Work-stealing

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Context

- A parallel platform with p processors
- A task-graph G to be executed
- Non-clairvoyant setting: the structure of G and/or the execution times of its constitutive tasks are discovered online

Work-sharing approach

Centralized scheduling

- A single list stores all ready tasks
- All processors retrieve work from that list

Advantage(s)

- Global view and knowledge

Drawback(s)

- Does not scale (contentions, etc.)

Work-stealing approach

Distributed scheduling

- Each processor owns a list of “its” ready tasks

Advantage(s)

- No contention problem
- Scalable solution

Drawback(s)

- Processors with empty lists do not know where to retrieve work from

Stealing policies 1/2

Global round-robin

- A global variable holds the identity of the next processor to steal from
- Variable incremented after each steal (successful or not)
- Advantage: eventual progress
- Drawback: centralized solution...

Local round-robin

- Each processor has its own variable indicating the next processor it should try to steal from
- Variable incremented after each steal (successful or not)
- Advantage: eventual progress; solution is scalable
- Drawback: all stealing processors may attempt to steal from the same processor; arbitrary notion of “distance” between processors

Stealing policies 2/2

Random stealing (Blumofe and Leiserson)

- The processor to steal from is randomly and uniformly chosen
- Advantage: decentralized; scalable; no notion of “distance”; low probability of simultaneous steal from same processor
- Drawback: performance???

Execution time as a function of number of steals

Assumption

- A steal takes a unit time (whether it is successful or not) (Hence, contentions are taken into account)

Notation

- Overall work: W (execution time on a single processor)
- Overall execution time: T_p
- Number of steal attempts: S

$$p \times T_p = W + S$$

(A processor is either working or doing a steal attempt)

The work-stealing algorithm

Principle

- Deepest-first order for execution
- Breadth-first order for steals

Specification

- Each processor stores ready tasks in a deque (double-ended queue)
- A new ready task is stored at the bottom of the deque
- The next task to be (locally) processed is taken from the bottom of the deque
- A task is stolen from the top of a randomly picked deque

Algorithm performance

Assumptions

- The DAG has a single entry node
- The DAG has a depth D
- The maximum out-degree of a node is 2
- A node has a unit execution time

Number of steal attempts

$$\mathbb{E}[S] = O(p \times D)$$

With probability at least $1 - \epsilon$, the number of steals is bounded by

$$S = O\left(p\left(D + \log\left(\frac{1}{\epsilon}\right)\right)\right)$$

Enabling tree

- If execution of node u enables node v
  - (u, v) is an **enabling edge**
  - u designated parent of v
  - Every node (except the root) has exactly one designated parent
- The enabling edges generate an **enabling tree**
- Node u of depth $d(u)$ in the enabling tree has **weight** $w(u) = D - d(u)$

Structural lemma 1/2

Theorem

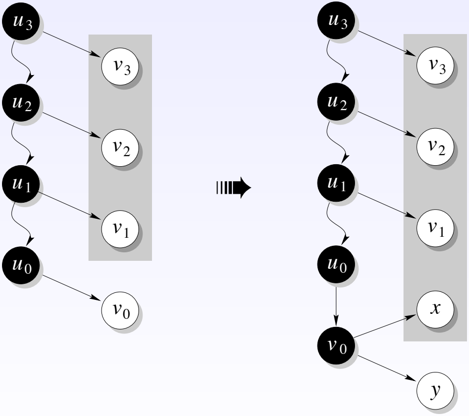
The designated parents of the nodes in the deque lie on some root-to-leaf path in the enabling tree

Proof by induction

- Initial case: trivial when deque is empty
- Induction
  - Trivial in the case of a steal
  - Trivial when an execution complete when the deque was empty: at most two ready nodes, at most one in the deque
  - What about execution completion when the deque was not previously empty?
    If a single new ready node: no problem (this node is processed right away)

Structural lemma 2/2

Execution completion when the deque was not previously empty



(a) Before. **(b)** After.

Amortized analysis using a potential function

Potential function

- Let R_i be the set of ready nodes at the beginning of round i
- Each ready node $u \in R_i$ has a potential $\phi_i(u)$:

$$\phi_i(u) = \begin{cases} 3^{2w(u)-1} & \text{if } u \text{ is processed} \\ 3^{2w(u)} & \text{otherwise } (u \text{ is in a deque}) \end{cases}$$

- Potential at round i:

$$\Phi_i = \sum_{u \in R_i} \phi_i(u)$$

- Initial potential: $\Phi_0 = 3^{2D-1}$
- Final potential: 0

Potential never increases

Two potential-changing actions

- Node u is removed from a deque (either through work-stealing or because the completion of the previous processing did not enable any node)
  $\phi_i(u) - \phi_{i+1}(u) = 3^{2w(u)} - 3^{2w(u)-1} = \frac{2}{3}3^{2w(u)} = \frac{2}{3}\phi_i(u)$
- Completion of a processed node (enabling some nodes)
  Completion of node u enables nodes x and y: x is processed and y placed in the deque

$$
\begin{aligned}
\phi_i(u) - \phi_{i+1}(x) - \phi_{i+1}(y) &= 3^{2w(u)-1} - 3^{2w(x)-1} - 3^{2w(y)} \\
&= 3^{2w(u)-1} - 3^{2(w(u)-1)-1} - 3^{2(w(u)-1)} \\
&= 3^{2w(u)-1}\left(1 - \frac{1}{9} - \frac{1}{3}\right) \\
&= \frac{5}{9}\phi_i(u) > 0
\end{aligned}
$$

Partitioning processors

- q a processor: $R_i(q)$ set of ready nodes in q's deque plus the node it processes

$$\Phi(q) = \sum_{u \in R_i(q)} \phi_i(u)$$

- A_i: set of processors whose deque is empty
  D_i: set of other processors

$$\Phi_i = \Phi_i(A_i) + \Phi_i(D_i)$$

- Aim: prove that every p steal attempts the potential decreases by a constant fraction with constant probability

Top-heavy deques

Theorem

Let q be a processor in D_i. The topmost node u in q's deque is such that:

$$\phi_i(u) \geq \frac{3}{4}\Phi_i(q)$$

Proof

- Let y be the node processed by q
- Suppose u is the only node in the deque
- Suppose u and y have the same designated parent

$$\begin{aligned}
\Phi_i(q) &= \phi_i(u) + \phi_i(y) \\
&= 3^{2w(u)} + 3^{2w(y)-1} \\
&= 3^{2w(u)} + 3^{2w(u)-1} \\
&= \tfrac{4}{3}\phi_i(u)
\end{aligned}$$

Balls and weighted bins property

Theorem

Suppose that p balls are thrown independently and uniformly at random into p bins, where bin i has weight W_i, with $\sum_i W_i = W$. For each bin i we define the random variable X_i as:

$$X_i = \begin{cases} W_i & \textit{if some ball lands in bin } i \\ 0 & \textit{otherwise} \end{cases}$$

Let $X = \sum_i X_i$. For any β, $0 < \beta < 1$:

$$Pr(X \geq \beta W) > 1 - \frac{1}{(1-\beta)e}$$

Impact of p steal attempts 1/3

Theorem

Consider any round i and a later round j such that p steal attempts occurred from round i (included) to round j (excluded). Then:

$$Pr\left(\Phi_i - \Phi_j \geq \frac{1}{4}\Phi_i(D_i)\right) > \frac{1}{4}$$

Impact of p steal attempts 2/3

- Let $q \in D_i$
- Let u be the node at the top of q's deque at round i
- We assume one of the p steal attempts target q
- Cases
  1. u is stolen
  2. Another node is stolen: therefore u was assigned
  3. No node is stolen
     1. u was previously stolen
     2. q has started the processing of u

  In any case, at the very least u is processed and the potential decreased by at least $\frac{2}{3}\phi_i(u)$

$$\frac{2}{3}\phi_i(u) \geq \frac{2}{3}\frac{3}{4}\Phi_i(q) = \frac{1}{2}\Phi_i(q)$$

Impact of p steal attempts 3/3

We consider a series of p steal attempts

- If a steal attempt targets $q \in D_i$, the potential decreases by $\frac{1}{2}\Phi_i(q)$
- $\forall q \in D_i$, $W_q = \frac{1}{2}\Phi_i(q)$
- $\forall q \in A_i$, $W_q = 0$
- $W = \frac{1}{2}\Phi_i(D_i)$
- We use the “Balls and weighted bins theorem” with $\beta = \frac{1}{2}$
  The potential decreases by $\beta W = \frac{1}{4}\Phi_i(D_i)$ with a probability greater than $1 - \frac{1}{(1-\frac{1}{2})e} = 1 - \frac{2}{e} > \frac{1}{4}$

Estimating the number of steal attempts 1/2

- A phase is defined by a series of $\Theta(P)$ steal attempts
- Phase starting with round i and ending with round j (excluded)
- $\Phi_i = \Phi_i(A_i) + \Phi_i(D_i)$
- Potential loss due to the steal attempts: at least $\frac{1}{4}\Phi_i(D_i)$ with probability at least $\frac{1}{4}$
- Potential loss due to task completion on A_i
  If node u completes, potential drops by at least $\frac{5}{9}\phi(u) > \frac{1}{4}\phi(u)$.
  Overall: greater than $\frac{1}{4}\Phi_i(A_i)$
- $Pr(\Phi_i - \Phi_j > \frac{1}{4}\Phi_i) > \frac{1}{4}$

Estimating the number of steal attempts 2/2

- Phase is **successful** if potential drops by at least $\frac{1}{4}$
- Initial potential: $\Phi_0 = 3^{2D-1}$
- Final potential: 0
- Maximal number of successful phases: $\mathcal{S}$
  $\left(\frac{3}{4}\right)^{\mathcal{S}} \times 3^{2D-1} < 1 \quad \Rightarrow \quad \mathcal{S}$ is at most $(2D-1)\log_{\frac{4}{3}}(3) < 8D$
- The expected number of phases is then at most $32D$
- The expected number of steal attempts is then $O(p \cdot D)$
- The probability that the execution takes $64D + 16\ln\left(\frac{1}{\epsilon}\right)$ phases or more is less than ϵ
- The number of steal attempts is $O\left(\left(D + \log\left(\frac{1}{\epsilon}\right)\right)p\right)$ with probability at least $1 - \epsilon$

Algorithm performance

Assumptions

- The DAG has a single entry node
- The DAG has a depth D
- The maximum out-degree of a node is 2
- A node has a unit execution time

Number of steal attempts: $\mathbb{E}[S] = O(p \times D)$

With probability at least $1 - \epsilon$, number of steals is bounded by

$$S = O\left(p\left(D + \log\left(\frac{1}{\epsilon}\right)\right)\right)$$

$$\mathbb{E}(T_p) = \frac{\mathcal{W}}{p} + O(D)$$

and $$T_p = O\left(\frac{\mathcal{W}}{p} + D + \log\left(\frac{1}{\epsilon}\right)\right) \text{ with probability } \geq 1 - \epsilon$$

What about the assumptions?

- The DAG has a single entry node
  Transformation increases D by $\lceil \log_2(I) \rceil$ where I is the number of entry nodes.
- The maximum out-degree of a node is 2
  Transformation multiplies D by $\lceil \log_2(\delta) \rceil$
- A node has a unit execution time
  In fact: maximum execution time is unit time
  Generalization: multiply number of steal attempts by the duration of the longest task...

Conclusion

- Not a list scheduling approach: because there are no centralized scheduler a processor may be left idle when there is ready nodes
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$$\mathbb{E}(T_p) = \frac{\mathcal{W}}{p} + O(D) \qquad \Rightarrow \qquad \mathbb{E}(T_p) = O(T_{\text{opt}})$$
- Many existing variants of random work stealing: Try to take advantage of (data) locality, to avoid lengthy communications, etc.