 2c_1 - \tau + 1 + p_2 \\ c_1 + 2 + p_2 - \tau \end{cases}}{1+p_2} \end{cases} = \min \begin{cases} \dfrac{\tau + c_1}{c_1} \\ \dfrac{1+p_2}{c_1} \\ \dfrac{c_1 + 1 - \tau + p_2}{1+p_2} \end{cases}$$

Finding a lower bound on the competitiveness (3)

1. Numeric resolution
2. Characterization of the shape of the optimal: $\tau < c_1$, $p_1 = 0$, etc.
3. New system:

$$\min \begin{cases} \frac{\tau+c_1}{c_1} \\ \frac{1+p_2}{c_1} \\ \frac{\min \begin{cases} 3c_1 - \tau \\ c_1 + 1 - \tau + p_2 \\ 2c_1 - \tau + 1 + p_2 \\ c_1 + 2 + p_2 - \tau \end{cases}}{1+p_2} \end{cases} = \min \begin{cases} \dfrac{\tau + c_1}{c_1} \\ \dfrac{1 + p_2}{c_1} \\ \dfrac{c_1 + 1 - \tau + p_2}{1 + p_2} \end{cases}$$

4. Solution: $c_1 = 2(1 + \sqrt{2})$, $p_2 = \sqrt{2}c_1 - 1$, $\tau = 2$, $\rho = \sqrt{2}$.