Groups, Actions and von Neumann algebras

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Théorie ergodique des actions de groupes et algèbres de von Neumann

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Résumé

Dans cette thèse, on s'intéresse à la théorie mesurée des groupes, à l'entropie sofique et aux algèbres d'opérateurs ; plus précisément, on étudie les actions des groupes sur des espaces de probabilités, des propriétés fondamentales de leur entropie sofique (pour des groupes discrets), leurs groupes pleins (pour des groupes Polonais), et les algèbres de von Neumann et leurs sous-algèbres moyennables (pour des groupes à caractère hyperbolique et des réseaux de groupes de Lie). Cette thèse est constituée de trois parties.

Dans une première partie j'étudie l'entropie sofique des actions profinies. L'entropie sofique est un invariant des actions mesurées des groupes sofiques défini par L. Bowen qui généralise la notion d'entropie introduite par Kolmogorov. La définition d'entropie sofique nécessite de fixer une approximation sofique du groupe. Nous montrons que l'entropie sofique des actions profinies est effectivement dépendante de l'approximation sofique choisie dans le cas des groupes libres et certains réseaux de groupes de Lie.

La deuxième partie est un travail en collaboration avec François Le Maître. Elle est constituée d'un article prépublié dans lequel nous généralisons la notion de groupe plein aux actions préservant une mesure de probabilité des groupes polonais, et en particulier, des groupes localement compacts. On définit une topologie polonaise sur ces groupes pleins et on étudie leurs propriétés topologiques fondamentales, notamment leur rang topologique et la densité des éléments apériodiques.

La troisième partie est un travail en collaboration avec Rémi Boutonnet. Elle est constituée de deux articles prépubliés dans lesquels nous considérons la question de la maximalité de la sous-algèbre de von Neumann d'un sous-groupe moyennable maximal, dans celle du groupe ambiant. Nous résolvons la question dans le cas des groupes à caractère hyperbolique en utilisant les techniques de Sorin Popa. Puis, nous introduisons un critère dynamique à la Furstenberg, permettant de résoudre la question pour des sous-groupes moyennables de réseaux des groupes de Lie en rang supérieur.

Mots-clés

Théorie ergodique, algèbres de von Neumann, groupes polonais, groupes sofiques, groupes pleins, maximale moyennabilité.

Groups, Actions and von Neumann algebras

Abstract

This dissertation is about measured group theory, sofic entropy and operator algebras. More precisely, we will study actions of groups on probability spaces, some fundamental properties of their sofic entropy (for countable groups), their full groups (for Polish groups) and the amenable subalgebras of von Neumann algebras associated with hyperbolic groups and lattices of Lie groups. This dissertation is composed of three parts.

The first part is devoted to the study of sofic entropy of profinite actions. Sofic entropy is an invariant for actions of sofic groups defined by L. Bowen that generalize Kolmogorov's entropy. The definition of sofic entropy makes use of a fixed sofic approximation of the group. We will show that the sofic entropy of profinite actions does depend on the chosen sofic approximation for free groups and some lattices of Lie groups.

The second part is based on a joint work with François Le Maître. The content of this part is based on a prepublication in which we generalize the notion of full group to probability measure preserving actions of Polish groups, and in particular, of locally compact groups. We define a Polish topology on these full groups and we study their basic topological properties, such as the topological rank and the density of aperiodic elements.

The third part is based on a joint work with Rémi Boutonnet. The content of this part is based on two prepublications in which we try to understand when the von Neumann algebra of a maximal amenable subgroup of a countable group is itself maximal amenable. We solve the question for hyperbolic and relatively hyperbolic groups using techniques due to Popa. With different techniques, we will then present a dynamical criterion which allow us to answer the question for some amenable subgroups of lattices of Lie groups of higher rank.

Key-words

Ergodic Theory, von Neumann algebras, Polish groups, sofic groups, full groups, maximal amenability.

Contents

Introduction

By a probability measure preserving action of a countable group Γ on the standard probability space (X, μ), we mean an action of Γ on X such that every element of Γ induces a measurable bijection of X which preserves the measure. We will say that the action of Γ on X is *free* if the set of fixed points of each element of Γ has measure zero. We will say that the action is *aperiodic* if almost every orbit is infinite and we will say that the action is *ergodic* if every Γ-invariant measurable subset of X has measure zero or one. Ergodic actions are the fundamental pieces of measure preserving actions, every probability measure preserving action admits a unique decomposition (up to measure zero) into ergodic actions.

Every measure preserving action of a countable group Γ gives rise to a unitary representation on a Hilbert space: the *Koopman representation*. In fact, if Γ acts on the probability space (X, μ) preserving the measure, then we can define a unitary representation κ of Γ on $L^2(X, \mu)$ by $\kappa_\gamma f(x) = f(\gamma^{-1}x)$.

Let us give some examples of free, ergodic and probability measure preserving actions.

- Every countable infinite group Γ admits a free, ergodic and probability measure preserving action: the *Bernoulli shift*. Let (Y, ν) be a probability space and set $(X, \mu) = (Y^\Gamma, \nu^\Gamma)$. The group Γ acts on X by shifting the sequences $\gamma_0(y_\gamma)_\gamma = (y_{\gamma_0^{-1}\gamma})_\gamma$ and this action preserves the measure. It is easy to observe that the action is free and ergodic.

- Let K be a compact group and let $\Gamma < K$ be a countable subgroup. The action induced by the multiplication on the left of Γ on K is free and preserves the Haar measure. Moreover if the group Γ is dense in K, then the action is also ergodic.

- Let $\Gamma > \Gamma_1 > \Gamma_2 > \ldots$ be a chain of finite index subgroups. The group Γ acts on the finite quotients Γ/Γ_n and hence it acts on the *profinite limit* of the sequence $\{\Gamma/\Gamma_n\}_n$. The profinite limit can be identified with the space of ends of a tree and therefore it has a natural topology which makes it homeomorphic to a Cantor space. If we equip the finite spaces Γ/Γ_n with the renormalized uniform counting measure, then we obtain a measure on the profinite limit which is Γ-invariant. The action of Γ on this space is called the *profinite action* of Γ with respect to the chain $\{\Gamma_n\}_n$ and it is always ergodic. If the subgroups Γ_n are normal in Γ and their intersection is trivial, then the profinite action is free.

We will now briefly describe some important notions related to actions of groups. Our purpose is not to give an overview of the theory, which the interested reader can find in the surveys of Furman [Fur11] and Gaboriau [Gab10]. We just want to recall the basic definitions and some important theorems that one should have in mind before reading this dissertation.

Conjugacy and entropy

Two measure preserving actions of a countable group Γ on the probability spaces (X, μ) and (Y, ν) are *conjugate* if there are Γ-invariant full measure subsets $A \subset X$ and $B \subset Y$ and a measure preserving isomorphism $\varphi : A \to B$ such that $\varphi(\gamma x) = \gamma\varphi(x)$ for every $x \in A$ and $\gamma \in \Gamma$.

It is straightforward to check that if two actions of Γ are conjugate, then their Koopman representations are isomorphic, so any spectral invariant of the representation is a conjugacy invariant. However Koopman representations are not complete invariants, for example all Bernoulli shifts over a finite base are associated to the same unitary representation. However these actions are not conjugate, Bernoulli shifts may have different *entropy*.

Entropy for measure preserving actions of the integer group has been defined by Kolmogorov in the fifties. One of the first striking application is that entropy can be used to classify Bernoulli shifts. In fact, the entropy of a Bernoulli shift over a finite base is equal to the Shannon entropy of the base, namely $-\sum_{y \in Y} \nu(\{y\}) \log(\nu(\{y\}))$ and a deep theorem of Ornstein tells us that this non-negative number is in fact a complete invariant. The classification of Bernoulli shifts is only one of the many applications of entropy, which gives rise to a fascinating theory, outlined for example in Katok's survey [Kat07].

In order to classify Bernoulli shifts of more general groups, one is led to generalize the concept of entropy. This has been done in the context of amenable groups by Ornstein and Weiss, using the notion of tilings and quasi tilings. Using this new entropy, Ornstein and Weiss were able to completely classify Bernoulli shifts exactly as for actions of $\mathbb{Z}$, [OW87].

While the entropy of actions of amenable groups was widely studied, there were some evidences pointing out that it would not have been possible to extend the definition to non amenable groups: some of the crucial properties of the entropy can not be true in the more general setting. In particular the question about Bernoulli shifts was unsolved. Several years later in 2008, Bowen introduced a new concept of entropy for actions of sofic groups which extends the previous definition, see [Bow10a] and [Bow10b]. He was able to classify Bernoulli shifts of a large class of sofic groups as for amenable groups: the entropy of the base space is an invariant. This classification was extended to all sofic groups shortly later by Kerr and Li in [KL11], where they also proposed a definition of sofic topological entropy and stated a variational principle.

Weak containment

Weak containment in the context of measure preserving actions was introduced by Kechris in [Kec10]. An action of the countable group Γ on the probability space (X, μ) is *weakly contained* in an action of Γ on the probability space (Y, ν) if for every $\varepsilon > 0$, for every finite partition $\alpha = \{A_1, \ldots, A_n\}$ of X and for every finite subset $F \subset \Gamma$, there is a finite partition $\beta = \{B_1, \ldots, B_n\}$ of Y such that

$$\sum_{i,j \leq n} \sum_{f \in F} |\mu(A_i \cap fA_j) - \nu(B_i \cap fB_j)| < \varepsilon.$$

The definition is inspired by the notion of weak containment for representations and it is in fact stronger. If an action a is weakly contained in an action b, then the Koopman representation of a is weakly contained in the Koopman representation of b, [Kec10, Proposition 10.5].

We say that two actions are *weakly equivalent* if they are weakly contained one into the other. By definition, if two actions are conjugate, then they are weakly equivalent. Note however that weak equivalence is weaker than conjugacy. For example ergodicity is not an invariant of weak equivalence: every ergodic, non strongly ergodic action of a countable group is weakly equivalent to the product of the action with the trivial action.

Even though weak containment is a relatively new concept, in the last few years there has been a big interest around it and several results were obtained.

- All free and ergodic actions of an amenable group are weakly equivalent, [FW04] and [Kec10].
- Every free action of a countable group weakly contains the Bernoulli shift of the group with respect to any base, [AW13].
- More generally, every free action is weakly equivalent to the product of itself with a Bernoulli shift [TD12].

Abért and Elek studied deeply weak containment and weak equivalence for profinite actions in [AE12]. In particular, they proved that free groups, $\mathrm{SL}_n(\mathbb{Z})$ and many other groups admit an uncountable family of weakly inequivalent actions. It is still unknown whether the result holds for every non-amenable group.

Orbit equivalence

Two measure preserving actions of the countable groups Γ and Λ on the probability spaces (X, μ) and (Y, ν) are *orbit equivalent* if there are invariant full measure subsets $A \subset X$ and $B \subset Y$ and a measure preserving isomorphism $S : A \to B$ such that for every $x, y \in A$ we have that $x \in \Gamma y$ if and only if $S(x) \in \Lambda S(y)$. Orbit equivalence is clearly weaker than conjugacy, we do not even ask Γ and Λ to be isomorphic. We will say that two groups are orbit equivalent if they admit free, ergodic actions which are orbit equivalent.

- A pionier result of Dye [Dye59] states that all ergodic, free and probability measure preserving actions of the integer group $\mathbb{Z}$ on a standard probability space are orbit equivalent. Moreover all free, ergodic actions of all locally finite groups are orbit equivalent to an action (and hence all) of $\mathbb{Z}$.
- Ornstein-Weiss in [OW80] were able to describe the class of groups that are orbit equivalent to the integers: it is the class of amenable groups. They proved that all free, ergodic actions of an amenable group are orbit equivalent to an action (and hence all) of $\mathbb{Z}$. The result was shortly later generalized in [CFW81].
- Gaboriau proved that probability measure preserving actions of free groups of different rank are not orbit equivalent, [Gab00]. By a result of Hjorth, [Hjo06], the class of groups that are orbit equivalent to a free group is the class of *treeable groups*. This class contains various surface groups but it is still unclear how to characterize it in more group-theoretical terms, see [Gab05]. For example, it is unknown whether limit groups are treeable.
- All non-amenable groups admit a continuum of orbit inequivalent actions see [Ioa11] and [Eps07] which is based on [GL09].

Even though orbit equivalence is a weak notion, orbit equivalent groups share many properties. For example amenability, property (T) ([Zim81], [AD05]) and Haagerup's property ([Jol05]) are invariants of orbit equivalence and two orbit equivalent groups have the same ℓ^2 Betti numbers ([Gab02]).

In some cases, orbit equivalence turns out to be equivalent to conjugacy. Furman proved in [Fur99] that any action which is orbit equivalent to the standard action of $\mathrm{SL}_n(\mathbb{Z})$ on the torus for $n \geq 3$ is essentially conjugated to it. Actions which satisfy this property are often called *rigid* actions. Since Furman's result, there have been several results of rigidity which culminated in Popa's cocycle superrigidity theorems [Pop07], [Pop08].

von Neumann algebras

A *von Neumann algebra* is a weakly closed $*$-subalgebra of the bounded operators on a Hilbert space. For a countable group Γ, we denote by λ the left-regular representation of Γ on $\ell^2(\Gamma)$. The *von Neumann algebra* of Γ, denoted $\mathrm{L}\Gamma$, is the weak closure of the linear span of the unitary operators $\{\lambda(\gamma)\}_{\gamma\in\Gamma}$.

In a similar manner, one can also define the von Neumann algebra of an action. Suppose that the countable group Γ acts on the probability space (X, μ) freely and preserving the measure. Denote by λ_X the diagonal representation of Γ on $\mathrm{L}^2(X, \mu) \otimes \ell^2(\Gamma)$ and we let $\mathrm{L}^\infty(X, \mu)$ act on $\mathrm{L}^2(X, \mu)$. The von Neumann algebra of the action $\mathrm{L}^\infty(X, \mu) \rtimes \Gamma$, called *group measure space construction*, is the weak closure of the linear span of $\{\lambda_X(\gamma)\}_{\gamma\in\Gamma}$ and $\mathrm{L}^\infty(X, \mu)$.

Orbit equivalence has its roots in the theory of von Neumann algebras. Murray and von Neumann used probability measure preserving actions as a source of examples of von Neumann algebras. Singer proved in [Sin55] that the group measure space construction depends only on the orbit-equivalence class of the action. Feldmann and Moore then generalized Singer's theorem to the context of equivalence relations [FM77b].

The group measure space construction is not a complete invariant of orbit equivalence [CJ82]. By [Sin55] and [FM77b], we know exactly which isomorphisms of von Neumann algebras are induced by orbit equivalence relations: those that respect the inclusion $\mathrm{L}^\infty(X, \mu) < \mathrm{L}^\infty(X, \mu) \rtimes \Gamma$.

The subalgebra $\mathrm{L}^\infty(X, \mu) < \mathrm{L}^\infty(X, \mu) \rtimes \Gamma$ is a *Cartan subalgebra*, that is a maximal abelian subalgebra whose normalizer generates the von Neumann algebra. By [Sin55] and [FM77b] a von Neumann algebra that has a unique Cartan subalgebra (up to conjugation), is canonically attached to the orbit equivalence class and a series of remarkable theorems shows that there are several von Neumann algebras with this property, see [OP10a], [PV14a] and [PV14b]. In particular, all group measure space constructions associated to actions of free (non-abelian) groups have a unique Cartan, [PV14a] and hence these von Neumann algebras remember the rank of the group.

Von Neumann algebras arising from groups are far less understood, even in the case of free groups. Let us state two well-known open problems.

- Does the von Neumann algebra associated to a free group remember the rank of the group?
- Is every amenable subalgebra of the von Neumann algebra associated to a free group contained in a unique maximal amenable subalgebra?

The second question is known under the name of Peterson-Thom conjecture, [PT11] and it will be the main motivation for the study of amenable subalgebras of the fourth and fifth chapter.

Ultraproducts, weak containment and sofic entropy

Measure preserving actions of countable groups on standard probability spaces have been studied for more than a century. Recently there has been some interest in ultraproduct of actions and their connection with sofic groups, see for example [CKTD13], [AE11], [Pes08], [ES05] and [KL13]. Ultraproducts are a natural limit procedure and the measure preserving actions constructed in this way, remember many properties of the sequences of actions used in their construction. One of the main difficulties of this construction, is that ultraproduct actions are defined on the *Loeb probability space*, which as measure space is isomorphic to $\{0,1\}^{\mathbb{R}}$ equipped with the product measure (Theorem 1.1.10).

Some of the theory of measure preserving actions easily generalizes to general measure spaces. For example Dye worked without any assumption on the probability space in [Dye59]. Anyway not much is known in the general setting. In the first chapter, we will try to understand actions on general probability space under the point of view of weak containment. We will prove the following.

Theorem 1. *Every probability measure preserving action of a countable group on a diffuse space is weakly equivalent to an action on a standard probability space.*

More precisely, we will prove in Theorem 1.2.15 that every probability measure preserving action on a diffuse space has a standard diffuse factor which is weakly equivalent to the action.

The family of all actions of G on finite or diffuse probability spaces is to big to be a set. But the above theorem implies that the family of weakly equivalence classes of actions is a set and it is isomorphic to the set of classes of actions on $\{1,\dots,n\}$ for $n \in \mathbb{N}$ and on a fixed standard probability space, say $[0,1]$ with respect to the Lebesgue measure. Let us denote the set of classes by $\overline{\mathrm{Act}}(G)$. One of the avantages of working with $\overline{\mathrm{Act}}(G)$ is that it is closed under ultraproducts: for every sequence of (classes of) actions $(a_n)_n$ of $\overline{\mathrm{Act}}(G)$ and for every ultrafilter $\mathfrak{u}$, the (class of the) action on the ultraproduct space $a_{\mathfrak{u}}$ is still an element of $\overline{\mathrm{Act}}(G)$.

Abért and Elek defined in [AE11] a compact, metric topology on the space of weak equivalence classes of actions on a standard Borel space which, by Theorem 1, is isomorphic to $\overline{\mathrm{Act}}(G)$. Once we identify these two spaces, it is not hard to see that every converging sequence converges to the class of its ultraproduct (with respect to any ultrafilter). Since ultraproducts of sequences of actions always exist, the topology is necessarily compact and it is completely determined by this property. This compact space was later studied in [TD12], [Bur15b] and [Bur15a].

We introduce in Definition 1.2.9 a compact, metric topology on $\overline{\mathrm{Act}}(G)$, which is equivalent to the topology of Abért and Elek. This metric is essentially the metric used in [Bur15b]. A sequence is converging for this topology if the asymptotic of the *statistics* of the actions converges to the *statistics* of the limit action and as in the case of Abért and Elek's topology, every converging sequence converges to its ultraproduct, see Theorem 1.2.22.

The aim of Chapter 2 is to give a concise, simple and self-contained proof of the compactness of the space, Theorem 1 of [AE11]. We will then analyse limits of finite actions and we will obtain an interesting corollary in the context of sofic entropy.

It will follow easily from the definition of the topology on $\overline{\mathrm{Act}}(G)$, that if $\{H_n\}_n$ is a descending chain of finite index subgroups of G, then the limit of the sequence of the finite actions G/H_n is the (class of the) profinite action $a^{(H_n)}$. Since limits are always weakly equivalent to the ultraproducts of the sequences, we get the following interesting corollary.

Corollary 2. *Let G be a countable group and let (H_n) be a chain of finite index subgroups. Then the profinite action $a^{(H_n)}$ associated to the sequence $(H_n)_n$ is weakly equivalent to the ultraproduct of the sequence of finite actions on the quotients (G/H_n) with respect to any ultrafilter.*

We will give an application of Corollary 2 in the context of sofic entropy.

Sofic Entropy

A *sofic approximation* of a countable group G is given by sequence of natural numbers $(n_k)_k$ and a sequence of maps $\{\theta_k : G \to S_{n_k}\}_k$ which is asymptotically multiplicative and free in the sense that and for all $g, h \in G$

$$\lim_{k\to\infty} \frac{1}{n_k} |\{i \in \{1, \dots, n_k\} : \theta_k(g)\theta_k(h)i = \theta_k(gh)i\}| = 1$$

and for every $g, h \in G$ with $g \neq h$,

$$\lim_{k\to\infty} \frac{1}{n_k} |\{i \in \{1, \dots, n_k\} : \theta_k(g)i \neq \theta_k(h)i\}| = 1.$$

A group is *sofic* if it admits a sofic approximation. The class of sofic groups is a large class of groups and at the time of writing there is no group which is known to be non sofic. For example, all residually finite groups are sofic, in fact every chain of finite index normal subgroups $\{H_n\}_n$ of G such that $\cap_n H_n = \{1_G\}$ gives a sofic approximation $\{\theta_n : G \to Sym(G/H_n)\}_n$, where each θ_n is induced by the left multiplication of G on the quotients. Sofic groups also include amenable groups and is stable under various operations, see [Pes08] and [ES05].

On the other hand, many conjectures are known to hold for sofic groups, for example sofic groups are hyperlinear [ES05], they satisfy the Gottschak Surjunctivity conjecture [Gro99], Kaplansky's Direct Finitness conjecture [ES04], the Determinant conjecture [ES05] and others. We invite the interested reader to look at Pestov's survey [Pes08] and references therein.

Sofic entropy is a conjugacy invariant for actions of a sofic group G which is built using a fixed sofic approximation of G. This invariant depends on the sofic approximation and once the approximation is fixed, the entropy is only defined (as a non-negative number) for some actions, which we will call its *domain of definition*. For the others the entropy is just declared to be $-\infty$. This means that each sofic approximation gives us a possibly different notion of entropy which has its proper domain. Bowen proved in [Bow10b], see also [Ker13], that for Bernoulli shifts the entropy is always defined and its value does not depend on the sofic approximation. This phenomenon was later extended to algebraic actions see [Bow11], [KL11] and [Hay14].

At the end of the first chapter, we will try to clarify how the domain of definition of sofic entropy depends on the sofic approximation. The answer appears extremely simple when the sofic entropy is defined using a sofic approximation which comes from a chain of finite index subgroups. In fact if G is a residually finite group and $(H_n)_n$ is a chain of subgroups such that the associated profinite action is free, then the sequence of actions of G on the finite quotients is a sofic approximation of G, which we will denote by $\Sigma_{(H_n)}$. The following proposition is a consequence of Corollary 2.

Proposition 3. *Let G be a residually finite group and let $(H_n)_n$ be a chain of finite index subgroups such that the associated profinite action $a^{(H_n)}$ is free. Then for every measure preserving action b of G on a standard probability space (X, μ), we have that $\mathrm{h}_{\Sigma_{(H_n)}}(b) > -\infty$ if and only if the action b is weakly contained in the profinite action $a^{(H_n)}$.*

The proposition tells us that the domains of definition depend on the sofic approximation and there are actions that are in some domains but not in others. Abért and Elek in [AE12] proved an interesting result about rigidity of weak equivalence for profinite actions, which we can combine with the previous proposition to get the following result.

Theorem 4. *Let G be a countable free group or $\mathrm{PSL}_k(\mathbb{Z})$ for $k \geq 2$. Then there is a continuum of normal chains $\{(H^r_n)_n\}_{r\in\mathbb{R}}$ such that $\mathrm{h}_{\Sigma_{(H^r_n)}}(a^{(H^s_n)}) > -\infty$ if and only if $r = s$.*

Observe that the entropy of profinite actions has been calculated in [CZ14] and it is always 0, when it is defined. Since profinite actions have a generating partition with finite (actually arbitrarily small) entropy (Lemma 1.3.13), we can use Bowen's computation of entropy for products of actions with Bernoulli shifts [Bow10b] to get actions which have positive entropy with respect to some sofic approximations and $-\infty$ with respect to others, see Theorem 1.3.12.

We do not know any action for which the sofic entropy can have two different non-negative values.

More Polish full groups

The second chapter of this dissertation is based on a joint work with François Le Maître.

For every action of a countable group Γ on the standard probability space (X, μ), the *orbit equivalence relation* of the action $\mathcal{R}_\Gamma$ on X is defined by

$$\mathcal{R}_\Gamma = \{(x, y) \in X \times X : \text{ there is } \gamma \in \Gamma \text{ such that } \gamma x = y\}.$$

It is easy to observe that two actions are orbit equivalent if and only if their equivalence relations are isomorphic up to measure zero. The equivalence relations arising in this way are called countable pmp (probability measure preserving) equivalence relations. They have geometric and cohomogical interpretations as well as fruitful relations with von Neumann algebras. We refer the interested reader to the survey of Gaboriau [Gab10].

Another way of formulating orbit equivalence is due to Dye. Suppose that Γ acts on the standard probability space (X, μ). The *full group* induced by the Γ-action, is the group of all $T \in \mathrm{Aut}(X, \mu)$ such that for almost every $x \in X$, we have $T(x) \in \Gamma \cdot x$. This group still encodes orbit equivalence in the following sense: two actions are orbit equivalent if and only if their full groups are conjugate in $\mathrm{Aut}(X, \mu)$ and a theorem of Dye (see Theorem 2.2.31) implies that two actions are orbit equivalent if and only if their full groups are abstractly isomorphic.

As a consequence, one should be able to understand all the invariants of orbit equivalence in terms of full groups. This works well for ergodicity: an action is ergodic if and only if the associated full group is a simple group. Another example is given by aperiodicity: an action is aperiodic if and only if the full group contains an element which induces a free action of $\mathbb{Z}$.

In order to understand finer orbit equivalence invariants in terms of properties of full groups, one is led to introduce a Polish group topology on them. This topology is called the *uniform topology*, and it is induced by the *uniform metric* d_u defined on $\mathrm{Aut}(X,\mu)$ by

$$d_u(T,S) = \mu\left(\{x \in X : \; Tx \neq Sx\}\right).$$

For example, Giordano and Pestov proved in [GP07] that if Γ acts freely on (X,μ), then Γ is amenable if and only if the full group of the action is extremely amenable for the uniform topology. Another example is given by the topological rank, that is the minimal number of elements needed to generate a dense subgroup. Le Maître showed in [LM14a] that the topological rank of a full group can be expressed in terms of a fundamental invariant of orbit equivalence: the cost.

The aim of Chapter 2 is to generalize the notion of full groups to actions of arbitrary Polish groups. Given a measure preserving action of the Polish group G on the standard probability space (X,μ), we define the *orbit full group* of the action exactly as before: it is the set of $T \in \mathrm{Aut}(X,\mu)$ such that for almost every $x \in X$, we have $T(x) \in G \cdot x$. We will denote this full group by $[\mathcal{R}_G]$ to remember that it is the full group of the equivalence relation induced by the action of G. We should warn the reader that our definition needs a concrete action of G on X, and not just a morphism $G \to \mathrm{Aut}(X,\mu)$.

As we said before, in order to understand deeper orbit full groups, we have to introduce a Polish topology on them. All the orbit full groups are closed for the uniform topology, but they are separable if and only if they arise as full groups of countable pmp equivalence relations. This does not rule out the existence of a Polish topology on them, for instance a compact group acting on itself by translation generates the transitive equivalence relation, so the associated orbit full group is $\mathrm{Aut}(X,\mu)$, which is a Polish group for the weak topology.

The aim of Chapter 2 is to define a Polish group topology on *all* orbit full groups, which will not be in general the restriction to $[\mathcal{R}_G]$ of a topology on $\mathrm{Aut}(X,\mu)$. We will call this topology the *topology of convergence in measure*. When the action of G on X is free, we can associate to any element $T \in [\mathcal{R}_G]$ the function $f : X \to G$ uniquely defined by $T(x) = f(x) \cdot x$. Doing so, we embed $[\mathcal{R}_G]$ in the space of measurable functions from X to G, and the Polish topology we will define coincides with the restriction of the topology of convergence in measure.

Theorem 5. *Let G be a Polish group acting in a measure preserving Borel manner on a standard probability space (X,μ). Then the associated orbit full group*

$$[\mathcal{R}_G] = \{T \in \mathrm{Aut}(X,\mu) : \forall x \in X, T(x) \in G \cdot x\}$$

is a Polish group for the topology of convergence in measure.

Moreover, if the action is ergodic, then $[\mathcal{R}_G]$ has a unique Polish group topology, and if the action is free, then G embeds into $[\mathcal{R}_G]$.

Every locally compact group G admits a free ergodic measure-preserving action, [AEG94, Proposition 1.2]. Given a free, ergodic and measure preserving action of a locally compact, non-discrete and non-compact group on the probability space (X,μ), we remark that the topology of convergence in measure of $[\mathcal{R}_G]$ is neither the uniform topology nor the weak topology. In fact whenever G acts freely, G seen as a subset of $\mathrm{Aut}(X,\mu)$ is discrete for the uniform topology, hence $[\mathcal{R}_G]$ can not be separable for the uniform topology. Moreover we show in Corollary 2.2.28 that if G is not compact, then $[\mathcal{R}_G] \neq \mathrm{Aut}(X,\mu)$, so the topology of convergence in measure is not the weak topology either, by Corollary 2.2.14.

Dye in [Dye59] gave an abstract definition of full groups: a subgroup $\mathbb{G} \leq \mathrm{Aut}(X,\mu)$ is *full* if for every countable subgroup $\Gamma \leq \mathbb{G}$, the full group generated by Γ is still a subgroup of $\mathbb{G}$. Clearly the orbit full groups we have defined are full groups in the sense of Dye's definition.

We remark that not all full groups can have a Polish topology. In fact we show that if an ergodic full group admits a Polish topology, then such a topology is unique, refines the weak topology and is weaker than the uniform topology (Theorem 2.3.8). It follows that if an ergodic full group admits a Polish topology, then it is a Borel subset of $\mathrm{Aut}(X,\mu)$ (Corollary 2.3.9). This allows us to give examples of full groups which cannot carry a Polish group topology (Corollary 2.3.19). Note that such a phenomenon is actually common for topological full groups, as was recently shown by Ibarlucias and Melleray [IM13].

One of the main interests of full groups induced by actions of countable groups is that they are complete invariants of orbit equivalence. Similarly to the case of countable groups, we say that two actions of two Polish groups G and H on the standard probability spaces (X,μ) and (Y,ν) are *orbit equivalent* if there are full measure subsets $A \subseteq X$ and $B \subset Y$ and a measure preserving bijection $S : A \to b$ such that for all $x \in A$,

$$S(G \cdot x) \cap A = (H \cdot S(x)) \cap B.$$

It is clear from this definition that if two actions are orbit equivalent, then their orbit full groups are conjugate in $\mathrm{Aut}(X,\mu)$ and in particular they are isomorphic. The converse is however more complicated. Dye's Reconstruction Theorem (see Theorem 2.2.31) still holds, so any isomorphism of full groups is given by the conjugation by some $S \in \mathrm{Aut}(X,\mu)$. However this does not imply that the orbit equivalence of the groups are orbit equivalent. We will show in Theorem 2.2.30 that this is the case for locally compact groups.

Theorem 6. *Let G and H be two locally compact second countable groups acting in a Borel measure preserving ergodic manner on a standard probability space (X,μ). Suppose that $\psi : [\mathcal{R}_G] \to [\mathcal{R}_H]$ is an abstract group isomorphism. Then there is an orbit equivalence S between $\mathcal{R}_G$ and $\mathcal{R}_H$ such that for all $T \in [\mathcal{R}_G]$,*

$$\psi(T) = S^{-1}TS.$$

Orbit full groups arise as intermediate examples between full groups of countable pmp equivalence relations and $\mathrm{Aut}(X,\mu)$, so they should share the topological properties which are satisfied by both. One of the simplest of such properties is contractibility, and indeed it is not hard to see that orbit full groups are contractible using the same approach of Keane for $\mathrm{Aut}(X,\mu)$ in [Kea70] (see Corollary 2.3.3).

However $\mathrm{Aut}(X,\mu)$ and full groups of countable pmp equivalence relations have many opposite properties. For example, any aperiodic element has a dense conjugacy class in $\mathrm{Aut}(X,\mu)$, while in the full group of a countable pmp equivalence relation, the identity cannot be approximated by aperiodic elements. We can characterize which group actions induce an orbit full group for which the aperiodic elements have dense conjugacy classes.

Theorem 7. *For a Borel, measure preserving action of the Polish group G on the probability space (X, μ), the following are equivalent:*

(i) the set of aperiodic elements is dense in $[\mathcal{R}_G]$;

(ii) the conjugacy class of any aperiodic element of $[\mathcal{R}_G]$ is dense in $[\mathcal{R}_G]$;

(iii) for every free measure preserving action of a countable discrete group Γ on the probability space (X, μ), there is a dense G_δ in $[\mathcal{R}_G]$ of elements inducing a free action of $\Gamma * \mathbb{Z}$;

(iv) for all neighborhood of the identity U in G, the set of $x \in X$ such that $U \cdot x \neq \{x\}$ has full measure.

Note that condition *(iii)* is inspired by results that Törnquist obtained for $[\mathcal{R}_G] = \mathrm{Aut}(X, \mu)$ [Tör06]. Using condition *(iv)*, we get a nice dichotomy for measure-preserving ergodic actions of locally compact groups: either they generate a countable pmp equivalence relation, or all the above conditions are satisfied (see Corollary 2.3.6).

Characters of full groups

In the last section of Chapter 2 we classify character representations of ergodic orbit full groups.

Every unitary representation of a group G splits as direct sum of cyclic representations. These representations are encoded by positive type functions, that are the functions $f : G \to \mathbb{C}$ such that for all finite tuple $(g_1, ..., g_n)$ of elements of G, the matrix $(f(g_i g_j^{-1}))_{i,j=1,...,n}$ is positive semi-definite.

A positive type function $\chi : G \to \mathbb{C}$ is a *character* if it satisfies the following conditions:

- it is *conjugacy-invariant*: for all $g, h \in G$, we have $\chi(g^{-1}hg) = \chi(h)$ and
- it is *normalized*: $\chi(1_G) = 1$.

A *character representation* is a unitary representation of G which splits as a direct sum of cyclic representations whose corresponding positive definite functions are characters. Character representations are the representations into the unitary groups of *finite von Neumann algebras*, see [DM13, Section 2.3] for more details.

Every discrete group Γ has a faithful character representation, namely the *regular representation*. It is associated to the *regular character* χ_r defined by $\chi_r(\gamma) = 0$ if $\gamma \neq 1_\Gamma$ and $\chi_r(1_\Gamma) = 1$. The set of characters of Γ is convex and compact for the pointwise topology. Moreover, it is a *Choquet simplex*, meaning that every character can be written in a unique way as an integral of extremal characters. The problem of classifying extremal characters has a long history, starting with the work of Thoma who classified extremal characters of the group of permutations of the integers with finite support [Tho64]. Since then, many examples were studied, see for instance [PT13] and references therein.

The set of continuous characters of a locally compact group is again a Choquet simplex, but locally compact groups do not necessarily have a faithful character representation. For example, all the continuous character representations of connected semi-simple Lie groups are trivial by a result of Segal and von Neumann [SvN50]. Recently Creutz and Peterson have shown that the same is true for non discrete totally disconnected simple locally compact groups having the Howe-Moore property [CP13, Theorem. 4.2].

For Polish groups, the situation is more complicated. The set of continuous characters may cease to form a Choquet simplex. For example, the abelian group of measurable maps into the circle $\mathrm{L}^0(X, \mu, \mathbb{S}^1)$ has no continuous extremal character, although it has continuous characters (see [BdlHV08, Example C.5.10]). However, if the Polish group G contains a countable dense subgroup Γ which has only countably many extremal characters, then the continuous extremal characters of G are given by the extremal characters of Γ which extend continuously to G. It is then easy to see that the continuous characters of G form a Choquet subsimplex of the characters of Γ. This remark has been crucial for the understanding of continuous characters of several Polish groups. In particular, it was used by Dudko to give a complete description of the characters of the full group of the hyperfinite ergodic equivalence relation [Dud11]. We extend his result and classify all the characters of an arbitrary ergodic orbit full group which are continuous for the uniform topology.

Theorem 8. *For a Borel, measure preserving action of the Polish group G on the probability space (X, μ), we have the following dichotomy:*

1. *Either $[\mathcal{R}_G]$ is the full group of a countable pmp equivalence relation, and then all its continuous characters are (possibly infinite) convex combinations of the characters χ_k given by*

$$\chi_k(g) = \mu(\{x \in X : g \cdot x = x\})^k$$

   for $k \in \mathbb{N}$ and the constant character $\chi_0 \equiv 1$.

2. *or $[\mathcal{R}_G]$ does not have any nontrivial continuous character representation.*

Orbit full groups of locally compact groups

The third chapter of this dissertation is devoted to the study of orbit full groups of free actions of locally compact second countable unimodular groups. As for the second chapter, Chapter 3 is based on a joint work with François Le Maître.

Measure preserving actions of Polish groups can have some strange properties. For example Kolmogorov found an essentially transitive action which is not ergodic, see Example 2.2.15. These strange properties reflect to strange properties of the associated full groups. Indeed, suppose that the Polish group G acts on the probability space (X, μ) preserving the measure. One could hope that for every dense subgroup $H \subset G$, the orbit full group $[\mathcal{R}_H]$ is dense in $[\mathcal{R}_G]$. Similarly one could hope that the set of elements in $[\mathcal{R}_G]$ that can be written using only countably many elements of G is dense in $[\mathcal{R}_G]$. But both properties are false for the full group associated to the action of Example 2.2.15.

However, if we suppose that the acting group is locally compact, the above properties are always true.

Theorem 9. *For every ergodic, measure preserving action of a locally compact Polish group G on a probability space (X, μ) and for every dense subgroup $H \subset G$, the orbit full group $[\mathcal{R}_H]$ is dense in $[\mathcal{R}_G]$.*

We will show Theorem 9 under a more general assumption. Becker in [Bec13], defined the notion of *suitable* action of a Polish group. These are the actions that, in some sense, behave nicely and we will prove that Theorem 9 holds for every suitable action of a Polish group G.

Measure preserving actions of locally compact Polish groups can be, in some sense, reduced to actions of countable groups. For this, we need the notion of *cross-section* defined by Forrest in [For74] and shortly later generalized in [FHM78]. The following theorem is essentially a version of Proposition 2.13 of [For74] in our context, which we will prove for sake of completeness.

Theorem 10. *Let G be a unimodular locally compact non-compact and non-discrete Polish group. For a measure preserving, essentially free and ergodic action of G on the probability space (X, μ), there exist a countable group Γ and a probability measure preserving action of Γ on (Y, ν) such that the action of G is orbit equivalent to the product action $\mathbb{S}^1 \times \Gamma$ on $\mathbb{S}^1 \times Y$, where $\mathbb{S}^1$ acts on itself by translation.*

Moreover, G is amenable if and only if the orbit equivalence relation induced by Γ on (Y, ν) is amenable.

Using Theorem 9 and Theorem 10, we can now compute the topological rank of orbit full groups associated to actions of locally compact non-compact and non-discrete Polish groups. In fact, by Theorem 10, we can suppose that $G = \mathbb{S}^1 \times \Gamma$ and that the action of $\mathbb{S}^1$ is free (but not the action of Γ). Now take a dense subgroup $\mathbb{Z} \subset \mathbb{S}^1$ and consider the dense subgroup $\mathbb{Z} \times \Gamma \subset \mathbb{S}^1 \times \Gamma$. By Theorem 9, we know that $[\mathbb{Z} \times \Gamma]$ is dense in $[\mathcal{R}_G]$ and by [LM14a], the topological rank of $[\mathbb{Z} \times \Gamma]$ is 2.

Theorem 11. *Let G be a locally compact unimodular non-compact and non-discrete Polish group. For every measure preserving, essentially free and ergodic action of G, there is a dense G_δ of couples (T, U) in $[\mathcal{R}_G]^2$ which generate a dense free subgroup of $[\mathcal{R}_G]$ acting freely. In particular, the topological rank of $[\mathcal{R}_G]$ is 2.*

We remark that the result is already known for compact groups. In fact if G is compact and the action is ergodic, then the action is transitive and $[\mathcal{R}_G] = \mathrm{Aut}(X, \mu)$, hence we can apply Prasad's result [Pra81]. We do not know whether the result holds for Polish groups, even in the case of suitable actions.

Maximal amenable subalgebras of von Neumann algebras associated with hyperbolic groups

The forth chapter of this dissertation is based on a joint work with Rémi Boutonnet.

A (separable) finite von Neumann algebra $A \subset B(H)$ is said to be *amenable* if there is a state φ on $B(H)$, which is A-central, meaning that $\varphi(xT) = \varphi(Tx)$ for all $x \in A$ and all $T \in B(H)$. Moreover, this definition does not depend on the choice of the Hilbert space H on which A is represented.

Amenability has always played a central role in the study of von Neumann algebras. First it is a source of isomorphism, via the fundamental result of Connes [Con76] that amenable implies hyperfinite, and the uniqueness of the hyperfinite II_1-factor. This characterization implies in particular that all von Neumann subalgebras of an amenable tracial von Neumann algebra are completely described: they are hyperfinite. Amenability is also at the core of the concepts of solidity and strong solidity defined in [Oza04, OP10a]. It is hence very natural to try to understand the maximal amenable subalgebras of a given finite von Neumann algebra.

In this direction, Kadison asked in the 1960's the following question: is any maximal amenable subalgebra of a II_1-factor necessarily a factor? Popa solved this problem in [Pop83], producing an example of a maximal amenable subalgebra of the free group factor LF_n which is abelian. The subalgebra in question is generated by one of the free generators of the free group F_n. This striking result led to more questions, refining Kadison's question: what if the ambient II_1-factor is McDuff? has property (T)? More generally can one provide concrete examples of maximal amenable subalgebras in a given II_1-factor? Some progress on this topic have been made recently.

By considering infinite tensor products of free group factors, Shen constructed in [She06] an abelian, maximal amenable subalgebra in a McDuff II_1-factor. In [CFRW10], it is proved that the subalgebra of the free group factor generated by the symmetric laplacian operator (the *radial subalgebra*) is maximal amenable. In [Hou14a], Houdayer provided uncountably many non-isomorphic examples of abelian maximal amenable subalgebras in II_1-factors. In 2010, Jolissaint [Jol10] extended Popa's result, providing examples of maximal amenable subalgebras in factors associated to amalgamated free-product groups, over finite subgroups.

In Chapter 4, we intend to provide examples of maximal amenable subalgebras of factors associated with hyperbolic groups. At the group level, amenable subgroups of hyperbolic groups are completely understood: they are virtually cyclic, and they act in a nice way on the Gromov boundary of the group. At the level of von Neumann algebras, we can show the following, generalizing the main result of [Pop83].

Theorem 12. *Consider a hyperbolic group G and an infinite, maximal amenable subgroup $H < G$. Then the group von Neumann algebra LH is maximal amenable inside LG.*

This answers a question of Cyril Houdayer [Hou13, Problème 3.13].

Every maximal amenable subgroup H of a hyperbolic group is virtually cyclic, so the associated von Neumann algebra LH is far from being a factor. By Remark 4.2.6, we obtain many counterexamples to Kadison's question, even in property (T) factors. For instance factors of the form $L\Gamma$, with Γ a cocompact lattice in $\mathrm{Sp}(n,1)$, are counterexamples with property (T).

The proof of Theorem 12 is in the spirit of Popa's asymptotic orthogonality property [Pop83]. It relies on an analysis of LH-central sequences and property Gamma. By definition, a diffuse finite von Neumann algebra M has *property Gamma* if it admits a sequence of unitaries $(u_n)_n \subset M$ which tends weakly to 0 such that for every $x \in M$,

$$\lim_n \|xu_n - u_n x\|_2 = 0.$$

By [Con76], diffuse finite amenable von Neumann algebras have property Gamma. What we really show is that $LH \subset LG$ is maximal Gamma, that is, it is maximal among von Neumann subalgebras of LG with property Gamma. However amenability and property Gamma coincide for subalgebras of solid von Neumann algebras ([Oza04, Proposition 7]) and the main result of [Oza04] shows that LG is solid, whenever G is hyperbolic.

Using similar techniques, we can prove the following result for relatively hyperbolic groups.

Theorem 13. *Let G be a group which is hyperbolic relative to a family $\mathcal{G}$ of subgroups of G and consider an infinite subgroup $H \in \mathcal{G}$ such that LH has property Gamma. Then the group von Neumann algebra LH is maximal Gamma inside LG.*

Using results of Osin [Osi06b, Osi06a], we obtain the following corollary, which generalizes Theorem 12 and the main result of [Jol10].

Corollary 14. *Let G be a group which is hyperbolic relative to a family $\mathscr{G}$ of amenable subgroups and H be an infinite maximal amenable subgroup of G. Then the group von Neumann algebra LH is maximal amenable inside LG.*

By the comments after Proposition 12 in [Oza06], LG is solid for G as in Corollary 14, so maximal amenable is equivalent to maximal Gamma.

Limit groups are examples of groups G covered by this corollary.

It is also possible to prove similar results in the context of hyperbolically embedded subgroups, in the sense of [DGO11]: generalizing our techniques one can show that if $H < G$ is an infinite amenable subgroup which is hyperbolically embedded then LH is maximal amenable inside LG.

Finally, we extend our results to products of groups as above. We also allow the groups to act on an amenable von Neumann algebra, and we get a similar result about the crossed product von Neumann algebra. Such a product situation were already investigated in [She06] and [CFRW10]. We thank Stuart White for suggesting us to study this case.

Theorem 15. *Let $n \geq 1$, and consider for all $i = 1, \dots, n$ an inclusion of groups $H_i < G_i$ as in Theorem 13. Put $G = G_1 \times \cdots \times G_n$ and $H = H_1 \times \cdots \times H_n$.*

Then for any trace-preserving action of G on a finite amenable von Neumann algebra (Q, τ), the crossed-product $Q \rtimes H$ is maximal amenable inside $Q \rtimes G$.

In particular, when G and H are as above, for any free measure preserving action on a probability space $G \curvearrowright (X, \mu)$, the equivalence relation on (X, μ) given by the H-orbits is maximal hyperfinite inside the equivalence relation given by the G-orbits.

In Theorem 15, note that $Q \rtimes H \subset Q \rtimes G$ is not maximal Gamma in general. We will in fact use Houdayer's relative version of the asymptotic orthogonality property to conclude ([Hou14b]). The argument relies on the same analysis of LH-central sequences.

Maximal amenable von Neumann subalgebras arising from maximal amenable subgroups

The last chapter of this dissertation is also based on a join work with Rémi Boutonnet.

In Chapter 4, we showed that any infinite maximal amenable subgroup in a hyperbolic group Γ gives rise to a maximal amenable von Neumann subalgebra of $L\Gamma$.

Question. Assume that $\Lambda < \Gamma$ is a maximal amenable subgroup. Under which conditions is $L\Lambda$ maximal amenable inside $L\Gamma$?

In Chapter 5, we will provide a general sufficient condition ensuring this rigidity phenomenon.

Definition. Consider an amenable subgroup Λ of a discrete countable group Γ. Suppose that Γ acts continuously on the compact space X. We say that Λ is *singular* in Γ (with respect to X) if for any Λ-invariant probability measure μ on X and for every $g \in \Gamma \setminus \Lambda$, we have that the measure $g \cdot \mu$ is singular with respect to the measure μ.

Theorem 16. *Let Γ be a countable group and $\Lambda < \Gamma$ be an amenable singular subgroup as in the previous definition. Then for any trace preserving action $\Gamma \curvearrowright (Q, \tau)$ on a finite amenable von Neumann algebra, $Q \rtimes \Lambda$ is maximal amenable inside $Q \rtimes \Gamma$.*

The conclusion of the above theorem implies in particular that

- $L\Lambda$ is maximal amenable inside $L\Gamma$ (case where $Q = \mathbb{C}$);
- for any free measure preserving action on a probability space $\Gamma \curvearrowright (Y, \nu)$, the orbit equivalence relation $\mathcal{R}(\Lambda \curvearrowright (Y, \nu))$ is maximal hyperfinite inside $\mathcal{R}(\Gamma \curvearrowright (Y, \nu))$ (case where $Q = L^\infty(Y, \nu)$).

Corollary 17. *In the following examples, Λ is singular inside Γ, so that the conclusion of Theorem 16 holds.*

(i) *Γ is a hyperbolic group and Λ is an infinite maximal amenable subgroup;*

(ii) *Λ is any amenable group with an infinite index subgroup Λ_0, and $\Gamma = \Lambda *_{\Lambda_0} \Lambda'$, for some other group Λ' containing Λ_0;*

(iii) *$\Gamma = \mathrm{SL}_n(\mathbb{Z})$ and Λ is the subgroup of upper triangular matrices.*

Point (ii) above was proved independently by B. Leary [Leaon] for more general von Neumann algebras (not only group algebras).

Regarding the question of providing abelian, maximal amenable subalgebras in a given von Neumann algebra, we can prove the following. The result is not as explicit as the above examples, but it is quite general. We are grateful to Jesse Peterson for stimulating our interest in this question in the setting of lattices in Lie groups.

Theorem 18. *Consider a lattice Γ in a connected semi-simple real Lie group G with finite center. Then Γ admits a singular subgroup which is virtually abelian.*

As we explain in Remark 5.2.6, if moreover G has no compact factors and Γ is torsion free and co-compact in G, then Γ admits an abelian singular subgroup.

At this point, let us mention that all the former results on maximal amenability followed Popa's strategy of proving the maximal amenability of $Q \subset M$ by studying Q-central sequences in M. Namely the inclusion $Q \subset M$ was usually shown to satisfy the so-called "asymptotic orthogonality property". In contrast, our result relies on a new strategy, more specific to group von Neumann algebras, and completely different from Popa's approach.

The general idea in our approach is the following. Assume that Γ acts on some compact space X. Then the maximal amenable subgroups of Γ are stabilizers of probability measures on X. In non-commutative terms, one can more generally say that amenable subalgebras of $L\Gamma$ centralize states on the reduced C*-algebraic crossed-product $C(X) \rtimes_r \Gamma$. The advantage of focusing our study on this crossed-product C*-algebra is that it allows concrete computations. We will see at the end of this paper that this point of view also has a theoretical interest, providing new insight on solidity and strong solidity.

Finally, let us mention nice characterizations of singularity communicated to us by Narutaka Ozawa.

Theorem 19 (Ozawa). *Consider an amenable subgroup Λ of a discrete countable group Γ. The following are equivalent.*

1. *Λ is a singular subgroup of Γ;*
2. *Every Λ-invariant state on $C_r^*(\Gamma)$ vanishes on $\Gamma \setminus \Lambda$;*
3. *For every $g \in \Gamma \setminus \Lambda$, we have that $0 \in \overline{\mathrm{conv}}^{\|\cdot\|}(\{\lambda(tgt^{-1})\,,\, t \in \Lambda\}) \subset B(\ell^2\Gamma)$;*
4. *For any net (ξ_n) of almost Λ-invariant unit vectors in $\ell^2\Gamma$ and all $g \in \Gamma \setminus \Lambda$, the inner product $\langle \lambda_g \xi_n, \xi_n \rangle$ goes to 0.*

Note that the last characterization is in the spirit of Popa's Asymptotic Orthogonality Property.

Chapter 5 was originally born as an attempt to give a geometric proof of the maximality of the radial subalgebra in the free group factor [CFRW10]. We wanted to apply the theory of entropy of random walks developed in [KV83] and [Kai00]. In fact using this entropy, one can well understand the action of $\ell^1\Gamma$ on the space of measure on the boundary of the group by convolution. This action has some hyberbolic behavior, so one could hope to use it to prove the maximality of the radial subalgebra.

Question. Is it possible to give a geometric proof of [CFRW10] in the spirit of Chapter 4 and 5?

Chapter 1

Ultraproducts, weak equivalence and sofic entropy

Abért and Elek defined a metrizable and compact topology on the space of weakly equivalence classes of probability measure preserving actions of a countable group. We propose here an equivalent metric and we will give a simple proof of the compactness of the space. We will prove that any probability measure preserving action of a countable group on any diffuse space is weakly equivalent to an action on a standard diffuse space. We will analyse ultraproduct of finite actions. For a residually finite group, we will show that the profinite action associated to a sequence of finite index subgroups is weakly equivalent to the ultraproduct action of the sequence of actions on the quotients. Finally, we will obtain a corollary about sofic entropy, we will show that for free groups and some property (T) groups, sofic entropy of profinite actions depends crucially on the sofic approximation used for computing the sofic entropy.

Contents

1.1 Ultrapoducts of probability spaces

In this section, we describe the ultraproduct of probability measure spaces. These probability spaces were introduced by Loeb in [Loe75] in the language of non-standard analysis and they are often called Loeb spaces. All the material presented here is well-known and a recent exposition can be found in [CKTD13] and [ES12].

Let us fix a non-principal ultrafilter $\mathfrak{u}$ on $\mathbb{N}$.

1.1.1 Set-theoretic ultraproducts

Definition 1.1.1. Let $\{X_n\}_{n\in\mathbb{N}}$ be a family of sets and let X be their product $X := \prod_{n\in\mathbb{N}} X_n$. We define the **ultraproduct** of the family $\{X_n\}_n$ to be the following quotient of X

$$X_{\mathfrak{u}} := X/\sim_{\mathfrak{u}} \quad \text{ where } \quad (x_n)_n \sim_{\mathfrak{u}} (y_n)_n \text{ if } \{n : x_n = y_n\} \in \mathfrak{u}.$$

We will denote by $x_{\mathfrak{u}}$ and $A_{\mathfrak{u}}$ elements and subsets of $X_{\mathfrak{u}}$. For a sequence $(x_n)_n \in X$, we will denote by $[x_n]_{\mathfrak{u}}$ its class in $X_{\mathfrak{u}}$ and similarly for a sequence of subsets $\{A_n \subset X_n\}_n$, we will denote by $[A_n]_{\mathfrak{u}}$ the class of $(A_n)_n$.

It is easy to observe that

$$[A_n]_{\mathfrak{u}} \cap [B_n]_{\mathfrak{u}} = [A_n \cap B_n]_{\mathfrak{u}}, \quad [A_n]_{\mathfrak{u}} \cup [B_n]_{\mathfrak{u}} = [A_n \cup B_n]_{\mathfrak{u}}.$$

Remark 1.1.2. We remark that if $\{X_n\}_n$ is a sequence of finite sets such that $\lim_{\mathfrak{u}} |X_n| = \infty$ or if it is a sequence of countable non-finite sets, the ultraproduct $X_{\mathfrak{u}}$ has the cardinality of the continuum.

In fact, it is easy to construct a surjective map from $X_{\mathfrak{u}}$ to interval $[0,1]$. For example, if $X_n = \{1, \ldots, n\}$ then the map can be defined as

$$\varphi : X_{\mathfrak{u}} \to [0,1], \quad \varphi([a_n]_n) =: \lim_{n\in\mathfrak{u}} \frac{a_n}{n},$$

where the limit on the right is the limit with respect to the Euclidean topology. Since the rationals are dense in the interval, the map φ has to be surjective and a similar argument works for the general case.

1.1.2 Metric ultraproducts

Definition 1.1.3. Let $\{(M_n, d_n)\}_{n\in\mathbb{N}}$ be a family of uniformly bounded metric spaces. We define the pseudo-metric $d_{\mathfrak{u}}$ on $M := \prod_{n\in\mathbb{N}} M_n$ by

$$d_{\mathfrak{u}}((x_n)_n, (y_n)_n) := \lim_{n\in\mathfrak{u}} d_n(x_n, y_n).$$

We define the **metric ultraproduct** of the family $\{(M_n, d_n)\}_n$ with respect to the ultrafilter $\mathfrak{u}$ to be the metric space associated to the pseudo-metric $d_{\mathfrak{u}}$, that is $M_{\mathfrak{u}} := M/\{d_{\mathfrak{u}} = 0\}$.

Remark 1.1.4. Let $\{G_n\}_n$ be a sequence of groups and let d_n be a bounded bi-invariant metric on G_n. It is easy to check that the subgroup

$$K_{\mathfrak{u}} := \left\{ (g_n)_n \in \prod_n G_n : \; d_{\mathfrak{u}}((g_n)_n, (1_G)_n) = 0 \right\}$$

is normal, so the metric ultraproduct $G_{\mathfrak{u}}$ is a topological group and the metric $d_{\mathfrak{u}}$ is bi-invariant. For more on ultraproduct of groups, see [Pes08].

1.1.3 Measure Spaces

We will now define the ultraproduct of a sequence of probability spaces using Carathéodory's method. Let $\{(X_n, \mathscr{B}_n, \mu_n)\}_{n\in\mathbb{N}}$ be a family of probability spaces and let X_{u} be their ultraproduct. We define

$$\theta : \mathscr{P}(X_{\mathrm{u}}) \to [0, \infty],$$
$$\theta(A_{\mathrm{u}}) := \inf\left\{ \sum_{i\in\mathbb{N}} \lim_{n\in\mathrm{u}} \mu_n(B_n^i) : A_{\mathrm{u}} \subset \bigcup_{i\in\mathbb{N}} [B_n^i]_{\mathrm{u}},\ B_n^i \in \mathscr{B}_n\ \forall n, i \in \mathbb{N} \right\}.$$

Proposition 1.1.5. *The function θ defined above is an outer measure.*

Proof. For this, we have to check that $\theta(\emptyset) = 0$, that if $A_{\mathrm{u}} \subset C_{\mathrm{u}}$ then $\theta(A_{\mathrm{u}}) \leq \theta(C_{\mathrm{u}})$ and that for every sequence $\{A_{\mathrm{u}}^j \subset X_{\mathrm{u}}\}_j$ we have $\theta(\cup_j A_{\mathrm{u}}^j) \leq \sum_j \theta(A_{\mathrm{u}}^j)$. This can be done exactly as for the Lebesgue measure, see [Fre04a, 114D].

- Since $\emptyset \subset [\emptyset]_{\mathrm{u}}$, we must have that $\theta(\emptyset) = 0$.
- Suppose $A_{\mathrm{u}} \subset C_{\mathrm{u}}$. For every family $\{B_n^i \in \mathscr{B}_n\}_{i,n}$ such that $C_{\mathrm{u}} \subset \cup_i [B_n^i]_{\mathrm{u}}$, we have that $A_{\mathrm{u}} \subset \cup_i [B_n^i]_{\mathrm{u}}$ so that $\theta(A_{\mathrm{u}}) \leq \theta(C_{\mathrm{u}})$.
- Let $\{A_{\mathrm{u}}^j\}_{j\in\mathbb{N}}$ be a sequence of subsets of X_{u} and fix $\varepsilon > 0$. For every $j \in \mathbb{N}$, fix a family $\{B_n^{j,i} \in \mathscr{B}_n\}_{i,n}$ such that

$$A_{\mathrm{u}}^j \subset \bigcup_{i\in\mathbb{N}} [B_n^{j,i}]_{\mathrm{u}} \text{ and } \sum_{i\in\mathbb{N}} \lim_{n\in\mathrm{u}} \mu_n(B_n^{j,i}) \leq \theta(A_{\mathrm{u}}^j) + 2^{-j}\varepsilon.$$

  Then $\cup_j A_{\mathrm{u}}^j \subset \cup_{j,i} [B_n^{j,i}]_{\mathrm{u}}$, so

$$\theta(\cup_j A_{\mathrm{u}}^j) \leq \sum_{i,j} \lim_{n\in\mathrm{u}} \mu_n(B_n^{j,i}) \leq \sum_j \theta(A_{\mathrm{u}}^j) + \varepsilon. \qquad \square$$

Whenever we have an outer measure, Carathéodory's Theorem gives us a way of constructing a measure.

Definition 1.1.6. The **measure ultraproduct** of a family of probability spaces $\{(X_n, \mathscr{B}_n, \mu_n)\}_{n\in\mathbb{N}}$ is the probability space $(X_{\mathrm{u}}, \mathscr{B}_{\mathrm{u}}, \mu_{\mathrm{u}})$, where

$$\mathscr{B}_{\mathrm{u}} := \{A_{\mathrm{u}} \subset X_{\mathrm{u}} : \theta(B_{\mathrm{u}}) \geq \theta(B_{\mathrm{u}} \cap A_{\mathrm{u}}) + \theta(B_{\mathrm{u}} \setminus A_{\mathrm{u}}) \text{ for every } B_{\mathrm{u}} \subset X_{\mathrm{u}}\}$$
$$\mu_{\mathrm{u}}(A_{\mathrm{u}}) := \theta(A_{\mathrm{u}}) \quad \text{for every } A_{\mathrm{u}} \in \mathscr{B}_{\mathrm{u}}.$$

Carathéodory's Theorem, see for example [Fre04a, 113C], tells us that $(X_{\mathrm{u}}, \mathscr{B}_{\mathrm{u}}, \mu_{\mathrm{u}})$ is a measure space. In the following proposition we describe which subsets of the ultraproduct are measurable and we show how to compute their measure.

Proposition 1.1.7. *Let $\{(X_n, \mathscr{B}_n, \mu_n)\}_{n\in\mathbb{N}}$ be a family of probability spaces and let $(X_{\mathrm{u}}, \mathscr{B}_{\mathrm{u}}, \mu_{\mathrm{u}})$ be the measure space associated to θ via the Carathéodory's method, that is the measure ultraproduct of the family of probability spaces.*

1. *For every sequence $\{A_n \in \mathscr{B}_n\}_n$ we have $[A_n]_{\mathrm{u}} \in \mathscr{B}_{\mathrm{u}}$ and $\mu_{\mathrm{u}}([A_n]_{\mathrm{u}}) = \lim_{n\in\mathrm{u}} \mu_n(A_n)$.*
2. *For every $A_{\mathrm{u}} \in \mathscr{B}_{\mathrm{u}}$ there is a sequence $\{B_n \in \mathscr{B}_n\}_n$ such that $\mu_{\mathrm{u}}(A_{\mathrm{u}} \Delta [B_n]_{\mathrm{u}}) = 0$.*

Proof. (1) Let us prove that for every family $\{A_n \in \mathscr{B}_n\}_n$, we have that $[A_n]_\mathfrak{u} \in \mathscr{B}_\mathfrak{u}$. Consider a subset $B_\mathfrak{u} \subset X_\mathfrak{u}$, a real number $\varepsilon > 0$ and a family $C_n^i \in \mathscr{B}_n$ such that

$$B_\mathfrak{u} \subset \cup_i [C_n^i]_\mathfrak{u} \text{ and } \sum_i \theta([C_n^i]_\mathfrak{u}) \leq \theta(B_\mathfrak{u}) + \varepsilon.$$

So we have

$$\begin{aligned}
\theta(B_\mathfrak{u} \cap [A_n]_\mathfrak{u}) + \theta(B_\mathfrak{u} \setminus [A_n]_\mathfrak{u}) \leq & \theta(\cup_i([C_n^i]_\mathfrak{u} \cap [A_n]_\mathfrak{u})) + \theta(\cup_i([C_n^i]_\mathfrak{u} \setminus [A_n])) \\
= & \theta(\cup_i [C_n^i \cap A_n]_\mathfrak{u}) + \theta(\cup_i [C_n^i \setminus A_n]) \\
\leq & \sum_i \lim_{n \in \mathfrak{u}} \Big(\mu_n(C_n^i \cap A_n) + \mu_n(C_n^i \setminus A_n) \Big) \\
= & \sum_i \lim_{n \in \mathfrak{u}} \mu_n(C_n^i) \\
\leq & \theta(B_u) + \varepsilon.
\end{aligned}$$

As ε is arbitrary, $[A_n]_\mathfrak{u}$ is $\mu_\mathfrak{u}$-measurable.

As we have observed before, given two subsets $[A_n^1]_\mathfrak{u}$ and $[A_n^2]_\mathfrak{u}$ of $X_\mathfrak{u}$, we have that $[A_n^1]_\mathfrak{u} \cup [A_n^2]_\mathfrak{u} = [A_n^1 \cup A_n^2]_\mathfrak{u}$. We remark that the same property does not hold for countable unions but the following lemma shows that a similar property holds in the measurable setting.

Lemma 1.1.8. *For every countable family $\{B_n^i \in \mathscr{B}_n\}_{i,n \in \mathbb{N}}$ there is a family $\{C_n \in \mathscr{B}_n\}_{n \in \mathbb{N}}$ such that*

$$\cup_i [B_n^i]_\mathfrak{u} \subset [C_n]_\mathfrak{u} \text{ and } \lim_{n \in \mathfrak{u}} \mu_n(C_n) = \lim_{i \to \infty} \lim_{n \in \mathfrak{u}} \mu_n(\cup_{j=1}^i B_n^j).$$

Proof. The proof is a standard diagonal argument for ultraproducts. For every n and i, we set $D_n^i := \cup_{j=1}^i B_n^j$. For $i \geq 1$, put

$$L_i := \left\{ m \in \{i, i+1, \dots\} : \left| \lim_{n \in \mathfrak{u}} \mu_n(D_n^i) - \mu_m(D_m^i) \right| \leq \frac{1}{2^i} \right\}.$$

Observe that $L_i \in \mathfrak{u}$. We define the function

$$f : \mathbb{N} \to \mathbb{N} \quad \text{as } f(n) := \begin{cases} \max\{i : n \in L_i\} & \text{for } n \in \cup_i L_i \\ f(n) := 1 & \text{otherwise} \end{cases}$$

By construction $f(n) \leq n$, $f(n)$ tends to infinity as $n \to \mathfrak{u}$ and, for every m in a subset $I_0 \in \mathfrak{u}$, we have $|\lim_{n \in \mathfrak{u}} \mu_n(D_n^{f(m)}) - \mu_m(D_m^{f(m)})| \leq 2^{-f(m)}$.

We set $C_n := D_n^{f(n)}$. For every $i \in \mathbb{N}$ and for every $n \in L_i$, we have that $f(n) \geq i$, hence $C_n \supset D_n^i$. Since this is true for every i, we obtain that $[C_n]_\mathfrak{u} \supset \cup_i [D_n^i]_\mathfrak{u} = \cup_i [B_n^i]_\mathfrak{u}$ which implies that

$$\lim_{n \in \mathfrak{u}} \mu_n(C_n) \geq \lim_{i \to \infty} \lim_{n \in \mathfrak{u}} \mu_n(\cup_{j=1}^i B_n^j).$$

Finally

$$\mu_m(C_m) = \mu_m(D_m^{f(m)}) \leq \lim_{n \in \mathfrak{u}} \mu_n(D_n^{f(m)}) + \frac{1}{2^{f(m)}} \leq \lim_{i \to \infty} \lim_{n \in \mathfrak{u}} \mu_n(\cup_{j=1}^i B_n^j) + \frac{1}{2^{f(m)}}. \qquad \square$$

Let us now compute the measure of $[A_n]_\mathfrak{u}$. By the definition of θ, we must have $\theta([A_n]_\mathfrak{u}) \leq \lim_\mathfrak{u} \mu_n(A_n)$. For the reverse inequality, fix $\varepsilon > 0$ and consider a countable family $\{B_n^i \in \mathscr{B}_n\}_{i,n}$ such that

$$\left|\theta([A_n]_\mathfrak{u}) - \sum_{i\in\mathbb{N}} \lim_{n\in\mathfrak{u}} \mu_n(B_n^i)\right| < \varepsilon.$$

By Lemma 1.1.8, there is a family $\{C_n \in \mathscr{B}_n\}_n$ such that $[C_n]_\mathfrak{u} \supset \cup_i [B_n^i]_\mathfrak{u} \supset [A_n]_\mathfrak{u}$ which satisfies

$$\lim_{n\in\mathfrak{u}} \mu_n(A_n) \leq \lim_{n\in\mathfrak{u}} \mu_n(C_n) \leq \lim_{i\to\infty} \lim_{n\in\mathfrak{u}} \mu_n(\cup_{j=1}^i B_n^j) \leq \sum_{j=0}^{\infty} \lim_{n\in\mathfrak{u}} \mu_n(B_n^j) \leq \theta([A_n]_\mathfrak{u}) + \varepsilon.$$

Since ε is arbitrary, we obtain that $\theta([A_n]_\mathfrak{u}) = \lim_\mathfrak{u} \mu_n(A_n)$.

(2) Let $A_\mathfrak{u} \in \mathscr{B}_\mathfrak{u}$ be a measurable subset. By definition of θ, for every $j \in \mathbb{N}$, there is a countable family $\{B_n^{i,j} \in \mathscr{B}_n\}_{n,i}$ such that

$$A_\mathfrak{u} \subset \cup_i [B_n^{i,j}]_\mathfrak{u} \text{ and } \sum_{i\in\mathbb{N}} \lim_\mathfrak{u} \mu_n(B_n^{i,j}) - \mu_\mathfrak{u}(A_\mathfrak{u}) \leq 2^{-j}.$$

By Lemma 1.1.8, for every $j \in \mathbb{N}$, there is a family $\{C_n^j \in \mathscr{B}_n\}_n$ such that

$$A_\mathfrak{u} \subset \cup_i [B_n^{i,j}]_\mathfrak{u} \subset [C_n^j]_\mathfrak{u} \text{ and } \mu_\mathfrak{u}([C_n^j]_\mathfrak{u}) - \mu_\mathfrak{u}(A_\mathfrak{u}) \leq 2^{-j}.$$

Observe that

$$X_\mathfrak{u} \setminus \bigcap_{j\in\mathbb{N}} [C_n^j]_\mathfrak{u} = \bigcup_{j\in\mathbb{N}} X_\mathfrak{u} \setminus [C_n^j]_\mathfrak{u} = \bigcup_{j\in\mathbb{N}} [X_n \setminus C_n^j]_\mathfrak{u},$$

so again by Lemma 1.1.8, there is a family $\{D_n \in \mathscr{B}_n\}_n$ such that

$$\mu_\mathfrak{u}\left([D_n]_\mathfrak{u} \Delta (\cup_j [X_n \setminus C_n^j]_\mathfrak{u})\right) = 0.$$

Hence if we define $B_n := X_n \setminus D_n$, we have $\mu_\mathfrak{u}([B_n]_\mathfrak{u} \Delta(\cap_j [C_n^j]_\mathfrak{u})) = 0$ and

$$\mu_\mathfrak{u}(A_\mathfrak{u} \Delta [B_n]_\mathfrak{u}) \leq \lim_i \mu_\mathfrak{u}(\cap_{j=1}^i [C_n^j]_\mathfrak{u} \setminus A_\mathfrak{u}) = 0.$$ □

Remark 1.1.9. Proposition 1.1.7 implies that the measure algebra of the ultraproduct of a family of probability spaces is the metric ultraproduct of their measure algebras. See [Fre04b, Section 328].

1.1.4 Maharam-type

We now prove that the ultraproduct of a family of finite or standard probability spaces is a nice, homogeneous probability space. The following theorem is a special case of [JK00] (which is written in the language of non-standard analysis).

Theorem 1.1.10. *Let $\{(X_n, \mathscr{B}_n, \mu_n)\}$ be a sequence of diffuse standard probability spaces or a sequence of finite spaces equipped with their uniform counting measure such that $\lim_{n\in\mathfrak{u}} |X_n| = \infty$. Then the measure ultraproduct $(X_\mathfrak{u}, \mathscr{B}_\mathfrak{u}, \mu_\mathfrak{u})$ is measurably isomorphic to $(\{0,1\}^\mathbb{R}, \nu^\mathbb{R})$ where ν is the normalized counting measure on $\{0,1\}$ and $\nu^\mathbb{R}$ is the product measure. That is, the measure algebras $\mathrm{MAlg}(X_\mathfrak{u}, \mu_\mathfrak{u})$ and $\mathrm{MAlg}(\{0,1\}^\mathbb{R}, \nu^\mathbb{R})$ are isomorphic.*

Observe that $X_\mathfrak{u}$ and $\{0,1\}^{\mathbb{R}}$ are not isomorphic as sets: they do not have the same cardinality. To prove the theorem, we recall the notion of Maharam type, see [Fre04b, 331F].

Definition 1.1.11. Let (X,μ) be a probability space and let us denote by $\mathfrak{A} = \mathrm{MAlg}(X,\mu)$ its measure algebra.

- A subset $\mathcal{A} \subset \mathfrak{A}$ **σ-generates**, if $\mathfrak{A}$ is the smallest σ-subalgebra of $\mathfrak{A}$ containing $\mathcal{A}$.
- The **Maharam type** of the measure algebra $\mathfrak{A}$ is the smallest cardinal of any subset of $\mathfrak{A}$ which σ-generates $\mathfrak{A}$.
- A measure algebra $\mathfrak{A}$ is **homogeneous** if the Maharam type of $\mathfrak{A}$ is equal to the Maharam type of $\mathrm{MAlg}(A,\mu/\mu(A))$ for every $A \in \mathfrak{A}$.

All the probability measure algebras which have the same Maharam type are isomorphic, see [Fre04b, 331L].

Theorem 1.1.12. *Every homogeneous probability measure algebra $\mathfrak{A}$ is isomorphic to the measure algebra of $(\{0,1\}^Z, \nu^Z)$ for a set Z which has the cardinality of the Maharam type of $\mathfrak{A}$.*

We can now prove the theorem.

Proof of Theorem 1.1.10. First observe that $\mathrm{MAlg}(X_\mathfrak{u},\mu_\mathfrak{u})$ has at most the cardinality of the continuum, because by Remark 1.1.9, $\mathrm{MAlg}(X_\mathfrak{u},\mu_\mathfrak{u})$ is the metric ultraproduct of a family of separable metric spaces. So we have to show that the Maharam type of $\mathrm{MAlg}(A_\mathfrak{u},\mu_\mathfrak{u}/\mu_\mathfrak{u}(A))$ is at least the continuum for every $A_\mathfrak{u} \subset X_\mathfrak{u}$ measurable and non negligible.

We start showing the result when $(X_n,\mathscr{B}_n,\mu_n)$ is a diffuse standard probability space for every n. By Proposition 1.1.7, there is a sequence $\{A_n \in \mathscr{B}_n\}_n$ such that $A_\mathfrak{u} = [A_n]_\mathfrak{u}$ up to measure 0. Since for every n, the measure space $(A_n, \mathscr{B}_n|_{A_n}, \mu_n/\mu_n(A_n))$ is also a standard probability space, it is enough to show that the Maharam type of $\mathrm{MAlg}(X_\mathfrak{u},\mu_\mathfrak{u})$ is at least the continuum. For this we will use the following standard result, which is proved in [Fre04b, 331J].

Lemma 1.1.13. *Let $\mathfrak{A}$ be a measure algebra and let Z be a set. Suppose that there is a family $\{A_z\}_{z\in Z}$ of measurable mutually independent sets of measure $\mu(A_z) = 1/2$. Then the Maharam type of $\mathfrak{A}$ is greater or equal to the cardinality of Z.*

We now exhibit a continuum family of independent sets of $(X_\mathfrak{u},\mathscr{B}_\mathfrak{u},\mu_\mathfrak{u})$. For every n, take a countable family $\mathcal{B}_n = \{B_n^i\}_i$ of measurable mutually independent set of X_n of measure $1/2$. For every function $f:\mathbb{N}\to\mathbb{N}$, put $B_\mathfrak{u}^f := [B_n^{f(n)}]_\mathfrak{u}$. If we denote with $\mathbb{N}_\mathfrak{u}$ the ultraproduct of $\{\mathbb{N},\mathbb{N},\ldots\}$, then for every $f \in \mathbb{N}_\mathfrak{u}$ the measurable subset $B_\mathfrak{u}^f$ is well-defined, since it does not depend on the values of f on subsets outside $\mathfrak{u}$. Observe also that if $f_1,\ldots,f_k$ differ $\mathfrak{u}$-almost always, then $B_\mathfrak{u}^{f_1},\ldots,B_\mathfrak{u}^{f_k}$ are independent. Therefore the family $\{B_\mathfrak{u}^f\}_{f\in\mathbb{N}_\mathfrak{u}}$ is a family of measurable mutually independent sets of measure $1/2$. The cardinality of this family is the cardinality of $\mathbb{N}_\mathfrak{u}$ which is the continuum by Remark 1.1.2. Hence Lemma 1.1.13 concludes the proof of the theorem in the diffuse case.

The same strategy works for finite uniform spaces. Suppose that for every n, the measure space (X_n,μ_n) is a finite uniform space and suppose that $\lim_{n\in\mathfrak{u}}|X_n| = \infty$. For every $A_\mathfrak{u} \in \mathscr{B}_\mathfrak{u}$, by Proposition 1.1.7, there is a sequence $\{A_n\}_n$ such that $A_\mathfrak{u} = [A_n]_\mathfrak{u}$ up to measure 0. Let us denote by $g:\mathbb{N}\to\mathbb{N}$ the function such that $2^{g(n)} \le |A_n| \le 2^{g(n)+1}$. For every n, consider

$C_n \subset A_n$ a subset of $2^{g(n)}$-elements. Observe that $\lim_{n\in\mathfrak{u}} g(n) = \infty$ and $\mu_\mathfrak{u}([C_n]_\mathfrak{u}) \geq \mu_\mathfrak{u}(A_\mathfrak{u})/2$. For every n, there is a family $\mathcal{B}_n = \{B_n^1, \dots, B_n^{g(n)}\}$ of mutually independent sets such that $|B_n^i| = |C_n|/2$ for every n and $i \leq g(n)$. As before, for every function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) \leq g(n)$, we can define $B_\mathfrak{u}^f := [B_n^{f(n)}]_\mathfrak{u}$. If we denote with $Z_\mathfrak{u}$ the ultraproduct of $Z_n = \{1, \dots, g(n)\}$, then, as before, for every $f \in Z_\mathfrak{u}$ the subset is well defined $B_\mathfrak{u}^f$ and if $f_1, \dots, f_k$ differ $\mathfrak{u}$-almost always, then $B_\mathfrak{u}^{f_1}, \dots, B_\mathfrak{u}^{f_k}$ are independent. Hence the family $\{B_\mathfrak{u}^f\}_{f\in Z_\mathfrak{u}}$ is a family of measurable mutually independent sets. Again by Remark 1.1.2, the cardinality of $Z_\mathfrak{u}$ is the continuum, so Lemma 1.1.13 implies that the Maharam type of $[C_n]_\mathfrak{u}$ is the continuum. Observe that the Maharam type is monotone under taking ideals [Fre04b, 331H(c)], hence also the Maharam type of $\mathrm{MAlg}(A_\mathfrak{u}, \mu_\mathfrak{u}/\mu_\mathfrak{u}(A))$ is the continuum. So the proof theorem is concluded. ☐

1.1.5 Automorphisms

Let (X, μ) be a probability space and let $\mathrm{Aut}(X, \mu)$ be its group of measure preserving automorphisms.

- The **uniform topology** on $\mathrm{Aut}(X, \mu)$ is the topology defined by the metric

$$\delta(S, T) := \mu(\{x \in X : Tx \neq Sx\}).$$

- The **weak topology** on $\mathrm{Aut}(X, \mu)$ is the topology for which T_n tends to T if

$$\mu(T_n(A)\Delta T(A)) \to 0, \quad \forall A \subset X \text{ measurable.}$$

Example 1.1.14. Let $X = \{1, \dots, n\}$ and let μ_n be the normalized counting measure on X. The group $\mathrm{Aut}(X, \mu_n)$ is the symmetric group over n elements S_n. The uniform topology is induced by the metric

$$\delta(\sigma, \tau) = \frac{1}{n} |\{i : \sigma(i) \neq \tau(i)\}| .$$

The metric δ is also called the Hamming distance.

Proposition 1.1.15. *Let $\{(X_n, \mu_n)\}_{n\in\mathbb{N}}$ be a family of probability spaces. Then the metric-ultraproduct of the family $\{(\mathrm{Aut}(X_n, \mu_n), \delta_n)\}_n$ embeds isometrically in $(\mathrm{Aut}(X_\mathfrak{u}, \mu_\mathfrak{u}), \delta_\mathfrak{u})$.*

Proof. Set $G := \prod_n \mathrm{Aut}(X_n, \mu_n)$ and define

$$T : G \to \mathrm{Aut}(X_\mathfrak{u}) \ \text{ as } \ T(g_n)_n[x_n]_\mathfrak{u} := [g_n x_n]_\mathfrak{u}.$$

Given $(g_n)_n$ and $(h_n)_n$ in G, we have

$$\delta_\mathfrak{u}(T(g_n)_n, T(h_n)_n) = \lim_{n\in\mathfrak{u}} \mu_n(\{x \in X_n : \ g_n x \neq h_n x\}) = \lim_{n\in\mathfrak{u}} \delta_n(g_n, h_n),$$

hence T factorizes to an isometry from the metric ultraproduct of $\{(\mathrm{Aut}(X_n, \mu_n), \delta_n)\}_n$ to $\mathrm{Aut}(X_\mathfrak{u}, \mu_\mathfrak{u})$. ☐

1.2 Limit of actions

In this section we will study measure preserving actions on general probability spaces under the point of view of weak containment. We will prove that any measure preserving action on a diffuse probability space is weakly equivalent to an action on a standard probability space. This will be the key tool for understanding ultraproducts of sequences of probability measure preserving actions of a countable group G. We will introduce a compact, metric topology on the space of weak equivalence classes of actions which is equivalent to the topology defined in [AE11], a sequence of (classes of) actions converges if all its ultraproducts are weakly equivalent and in this case, the ultraproduct is the limit.

We will denote by a, b and c the probability measure preserving actions (pmp) of G on a probability space, denoted by $(X_a, \mu_a), (X_b, \mu_b)$ and (X_c, μ_c) (which will not be standard in general). We will denote by $\mathrm{Act}_d(G)$ the set of the pmp actions of G on a (fixed) standard diffuse probability space and with $\mathrm{Act}_f(G)$ the set of actions of G on the finite sets $\{1, \ldots, n\}$ for $n \in \mathbb{N}$, which we equip with their counting measure. We set $\mathrm{Act}(G) := \mathrm{Act}_d(G) \sqcup \mathrm{Act}_f(G)$.

Definition 1.2.1. Let a be a pmp action of G on the probability space (X_a, μ_a). An action b of G is a **factor** of a, denoted $b \sqsubseteq a$, if there is a G-invariant isometric embedding of σ-algebras $\mathrm{MAlg}(X_b, \mu_b) \hookrightarrow \mathrm{MAlg}(X_a, \mu_a)$.

More concretely factors of a are exactly the restriction of a to G-invariant σ-subalgebras of $\mathrm{MAlg}(X_a, \mu_a)$.

Remark 1.2.2. By Theorem 343B of [Fre04b], if b is a pmp action of G on the standard Borel probability space (X_b, μ_b) and b is a factor of a, then there is a G-invariant measure preserving map $\pi : X_a \to X_b$. However, we will never use this theorem.

Let (X, μ) be a probability space. We denote by $\mathrm{Part}_k(X)$ the set of partitions of X with k atoms and by $\mathrm{Part}_f(X)$ the set of finite partitions of X (in what follows, f will never be a natural number). For $\alpha \in \mathrm{Part}_f(X)$, we will denote by $|\alpha|$ the number of atoms of α.

Given a pmp action a of G, a finite subset $F \subset G$ and a partition $\alpha \in \mathrm{Part}_f(X_a)$, we set

$$\mathfrak{c}(a, F, \alpha) := (\mu(A^i \cap gA^j))_{i,j \leq |\alpha|, g \in F}.$$

Given two pmp actions of G (on the probability spaces $(X_a, \mu_a), (X_b, \mu_b)$), a finite subset $F \subset G$ and two finite partitions $\alpha = \{A_1, \ldots, A_k\} \in \mathrm{Part}_f(X_a)$ and $\beta = \{B_1, \ldots, B_k\} \in \mathrm{Part}_{|\alpha|}(X_b)$, we put

$$\|\mathfrak{c}(a, F, \alpha) - \mathfrak{c}(b, F, \beta)\|_1 := \sum_{i,j \leq |\alpha|} \sum_{f \in F} |\mu_a(A_i \cap fA_j) - \mu_b(B_i \cap fB_j)|.$$

The following definition is due to Kechris, [Kec10].

Definition 1.2.3. Let a, b two pmp actions of G. We say that a is **weakly contained** in b, and we will write $a \prec b$, if for every $\varepsilon > 0$, for every finite subset $F \subset G$ and for every finite partition $\alpha \in \mathrm{Part}_f(X_a)$ there is $\beta \in \mathrm{Part}_{|\alpha|}(X_b)$ such that

$$\|\mathfrak{c}(a, F, \alpha) - \mathfrak{c}(b, F, \beta)\|_1 \leq \varepsilon.$$

Two actions a and b are **weakly equivalent**, denoted by $a \sim b$, if $a \prec b$ and $b \prec a$.

Definition 1.2.4. The **weak topology** on $\mathrm{Act}_d(G)$ is the weakest topology for which the following sets form a base of open neighborhoods of $a \in \mathrm{Act}_d(G)$:

$$\{b \in \mathrm{Act}_d(G) : \ \|\mathfrak{c}(a,F,\alpha) - \mathfrak{c}(b,F,\alpha)\|_1 < \varepsilon\}$$
$$\text{for } \alpha \in \mathrm{Part}_f(X_a),\ F \subset G \text{ finite and } \varepsilon > 0.$$

For a standard probability space (X,μ), we have an injective map $\mathrm{Act}_d(G) \hookrightarrow \mathrm{Aut}(X,\mu)^G$. The weak topology of $\mathrm{Act}_d(G)$ corresponds to the product topology of the weak topology of $\mathrm{Aut}(X,\mu)$.

1.2.1 WC topology

We now define a topology equivalent to the topology defined in [AE11]. This topology will play a central role in the understanding of ultraproducts of actions.

Definition 1.2.5. Given two pmp actions a, b of G, a finite subset $F \subset G$ and $k \in \mathbb{N}$, we define

$$d_{F,\alpha}(a,b) := \inf_{\beta \in \mathrm{Part}_k(X_b)} \|\mathfrak{c}(a,F,\alpha) - \mathfrak{c}(b,F,\beta)\|_1 \quad \text{for every } \alpha \in \mathrm{Part}_k(X_a),$$
$$d_{F,k}(a,b) := \sup_{\alpha \in \mathrm{Part}_k(X_a)} d_{F,\alpha}(a,b).$$

Clearly $a \prec b$ if and only if for every finite subset $F \subset G$ and $k \in \mathbb{N}$, we have $d_{F,k}(a,b) = 0$.

Remark 1.2.6. Given two partitions α and β of the probability space (X,μ), we say that α *refines* β if each atom of β is (up to measure 0) a union of atoms of α. For every pmp actions a, b of G, for every finite subset $F \subset G$ and finite partitions $\alpha, \beta \in \mathrm{Part}_f(X_a)$

$$\text{if } \alpha \text{ refines } \beta \text{ then } d_{F,\alpha}(a,b) \geq d_{F,\beta}(a,b).$$

Remark 1.2.7. Let a and b be two pmp actions of G. Let $\alpha_n \in \mathrm{Part}_f(X_a)$ be an increasing sequence of partitions such that the algebra generated by $\cup_n \alpha_n$ is dense in $\mathrm{MAlg}(X_a,\mu_a)$. Then $a \prec b$ if and only if for every $F \subset G$ and $n \in \mathbb{N}$, we have $d_{F,\alpha_n}(a,b) = 0$.

In fact, we have to show that for every finite partition $\alpha \in \mathrm{Part}_f(X_a)$ and finite subset $F \subset G$ we have $d_{F,\alpha}(a,b) = 0$. Once α and F are fixed, for every $\varepsilon > 0$ there are $n \geq 0$ and a partition $\beta \in \mathrm{Part}_{|\alpha|}(X_a)$ refined by α_n such that $\|\mathfrak{c}(a,F,\alpha) - \mathfrak{c}(a,F,\beta)\|_1 < \varepsilon$. So

$$\begin{aligned}
d_{F,\alpha}(a,b) &= \inf_{\gamma \in \mathrm{Part}_{|\alpha|}(X_b)} \|\mathfrak{c}(a,F,\alpha) - \mathfrak{c}(b,F,\gamma)\|_1 \\
&\leq \inf_{\gamma \in \mathrm{Part}_{|\alpha|}(X_b)} \|\mathfrak{c}(a,F,\beta) - \mathfrak{c}(b,F,\gamma)\|_1 + \varepsilon \\
&\leq d_{F,\beta}(a,b) + \varepsilon \\
&\leq d_{F,\alpha_n}(a,b) + \varepsilon = \varepsilon.
\end{aligned}$$

Proposition 1.2.8. *Given three pmp actions a, b and c of G for every $\alpha \in \mathrm{Part}_f(X_a)$, we have $d_{F,\alpha}(a,c) \leq d_{F,\alpha}(a,b) + d_{F,|\alpha|}(b,c)$.*

Proof. Put $k = |\alpha|$. The proof is a straightforward computation:

$$\begin{aligned}
d_{F,\alpha}(a,c) =& \inf_{\gamma \in \mathrm{Part}_k(X_c)} \|\mathfrak{c}(a,F,\alpha) - \mathfrak{c}(c,F,\gamma)\|_1 \\
\leq& \inf_{\beta \in \mathrm{Part}_k(X_b)} \inf_{\gamma \in \mathrm{Part}_k(X_c)} (\|\mathfrak{c}(a,F,\alpha) - \mathfrak{c}(b,F,\beta)\|_1 + \|\mathfrak{c}(b,F,\beta) - \mathfrak{c}(c,F,\gamma)\|_1) \\
\leq& \inf_{\beta \in \mathrm{Part}_k(X_b)} \left(\|\mathfrak{c}(a,F,\alpha) - \mathfrak{c}(b,F,\beta)\|_1 + \inf_{\gamma \in \mathrm{Part}_k(X_c)} \|\mathfrak{c}(b,F,\beta) - \mathfrak{c}(c,F,\gamma)\|_1 \right) \\
\leq& d_{F,\alpha}(a,b) + \sup_{\beta \in \mathrm{Part}_k(X_b)} \inf_{\gamma \in \mathrm{Part}_k(X_c)} \|\mathfrak{c}(b,F,\beta) - \mathfrak{c}(c,F,\gamma)\|_1 \\
\leq& d_{F,\alpha}(a,b) + d_{F,|\alpha|}(b,c). \qquad \square
\end{aligned}$$

Definition 1.2.9. The **WC-topology** on $\mathrm{Act}(G)$ is the topology generated by the family of pseudo-metrics $\overline{d}_{F,k}(a,b) := d_{F,k}(a,b) + d_{F,k}(b,a)$, where $F \subset G$ is any finite subset and $k \in \mathbb{N}$.

The topology is not T_1 and two actions have the same closure if and only if they are weakly equivalent. We denote by $\overline{\mathrm{Act}}(G)$ the space of weakly-equivalent classes of actions. The WC-topology descends to a metric topology on $\overline{\mathrm{Act}}(G)$. The definition of the WC-topology is similar to the definition given by Burton in [Bur15b]. In the same paper he proved that the topology is equivalent to the topology of [AE11] and we will give a simpler and different proof in Theorem 1.2.22.

The following proposition will be crucial to understand limits for the WC-topology.

Proposition 1.2.10. *Let $\{a, a_1, a_2, \ldots\}$ be a family of actions of G. Then for every finite subset $F \subset G$, the following conditions are equivalent*

1. *for every finite partition $\alpha \in \mathrm{Part}_f(X_a)$, we have $\lim_n d_{F,\alpha}(a,a_n) = 0$,*
2. *for every $k \in \mathbb{N}$, we have $\lim_n d_{F,k}(a,a_n) = 0$.*

Proof. Condition (2) is by definition stronger than condition (1), so let us suppose that (1) holds. Fix $\varepsilon > 0$. For $k \in \mathbb{N}$ set

$$C := \{\mathfrak{c}(\beta, F, a) : \beta \in \mathrm{Part}_k(X_a)\} \subseteq [0,1]^{|F|k^2}.$$

By compactness, there are partitions $\alpha_1, \ldots, \alpha_j \in \mathrm{Part}_k(X_a)$ such that

$$\forall x \in C \text{ there is } i \leq j \text{ such that } \|\mathfrak{c}(\alpha_i, F, a) - x\|_1 \leq \varepsilon.$$

Consider the finite partition α generated by $\alpha_1, \ldots, \alpha_j$. By hypothesis there is $N \in \mathbb{N}$ such that for every $n \geq N$, we have that $d_{F,\alpha}(a,a_n) < \varepsilon$. Since α refines α_i for every i, we also have that $d_{F,\alpha_i}(a,a_n) < \varepsilon$ for every $i \leq j$ and $n \geq N$. So for $n \geq N$ and for every $\beta \in \mathrm{Part}_f(X_a)$, there is $i \leq j$ such that $\|\mathfrak{c}(\beta,F,a) - \mathfrak{c}(\alpha_i,F,a)\|_1 \leq \varepsilon$, therefore

$$d_{F,\beta}(a,a_n) \leq \|\mathfrak{c}(\beta,F,a) - \mathfrak{c}(\alpha_i,F,a)\|_1 + d_{F,\alpha_i}(a,a_n) \leq 2\varepsilon. \qquad \square$$

The following proposition is inspired by Theorem 5.3 of [CKTD13].

Proposition 1.2.11. *For a sequence of actions $a_n \in \mathrm{Act}_d(G)$, the following are equivalent:*

1. *for every finite subset $F \subset G$ and $\alpha \in \mathrm{Part}_f(X_a)$, we have $d_{F,\alpha}(a,a_n) \to 0$,*

2. *there is a family of automorphisms $T_n \in \mathrm{Aut}(X_{a_n})$ such that $T_n a_n T_n^{-1}$ converges to the action a in the weak topology.*

Proof. The fact that (2) implies (1) follows directly from the definitions, so we can suppose that (1) holds. By a diagonal argument, we can find an increasing sequence of finite partitions $(\alpha_n)_n = (\{A_1^n, \ldots, A_{k_n}^n\})_n$ and an increasing sequence of finite subsets F_n of G such that $d_{F_n,\alpha_n}(a, a_n)$ tends to 0, $\cup_n F_n = G$ and the algebra generated by $\cup_n \alpha_n$ is dense in $\mathrm{MAlg}(X, \mu)$. By (1), there is a sequence of partitions $(\beta_n)_n = (\{B_1^n, \ldots, B_{k_n}^n\})_n$ such that $\|\mathfrak{c}(a, F_n, \alpha_n) - \mathfrak{c}(a_n, F_n, \beta_n)\|_1$ tends to 0, which we can choose to satisfy $\mu(A_i^n) = \mu(B_i^n)$. For every n, there is $T_n \in \mathrm{Aut}(X, \mu)$ such that $\alpha_n = T_n \beta_n$. Now observe that $\mathfrak{c}(T_n a_n T_n^{-1}, F_n, T_n \beta_n) = \mathfrak{c}(a_n, F_n, \beta_n)$, so (2) holds. □

The following corollary is well-known (in the standard setting).

Corollary 1.2.12. *For every pmp action b on any probability space, the set of $\{a \in \mathrm{Act}_d(G) : a \prec b\}$ is weakly closed.*

Proof. We use Proposition 1.2.8. Let $(a_n)_n$ be a sequence which converges weakly to a such that $a_n \prec b$ for every n. By the (easy part of the) previous proposition, for every $\alpha \in \mathrm{Part}_f(X_a)$ and $F \subset G$ finite, we have that $d_{F,\alpha}(a, a_n) \to 0$. Hence

$$d_{F,\alpha}(a, b) \leq d_{F,\alpha}(a, a_n) + d_{F,k}(a_n, b) = d_{F,\alpha}(a, a_n) \to 0. \qquad \square$$

Definition 1.2.13. For every pmp action a of G and for every $g \in G$, we set

$$\begin{aligned} \mathrm{Fix}_g(a) :=& \{x \in X_a : \; gx = x\}, \\ |\mathrm{Fix}_g(a)| :=& \mu_a(\mathrm{Fix}_g(a)). \end{aligned}$$

Proposition 1.2.14. *For every $g \in G$, the map $|\mathrm{Fix}_g(\cdot)| : \overline{\mathrm{Act}}(G) \to [0,1]$ is well-defined and continuous.*

Proof. Let $a, b \in \mathrm{Act}(G)$. By Rokhlin Lemma, for every $\varepsilon > 0$, there are $A_\varepsilon, B_\varepsilon \subset X_a$ and $N \geq 1$, such that $\alpha := \{\mathrm{Fix}_g(a), A_\varepsilon, gA_\varepsilon, \ldots, g^N A_\varepsilon, B_\varepsilon\}$ is a partition of X_a and $\mu(B_\varepsilon) \leq \varepsilon$. Put $F := \{1_G, g, \ldots, g^N\}$ and observe that if $d_{F,\alpha}(a, b) \leq \eta$, then

$$|\mathrm{Fix}_g(b)| \leq |\mathrm{Fix}_g(a)| + \eta + \varepsilon. \qquad \square$$

1.2.2 Every action is weakly equivalent to a standard one

Theorem 1.2.15. *Every pmp action a of the countable group G on a diffuse space has a standard factor which is weakly equivalent to a. In particular every pmp action of G is weakly equivalent to an action on a standard Borel probability space.*

We remark that the theorem was also essentially proved for ultraproduct actions in the proof of the main theorem of [AE11]. We start showing that any pmp actions has at least a diffuse standard factor.

Lemma 1.2.16. *Every pmp action a of G on a diffuse space has a standard diffuse factor.*

Proof. If (X_a, μ_a) does not have any atom, we can find an increasing sequence of finite partitions $(\alpha_n)_n \subset \mathrm{Part}_f(X_a)$ such that the measure of each atom in α_n is less than $1/n$ for every n. Then observe that the G-invariant σ-algebra generated by $\cup_n G\alpha_n$ is a separable measure algebra without atoms, so the factor associated is a factor of a on a diffuse, standard probability space. ☐

The theorem follows from two facts: the weak topology on $\mathrm{Act}_d(G)$ is separable and the following easy lemma.

Lemma 1.2.17. *For two pmp actions a and b, the following are equivalent.*

1. *The action a is weakly contained in b, $a \prec b$.*
2. *We have $\{c \in \mathrm{Act}(G) : \ c \prec a\} \subseteq \{c \in \mathrm{Act}(G) : \ c \prec b\}$.*

Moreover if (X_a, μ_a) does not have any atom, then we can take c in (2) to be in $\mathrm{Act}_d(G)$.

Proof. The fact that (1) implies (2) follows from the transitivity of the weak containment. For the converse take a finite partition $\alpha \in \mathrm{Part}_f(X_a)$ and a finite subset $F \subset G$. The σ-closure of the G-invariant algebra generated by α is a factor of a which we denote by $c \in \mathrm{Act}(G)$. By construction $d_{F,\alpha}(a, c) = 0$ and by (2), we have $c \prec b$. So $d_{F,\alpha}(a, b) \leq d_{F,\alpha}(a, c) + d_{F,|\alpha|}(c, b) = 0$. For the moreover part, we can consider the factor c' associated to the σ-closure of the G-invariant algebra generated by α and the standard factor constructed in Lemma 1.2.16. ☐

Proof of Theorem 1.2.15. By Corollary 1.2.12, the set $A := \{c \in \mathrm{Act}_d(G) : c \prec a\}$ is weakly closed. Let $\{b_n\}_{n\in\mathbb{N}}$ be a countable weakly-dense subset of A. For every n, let $\{\beta_n^k\}_{k\in\mathbb{N}}$ be an increasing sequence of finite partitions of X_{b_n} which generate the σ-algebra. Let $\{F_n\}_n$ be an increasing sequence of finite subsets of G. For every $n, m, k \in \mathbb{N}$, let $\alpha_n^{k,m}$ be a partition of X_a such that

$$\|\mathfrak{c}(b_n, F_m, \beta_n^k) - \mathfrak{c}(a, F_m, \alpha_n^{k,m})\|_1 \leq \frac{1}{m}.$$

Consider the G-invariant σ-algebra $\mathfrak{A}$ generated by the partitions $\{\alpha_n^{k,m}\}_{n,k,m}$. Then $\mathfrak{A}$ is separable, since it is generated by finite partitions and G is countable, so the associated factor b is a factor of a on a standard diffuse probability space which by construction weakly contains b_n for every n. Corollary 1.2.12 implies that

$$\{c \in \mathrm{Act}_d(G) : c \prec a\} = \overline{\{b_n\}_n} \subseteq \{c \in \mathrm{Act}_d(G) : \ c < b\}$$

therefore (2) of Lemma 1.2.17 holds, hence $a \prec b$. ☐

From now on, we will identify $\overline{\mathrm{Act}}(G)$ with the set of weak equivalence classes of actions of G on any diffuse of finite uniform probability space.

1.2.3 Ultraproduct and weak equivalence

Given a pmp action a of G, a partition $\alpha \in \mathrm{Part}_f(X_a)$ and a finite subset $F \subset G$ we denote by α_F the partition generated by the F-translates of α.

Definition 1.2.18. Consider two pmp actions a and b of G and let us fix a partition $\alpha \in \mathrm{Part}_f(X_a)$, a finite subset $F \subset G$ and $\delta > 0$. A (α, δ, F)-**homomorphism** φ from a to b, is a homomorphism from the measure algebra of α_F, to the measure algebra $\mathrm{MAlg}(X_b, \mu_b)$, which satisfies

- $\mu_b(f\varphi(A)\Delta\varphi(f(A))) < \delta$ for every $A \in \alpha$ and $f \in F$,
- $\sum_{A\in\alpha_F} |\mu_b(\varphi(A)) - \mu_a(A)| < \delta$.

We denote by $\mathrm{Hom}(a, \alpha, F, \delta, b)$ the set of (α, δ, F)-homomorphisms from a to b

Proposition 1.2.19. *An action a is weakly contained in b if and only if for every $\alpha \in \mathrm{Part}_f(X_a)$, for every finite subset $F \subset G$ and for every $\delta > 0$, the set $\mathrm{Hom}(a, \alpha, F, \delta, b)$ is not empty.*

Proof. Suppose that $a \prec b$. Given $\alpha \in \mathrm{Part}_k(X_a)$, a finite subset $F \subset G$ which contains the identity and $\varepsilon > 0$, we consider $\alpha_F = \{A_1, \dots, A_k\}$. By hypothesis there is a partition $\beta = \{B_1, \dots, B_k\} \in \mathrm{Part}_k(X_b)$ such that $\|\mathfrak{c}(a, F, \alpha_F) - \mathfrak{c}(b, F, \beta)\|_1 < \varepsilon$. Set $\varphi(A_i) = B_i$. Given $A \in \alpha$ and $f \in F$ there are $I, J \subset \{1, \dots, k\}$ such that $A = \sqcup_{i\in I} A_i$ and $fA = \sqcup_{j\in J} A_j$. Then

$$\begin{aligned}
\mu_b(f\varphi(A)\Delta\varphi(fA)) =& \mu_b(\varphi(A)) + \mu_b(\varphi(fA)) - 2\mu_b((f(\sqcup_{i\in I} B_i)) \cap (\sqcup_{j\in J} B_j)) \\
\leq& \mu_b(\sqcup_{i\in I} B_i) + \mu_b(\sqcup_{j\in J} B_j) - 2\sum_{i\in I}\sum_{j\in J} \mu_b(fB_i \cap B_j) \\
\leq& \mu_a(\sqcup_{i\in I} A_i) + \mu_a(\sqcup_{j\in J} A_j) - 2\sum_{i\in I}\sum_{j\in J} \mu_a(fA_i \cap A_j) + 4\varepsilon \\
\leq& 2\mu_a(A) - 2\mu_a(fA \cap fA) + 4\varepsilon = 4\varepsilon.
\end{aligned}$$

For the converse fix $\alpha = \{A_1, \dots, A_k\} \in \mathrm{Part}_k(X_a)$, a finite subset $F \subset G$ which contains the identity and $\delta > 0$. Take $\varphi \in \mathrm{Hom}(a, \alpha, F, \delta, b)$. Define $B_i = \varphi(A_i)$ and $\beta = \varphi(\alpha)$. For $i, j \in \{1, \dots, k\}$ and $f \in F$, we have

$$\begin{aligned}
|\mu_a(A_i \cap fA_j) - \mu_b(B_i \cap fB_j)| =& |\mu_a(A_i \cap fA_j) - \mu_b(\varphi(A_i) \cap f\varphi(A_j))| \\
\leq& |\mu_a(A_i \cap fA_j) - \mu_b(\varphi(A_i \cap fA_j))| + \delta \\
\leq& |\mu_a(A_i \cap fA_j) - \mu_a(A_i \cap fA_j)| + 2\delta = 2\delta.
\end{aligned}$$ □

Definition 1.2.20. Let $(a_n)_n$ be a sequence of pmp actions of G. The **ultraproduct** of the sequence $(a_n)_n$ is the action of G on the ultraproduct measure space of the sequence $\{(X_{a_n}, \mu_{a_n})\}_n$ given by $g[x_n]_{\mathfrak{u}} := [gx_n]_{\mathfrak{u}}$, see Proposition 1.1.15.

Proposition 1.2.21 (Theorem 5.3 of [CKTD13])**.** *Let $a \in \mathrm{Act}(G)$, let $(b_n)_n$ be a sequence of actions of G and let $b_{\mathfrak{u}}$ be its ultraproduct. Then $a \prec b_{\mathfrak{u}}$ if and only if $a \sqsubseteq b_{\mathfrak{u}}$.*

Proof. Let us suppose that $a \prec b_{\mathfrak{u}}$. Let $(\alpha_n)_n$ be an increasing sequence of partitions of X_a, such that the algebra generated by them $\mathcal{A}$ is a dense G-invariant subalgebra of $\mathrm{MAlg}(X_a, \mu_a)$. Let $F_n \subset G$ be an increasing sequence of finite subsets which contain the identity and such that $\cup_n F_n = G$. By Proposition 1.2.19, for every n we can take $\varphi_n \in \mathrm{Hom}(a, \alpha_n, F_n, 1/n, b)$. We denote by $\varphi_{\mathfrak{u}} : \mathcal{A} \to \mathrm{MAlg}(X_{\mathfrak{u}}, \mu_{\mathfrak{u}})$ the map defined by $\varphi_{\mathfrak{u}}(A) := [\varphi_n(A)]_{\mathfrak{u}}$. It is clear that $\varphi_{\mathfrak{u}}$ is a G-invariant homomorphism which respect the measure, hence it is an isometry with respect to the natural metric on MAlg. Therefore we can extend $\varphi_{\mathfrak{u}}$ to a G-invariant isometric embedding of σ-algebras $\mathrm{MAlg}(X_a, \mu_a)$ to $\mathrm{MAlg}(X_{\mathfrak{u}}, \mu_{\mathfrak{u}})$. □

1.2.4 WC-compactness

We now show that the ultraproduct of a sequence of actions defined in Definition 1.2.20 is the limit with respect to the ultrafilter $\mathfrak{u}$ for the WC-topology. Observe that the ultraproduct of a sequence of actions always exists, so Theorem 1.2.22 implies that the topology is sequentially compact. Since the topology is metrizable, the topology is also compact, so we obtain Theorem 1 of [AE11]. On the other hand the theorem characterizes the topology in terms of ultraproducts of actions, and the same characterization holds for the topology in [AE11]. Therefore the two topology are equivalent.

Theorem 1.2.22. *For every sequence of actions $(a_n)_n \subset \overline{\mathrm{Act}}(G)$ the $\mathfrak{u}$-WC-limit of the sequence exists and is weakly equivalent to $a_\mathfrak{u}$. In particular a sequence $(a_n)_n$ WC-converges to a if and only if a is weakly equivalent to the ultraproduct action $a_\mathfrak{u}$ with respect to every ultrafilter $\mathfrak{u}$.*

Proof. Let $(a_n)_n$ be a sequence in $\overline{\mathrm{Act}}(G)$. By Theorem 1.2.15, and a little abuse of notations, we have that the ultraproduct of the sequence $a_\mathfrak{u}$ is an element of $\overline{\mathrm{Act}}(G)$. We want to show that the $\mathfrak{u}$-WC-limit of the sequence $(a_n)_n$ is $a_\mathfrak{u}$, so by Proposition 1.2.10, we have to show that for every $\alpha \in \mathrm{Part}_f(X_{a_\mathfrak{u}})$, for every finite subset $F \subset G$ and $k \in \mathbb{N}$, we have

$$\lim_{n\in\mathfrak{u}} d_{F,k}(a_n, a_\mathfrak{u}) = \lim_{n\in\mathfrak{u}} d_{F,\alpha}(a_\mathfrak{u}, a_n) = 0.$$

For every finite partition $\alpha_\mathfrak{u} = \{[A_n^1]_\mathfrak{u}, \ldots, [A_n^k]_\mathfrak{u}\} \in \mathrm{Part}_k(X_{a_\mathfrak{u}})$, consider the family of partitions $\alpha_n := \{A_n^1, \ldots, A_n^k\} \in \mathrm{Part}_f(X_{a_n})$. Then for every finite subset $F \subset G$, we have

$$\lim_{n\in\mathfrak{u}} \|\mathfrak{c}(a_n, F, \alpha_n) - \mathfrak{c}(a_\mathfrak{u}, F, \alpha_\mathfrak{u})\|_1 = 0,$$

and hence $\lim_{n\in\mathfrak{u}} d_{F,\alpha_\mathfrak{u}}(a_\mathfrak{u}, a_n) = 0$. On the other hand, suppose that there are a finite subset $F \subset G$, an integer $k \in \mathbb{N}$ and $\varepsilon > 0$ such that $\lim_{n\in\mathfrak{u}} d_{F,k}(a_n, a_\mathfrak{u}) > \varepsilon$. Then for every n in a set $I \in \mathfrak{u}$, there is a partition $\alpha_n = \{A_n^1, \ldots, A_n^k\} \in \mathrm{Part}_k(X_{a_n})$ such that $d_{F,\alpha_n}(a_n, a_\mathfrak{u}) > \varepsilon$. So if we take the partition $\alpha_\mathfrak{u} := \{[A_n^1]_\mathfrak{u}, \ldots, [A_n^k]_\mathfrak{u}\}$ we observe that

$$\lim_{n\in\mathfrak{u}} \|\mathfrak{c}(a_n, F, \alpha_n) - \mathfrak{c}(a_\mathfrak{u}, F, \alpha_\mathfrak{u})\|_1 \geq \varepsilon,$$

which is a contradiction. ☐

The following interesting corollary was remarked in both [AE11, Corollary 3.1] and [CKTD13, Proposition 5.7].

Corollary 1.2.23. *Let a be a pmp action of G and let $b \in \mathrm{Act}(G)$ be an action which is weakly contained in a. Then there is an action a' weakly equivalent to a such that b is a factor of a'.*

Increasing sequences of actions always admit limits and such limits are easily described.

Proposition 1.2.24. *Let $(a_n)_n$ be an upward directed sequence of actions in $\overline{\mathrm{Act}}(G)$.*

(1) The sequence converges to an action $a \in \overline{\mathrm{Act}}(G)$.

(2) For every $n \in \mathbb{N}$, we have $a_n \prec a$.

(3) If $b \in \overline{\mathrm{Act}}(G)$ satisfies that $a_n \prec b$ for every $n \in \mathbb{N}$, then $a \prec b$.

Proof. By compactness, there is a WC-converging subsequence $(a_{n_k})_k$ and let a be its limit. We claim that (2) and (3) holds for a. For this fix $n > 0$, a finite subset $F \subset G$ and $\alpha \in \mathrm{Part}_f(X_{a_n})$. Since $(a_{n_k})_k$ WC-converges to a,

$$d_{F,\alpha}(a_n, a) \leq d_{F,\alpha}(a_n, a_{n_k}) + d_{F,|\alpha|}(a_{n_k}, a) = d_{F,|\alpha|}(a_{n_k}, a) \xrightarrow{k} 0$$

hence $a_n \prec a$ for every n. Let $b \in \overline{\mathrm{Act}}(G)$ an action such that $a_n \prec b$ for every $n \in \mathbb{N}$. Then for every partition $\alpha \in \mathrm{Part}_f(X_a)$ and finite subset $F \subset G$,

$$d_{F,\alpha}(a,b) \leq d_{F,\alpha}(a,a_{n_k}) + d_{F,|\alpha|}(a_{n_k},b) = d_{F,\alpha}(a,a_{n_k}) \xrightarrow{k} 0,$$

hence $d_{F,\alpha}(a,b) = 0$ for every α and F, which implies $a \prec b$.

Let a' and a'' two different cluster points of $(a_n)_n$. Then by (2) we have that $a_n \prec a'$ and $a_n \prec a''$ for every n and by (3) we get that $a' \prec a''$ and $a'' \prec a'$, that is a' is weakly equivalent to a'' and hence they represent the same element of $\overline{\mathrm{Act}}(G)$. $\square$

Corollary 1.2.25. *Let $(a_n)_n$ be an increasing sequence of finite actions and let a be the associated profinite action. Then $(a_n)_n$ WC-converges to a. In particular the profinite action a is weakly equivalent to the ultraproduct action a_{u}.*

Proof. By Proposition 1.2.24, it is enough to show that for every action $b \in \overline{\mathrm{Act}}(G)$ such that $a_n \prec b$ for every n, we have that $a \prec b$. Fix such an action b. For every n, we denote by $\alpha_n \in \mathrm{Part}_f(X_a)$ the partition on clopen sets such that $a\big|_{\alpha_n} = a_n$. By Remark 1.2.7, it is enough to show that for every finite subset $F \subset G$ and $n \in \mathbb{N}$, we have $d_{F,\alpha_n}(a,b) = 0$. This is straightforward

$$d_{F,\alpha_n}(a,b) \leq d_{F,\alpha_n}(a,a_n) + d_{F,|\alpha_n|}(a_n,b) = 0.$$ $\square$

1.3 Sofic entropy

In this section we will show that for free groups and $\mathrm{PSL}_k(\mathbb{Z})$ the sofic entropy of profinite actions depends on the sofic approximation.

1.3.1 Sofic actions

Let G be a countable group, let $\mathbf{F}$ be a countable free group and let $\pi : \mathbf{F} \to G$ be a surjective homomorphism. Let us fix a section $\rho : G \to \mathbf{F}$ which maps the identity to the identity. Given any action a of G, we denote by $a^{\mathbf{F}}$ the action of $\mathbf{F}$ defined by $a^{\mathbf{F}}(g) := a(\pi(g))$. For an action a, recall that $|\operatorname{Fix}_g(a)|$ is the measure of the fixed point of g, (Definition 1.2.13).

Definition 1.3.1. A **sofic approximation** $\Sigma = (a_n)_n$ of G is a sequence of finite actions $a_n \in \mathrm{Act}_f(\mathbf{F})$ such that

- for every $g \in \ker\pi$, we have that $\lim_n |\operatorname{Fix}_g(a_n)| = 1$,
- for every $g \notin \ker\pi$, we have that $\lim_n |\operatorname{Fix}_g(a_n)| = 0$.

A group is *sofic* if it has a sofic approximation.

Definition 1.3.2. Given a sofic approximation $\Sigma = (a_n)_n$ of G, the ultraproduct action a_{u} of the sequence (a_n) is an action of $\mathbf{F}$ for which $\ker\pi$ acts trivially. Hence we can see the action a_{u} as a G-action, which we will denote by a_{u}^{Σ} and we will call it the **sofic action** associated to Σ.

Definition 1.3.3. An action a of the group G is **sofic** if there exists a sequence of finite actions $(a_n)_n \subset \mathrm{Act}_f(\mathbf{F})$ such that

- for every $\alpha \in \mathrm{Part}_f(X_a)$ and $F \subset G$ finite, we have $\lim_n d_{\rho(F),\alpha}(a^{\mathbf{F}}, a_n) = 0$,
- for every $g \in \ker \pi$, we have $\lim_n |\mathrm{Fix}_g(a_n)| = 1$.

We observe that the definition does not depend on the choice of ρ. Moreover we could also ask that $d_{F,\alpha}(a^{\mathbf{F}}, a_n) \to 0$ for every finite subset F of the free group $\mathbf{F}$. Observe also that if an action a of G is sofic, then the sequence $(a_n)_n$ as in Definition 1.3.3 is a sofic approximation, so any group which admits a sofic free action is sofic.

Proposition 1.3.4. *An action $a \in \mathrm{Act}(G)$ of the countable group G is sofic if and only if there is a sofic approximation Σ of G such that a is a factor of the sofic action a_{u}^{Σ}.*

Proof. If the action a is sofic, then by construction $a^{\mathbf{F}}$ is weakly contained in a_{u} and hence by Proposition 1.2.21, we have that a is a factor of a_{u}^{Σ}. On the other hand, if a is a factor of a_{u}^{Σ}, then $d_{\rho(F),\alpha}(a^{\mathbf{F}}, a_n) = d_{\rho(F),\alpha}(a_{\mathrm{u}}, a_n)$, which tends to zero by Theorem 1.2.22. □

Remark 1.3.5. It is not known whether every sofic action of a sofic group is of the form a_{u}^{Σ} for a sofic approximation Σ of G. The question is even open for Bernoulli shifts. They are sofic by [EL10] but we do not know if for non-amenable groups they are of the form a_{u}^{Σ}.

One can show that Definition 1.3.3 is equivalent to the definition of Elek and Lippner [EL10] in terms of colored graphs and to the (unpublished) definition of Ozawa of soficity of pseudo full groups, see Definition 10.1 in [CKTD13]. Remark that the authors in [CKTD13] prove that Definition 1.3.3 implies the soficity of the pseudo full group in the proof of Theorem 10.7.

1.3.2 Sofic entropy

In what follows, we use the definitions and notations of Kerr [Ker13] with the only exception that we will use ultralimits instead of limsup in the definition.

Let G be a countable sofic group and let $a \in \mathrm{Act}(G)$ be an action of G on a standard probability space. Let $\mathbf{F}$ be a free group, let $\pi : \mathbf{F} \to G$ be a surjective homomorphism and let $\rho : G \to \mathbf{F}$ be a section of π which maps the identity to the identity. Fix a sofic approximation $\Sigma = (a_n)_n$ as in Definition 1.3.1. Consider two partitions $\xi \leq \alpha \in \mathrm{Part}_f(X_a)$, a finite subset $F \subset G$ and $\delta > 0$. We put $\mathrm{Hom}(a, \alpha, F, \delta, a_n) := \mathrm{Hom}(a^{\mathbf{F}}, \alpha, \rho(F), \delta, a_n)$ (see Definition 1.2.18), where $a^{\mathbf{F}}(g) = a(\pi(g))$. We denote by $|\mathrm{Hom}(a, \alpha, F, \delta, a_n)|_{\xi}$ the cardinality of the set of $(\alpha, \delta, \rho(F))$-homomorphisms from a to a_n restricted to ξ, as explained in [Ker13].

We can now define the entropy of a with respect to Σ

$$\begin{aligned}
\mathrm{h}_{\Sigma}^{\xi}(\alpha, F, \delta, a) &:= \lim_{n \in \mathrm{u}} \frac{1}{|X_{a_n}|} \log\left(|\mathrm{Hom}(a, \alpha, F, \delta, a_n)|_{\xi}\right), \\
\mathrm{h}_{\Sigma}^{\xi}(\alpha, F, a) &:= \inf_{\delta > 0} \mathrm{h}_{\Sigma}^{\xi}(\alpha, F, \delta, a), \\
\mathrm{h}_{\Sigma}^{\xi}(\alpha, a) &:= \inf_{F \subset G} \mathrm{h}_{\Sigma}^{\xi}(\alpha, F, a), \\
\mathrm{h}_{\Sigma}^{\xi}(a) &:= \inf_{\alpha > \xi} \mathrm{h}_{\Sigma}^{\xi}(\alpha, a), \\
\mathrm{h}_{\Sigma}(a) &:= \sup_{\xi} \mathrm{h}_{\Sigma}^{\xi}(a).
\end{aligned}$$

where ξ and α are finite partitions of X_a with $\xi < \alpha$, $F \subset G$ is a finite subset and $\delta > 0$ is a real number. Observe that the definition does not depend on the section $\rho : G \to \mathbf{F}$, since for every $g \in \ker \pi$, we have that $\lim_n |\operatorname{Fix}_g(a_n)| = 1$. If for some α, δ, F and n the set $\operatorname{Hom}(\alpha, F, \delta, a_n)$ is empty, we will set $\mathrm{h}^{\xi}_{\Sigma}(\alpha, F, \delta, a) = -\infty$.

Proposition 1.3.6. *Let G be a countable sofic group and let $a \in \operatorname{Act}(G)$ be an action of G. Fix a sofic approximation Σ and let $a^{\Sigma}_{\mathfrak{u}}$ be the sofic action as in Definition 1.3.2. Then $\mathrm{h}_{\Sigma}(a) > -\infty$ if and only if $a \prec a^{\Sigma}_{\mathfrak{u}}$.*

This proposition is a corollary of Proposition 1.2.19. We observe that it is also a special case of Proposition 6 of [GP14].

Proof. Let $\mathbf{F}$ be a free group, let $\pi : \mathbf{F} \to G$ be a surjective homomorphism, let $\rho : G \to \mathbf{F}$ be a section and let $\Sigma = (a_n)_n$ be a sofic approximation.

Suppose that $\mathrm{h}_{\Sigma}(a) > -\infty$. Then there is a finite partition $\xi \in \operatorname{Part}_f(X_a)$ such that for every $\alpha \in \operatorname{Part}_f(X_a)$ with $\alpha > \xi$, for every finite subset $F \subset G$ and for every $\delta > 0$, we have

$$\left\{n \in \mathbb{N} : \operatorname{Hom}(a^{\mathbf{F}}, \alpha, \rho(F), \delta, a_n) \neq \emptyset\right\} \in \mathfrak{u}.$$

Take $\varphi_n \in \operatorname{Hom}(a^{\mathbf{F}}, \alpha, \rho(F), \delta, a_n)$ and define $\varphi_{\mathfrak{u}}(A) := [\varphi_n(A)]_{\mathfrak{u}}$. By construction we have that $\varphi_{\mathfrak{u}} \in \operatorname{Hom}(a, \alpha, F, \delta, a^{\Sigma}_{\mathfrak{u}})$ and hence the set is not empty. Therefore Proposition 1.2.19 implies that $a \prec a^{\Sigma}_{\mathfrak{u}}$.

Conversely, if we suppose that $a \prec a^{\Sigma}_{\mathfrak{u}}$, Proposition 1.2.19 tells us that for every finite partition $\alpha = \{A^1, \ldots, A^k\} \in \operatorname{Part}_f(X_a)$, for every finite subset $F \subset G$ and for every $\delta > 0$ the set $\operatorname{Hom}(a, \alpha, F, \delta, a^{\Sigma}_{\mathfrak{u}})$ is not empty. Take an element $\varphi_{\mathfrak{u}} \in \operatorname{Hom}(a^{\mathbf{F}}, \alpha, \rho(F), a_{\mathfrak{u}})$. Choose a family of subsets $\{B^i_n\}_{i,n}$ such that $\varphi_{\mathfrak{u}}(A^i) = [B^i_n]_{\mathfrak{u}}$ and set $\varphi_n(A^i) := B^i_n$. Then, we observe that for every $\varepsilon > 0$, the set of $n \in \mathbb{N}$ such that $\varphi_n \in \operatorname{Hom}(a, \alpha, F, \delta + \varepsilon, a_n)$ is in $\mathfrak{u}$, hence $\mathrm{h}_{\Sigma}(a) > -\infty$. ☐

Let G be a residually finite group and let $(H_n)_n$ be a chain of finite index subgroups of G. We denote by $a^{(H_n)}$ the profinite action associated to the sequence which we will always assume to be free. If the profinite action $a^{(H_n)}$ is free, then the sequence of finite actions gives us a sofic approximation of the group which we will denote by $\Sigma_{(H_n)}$.

Combining Proposition 1.3.6 with Corollary 1.2.25, we get the following interesting result.

Corollary 1.3.7. *Let G be a residually finite group and let $(H_n)_n$ be a chain of finite index subgroups of G such that the associated profinite action is free. Then for every action $a \in \operatorname{Act}(G)$ we have that $\mathrm{h}_{\Sigma_{(H_n)}}(a) > -\infty$ if and only if $a \prec a^{(H_n)}$.*

Since the corollary holds for every ultrafilter, it is still true for the usual definition of entropy with $\limsup$. In particular the sofic entropy of a non-strongly ergodic action with respect to a sofic approximation given by expanders is always $-\infty$.

Corollary 1.3.8. *Let G be a residually finite group let $(K_n)_n$ be a chain of finite index subgroups of G which has property (τ). For every non-strongly ergodic action a of G, we have $\mathrm{h}_{\Sigma_{(K_n)}}(a) = -\infty$.*

Proof. It is enough to observe that if $(K_n)_n$ has property (τ), then $a^{(K_n)}$ is strongly ergodic, as explained for example in Lemma 2.2 of [AE12], and an action weakly contained in a strongly ergodic action is also strongly ergodic (cf Lemma 5.1 [AE12]). ☐

1.3.3 Sofic entropy of profinite actions

Combining Corollary 1.3.7 with [AE12], we can now show that for some groups sofic entropy of profinite actions crucially depends on the sofic approximation.

Theorem 1.3.9. *Let G be a countable free group or $\mathrm{PSL}_k(\mathbb{Z})$ for $k \geq 2$. Then there is a continuum of normal chains $\{(H_n^r)_n\}_{r\in\mathbb{R}}$ such that $\mathrm{h}_{\Sigma_{(H_n^r)}}(a^{(H_n^s)}) > -\infty$ if and only if $r = s$.*

Note that the sofic entropy of profinite actions is either 0 or $-\infty$ as shown in Section 4 of [CZ14], see also Lemma 1.3.13. Theorem 1.3.9 follows from Corollary 1.3.7 and the following theorem.

Theorem 1.3.10 (Abért-Elek, [AE12])**.** *Let G be a countable free group or $\mathrm{PSL}_k(\mathbb{Z})$ for $k \geq 2$. Then there is a continuum of normal chains $\{(H_n^r)_n\}_{r\in\mathbb{R}}$ such that $a^{(H_n^s)} \prec a^{(H_n^r)}$ if and only if $r = s$.*

Sketch of the Proof for $G = \mathrm{PSL}_k(\mathbb{Z})$, $k \geq 3$. Let $(H_n)_n$ be the sequence of congruence subgroups of G, so that the family $\{G/H_n\}$ is a family of pairwise-non isomorphic finite non Abelian simple groups. For $I = \{i_1, i_2, i_3, \dots\} \subset \mathbb{N}$ infinite, we denote by a^I the profinite action associated to the normal chain $(\cap_{i\leq n} H_i)_n$. Observe that for an infinite I, the profinite action a^I is free and moreover, by property (T), it is strongly ergodic. Therefore we can apply Lemma 5.2 of [AE12] to get that $a^I \prec a^J$ if and only if $I \subset J$. So if we take any continuum of incomparable infinite subset of $\mathbb{N}$, then the associated profinite actions $\{a^I\}_I$ are weakly incomparable.

In order to see that Theorem 1.3.10 holds for $\mathrm{PSL}_2(\mathbb{Z})$ and for free groups, we can use that the congruence subgroups in $\mathrm{PSL}_2(\mathbb{Z})$ have property (τ) and that the proof above passes to finite index subgroups, see the proof of Theorem 3 in [AE12]. $\square$

Remark 1.3.11. Theorem 1.3.10 holds for a large variety of groups. In fact the Strong Approximation Property claims that any Zariski dense subgroup of the rational point of a rational algebraic linear group, has infinitely many pairwise non-isomorphic simple non-Abelian finite quotients, see [LS03, Window 9]. This was used in [AE12] to find family of pairwise inequivalent free actions of linear property (T) groups. One can then combine this fact with Margulis normal subgroup theorem to show that Theorem 1.3.9 holds for many lattices of higher rank algebraic linear groups.

We know give an example of an action which has positive entropy with respect to a sofic approximation and $-\infty$ with respect to another. We will do this considering the examples of Theorem 1.3.9 and taking the diagonal product with respect to a Bernoulli shift. Then Bowen's computation for such actions will allow us to conclude.

Theorem 1.3.12. *Let G be a countable free group or $\mathrm{PSL}_k(\mathbb{Z})$ for $k \geq 2$. For every $r \geq 0$, there is an action a of G and two sofic approximations Σ_1 and Σ_2 such that $\mathrm{h}_{\Sigma_1}(a) = r$ and $\mathrm{h}_{\Sigma_2}(a) = -\infty$.*

In the proof of the theorem, we will need the following easy lemma which was point out to us by L. Bowen.

Lemma 1.3.13. *Let $(H_n)_n$ be a chain of finite index subgroups of G and denote by $a = a^{(H_n)}$ the associated profinite action of G. For every $\varepsilon > 0$, there is a generating partition α of X_a with $H(\alpha) \leq \varepsilon$.*

Proof. Put $i_0 := [G : H_1]$ and $i_n := [H_n : H_{n+1}]$, without lost of generality we can suppose that $i_n \geq 2$ for every $n \geq 0$. Let us fix $\varepsilon > 0$ and take $N \in \mathbb{N}$ such that $2^{-(N-1)} + \sum_{n \geq N} n 2^{-(n-1)} < \varepsilon$. For every $n \geq N$, take a clopen $A_n \subset X_n$ such that

- $A_n \cap A_m = \emptyset$ if $n \neq m$,
- A_n is a clopen set associated to a conjugate H_n^g of H_n, that is it has measure $1/[G : H_n]$ and it is H_n^g-invariant.

Set $A_0 := X \setminus \cup_{n \geq N} A_n$ and $\alpha := \{A_0, A_N, A_{N+1}, \ldots\}$. The partition α is generating and observe that $\mu(A_n) \leq 2^{-n}$ and $\mu(A_0) \geq 1 - 2^{-(N-1)}$. We now compute the entropy of α,

$$
\begin{aligned}
H(\alpha) &= -\mu(A_0)\log(\mu(A_0)) - \sum_{n \geq N} \mu(A_n)\log(\mu(A_n)) \\
&\leq -\log(1 - 2^{-(N-1)}) + \sum_{n \geq N} \frac{\log(i_1 \ldots i_n)}{i_1 \ldots i_n} \\
&\leq 2^{-(N-1)} + \sum_{n \geq N} 2^{-(n-1)} \sum_{j=1}^{n} \frac{\log(i_j)}{i_j} \\
&\leq 2^{-(N-1)} + \sum_{n \geq N} n 2^{-(n-1)} < \varepsilon.
\end{aligned}
$$

□

Proof of Theorem 1.3.12. Let (X, μ) be a finite probability space with $H(\mu) = r$ and denote by b the Bernoulli shift of G on (X^G, μ^G). By Theorem 1.3.10, there are two normal chains of finite index subgroups $(H_n)_n$ and $(K_n)_n$ such that the actions $a^{(H_n)}$ and $a^{(K_n)}$ are weakly incomparable and so the diagonal action $a^{(H_n)} \times b$ is not weakly contained in $a^{(K_n)}$. Lemma 1.3.13 and Bowen's Theorem [Bow10b, Theorem 8.1] tell us that $\mathrm{h}_{\Sigma_{(H_n)}}(a^{(H_n)} \times b) = H(\mu) = r$ and by Corollary 1.3.7 we have that $\mathrm{h}_{\Sigma_{(K_n)}}(a^{(H_n)} \times b) = -\infty$. □

Chapter 2

More full groups

The following chapter is based on a joint work with François Le Maître.

We associate to every action of a Polish group on a standard probability space a Polish group that we call the *orbit full group*. For discrete groups, we recover the well-known full groups of pmp equivalence relations equipped with the uniform topology. However, there are many new examples, such as orbit full groups associated to measure preserving actions of locally compact groups. We also show that such full groups are complete invariants of orbit equivalence.
We give various characterizations of the existence of a dense conjugacy class for orbit full groups, and we show that the ergodic ones actually have a unique Polish group topology. Furthermore, we characterize ergodic full groups of countable pmp equivalence relations as those admitting non-trivial continuous character representations.

Contents

2.1 Preliminaries

2.1.1 Polish spaces

A **Polish space** is a separable topological space which admits a compatible complete metric. A countable intersection of open subsets of a topological spaces is called a G_δ.

Proposition 2.1.1 ([Kec95, Theorem 3.11]). *Let (X, τ) be a Polish space. A subset $Y \subset X$ is Polish for the induced topology if and only if Y is a G_δ.*

A **standard Borel space** is an uncountable set X equipped with a σ-algebra $\mathfrak{B}$ such that there exists a Polish topology on X for which $\mathfrak{B}$ is the σ-algebra of Borel subsets. A fundamental fact is that all the standard Borel spaces are isomorphic [Kec95, Theorem 15.6], and that every uncountable Borel subset of a standard Borel space is a standard Borel space when equipped with the induced σ-algebra [Kec95, Corollary 13.4].

Theorem 2.1.2 (Luzin-Suslin, see [Kec95, Theorem 15.1]). *Let X and Y be two standard Borel spaces and let $f : X \to Y$ be an injective Borel map. Then for every Borel subset A of X, $f(A)$ is Borel.*

A subset A of a Polish space X is **analytic** if there is a standard Borel space Y, a Borel subset B of Y and a Borel function $f : Y \to X$ such that $A = f(B)$. In general, analytic sets are not Borel, however they are Lebesgue-measurable (see Theorem 4.3.1 in [Sri98]). If X and Y are two Polish spaces, a map $f : X \to Y$ will be called analytic if the preimage of any open set is analytic. Note that analytic maps are Lebesgue-measurable by the aforementioned result.

2.1.2 Polish groups

A topological group whose topology is Polish is called a **Polish group**. Polish groups have several good properties. We list three of them, for proofs see Section 1.2 of [BK96].

Properties 2.1.3.

(α) *Let G be a Polish group, and let H be a subgroup of G. Then H is Polish for the induced topology if and only if H is closed in G.*

(β) *Let G be a Polish group, and let $H \lhd G$ be a closed normal subgroup. Then G/H is a Polish group for the quotient topology.*

(γ) *Let $\varphi : G \to H$ be an analytic homomorphism between two Polish groups G and H. Then φ is continuous. If moreover φ is surjective, then φ induces a topological isomorphism between $G/\ker(\varphi)$ and H.*

Let us end this section by citing a deep result of Becker and Kechris, which will be crucial in the proof of our main theorem.

Theorem 2.1.4 ([BK93]). *Let a Polish group G act in a Borel manner on a standard Borel space X. Then there is a Polish topology τ on X inducing its Borel structure such that the action of G on (X, τ) is continuous.*

2.1.3 The Polish group $\mathrm{Aut}(X, \mu)$

A **standard probability space** is a standard Borel space equipped with a non atomic probability measure. All standard probability spaces are isomorphic (see [Kec95, Theorem 17.41]). The **measure algebra** of the standard probability space (X, μ) is the σ-algebra of measurable subsets of X, where two such subsets are identified if their symmetric difference has measure zero. We will denote the measure algebra with $\mathrm{MAlg}(X, \mu)$ and recall that it is a Polish space when equipped with the topology induced by the complete metric $d_\triangle$ defined by $d_\triangle(A, B) = \mu(A \triangle B)$.

Definition 2.1.5. Let (X, μ) be a standard probability space. The group $\mathrm{Aut}(X, \mu)$ of measure preserving Borel bijections of (X, μ), identified up to measure zero, carries two natural metrizable topologies :

- the **weak topology**, for which a sequence $(T_n)_n$ converges to T if for every measurable subset $A \subset X$, we have $\mu(T_n(A)\Delta T(A)) \to 0$.
- the **uniform topology**, induced by the **uniform metric** d_u defined by

$$d_u(T, S) := \mu\left(\{x \in X : \ Tx \neq Sx\}\right).$$

Proposition 2.1.6 ([Hal60]). *The group* $\mathrm{Aut}(X, \mu)$ *is a Polish group with respect to the weak topology, and the uniform metric* d_u *is complete.*

2.1.4 Spaces of measurable maps

Definition 2.1.7. Let (X, μ) be a standard probability space, and let (Y, τ) be a Polish space. We denote by $\mathrm{L}^0(X, \mu, (Y, \tau))$ (and by $\mathrm{L}^0(X, \mu, Y)$ whenever it is clear which topology we fix on Y) the space of Lebesgue-measurable maps from X to Y, identified up to measure 0. Any compatible bounded metric d on (Y, τ) induces a metric $\tilde{d}$ on $\mathrm{L}^0(X, \mu, (Y, \tau))$ defined by

$$\tilde{d}(f, g) := \int_X d(f(x), g(x)) d\mu(x).$$

The topology induced by $\tilde{d}$ is called the topology of **convergence in measure**.

This topology only depends on the topology of Y by the following proposition.

Proposition 2.1.8 ([Moo76, Proposition 6]). *Let* (f_n) *be a sequence of elements of* $\mathrm{L}^0(X, \mu, (Y, \tau))$ *and* $f \in \mathrm{L}^0(X, \mu, (Y, \tau))$. *Then the following are equivalent:*

(a) the sequence $(f_n)_n$ *converges to* f, *that is* $\tilde{d}(f_n, f) \to 0$,

(b) for all $\varepsilon > 0$, $\mu(\{x \in X : d(f(x), f_n(x)) > \varepsilon\}) \to 0$,

(c) every subsequence of $(f_n)_{n\in\mathbb{N}}$ *admits a subsequence* $(f_{n_k})_{k\in\mathbb{N}}$ *such that for almost all* $x \in X$ *we have*

$$f_{n_k}(x) \to f(x).$$

Remark 2.1.9. In a topological space, a sequence converges to a point if and only if all its subsequences have a subsequence converging to this point, so item (c) of the previous proposition implies that the topology of convergence in measure is the coarsest metrizable topology τ for which $f_n \to f$ a.s. implies that $f_n \to f$ with respect to τ.

The topology of convergence in measure on $\mathrm{L}^0(X,\mu,Y)$ is a Polish topology. A dense countable subset is constructed as follows. Fix a dense countable subset $D \subset Y$ and a dense countable subalgebra $\mathcal{A}$ of $\mathrm{MAlg}(X,\mu)$. Then, the family of $\mathcal{A}$-measurable functions from X to D that take only finitely many values is dense in $\mathrm{L}^0(X,\mu,Y)$.

When $Y = G$ is a Polish group, we equip $\mathrm{L}^0(X,\mu,G)$ with the group structure given by the pointwise product, that is for $f,g \in \mathrm{L}^0(X,\mu,G)$ we put

$$f \cdot g(x) := f(x)g(x).$$

Proposition 2.1.10. *The Polish space $\mathrm{L}^0(X,\mu,G)$ is a Polish group for the topology of convergence in measure and the pointwise product.*

Let (Y,τ) be a Polish space. Then $\mathrm{Aut}(X,\mu)$ acts on the right on $\mathrm{L}^0(X,\mu,(X,\tau))$ by precomposition:

$$\text{for } T \in \mathrm{Aut}(X,\mu) \text{ and } f \in \mathrm{L}^0(X,\mu,(Y,\tau)) \text{ we define } (f \cdot T)(x) := f(Tx).$$

Note that this is an action by isometries. Moreover, when $Y = X$, we may view $\mathrm{Aut}(X,\mu)$ as a subset of $\mathrm{L}^0(X,\mu,(X,\tau))$ identifying a transformation T with the function $f_T(x) := T(x)$, which corresponds to identify $\mathrm{Aut}(X,\mu)$ with the orbit of $\mathrm{id}_X \in \mathrm{L}^0(X,\mu,(X,\tau)$.

Proposition 2.1.11. *Let (X,μ) be a standard probability space equipped with a compatible Polish topology τ_X, and let (Y,τ_Y) be a Polish space.*

(1) The action of $\mathrm{Aut}(X,\mu)$ on $\mathrm{L}^0(X,\mu,(Y,\tau_Y))$ is continuous.

(2) The inclusion $\mathrm{Aut}(X,\mu) \hookrightarrow \mathrm{L}^0(X,\mu,(X,\tau_X))$ is an embedding.

Proof. (1). Fix a compatible complete bounded metric d_Y on Y. Suppose now that $T_n \to T$ and $f_n \to f$, we want to prove that $\tilde{d}_Y(f_nT_n, fT) \to 0$. Since each T_n is an isometry,

$$\tilde{d}_Y(f_nT_n, fT) = \tilde{d}_Y(f_n, fTT_n^{-1}) \leq \tilde{d}_Y(f_n, f) + \tilde{d}_Y(f, fTT_n^{-1}),$$

hence it is enough to show that if $T_n \to \mathrm{id}_X$ in $\mathrm{Aut}(X,\mu)$, then for every $f \in \mathrm{L}^0(X,\mu,(Y,\tau))$ we have $fT_n \to f$ in measure. Moreover we can suppose that f has finite range, because the set of such functions is dense. For such a function f, we can consider the finite partition of the space given by $\{f^{-1}(a)\}_{a\in f(X)}$ and by definition of weak convergence $\mu(T_nf^{-1}(a)\Delta f^{-1}(a)) \to 0$ for every $a \in f(X)$. So $\mu(\{x \in X : fT_n(x) \neq f(x)\}) \to 0$, in particular $fT_n \to f$ in measure.

(2). Fix a Polish topology on X induced by the complete metric d_X. Since $\mathrm{Aut}(X,\mu)$ can be identified with the orbit of id_X in $\mathrm{L}^0(X,\mu,X)$, the first part implies that the inclusion is continuous.

To see that it is an embedding, we first note that since $\mathrm{Aut}(X,\mu)$ acts by homeomorphisms on $\mathrm{L}^0(X,\mu,\tau)$, a sequence $(T_n)_n$ converges to T in measure if and only if $T_nT^{-1} \to \mathrm{id}_X$ in measure. So it is enough to show that every weak neighborhood of the identity contains a neighborhood of the identity for the topology of convergence in measure.

Given $r > 0$ and a subset A of X, we let $(A)_r := \{y \in X : \exists x \in A, d_r(x,y) < r\}$ be its r-neighborhood. Since the measure μ is regular, a pre-basis of neighborhoods of the identity for the weak topology on $\mathrm{Aut}(X,\mu)$ is given by the sets

$$\mathcal{V}_{F,\varepsilon} = \{T \in \mathrm{Aut}(X,\mu) : \mu(F \triangle T(F)) < \varepsilon\},$$

where F is closed and ε is positive. So fix such a closed set F and $\varepsilon > 0$. Since F is closed, we have that $F = \bigcap_{n\in\mathbb{N}} (F)_{\frac{1}{n}}$, so there is $\delta > 0$ with $\delta < \varepsilon$, such that $\mu((F)_\delta \setminus F) < \varepsilon$.

Now suppose that $\tilde{d}_X(T, \mathrm{id}_X) \leq \delta^2$, which implies that $\mu(\{x \in X : d_X(Tx, x) > \delta\}) < \delta$. Then $\mu(T(F) \setminus (F)_\delta) < \delta < \varepsilon$ and since T preserves the measure,

$$\mu(T(F)\Delta F) \leq \mu(T(F)\Delta(F)_\delta) + \mu((F)_\delta \Delta F) \leq 3\varepsilon.$$

We conclude that $T \in \mathcal{V}_{F,3\epsilon}$, and so the topology of convergence in measure refines the weak topology. □

Combining the above proposition and Proposition 2.1.1, we deduce that $\mathrm{Aut}(X,\mu)$ is a G_δ in $\mathrm{L}^0(X,\mu,(X,\tau))$. However, it is not closed, and one can actually show that its closure consists in the monoid of all measure preserving maps $(X,\mu) \to (X,\mu)$.

2.2 Full groups

2.2.1 Definition and fundamental facts

Let us start by recalling the original definition of full groups introduced by Dye in his pioneering work [Dye59], which is the starting point of our paper.

Definition 2.2.1. Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathrm{Aut}(X,\mu)$. We say that $T \in \mathrm{Aut}(X,\mu)$ is obtained by **cutting and pasting** the sequence $(T_n)_n$ if there is a partition $(A_n)_{n\in\mathbb{N}}$ of X such that for every $n \in \mathbb{N}$,

$$T_{\restriction A_n} = T_{n \restriction A_n}.$$

We will also say that T is obtained by cutting and pasting $(T_n)_{n\in\mathbb{N}}$ along $(A_n)_{n\in\mathbb{N}}$.

Definition 2.2.2 (Dye)**.** A subgroup $\mathbb{G}$ of $\mathrm{Aut}(X,\mu)$ is a **full group** if it is stable under cutting and pasting along any sequence of elements of $\mathbb{G}$.

Let us point out a fundamental fact.

Proposition 2.2.3 ([Dye59, Lemma 5.4])**.** *The restriction of the uniform metric d_u to any full group is complete.*

Definition 2.2.4. Suppose $\mathcal{R}$ is an equivalence relation on a standard probability space (X,μ). The **full group of the equivalence relation** $\mathcal{R}$, denoted by $[\mathcal{R}]$, is the set of all $T \in \mathrm{Aut}(X,\mu)$ such that for all $x \in X$, $(x, T(x)) \in \mathcal{R}$.

Remark 2.2.5. In the previous definition, we require that $(x, T(x)) \in \mathcal{R}$ for *all* $x \in X$, but up to modifying T on a measure zero set we could as well ask that this holds for *almost* all $x \in X$. Indeed, if T satisfies the latter condition, let A be the full measure set of $x \in X$ such that $(x, T(x)) \in \mathcal{R}$. Then A contains the full measure T-invariant set $B := \bigcap_{n\in\mathbb{Z}} T^n(A)$ and T coincides up to measure zero with the bijection T' defined by

$$T'(x) := \begin{cases} T(x) & \text{if } x \in B, \\ x & \text{else.} \end{cases}$$

It is then clear that for all $x \in X$, we have $(x, T'(x)) \in \mathcal{R}$.

A special and important case of the previous definition is when the equivalence relation is given by the action of a group.

Definition 2.2.6. Let G be a Polish group acting in a Borel manner on a standard probability space (X, μ). The associated **orbit full group** is the set of all

$$T \in \mathrm{Aut}(X, \mu) \text{ such that } T(x) \in G \cdot x, \text{ for all } x \in X.$$

In other words, it is the full group of the orbit equivalence relation $\mathcal{R}_G$ defined by $(x, y) \in \mathcal{R}_G$ whenever $\exists g \in G : g \cdot x = y$, and we will accordingly denote it by $[\mathcal{R}_G]$.

We want to stress that the previous definition makes sense only for *spatial* G-actions: we need a genuine G-action on X in order to define $[\mathcal{R}_G]$, and not just a morphism $G \to \mathrm{Aut}(X, \mu)$. The orbit full group should not be confused with the following one.

Definition 2.2.7. Let G be a subgroup of $\mathrm{Aut}(X, \mu)$. There is a smallest full group containing G, whose elements are obtained by cutting and pasting elements of G. This is the **full group generated** by G, denoted by $[G]_D$.

If G acts faithfully on (X, μ) and preserves the probability measure μ, then we have an injective map $G \hookrightarrow \mathrm{Aut}(X, \mu)$ and we clearly have $[G]_D \subset [\mathcal{R}_G]$. The inclusion is in general strict, as shown in Example 2.2.10 and Example 2.2.15.

Let us now give some concrete examples. We start with full groups of countable measure preserving equivalence relations, which are exactly the countably generated full groups.

Example 2.2.8. If Γ is a countable group acting on the standard probability space (X, μ) by measure preserving automorphisms, then one can easily check that the two full groups defined above coincide: $[\Gamma]_D = [\mathcal{R}_\Gamma]$. Moreover, the orbit full group $[\mathcal{R}_\Gamma]$ is separable (see [Kec10, Proposition 3.2]) and hence it is Polish with respect to the uniform topology by Proposition 2.2.3 (this follows also from the proof of Theorem 2.2.18).

As a matter of fact, the full groups given by the previous example are the only full groups that are Polish for the uniform topology.

Proposition 2.2.9. *A full group $\mathbb{G} < \mathrm{Aut}(X, \mu)$ is Polish with respect to the uniform topology if and only if it is the full group of a countable probability measure preserving equivalence relation.*

Proof. If $\mathbb{G} \leq \mathrm{Aut}(X, \mu)$ is a full group separable for the uniform topology, we can choose a dense countable subgroup $\Lambda \subset \mathbb{G}$ so that $[\Lambda]$ is a closed subgroup of $\mathbb{G}$ that contains Λ, hence it has to be equal to $\mathbb{G}$. On the other hand, any countable pmp equivalence relation comes from the action of a countable group by a result of Feldman and Moore [FM77a], hence it is Polish for the uniform topology. □

Example 2.2.10. Let G be an infinite compact group and denote its Haar measure by μ. The action of G on itself by left translation generates the transitive equivalence relation on G, so its orbit full group is by definition $[\mathcal{R}_G] = \mathrm{Aut}(G, \mu)$. In particular, it is Polish for the weak topology.

We remark now that the full group $[G]_D$ generated by G may be strictly smaller than $[\mathcal{R}_G]$. Indeed, let us consider the circle group $G = \mathbb{S}^1$ acting on itself by translation, and suppose that the map $T : g \mapsto g^{-1}$ was obtained by cutting and pasting the translations by some g_i along A_i. Fix an index i and $h, h' \in A_i$ such that $h^2 \neq h'^2$. Since $h \in A_i$, we have that $h^{-1} = g_i h$ and since $h' \in A_i$, we also get $h'^{-1} = g_i h'$, hence $h^2 = g_i^{-1} = h'^2$, a contradiction.

We say that a subgroup G of $\mathrm{Aut}(X,\mu)$ is **ergodic** if for every $A \subset X$ such that $\mu(A \Delta gA) = 0$ for every $g \in G$, we have that A has either measure zero or full measure. This is equivalent to say that the only G-invariant elements of $\mathrm{MAlg}(X,\mu)$ are X and $\emptyset$. Note that G is ergodic if and only if $[G]_D$ is. Ergodic full groups have the following very useful property.

Proposition 2.2.11. *Let $\mathbb{G}$ be an ergodic full group and let $A, B \in \mathrm{MAlg}(X,\mu)$ such that $\mu(A) = \mu(B)$. Then there is an involution $T \in \mathbb{G}$ such that $T(A) = B$.*

This proposition was used used by Fathi to show the following result (see [Fat78]; he proves this for $\mathrm{Aut}(X,\mu)$, but the same argument works for any ergodic full group).

Theorem 2.2.12. *A full group is ergodic if and only if it is simple.*

Proposition 2.2.11 can also be used to show the following (see [Kec10, Proposition 3.1]).

Proposition 2.2.13. *A full group is weakly dense in $\mathrm{Aut}(X,\mu)$ if and only if it is ergodic.*

The following result is a consequence of Property 2.1.3 (α) and the above proposition.

Corollary 2.2.14. *The group $\mathrm{Aut}(X,\mu)$ is the unique ergodic full group that is Polish for the weak topology.*

The previous corollary may be extended to the non ergodic case (see [LM14b, Proposition 1.17 and Theorem D.6]), which yields a complete description of the full groups which are Polish for the weak topology.

Let us now give a pathological example.

Example 2.2.15. Let $G = S_\infty$ be the Polish group of all permutations of the natural numbers. The action of S_∞ on $X = \{0,1\}^{\mathbb{N}}$ is faithful, has only countably many orbits and all the orbits but one are countable. Kolmogorov observed, see [Dan00, Example 9], that there is a measure μ on X for which the action of G is not ergodic. This implies that $[G]_D$ is not ergodic and by Proposition 2.2.13, $[G]_D$ is not weakly dense in $[\mathcal{R}_G] = \mathrm{Aut}(X,\mu)$.

2.2.2 Orbit full groups have a Polish group topology

So far, the examples of Polish full groups that we have seen are full groups of countable pmp equivalence relations, which are exactly the full groups for which the uniform topology is Polish by Proposition 2.2.9, and $\mathrm{Aut}(X,\mu)$, which is the unique ergodic Polish full group for the weak topology by Corollary 2.2.14. Note that both are instances of orbit full groups. Our main result is that *all* orbit full groups carry a natural Polish group topology.

Before defining this topology, we need a different description of orbit full groups. So we start with a Polish group G acting in a Borel manner on a standard probability space (X,μ).

Consider the Polish space $\mathrm{L}^0(X,\mu,G)$ endowed with the topology of convergence in measure, and define $\Phi : \mathrm{L}^0(X,\mu,G) \to \mathrm{L}^0(X,\mu,X)$ by

$$\Phi(f)(x) := f(x) \cdot x.$$

In what follows, we will always see the group $\mathrm{Aut}(X,\mu)$ as a subspace of $\mathrm{L}^0(X,\mu,X)$, see Proposition 2.1.11. Put $\widetilde{[\mathcal{R}_G]} := \Phi^{-1}(\mathrm{Aut}(X,\mu))$.

Lemma 2.2.16. *We have the equality* $\Phi\left(\widetilde{[\mathcal{R}_G]}\right) = [\mathcal{R}_G]$.

Proof. The inclusion $\Phi\left(\widetilde{[\mathcal{R}_G]}\right) \subseteq [\mathcal{R}_G]$ follows directly from the definition. For the reverse inclusion, take $T \in [\mathcal{R}_G]$ and consider the Borel set

$$A := \{(x,g) \in X \times G \;:\; gx = Tx\}.$$

By the Jankov-von Neumann Uniformisation Theorem (see [Kec95, Theorem 18.1]), we can find an analytic uniformisation of A, i.e. an analytic, hence Lebesgue-measurable, map $f : X \to G$ such that for every $x \in X$, we have $T(x) = f(x)x$. In other words, we can find $f \in \mathrm{L}^0(X,\mu,G)$ such that $T = \Phi(f)$, so $[\mathcal{R}_G] \subseteq \Phi(\widetilde{[\mathcal{R}_G]})$. □

Definition 2.2.17. The **topology of convergence in measure** on an orbit full group $[\mathcal{R}_G]$ is the quotient topology induced by $\widetilde{[\mathcal{R}_G]} \subseteq \mathrm{L}^0(X,\mu,G)$, where we put on $\mathrm{L}^0(X,\mu,G)$ the topology of convergence in measure (see Section 2.1.4).

We say that the action of G is **essentially free** if there is a full measure G-invariant subset A of X such that for every $g \in G \setminus \{1_G\}$ and every $x \in A$, $gx \neq x$. Note that this is stronger than asking that all elements of $G \setminus \{1_G\}$ have a set of fixed points of measure zero, even when G is locally compact.

Whenever the G action is essentially free, $\Phi : \widetilde{[\mathcal{R}_G]} \to [\mathcal{R}_G]$ is a bijection and the topology on the orbit full group is just the topology induced by the topology of convergence in measure on $\mathrm{L}^0(X,\mu,G)$. We will give later a more precise description of the topology of convergence in measure on $[\mathcal{R}_G]$ when the action is non free (see Corollary 2.2.23).

Theorem 2.2.18. *Let G be a Polish group acting in a Borel manner on a standard probability space (X,μ). Then the associated orbit full group*

$$[\mathcal{R}_G] = \{T \in \mathrm{Aut}(X,\mu) : \forall x \in X, T(x) \in G \cdot x\}$$

is a Polish group for the topology of convergence in measure. This topology is weaker than the uniform topology and refines the weak topology.

Moreover if the G-action is essentially free and measure preserving, then G embeds into $[\mathcal{R}_G]$.

Proof. We start by showing that the topology of convergence in measure on $[\mathcal{R}_G]$ is a Polish group topology.

By Theorem 2.1.4, we may and do fix a Polish topology τ on X such that $G \curvearrowright (X,\tau)$ is a continuous action. Now, the characterization of the convergence in measure in terms of pointwise converging subsequences (cf. ((c)) in Proposition 2.1.8) yields that Φ is continuous. Then, combining Proposition 2.1.11 and Proposition 2.1.1, we get that $\mathrm{Aut}(X,\mu) \subseteq \mathrm{L}^0(X,\mu,(X,\tau))$ is a G_δ and so $\widetilde{[\mathcal{R}_G]} = \Phi^{-1}(\mathrm{Aut}(X,\mu))$ is also a G_δ. Therefore using again Proposition 2.1.1 we see that $\widetilde{[\mathcal{R}_G]}$ is Polish.

We now equip $\widetilde{[\mathcal{R}_G]}$ with the group operation $*$ defined by

$$(f * g)(x) := f(\Phi(g)(x))g(x).$$

The inverse is given by $f^{-1}(x) := f(\Phi(f)^{-1}x)^{-1}$. These group operations are continuous by Proposition 2.1.11, Proposition 2.1.10 and the fact that Φ is continuous. So $(\widetilde{[\mathcal{R}_G]}, *)$ is a Polish group and it is easy to check that the restriction $\Phi_{\upharpoonright \widetilde{[\mathcal{R}_G]}}$ is a group homomorphism. Hence we deduce that

$$[\mathcal{R}_G] = \Phi(\widetilde{[\mathcal{R}_G]}) \cong \widetilde{[\mathcal{R}_G]} / \ker(\Phi),$$

is itself a Polish group for the quotient topology by Property 2.1.3 (β).

Remark 2.2.19. The Polish group $\widetilde{[\mathcal{R}_G]}$ can be thought of as the full group of the groupoid associated to the action.

Let us now check the that the topology of convergence in measure is intermediate between the uniform and the weak topology. Since Φ is continuous, clearly we have that the topology on the orbit full group refines the weak topology, which also yields that $[\mathcal{R}_G]$ is a Borel subgroup of $\mathrm{Aut}(X, \mu)$ by Theorem 2.1.2.

If the action of G is essentially free, then $\Phi(f_n) \to \mathrm{id}_X$ uniformly implies that $\mu(f_n^{-1}(1_G)) \to 1$ and hence the topology of convergence in measure is weaker than the uniform topology. The proof for non-free actions follows the same lines. We postpone the proof, because it will be a more direct consequence of the description of the quotient topology that we will give in Proposition 2.2.22.

Finally, when the G-action is essentially free, Φ restricts to a topological isomorphism between $\widetilde{[\mathcal{R}_G]}$ and $[\mathcal{R}_G]$. The "moreover" part of the theorem then follows from the fact that G embeds into $\widetilde{[\mathcal{R}_G]} \subseteq \mathrm{L}^0(X, \mu, G)$ by identifying G with the set of constant maps. $\square$

Remark 2.2.20. We point out that we will show in Theorem 2.3.8 that the topology we have defined is the unique possible Polish group topology for every ergodic orbit full group.

Topology for non-free actions

Let G be a Polish group acting on (X, μ) as in Theorem 2.3.8.

Recall that every Polish group admits a compatible right-invariant metric (see for instance [Gao09, Theorem 2.1.1]). The proof of the following proposition can be found in [Gao09, Lemma 2.2.8].

Proposition 2.2.21. *Let G be a Polish group, and let d_G be a compatible right-invariant metric on G. Suppose that $H \leq G$ is a closed subgroup, then the **quotient metric** $d_{G/H}$ on G/H defined by, for $gH, g'H \in G/H$,*

$$d_{G/H}(gH, g'H) := \inf_{h \in H} d_G(gh, g')$$

induces the quotient topology on G/H.

For $x \in X$, let $G_x := \mathrm{Stab}_G(x)$. Then G_x is a closed subgroup of G by a result of Miller (see [Kec95, Theorem 9.17]; this also follows from Theorem 2.1.4). We now prove an analogous statement of the previous proposition for orbit full groups.

Proposition 2.2.22. *Let G be a Polish group acting in a Borel manner on a standard probability space (X, μ). Let d_G be a compatible bounded right-invariant metric on G. Then the quotient metric $d_{[\mathcal{R}_G]}$ induced by $\tilde{d}_G$ on $[\mathcal{R}_G] = \widetilde{[\mathcal{R}_G]} / \ker \Phi$ is given by*

$$d_{[\mathcal{R}_G]}(T, U) = \int_X d_x(T(x), U(x)) d\mu(x),$$

where for all $x \in X$, we identify the G-orbit of x to the homogenous space G/G_x, equipped with the quotient metric d_x defined by $d_x(gG_x, g'G_x) := \inf_{h\in G_x} d_G(gh, g')$.

Before proving the proposition, we state the following corollary. Its proof is analogous to the proof of Proposition 2.1.8, hence we omit it.

Corollary 2.2.23. *Let (T_n) be a sequence of elements of an orbit full group $[\mathcal{R}_G]$, and let $T \in [\mathcal{R}_G]$. Then the following are equivalent:*

(a) $T_n \to T$ in measure,

(b) for all $\varepsilon > 0$, $\mu(\{x \in X : d_x(T(x), T_n(x)) > \varepsilon\}) \to 0$,

(c) every subsequence of $(T_n)_{n\in\mathbb{N}}$ admits a subsequence $(T_{n_k})_{k\in\mathbb{N}}$ such that for almost all $x \in X$ we have $T_{n_k}(x) \to T(x)$, where the convergence holds in the orbit of x, identified to the homogeneous space G/G_x.

Proof of Proposition 2.2.22. Let K be the subgroup of $\mathrm{L}^0(X, \mu, G)$ consisting of all $f : X \to G$ such that for all $x \in X$, $f(x) \in G_x$. It is clear that K is the kernel of the restriction of Φ to $(\widetilde{[\mathcal{R}_G]}, *)$. Moreover, for all $f \in \widetilde{[\mathcal{R}_G]}$ and $g \in K$,

$$(f * g)(x) = f(\Phi(g)(x))g(x) = f(x)g(x).$$

So two elements of $\widetilde{[\mathcal{R}_G]}$ are in the same right K-coset for the group operation $*$ in $\widetilde{[\mathcal{R}_G]}$ if and only if they are in the same right K-coset with respect to the pointwise multiplication. This implies that the quotient metric induced by $\tilde{d}_G$ on $\widetilde{[\mathcal{R}_G]}/K$ and the quotient metric induced by $\tilde{d}_G$ on $\mathrm{L}^0(X, \mu, G)/K$ agree on $[\mathcal{R}_G]$.

The latter metric comes from the pseudo-metric ρ on $\mathrm{L}^0(X, \mu, G)$ defined by $\rho(g, g') := \inf_{k\in K} \tilde{d}_G(gk, g')$. In order to establish the proposition, we need to show that

$$\rho(g, g') = \int_X d_x(g(x), g'(x))d\mu(x).$$

Note that the integral is well-defined because the function $(x, g, g') \mapsto d_x(g, g')$ is analytic, hence Lebesgue-measurable. Fix $g, g' \in \mathrm{L}^0(X, \mu, G)$. For every $\epsilon > 0$, we apply the Jankov-von Neumann Uniformisation Theorem (see [Kec95, Theorem 18.1]) to the set

$$\{(x, h) \in X \times G : d_G(g(x)h, g'(x) < d_x(g(x), g'(x)) + \varepsilon \text{ and } h \cdot x = x\}$$

and we get a function $k \in K$ such that for all $x \in X$,

$$d_G\left(g(x)k(x), g'(x)\right) < d_x(g(x), g'(x)) + \varepsilon.$$

This implies that $\rho(g, g') \leq \int_X d_x(g(x), g'(x))d\mu(x)$. The reverse inequality is a direct consequence of the fact that for all $k \in K$ and all $x \in X$,

$$d_G(g(x)k(x), g'(x)) \geq d_x(g(x), g'(x)).$$

□

2.2.3 Full groups of measurable equivalence relations

In this section, we study full groups of measurable equivalence relations, which will lead us to a simple criterion for distinguishing the orbit full groups of the previous section from $\mathrm{Aut}(X,\mu)$ (see Corollary 2.2.28).

Proposition 2.2.24. *Let (X,μ) be a standard probability space, and A be a Borel subset of $X\times X$. Then the set*

$$[A] := \{T\in\mathrm{Aut}(X,\mu) : \forall x\in X, (x,T(x))\in A\}$$

is a Borel subset of $\mathrm{Aut}(X,\mu)$.

Proof. We may suppose that (X,d) is a Cantor set. By Proposition 2.1.11, the weak topology on $\mathrm{Aut}(X,\mu)$ is the same as the topology of convergence in measure induced by $\mathrm{L}^0(X,\mu,(X,d))$. We will show that given a Borel subset A of $X\times X$, the function $\Phi_A : \mathrm{L}^0(X,\mu,(X,d))\to \mathrm{MAlg}(X,\mu)$ defined by

$$\Phi_A(f) := \{x\in X : (x,f(x))\in A\}$$

is Borel. This will be enough because $[A]=\Phi_A^{-1}(\{X\})$. Let $\mathcal{F}$ be the class of subsets of $X\times X$ for which Φ_A is Borel. Because $X\times X$ is again a Cantor set, we only need to show that $\mathcal{F}$ is a σ-algebra containing the clopen sets.

So suppose that A is a clopen subset of $X\times X$. We will actually show that Φ_A is continuous.

Fix $f\in\mathrm{L}^0(X,\mu,(X,d))$ and $\varepsilon>0$. Since A and $X^2\setminus A$ are open, there is $\delta>0$ smaller than $\varepsilon/2$ and a set B of measure less than $\varepsilon/2$ such that for all $x\in X\setminus B$ and $y\in X$ such that $d(y,f(x))\leq\delta$, we have

$$(x,f(x))\in A\Leftrightarrow (x,y)\in A.$$

It is then clear that for every g in the open neighborhood U of f given by

$$\mathcal{U}_\delta(f)=\left\{g\in\mathrm{L}^0(X,\mu,X) : \mu\left\{x\in X : d(f(x),g(x))>\delta\right\}\right)<\delta\right\},$$

we have $\mu(\Phi_A(f)\Delta\Phi_A(g))<\varepsilon$.

So the class $\mathcal{F}$ contains the clopen sets, and we now have to show that it is a σ-algebra. First, $\mathcal{F}$ is stable under complementation, since given $T\in\mathrm{Aut}(X,\mu)$, we have that $\Phi_{X^2\setminus A}(T)=X\setminus\Phi_A(T)$. Moreover if (A_n) is a countable family of elements of $\mathcal{F}$, then

$$\Phi_{\cup_n A_n}(T)=\bigcup_{n\in\mathbb{N}}\Phi_{A_n}(T),$$

and since taking a countable union is a Borel operation on $\mathrm{MAlg}(X,\mu)$, we get that $\bigcup_{n\in\mathbb{N}}A_n$ belongs to $\mathcal{F}$. ☐

Corollary 2.2.25. *Let $\mathcal{R}$ be a Borel equivalence relation on a standard probability space (X,μ). Then its full group*

$$[\mathcal{R}]=\{T\in\mathrm{Aut}(X,\mu) : \forall x\in X, (x,T(x))\in\mathcal{R}\}$$

is a Borel group for the Borel structure induced by the weak topology on $\mathrm{Aut}(X,\mu)$.

In the previous section, we saw that the orbit full group of a Borel action of Polish group is also Borel. Note however that there are actions of Polish groups inducing analytic, non Borel equivalence relations, so there are non Borel analytic equivalence relations whose full group is Borel. Using Theorem 2.3.16, one can show that there also are analytic equivalence relations whose full group is not Borel.

Let us now give a general criterion which allows us to distinguish full groups of measurable equivalence relations and $\mathrm{Aut}(X, \mu)$.

Proposition 2.2.26. *Let $\mathcal{R}$ be a Lebesgue-measurable equivalence relation on a standard probability space. Then the following are equivalent:*

(i) $[\mathcal{R}] = \mathrm{Aut}(X, \mu)$,

(ii) *$\mathcal{R}$ has full measure in $X \times X$,*

(iii) *$\mathcal{R}$ has an equivalence class of full measure.*

Proof. Both implications *(iii)⇒(i)* and *(iii)⇒(ii)* are obvious.

The implication *(ii)⇒ (iii)* is also easy: suppose that $\mathcal{R}$ has full measure, then by Fubini's Theorem for almost all $x \in X$, the $\mathcal{R}$-equivalence class of x has full measure, so that in particular there is one equivalence class of full measure.

For the remaining implication *(i)⇒(iii)*, assume that $[\mathcal{R}] = \mathrm{Aut}(X, \mu)$. We may suppose that X is a compact group equipped with the Haar measure. By assumption, for all $z \in X$ the left multiplication by z is in the full group $[\mathcal{R}]$, which means that for all $z \in X$ and almost $x \in X$, $(zx, x) \in \mathcal{R}$. Again by Fubini's theorem, we deduce that there exists $x \in X$ such that for almost all $z \in X$, $(zx, x) \in \mathcal{R}$, hence *(iii)* holds. □

Remark 2.2.27. The arguments used in the previous proof shows that the full group generate by the circle group acting on itself by left translation $[\mathbb{S}^1]_D$ cannot be the full group of any Lebesgue-measurable equivalence relation. Indeed, such an equivalence relation would have to be transitive by the previous proof, so that $[\mathbb{S}^1]_D$ would be equal to $\mathrm{Aut}(X, \mu)$, contradicting Example 2.2.10.

Corollary 2.2.28. *Suppose a Polish group G acts essentially freely on (X, μ), and that $[\mathcal{R}_G] = \mathrm{Aut}(X, \mu)$. Then G is compact.*

Proof. If $[\mathcal{R}_G] = \mathrm{Aut}(X, \mu)$, then by the previous proposition there is a G-orbit of full measure. The freeness of the G-action allow us to identify such an orbit to G, which then carries a left-invariant Borel probability measure, hence G is compact by Ulam's Theorem (see [GTW05, Theorem B.1]). □

2.2.4 Orbit equivalence and full groups

Let $\mathcal{R}$ and $\mathcal{R}'$ be two equivalence relations on a standard probability spaces (X, μ). We say that $\mathcal{R}$ and $\mathcal{R}'$ are **orbit equivalent** if there are a full measure subset $A \subseteq X$ and a Borel measure preserving bijection $S : A \to A$ such that for all $(x, y) \in A^2$, $(x, y) \in \mathcal{R}$ if and only if $(S(x), S(y)) \in \mathcal{R}'$. We also say that such an S is an **orbit equivalence** between $\mathcal{R}$ and $\mathcal{R}'$. It is easy to see that if S is an orbit equivalence, then it conjugates the full groups $[\mathcal{R}]$ and $[\mathcal{R}']$, that is, we have the relation $S[\mathcal{R}]S^{-1} = [\mathcal{R}']$. In the case of orbit full groups, one can say a bit more.

Lemma 2.2.29. *Let G and H be two Polish groups acting in a Borel manner on (X, μ). Let $S \in \mathrm{Aut}(X, \mu)$ is an orbit equivalence between $\mathcal{R}_G$ and $\mathcal{R}_H$. Then the conjugation by S is a group homeomorphism between the orbit full groups $[\mathcal{R}_G]$ and $[\mathcal{R}_H]$.*

Proof. By Theorem 2.2.18, $[\mathcal{R}_G]$ and $[\mathcal{R}_H]$ are Borel subgroups of $\mathrm{Aut}(X,\mu)$ and the conjugation by S induces a Borel isomorphism between $[\mathcal{R}_G]$ and $[\mathcal{R}_H]$. This isomorphism is a homeomorphism by Property 2.1.3 (γ). ☐

A fancier restatement of the previous lemma is that orbit full groups, seen as topological groups, are invariants of orbit equivalence. Now, we show that for ergodic measure preserving actions of locally compact groups, the orbit full groups are complete invariants.

Theorem 2.2.30. *Let G and H be two Polish locally compact groups acting in a Borel measure preserving ergodic manner on a standard probability space (X,μ). Suppose that $\psi : [\mathcal{R}_G] \to [\mathcal{R}_H]$ is an abstract group isomorphism. Then there exists an orbit equivalence S between $\mathcal{R}_G$ and $\mathcal{R}_H$ such that for all $T \in [\mathcal{R}_G]$,*

$$\psi(T) = S^{-1}TS.$$

Dye's Reconstruction Theorem plays a fundamental role in our proof.

Theorem 2.2.31 ([Dye63, Theorem 2]). *Suppose $\mathbb{G}_1$ and $\mathbb{G}_2$ are two ergodic full groups on a standard probability space (X,μ). Then for every abstract group isomorphism $\psi : \mathbb{G}_1 \to \mathbb{G}_2$, there exists $S \in \mathrm{Aut}(X,\mu)$ such that for all $T \in \mathbb{G}_1$, we have $\psi(T) = STS^{-1}$.*

The theorem also uses the following proposition, whose proof is inspired by Proposition $B.2$ of [Zim84].

Proposition 2.2.32. *Let G and H be two Polish locally compact groups acting in a Borel measure preserving manner on a standard probability space (X,μ), and suppose that $[\mathcal{R}_G] \subseteq [\mathcal{R}_H]$. Then there exists a full measure subset $X_0 \subseteq X$ such that*

$$\mathcal{R}_G \cap (X_0 \times X_0) \subseteq \mathcal{R}_H.$$

Proof. Let ν be the Haar measure on G. The fact that $[\mathcal{R}_G] \subseteq [\mathcal{R}_H]$ implies that for all $g \in G$ and almost all $x \in X$, we have $gx \in Hx$. By Fubini's Theorem, this implies that the set

$$X_0 := \{x \in X : \text{ for } \nu\text{-almost all } g \in G, \text{ we have } gx \in Hx\}$$

has full measure. Now let $x \in X_0$, and let $g_1 \in G$ be such that $g_1x \in X_0$. We want to show that $g_1x \in Hx$.

Since x and g_1x are in X_0, the sets $A := \{g \in G : gx \in Hx\}$ and $B := \{g \in G : gx \in Hg_1x\}$ have full measure and so $A \cap B$ has full measure. Take $g \in A^{-1} \cap B$, then $gx \in Hx \cap Hg_1x$, so the two orbits Hx and Hg_1x intersect, hence $g_1x \in Hx$. ☐

Proof of Theorem 2.2.30. By Dye's Reconstruction Theorem, ψ is the conjugation by some $S \in \mathrm{Aut}(X,\mu)$ and by the previous proposition applied two times, such an S has to be an orbit equivalence. ☐

Question 2.2.33. Can the previous theorem be extended to (some) non locally compact Polish groups?

2.3 Topological properties of full groups

2.3.1 Aperiodic elements and free actions

Now we study some topological properties of orbit full groups. Let us fix a Polish group G acting on a Borel manner on the standard probability space (X, μ). We will denote by $\mathbb{G} = [\mathcal{R}_G]$ the associated orbit full group. Recall that its Polish topology is weaker than the uniform topology and refines the weak topology.

We will first prove that orbit full groups are contractible using the continuity of the first return map.

Definition 2.3.1. Let $T \in \mathrm{Aut}(X, \mu)$ and $A \in \mathrm{MAlg}(X, \mu)$, Poincaré's Recurrence Theorem states that for almost every $x \in A$ there is a smaller $n_x \in \mathbb{N}$ such that $T^{n_x}(x) \in A$. The **first return map** T_A is then defined by $T_A(x) = x$ for all $x \notin A$, and by $T_A(x) = T^{n_x}(x)$ for all $x \in A$.

Proposition 2.3.2 and Corollary 2.3.3 are straightforward generalizations of Keane's results for $\mathrm{Aut}(X, \mu)$, [Kea70]. We will however give a detailed proof to show how the compatible metric on orbit full groups defined in Proposition 2.2.22 can be used.

Proposition 2.3.2. *Let $\mathbb{G}$ be an orbit full group. Then the function which maps $(T, A) \in \mathbb{G} \times \mathrm{MAlg}(X, \mu)$ to the first return map $T_A \in \mathbb{G}$ is continuous.*

Proof. Given $T \in \mathrm{Aut}(X, \mu)$, $A \in \mathrm{MAlg}(X, \mu)$ and $n > 0$, we let

$$C_n(T, A) := \{x \in X : T^n(x) \in A\},$$

we put $B_n(T, A) := C_n(T, A) \setminus \bigcup_{m<n} C_m(T, A)$ and $B_0(T, A) := X \setminus A$. Then $(B_n(T, A))_{n \geq 0}$ is a partition of X and $T_A(x) = T^n(x)$ for all $x \in B_n(T, A)$. Note that $B_n(T, A)$ depends continuously on $(T, A) \in \mathrm{Aut}(X, \mu) \times \mathrm{MAlg}(X, \mu)$, where $\mathrm{Aut}(X, \mu)$ is equipped with the weak topology.

Now, let $\varepsilon > 0$, and fix $(\tilde{T}, \tilde{A}) \in \mathbb{G} \times \mathrm{MAlg}(X, \mu)$. Since $(B_n(\tilde{T}, \tilde{A}))_{n \geq 0}$ is a partition of X, we may find $N > 0$ such that $\mu(X \setminus \bigcup_{m<N} B_m(\tilde{T}, \tilde{A})) < \varepsilon$. Let $\mathcal{U}$ be the set of couples $(T, A) \in \mathbb{G} \times \mathrm{MAlg}(X, \mu)$ such that

(1) $\sum_{m=0}^{N} d_{[\mathcal{R}_G]}(\tilde{T}^m, T^m) < \varepsilon$ and

(2) $\sum_{m=0}^{N} \mu(B_m(T, A) \Delta B_n(\tilde{T}, \tilde{A})) < \varepsilon$.

By continuity of $T \mapsto T^m$ and of $B_m(\cdot, \cdot)$, the set $\mathcal{U}$ is open. Let $(T, A) \in \mathcal{U}$, we now compute the distance between $\tilde{T}_{\tilde{A}}$ and T_A.

$$\begin{aligned} d_{[\mathcal{R}_G]}(\tilde{T}_{\tilde{A}}, T_A) &= \int_X d_x(\tilde{T}_{\tilde{A}}(x), T_A(x)) d\mu(x) \\ &\leq \sum_{m=0}^{N} \int_{B_m(\tilde{T}, \tilde{A}) \Delta B_m(T, A)} d_x(\tilde{T}_{\tilde{A}}(x), T_A(x)) d\mu(x) \\ &+ \sum_{m=0}^{n} \int_{B_m(\tilde{T}, \tilde{A}) \cap B_m(T, A)} d_x(\tilde{T}_{\tilde{A}}(x), T_A(x)) d\mu(x) \\ &+ \int_{\bigcup_{m>N} B_m(\tilde{T}, \tilde{A})} d_x(\tilde{T}_{\tilde{A}}(x), T_A(x)) d\mu(x). \end{aligned}$$

Since the metric d_x is bounded by 1, we then get the following inequality:

$$\begin{aligned} d_{[\mathcal{R}_G]}(\tilde{T}_{\tilde{A}}, T_A) &< \sum_{m=0}^{N} \int_{B_m(\tilde{T},\tilde{A})\cap B_m(T,A)} d_x(\tilde{T}_{\tilde{A}}(x), T_A(x)) d\mu(x) + 2\varepsilon \\ &< \sum_{m=0}^{N} \int_{B_m(\tilde{T},\tilde{A})\cap B_m(T,A)} d_x(\tilde{T}^m(x), T^m(x)) d\mu(x) + 2\varepsilon \\ &< 3\varepsilon. \end{aligned}$$ □

Corollary 2.3.3. *Orbit full groups are contractible.*

Proof. We may suppose that $X = [0,1]$ and that μ is the Lebesgue measure. The continuous map $H : [0,1] \times \mathbb{G} \to \mathbb{G}$ defined by $H(s,T) := T_{[s,1]}$ is a homotopy between the identity function $T \mapsto T$ and the constant function $T \mapsto \mathrm{id}_X$. □

The **support** of $T : X \to X$, denoted by $\operatorname{supp} T$, is defined by

$$\operatorname{supp} T := \{x \in X : T(x) \neq x\}.$$

We now state the main theorem of this section. Note that condition *(v)* is a generalization of [Tör06, The Category Lemma] to the case of orbit full groups.

Theorem 2.3.4. *For a Borel, measure preserving action of the Polish group G on the probability space (X,μ), we denote by $\mathbb{G} = [\mathcal{R}_G]$. The following are equivalent:*

(i) the set of aperiodic elements is dense in $\mathbb{G}$;

(ii) the conjugacy class of any aperiodic element of $\mathbb{G}$ is dense in $\mathbb{G}$;

(iii) there is a sequence (T_n) of aperiodic elements of $\mathbb{G}$ such that $T_n \to \mathrm{id}_X$;

(iv) for all $A \in \mathrm{MAlg}(X,\mu)$, there is a sequence (T_n) of elements of $\mathbb{G}$ such that $T_n \to \mathrm{id}_X$ and for all $n \in \mathbb{N}$, $A = \operatorname{supp} T_n$;

(v) whenever $\Gamma \curvearrowright (X,\mu)$ is a free measure preserving action of a countable discrete group Γ, there is a dense G_δ in $\mathbb{G}$ of elements inducing a free action of $\Gamma * \mathbb{Z}$;

(vi) for all $n \in \mathbb{N}$, there is a dense G_δ of $(T_1, ..., T_n)$ in $\mathbb{G}^n$ which induce a free action of $\mathbb{F}_n$.

(vii) for all neighborhood of the identity U in G, $\cup_{g\in U} \operatorname{supp} g$ has full measure.

Proof. Let us first prove that *(i)* implies *(ii)*, using the same argument of the case of $\mathrm{Aut}(X,\mu)$ (see [Kec10, Theorem 2.4]). By Rokhlin's Lemma, the conjugacy class of any aperiodic element of $\mathbb{G}$ is dense in the set of aperiodic elements of $\mathbb{G}$ for the uniform topology (see [LM14b, Corollary 5.11] for details). Since the topology of $\mathbb{G}$ is weaker than the uniform topology, we get that *(i)* implies that the conjugacy class of any aperiodic element is dense in $\mathbb{G}$, in other words *(i)⇒(ii)*. Clearly *(ii)* implies *(iii)* and the continuity of the first-return map (Proposition 2.3.2) yields that *(iii)* implies *(iv)*.

Törnquist proved in [Tör06, The Category Lemma] that *(v)* holds for $\mathbb{G} = \mathrm{Aut}(X,\mu)$, and the only topological fact he used was precisely *(iv)* so we can repeat his entire argument to get *(iv)⇒(v)* (see the observation before the proof of Lemma 2 in [Tör06]; the fact that P is an involution is not relevant here).

The implication *(v)⇒(vi)* is proven by induction, and *(i)* is a reformulation of *(vi)* for $n = 1$, so the implication *(vi)⇒(i)* also holds.

So all the statements from *(i)* to *(vi)* are equivalent, and we now only have to prove that *(vii)* is equivalent to them. For this, we will prove that *(iii)* implies *(vii)*, and then that *(vii)* implies *(iv)*.

Let us show that *(iii)* implies *(vii)*. Assuming that *(vii)* is not satisfied, we can find an open neighborhood of the identity U in G such that $\mu(\bigcup_{g\in U}\operatorname{supp} g) = 1-\delta$ for some $\delta > 0$. We define a neighborhood of the identity $\mathcal{U}$ in $\widetilde{[\mathcal{R}_G]}$ by

$$\mathcal{U} := \{f \in \widetilde{[\mathcal{R}_G]} : \mu(\{x \in X : f(x) \notin U\}) < \delta/2\}.$$

For every $f \in \mathcal{U}$, we have $d_u(\Phi(f), \mathrm{id}_X) < 1$, hence $\Phi(f)$ is not aperiodic. The projection of $\mathcal{U}$ on $[\mathcal{R}_G]$ is a neighborhood of the identity in $[\mathcal{R}_G]$ consisting of non-aperiodic elements, contradicting *(iii)*.

For the remaining implication *(vii)⇒(iv)*, we first use Theorem 2.1.4 and fix a topology τ on X such that the action of G on (X, τ) is continuous.

Let V be a neighborhood of the identity in G. We say that $T \in [\mathcal{R}_G]$ is *uniformly small* if there is $f \in \mathrm{L}^0(X, \mu, G)$ such that $f(X) \subset V$ and $T(x) = f(x)x$ for every $x \in X$. Observe that if we prove that for every $A \in \mathrm{MAlg}(X, \mu)$ there is a uniformly small $T \in [\mathcal{R}_G]$ such that $\mathrm{supp}(T) \subset A$, then by a maximality argument we would get that for every $A \in \mathrm{MAlg}(X, \mu)$ there is a uniformly small $T \in [\mathcal{R}_G]$ such that $\mathrm{supp}(T) = A$. And so condition *(iv)* would be satisfied.

Let us fix an open set $B \supseteq A$ such that $\mu(B \setminus A) < \mu(A)$ and an open neighborhood of the identity $U \subseteq G$ such that $U^{-1} = U$ and $U^2 \subseteq V$.

Claim. There exists a countable family $(g_i)_{i\in\mathbb{N}}$ of elements of U and an a.e. partition $(A_i)_{i\in\mathbb{N}}$ of A such that for all $i \in \mathbb{N}$, $g_i(A_i)$ is a subset of B which is disjoint from A_i.

Proof of the claim. Let $(U_n)_{n\in\mathbb{N}}$ be a countable basis of open neighborhoods of the identity in G, and let

$$S := \bigcap_{n\in\mathbb{N}} \bigcup_{g\in U_n} \operatorname{supp} g.$$

By hypothesis, S has full measure. Moreover since the action is continuous, for all $x \in S \cap B$ there is a $g \in U$ such that $g \cdot x \in B$ and $g \cdot x \neq x$. Always by continuity we can also find an open neighborhood $W_x \subseteq B$ of x such that $g(W_x)$ and W_x are disjoint. We can now define the partition $\{A_n\}$ to be a countable open subcover of $(W_x)_{x\in S\cap B}$ which exists by Lindelöf's Theorem. □

We now have two possible cases.

- If for some $i \in \mathbb{N}$, $g_i(A_i)$ is not disjoint from A, then we set $C := A_i \cap g_i^{-1}(g_i(A_i) \cap A)$ and the element T of $[\mathcal{R}_G]$ defined by

$$T(x) := \begin{cases} g_i \cdot x & \text{if } x \in C \\ g_i^{-1} \cdot x & \text{if } x \in g(C) \\ x & \text{otherwise} \end{cases}$$

  is uniformly small, non trivial, and supported in A.

- If for all $i \in \mathbb{N}$, $g_i(A_i)$ is disjoint from A, then since $\mu(B \setminus A) < \mu(A)$ and $\{A_i\}_i$ is a partition of A, there are two distinct indices $i, j \in \mathbb{N}$ such that $g_i(A_i) \cap g_j(A_j)$ has positive measure. Letting $C := g_i^{-1}(g_i(A_i) \cap g_j(A_j))$, we see that the element T of $[\mathcal{R}_G]$ defined by

$$T(x) := \begin{cases} g_i g_j \cdot x & \text{if } x \in C \\ g_j g_i^{-1} \cdot x & \text{if } x \in g_i g_j^{-1}(C) \\ x & \text{otherwise} \end{cases}$$

is uniformly small, non trivial, and supported in A. □

Remark 2.3.5. Condition (vii) is always satisfied as soon as G is non discrete and acts essentially freely and is never satisfied for countable discrete groups. In fact, it easy to see that the aperiodic elements form a closed proper subset of full groups of countable pmp equivalence relations.

Corollary 2.3.6. *Let G be a locally compact group acting ergodically and in a Borel measure preserving manner on (X, μ). Then we have the following dichotomy:*

(1) either $\mathcal{R}_G$ is a countable pmp equivalence relation,

(2) or the set of aperiodic elements is dense in $[\mathcal{R}_G]$.

Proof. Let $\{U_n\}_n$ be a sequence of open neighborhoods of the identity in G such that $\cap_n U_n = \{1_G\}$. We define the **core** of the action to be the intersection

$$\bigcap_{n \in \mathbb{N}} \bigcup_{g \in U_n} \operatorname{supp} g.$$

Note that the core is a Borel set by Theorem 2.1.4. Moreover the core is G-invariant because for all $g, h \in G$, $h(\operatorname{supp} g) = \operatorname{supp}(hgh^{-1})$ and the conjugation by h is a continuous isomorphism of G. By ergodicity, either the core has measure one, in which case (2) holds by a direct application of condition (vii) of the previous theorem, or the core has measure zero.

If the core has measure zero, then it is easy to check that the G-action yields a uniformly continuous morphism $G \to (\mathrm{Aut}(X, \mu), d_u)$. By separability of the group G, we can conclude from Proposition 2.2.9 that $[G]_D$ is the full group of a countable pmp equivalence relation and hence $[G]_D = [\mathcal{R}_G]$. Moreover G is locally compact, so we can apply Proposition 2.2.32 to deduce that the equivalence relation $\mathcal{R}_G$ is countable up to a measure zero set, that is (1) holds. □

Let us give a non-trivial example where condition (1) of the previous corollary is satisfied.

Example 2.3.7. Let $(\Gamma_n)_{n \in \mathbb{N}}$ be a sequence of finite groups and let $G := \prod_{n \in \mathbb{N}} \Gamma_n$ be their product. Let (X, μ) be a standard probability space, and fix a partition (A_n) of X such that each A_n has positive measure. For every $n \in \mathbb{N}$, we then fix a measure preserving action α_n of the finite group Γ_n which is free when restricted to A_n, and which is trivial outside of it.

We can now define an action of the group G on X by $(\gamma_n)_n \cdot x = \alpha_m(\gamma_m)x$ whenever $x \in A_m$. This action is faithful and its core is trivial. Note also that we can embed in this way any profinite group into any ergodic full group $\mathbb{G}$. It is in fact sufficient to take the actions α_n such that $\alpha_n(G_n) \subset \mathbb{G}$.

2.3.2 Uniqueness of the Polish topology

We now show that the topology of convergence in measure is the unique possible Polish group topology for ergodic orbit full groups.

Theorem 2.3.8. *Let $\mathbb{G}$ be an ergodic full group.*

(1) Any Polish group topology on $\mathbb{G}$ is weaker than the uniform topology .

(2) Any Polish group topology on $\mathbb{G}$ refines the weak topology.

(3) The group $\mathbb{G}$ carries at most one Polish group topology.

Item (1) was shown to hold by Kittrell and Tsankov for the full group of an ergodic countable pmp equivalence relation [KT10, Theorem 3.1], while item (2) was proven by Kallman for $\mathbb{G} = \mathrm{Aut}(X, \mu)$ in [Kal85]. We will not give a detailed proof of these points, but we will just indicate how to adapt the previous results to our broader setting.

Proof. (1) The proof of Theorem 3.1 in [KT10] only makes use of ergodicity via Proposition 2.2.11, and of the fact that in any full group, every element may be written as the product of at most three involutions [Ryz93]. So it adapts verbatim to obtain that $(\mathbb{G}, d_u)$ has the automatic continuity property[1]. We deduce that the identity map $(\mathbb{G}, d_u) \to (\mathbb{G}, \tau)$ is continuous, in other words the uniform topology refines τ.

(2) The arguments in [Kal85] also use only Proposition 2.2.11 and they adapt verbatim to obtain that the subsets of the form $\{g \in \mathbb{G} : \mu(A \bigtriangleup gA) < \varepsilon\}$ are analytic for any Polish group topology τ on $\mathbb{G}$. In particular, the identity map $(\mathbb{G}, \tau) \to (\mathrm{Aut}(X, \mu), w)$ is Baire-measurable, hence continuous by Property 2.1.3 (γ), so that τ refines w.

(3) Let τ and τ' be two Polish group topologies on $\mathbb{G}$. By (2), both topologies have to refine the weak topology, so the inclusion map $\mathbb{G} \hookrightarrow (\mathrm{Aut}(X, \mu), w)$ is Borel for both topologies. By Theorem 2.1.2, the Borel σ-algebras of $\mathbb{G}$ for τ and τ' are both equal to the Borel σ-algebra induced by the weak topology. So $\tau = \tau'$ by Property 2.1.3 (γ). ☐

We note that the proof of (3) yields the following corollary.

Corollary 2.3.9. *Let $\mathbb{G}$ be an ergodic full group admitting a Polish topology. Then it is a Borel subset of* $\mathrm{Aut}(X, \mu)$ *for the weak topology.*

We do not know whether the full group generated by the circle acting on itself by translation $[\mathbb{S}^1]_D$ is Polishable. Note however that this full group is Borel by Corollary 2.3.17. The following question has a positive answer when $\mathbb{G}$ is either $\mathrm{Aut}(X, \mu)$ ([BYBM13]), or the full group of a countable pmp ergodic equivalence relation [KT10].

Question 2.3.10. Let $\mathbb{G}$ be an ergodic orbit full group, then does it have the automatic continuity property?

Let us now observe that the third point of Theorem 2.3.8 still holds in the case of a full group with a countable ergodic decomposition, using a simple result from Le Maître's thesis [LM14b], which we prove for the reader's convenience.

1. A topological group G has the **automatic continuity property** if every morphism from G into a separable group H is continuous.

Lemma 2.3.11. *Let $(G_i)_{i\in\mathbb{N}}$ be a countable family of groups with trivial center admitting at most one Polish group topology. Then $\prod_{i\in\mathbb{N}} G_i$ has at most one Polish topology.*

Proof. Fix a Polish group topology on $\prod_{i\in\mathbb{N}} G_i$, and for $i\in\mathbb{N}$ denote by π_i the projection on G_i. Let $n\in\mathbb{N}$, set

$$G'_n := \{g\in\prod_{i\in\mathbb{N}} G_i : \forall k\neq n, \pi_k(g)=e\}.$$

Because each G_i has trivial center, G'_n is the commutator of $H_n := \{g\in\prod_{i\in\mathbb{N}} G_i : \pi_n(g)=e\}$, hence G'_n is closed in $\prod_{i\in\mathbb{N}} G_i$. Since G_n is isomorphic to G'_n, the induced topology on G'_n is the unique Polish group topology on G_n. In particular, for every open subset U of G_n, the set

$$\tilde{U} := \{e\}\times\cdots\times\{e\}\times U\times\{e\}\times\cdots\subset\prod_{i\in\mathbb{N}} G_i$$

is a G_δ.

Observe that H_n is closed, since it is the commutator of G'_n. So for any open subset U of G_n, the set

$$\tilde{U}\cdot H_n = G_1\times\cdots\times G_{n-1}\times U\times G_{n+1}\times\cdots$$

is analytic, which implies by Property 2.1.3 (γ) the uniqueness of the Polish topology of $\prod_{i\in\mathbb{N}} G_i$. ☐

A full group with countably many ergodic components is isomorphic to the product of the full groups of its restrictions to these ergodic components, so we can combine Theorem 2.3.8 *(3)* with the previous lemma to get the following corollary.

Corollary 2.3.12. *Let $\mathbb{G}$ be a full group with countably many ergodic components. Then it carries at most one Polish group topology.*

2.3.3 More (non) Borel full groups

In this section, we give more examples of Borel full groups, as well as examples of non Borel ones. But first, we need some background on the space of probability measures.

Let (Y,τ) be a Polish space. We equip the space $\mathcal{P}(Y)$ of probability measures on Y with the weak-* topology, that is, the coarsest topology making the maps

$$\mu\in\mathcal{P}(Y)\mapsto\int_X f d\mu$$

continuous for all bounded continuous functions $f: Y\to\mathbb{R}$. This is a Polish topology (see e.g. [Kec95, Section 17.E]).

Lemma 2.3.13. *Let (X,μ) be a standard probability space, and (Y,τ) be a Polish space. Then the following map*

$$\begin{array}{ll} \mathrm{L}^0(X,\mu,Y) & \to\mathcal{P}(Y)\\ f & \mapsto f_*\mu \end{array}$$

is continuous.

Proof. If a sequence $(f_n)_n$ converges to f in measure, then for almost every $x \in X$, we have that $f_n(x) \to f(x)$. For every continuous and bounded function $g : Y \to \mathbb{R}$ by Lebesgue Dominated Convergence Theorem we have,

$$\int_X g df_{n*}\mu = \int_X g(f_n(x))d\mu(x) \to \int_X g(f(x))d\mu(x) = \int_X g df_*\mu,$$

and hence the map is continuous. □

Proposition 2.3.14. *The set of completely atomic probability measures of a Polish space Y is a Borel subset of $\mathcal{P}(Y)$.*

Proof. We first note that whenever U is an open subset of Y, then the set

$$Atom(U) := \{\mu \in P(Y) : \text{there exists } a \in U \text{ such that } \mu(U) = \mu(\{a\})\}$$

is closed in $P(Y)$. Indeed if $\mu \notin Atm(U)$, then there are two positive functions f and g supported in U with disjoint supports such that $\int_X f d\mu$ and $\int_X g d\mu$ are strictly positive.

Let $(U_n)_{n\in\mathbb{N}}$ be a countable basis of open subsets of Y. Let $\mathcal{J}$ be the set of finite subsets $I \subset \mathbb{N}$ such that $(U_i)_{i\in I}$ is a disjoint family of open sets. Since a measure μ is atomic if and only if for all $\epsilon > 0$, there is an open set U of X of measure greater than $1-\epsilon$ containing finitely many atoms, the set of atomic measure is exactly

$$\bigcap_{n\in\mathbb{N}} \bigcup_{I\in\mathcal{J}} \left(\left\{ \mu \in \mathcal{P}(Y) : \mu\left(\bigcup_{i\in I} U_i\right) > 1 - \frac{1}{n} \right\} \cap \bigcap_{i\in I} Atom(U_i) \right),$$

and so it is Borel. □

Corollary 2.3.15. *Let (X, μ) be a standard probability space and (Y, τ) be a Polish space. Then the set $\mathrm{L}^0_D(X, \mu, Y)$ of elements of $\mathrm{L}^0(X, \mu, Y)$ with countable range is Borel.*

Proof. Observe that $f \in \mathrm{L}^0(X, \mu, Y)$ has countable range if and only if $f_*\mu$ is completely atomic. So $\mathrm{L}^0_D(X, \mu, Y)$ is the pre-image of a Borel set by a continuous function, hence it is Borel. □

The next theorem is a Borel version of Theorem 2.2.18. The proof in the Borel case is easier because we will not use a continuous model, that is, Theorem 2.1.4.

Theorem 2.3.16. *Let G be a Polish group acting in a Borel way by measure preserving transformations on a standard probability space (X, μ). Suppose that the action is essentially free, and let $H \leq G$ be a subgroup of G. Then the following are equivalent:*

1. *H is a Borel subgroup of G;*
2. *$[H]_D$ is a Borel subgroup of* $\mathrm{Aut}(X, \mu)$*;*
3. *$[\mathcal{R}_H]$ is a Borel subgroup of* $\mathrm{Aut}(X, \mu)$*.*

Proof. As we did for the proof of Theorem 2.2.18, we will consider the Polish space $\mathrm{L}^0(X, \mu, G)$, and we use the fact that the map $\Phi : \mathrm{L}^0(X, \mu, G) \to \mathrm{L}^0(X, \mu, X)$ defined by

$$\Phi(f)(x) := f(x) \cdot x$$

is Borel[2]. The homomorphism Φ is injective because the action is essentially free. Set $\widetilde{[H]_D} := \Phi^{-1}([H]_D)$ and $\widetilde{[\mathcal{R}_H]} := \Phi^{-1}([\mathcal{R}_H])$. Theorem 2.1.2 implies that $\widetilde{[H]_D}$ is Borel if and only if $[H]_D$ is, and that $\widetilde{[\mathcal{R}_H]}$ is Borel if and only if $[\mathcal{R}_H]$ is.

If we identify G with the Borel subset of constant maps in $\mathrm{L}^0(X,\mu,G)$, we have that $H = G \cap \widetilde{[H]_D} = G \cap \widetilde{[\mathcal{R}_H]}$ and so *2⇒1* and *3⇒1*. For the converse, first note that $\mathrm{L}^0(X,\mu,H)$ is a Borel subset of $\mathrm{L}^0(X,\mu,G)$, as shown in the corollary after Proposition 8 in [Moo76]. By Corollary 2.3.15, $\mathrm{L}^0_D(X,\mu,G)$ is also a Borel subset of $\mathrm{L}^0(X,\mu,G)$ and the implications *1⇒2* and *1⇒3* hold because

$$\widetilde{[\mathcal{R}_H]} = \widetilde{[\mathcal{R}_G]} \cap \mathrm{L}^0(X,\mu,H)$$
$$\widetilde{[H]_D} = \widetilde{[\mathcal{R}_H]} \cap \mathrm{L}^0_D(X,\mu,G).$$

□

Let us now point out two important consequences of the previous theorem. The first one is straightforward.

Corollary 2.3.17. *Suppose that G is a Polish group acting essentially freely and in a measure preserving Borel way on (X,μ). Then $[G]_D$ is Borel.*

Question 2.3.18. Can one remove the freeness assumption from the previous corollary?

Corollary 2.3.19. *There exists non Polishable ergodic full groups.*

Proof. Consider the free action of the circle $\mathbb{S}^1$ onto itself by translation. Let $H \leq \mathbb{S}^1$ be a non Borel subgroup which still acts ergodically. Then by Theorem 2.3.16 both $[H]_D$ and $[\mathcal{R}_H]$ will be non Borel full groups, and so by Corollary 2.3.9, they cannot have a Polish group topology.

The existence of such an H is a well-known consequence of the axiom of choice. Consider $\mathbb{R}$ as a $\mathbb{Q}$-vector space, and let $\tilde{H}$ be a hyperplane containing $\mathbb{Q}$. The subgroup $H := \tilde{H}/\mathbb{Z} \leq \mathbb{R}/\mathbb{Z} = \mathbb{S}^1$ is a proper subgroup of $\mathbb{S}^1$ with countable infinite index, which acts ergodically because it contains an irrational. Such a subgroup H can not be Lebesgue-measurable. Indeed, since $\mathbb{S}^1$ is covered by countably many translates of H, H has non zero-measure. But then, H must have finite-index, a contradiction. □

2.4 Character rigidity for full groups

In this section, we use a result of Dudko concerning characters of the full group of the hyperfinite equivalence relation $\mathcal{R}_0$ to classify characters of ergodic full groups admitting a Polish group topology. Recall that $\mathcal{R}_0$ is the countable pmp equivalence relation on $X = \{0,1\}^{\mathbb{N}}$ equipped with the product measure $\mu = (1/2\delta_0 + 1/2\delta_1)^{\otimes\mathbb{N}}$, defined by

$$(x_n)\mathcal{R}_0(y_n) \iff \exists p \in \mathbb{N} \mid \forall n \geq p, x_n = y_n.$$

Theorem 2.4.1. *For an ergodic Polish full group $\mathbb{G}$ we have the following dichotomy:*

1. *either $\mathbb{G}$ is the full group of a countable pmp equivalence relation, and all its continuous characters are (possibly infinite) convex combinations of the trivial character $\chi_0 \equiv 1$ and the characters $\{\chi_k\}_{k\geq 1}$ given by*

$$\chi_k(g) := \mu(\{x \in X : g\cdot x = x\})^k,$$

2. Note that $\mathrm{L}^0(X,\mu,X)$ is a standard Borel space, whose Borel structure does not depend on the Polish topology we put on X by [Moo76, Proposition 8].

2. *or $\mathbb{G}$ does not have any non trivial continuous character.*

The result of Dudko that we use may be stated as follows.

Theorem 2.4.2 ([Dud11]). *Let χ be a continuous character of the full group of the hyperfinite equivalence relation $\mathcal{R}_0$. Then there is a unique sequence non negative coefficients $(\alpha_k)_{k\geq 0}$ such that $\sum_{k=0}^{+\infty}\alpha_k = 1$ and $\chi = \sum_{k=0}^{+\infty}\alpha_k\chi_k$.*

Our result will follow from the description of the uniformly continuous characters of any ergodic full group, which was also observed by Gaboriau and Medynets (private communication).

Proposition 2.4.3. *Let $\mathbb{G}$ be an ergodic full group and let χ be a character of $\mathbb{G}$, continuous for the uniform topology. Then there is a unique sequence of non negative coefficients $(\alpha_k)_{k\geq 0}$ such that $\chi = \sum_{k=0}^{+\infty}\alpha_k\chi_k$.*

Proof. It is a standard fact that up to conjugating, we may assume that $[\mathcal{R}_0]\subset\mathbb{G}$ (for a proof in the general setting of ergodic full groups, see [LM14b, Theorem 2.19]). Let χ be a character of $\mathbb{G}$ continuous for the uniform topology. By Theorem 2.4.2, there is a sequence of non negative coefficients $(\alpha_k)_{k\geq 0}$ such that $\chi(g) = \sum_{k=0}^{+\infty}\alpha_k\chi_k(g)$ for all $g\in[\mathcal{R}_0]$. By Rokhlin's Lemma, the set $\mathcal{F}$ of elements of finite order of $\mathbb{G}$ is uniformly dense in $\mathbb{G}$. Since $\mathbb{G}$ is ergodic, every element of $\mathcal{F}$ is conjugate inside $\mathbb{G}$ to an element of $[\mathcal{R}_0]$. By definition χ is conjugacy-invariant, so $\chi(g) = \sum_{k=0}^{+\infty}\alpha_k\chi_k(g)$ for all $g\in\mathcal{F}$ and hence for any $g\in\mathbb{G}$ by continuity. □

Proof of theorem 2.4.1. Let $\mathbb{G}$ be an ergodic full group equipped with a Polish topology τ, and let χ be a τ-continuous non-trivial character of $\mathbb{G}$. By Theorem 2.3.8, τ is weaker than the uniform topology. So the character χ is continuous for the uniform topology and, by the previous proposition, there is a sequence of non negative coefficients $(\alpha_k)_{k\geq 0}$ such that $\chi = \sum_{k=0}^{+\infty}\alpha_k\chi_k$.

For all $T\in\mathrm{Aut}(X,\mu)$ and all $k\geq 1$, we have $\chi_k(T) = (1-d_u(T,\mathrm{id}_X))^k$, so $\chi_k(T_n)\to 1$ if and only if $d_u(T_n,\mathrm{id}_X)\to 0$. Since χ is not the trivial character, we deduce that $\chi(T_n)\to 1$ if and only if $d_u(T_n,\mathrm{id}_X)\to 0$. When T_n converges to id_X for τ, we have $\chi(T_n)\to 1$, which implies that T_n converges to id_X in the uniform topology. We deduce that τ refines the uniform topology, hence they are equal. Since τ is Polish, we can conclude from Proposition 2.2.9 that $\mathbb{G}$ is the full group of a countable pmp equivalence relation. □

Chapter 3

Full groups of locally compact groups

The following chapter is based on a joint work with François Le Maître.

We show that the topological rank of a full group generated by an ergodic, probability measure preserving action of a locally compact Polish group is two. For this, we will use the existence of a cross section and we will prove that for a locally compact Polish group, the full group generated by every dense subgroup is dense in the full group of the action of the group.

Contents

3.1 Preliminaries

Every measure preserving action α of a Polish group G on a probability space (X, μ) induces a homomorphism $\rho_\alpha : G \to \mathrm{Aut}(X, \mu)$. This homomorphism is always continuous with respect to the weak topology on $\mathrm{Aut}(X, \mu)$. For locally compact Polish [3] groups, measure preserving actions and weakly continuous homomorphisms from G to $\mathrm{Aut}(X, \mu)$ are the same in the following sense.

Theorem 3.1.1 (Mackey, [Mac62])**.** *Let G be a locally compact Polish group and let (X, μ) be a standard probability space. Then for every weakly continuous homomorphism $\rho : G \to \mathrm{Aut}(X, \mu)$ there exists a measure preserving action α of G on (X, μ) such that the induced homomorphism $\rho_\alpha : G \to \mathrm{Aut}(X, \mu)$ is equal to ρ. Moreover if α and β are two measure preserving actions of G such that the induced homomorphisms ρ_α and ρ_β are equal, then there is a Borel subset $A \subset G$ of full measure such that $\alpha\big|_A = \beta\big|_A$.*

If G is locally compact and Polish, then we will call the measure preserving action associated to a homomorphism $\rho : G \to \mathrm{Aut}(X, \mu)$ a **realization** of the action. Let us recall two important definitions.

Definition 3.1.2. A measure preserving action of a Polish group G on the probability measure space (X, μ) is

- **essentially free** if there is a subset of full measure $A \subset X$ such that for every $x \in A$ and every $g \in G$, we have that $gx \neq x$;
- **ergodic** if every Borel subset $A \subset X$ which is **almost G-invariant** (i.e. for all $g \in G$ we have $\mu(A \triangle g(A)) = 0$) has measure 0 or 1.

Remark 3.1.3.

- In the definition of freeness, we do not require A to be G-invariant. However for any such A, we can consider the measurable subset $A' := GA$. Then A' is clearly G-invariant and it satisfies the same condition.
- There are actions of compact groups such that for every $g \in G$, the set $\{x \in X : gx = x\}$ has measure 0 but which are not essentially free.
- Mackey's Theorem implies that if G is locally compact and separable then if a realization of an action is essentially free, then all Borel realizations are essentially free.
- Another easy consequence of Mackey's Theorem (see [Mac62, Theorem 3]) is that a measurable action of a locally compact Polish group is ergodic if and only if every Borel subset $A \subset X$ which is G-invariant (i.e. for all $g \in G$, we have $g(A) = A$) has measure 0 or 1. This is not true for general Polish groups, see Example 2.2.15

Every locally compact Polish group admits an essentially free measure preserving action (see e.g. Proposition 1.2 in [AEG94]). We will give now two concrete examples of actions.

Example 3.1.4. Suppose that G is totally disconnected. By van Dantzig's theorem, there exists a chain $(K_n)_{n \in \mathbb{N}}$ of compact open subgroups of G such that $\cap_n K_n = \{1_G\}$. The action of G on the countable set $\sqcup_n G/K_n$ gives a continuous injective embedding of G into the group of permutation of $\mathbb{N}$. So for each chain $\{K_n\}$, we can associate a Bernoulli shift on $[0,1]^{\mathbb{N}}$, which is ergodic and essentially free.

3. Recall that a locally compact group is Polish if and only if it is second-countable (see [Kec95, Theorem 5.3]).

Example 3.1.5. Suppose that G has a lattice $\Gamma < G$, let λ be a Haar measure on G and let D be a fundamental domain of the right Γ-action on G. Then any probability measure preserving action of Γ on (X, μ) induces a measure preserving action of G on $(X \times D, \mu \times \lambda|_D)$, see Definition 4.2.21 in [Zim84].

Observe that for the second example, the equivalence relation $\mathcal{R}_G$ is equal to the product equivalence relation $\mathcal{R}_\Gamma \times (D \times D)$. This property is actually shared by all measure preserving actions of locally compact Polish groups.

3.1.1 Cross-sections and product decomposition

We present now the most important property of measure preserving actions of locally compact Polish groups: the existence of a *cross-section*.

Definition 3.1.6. Consider an essentially free, measure preserving action of the locally compact Polish group G on the probability space (X, μ). A Borel subset $Y \subset X$ is a **cross-section** of the action if there exists a neighborhood of the identity $U \subset G$ such that the map $\theta : U \times Y \to X$ defined by $\theta(u, y) := uy$ is injective and such that $\mu(X \setminus GY) = 0$.

The existence of cross section was proved by Forrest in [For74, Proposition 2.10] in the more general context of *non-singular* actions. For a more recent proof, we invite the reader to check Theorem 4.2 of [KPV13]. The following theorem is essentially a version of [For74, Proposition 2.13] in the context of a measure preserving action of a unimodular locally compact group.

Theorem 3.1.7. *Let G be a unimodular, locally compact, non-compact and non-discrete Polish group. For a measure preserving, essentially free and ergodic action of G on the probability space (X, μ), there exists a countable group Γ and a probability measure preserving action of Γ on (Y, ν) such that the action of G is orbit equivalent to the product action $\mathbb{S}^1 \times \Gamma$ on $\mathbb{S}^1 \times Y$, where $\mathbb{S}^1$ acts on itself by translation.*

Moreover, G is amenable if and only if the orbit equivalence relation induced by Γ on (Y, ν) is amenable.

Proof. Fix a Haar measure λ on G. Let $Y \subset X$ be a cross section and let $U \subset G$ be a neighborhood of the identity as in Definition 3.1.6. We consider the restriction of $\mathcal{R}_G$ to Y,

$$\mathcal{R} := \{(y, y') \in Y \times Y : \exists g \in G, y' = gy\}.$$

We state here some well-known facts about the equivalence relation $\mathcal{R}$. For a proof of these properties, see Proposition 4.2 in [KPV13].

(1) $\mathcal{R}$ is a Borel, countable equivalence relation on Y,

(2) there exist a (unique) $\mathcal{R}$-invariant Borel probability measure ν on Y and a real number $0 < \mathrm{covol}(Y) < +\infty$ such that $\theta_*(\lambda|_U \times \nu) = \mathrm{covol}(Y)\mu$,

(3) $(\mathcal{R}, \nu)$ is ergodic if and only if the action of G is ergodic,

(4) $(\mathcal{R}, \nu)$ has infinite orbits almost everywhere if and only if G is non-compact,

(5) $(\mathcal{R}, \nu)$ is amenable if and only if G is amenable.

By property (4) above, we deduce that (Y, ν) is diffuse. Since $\mathcal{R}$ is countable and measure preserving, Feldman and Moore's result ([FM77a, Theorem 1]) gives us a measure preserving action of a countable group Γ on (Y, ν) which induces the equivalence relation $\mathcal{R}$.

Up to taking a open subset of U, we may assume that $\mu(\mathcal{U} \cdot Y) = \frac{1}{K}$ for some $K \in \mathbb{N}$. Set $A = \mathcal{U} \cdot Y$.

By ergodicity of G, we can find $T \in [\mathcal{R}_G]$ of order K such that $\{A, T(A), ..., T^{K-1}(A)\}$ is a partition of a full measure subset of X. Let us denote by c the counting measure on $\mathbb{Z}/K\mathbb{Z}$ and consider the equivalence relation $\mathcal{S}$ on $(\mathbb{Z}/K\mathbb{Z} \times U \times Y, c \times \lambda \times \nu)$ defined by

$$(k, u, y)\mathcal{S}(k', u', y') \text{ if } y\mathcal{R}y'.$$

Observe that for every $y \in Y$ the equivalence relation $\mathcal{S}$ restricted to $\mathbb{Z}/K\mathbb{Z} \times U \times \{y\}$ is transitive, so $\mathcal{S}$ is orbit equivalent to the product action $\mathbb{S}^1 \times \Gamma$ on $\mathbb{S}^1 \times Y$, where $\mathbb{S}^1$ acts on itself by translation. Moreover the measure preserving map

$$\begin{aligned}
&\Theta : (\mathbb{Z}/K\mathbb{Z} \times U \times Y, c \times \lambda \times \nu) \to (X, \mu) \\
&\Theta(k, u, y) := T^k(u \cdot y),
\end{aligned}$$

defines an orbit equivalence between $\mathcal{S}$ and $\mathcal{R}_G$. □

3.2 Dense subgroups in orbit full groups

The aim of this chapter is to describe some dense subgroups of the full group. We will prove the following theorem.

Theorem 3.2.1. *Let G be a locally compact Polish group. For every ergodic, measure preserving action of G on the probability space (X, μ) and for every dense subgroup $H \subset G$, we have that $[\mathcal{R}_H]$ is dense in $[\mathcal{R}_G]$.*

Let us show how we can compute the topological rank of a full group using Theorem 3.2.1.

Theorem 3.2.2. *Let G be a locally compact unimodular non-discrete and non-compact Polish group. For every measure preserving, essentially free and ergodic action of G, there is a dense G_δ of couples (T, U) in $[\mathcal{R}_G]^2$ which generate a dense free subgroup of $[\mathcal{R}_G]$ acting freely. In particular, the topological rank of $[\mathcal{R}_G]$ is 2.*

Proof. Let G be a locally compact unimodular, non-discrete and non-compact Polish group. Suppose that G acts on the probability space (X, μ) preserving the measure, essentially freely and ergodically. Let us denote by $\mathbf{F}_2$ the free group on two generators and observe that

$$\left\{(T, U) \in [\mathcal{R}_G]^2 : \overline{\langle T, U\rangle} = [\mathcal{R}_G] \text{ and } \langle T, U\rangle \cong \mathbf{F}_2\right\},$$

is a G_δ, so we have only to prove that it is dense.

By Theorem 3.1.7, there exists a (not necessarily free) action of a countable group Γ on a measure space (Y, ν), such that $\mathcal{R}_G$ is orbit equivalent to the product action of $\mathbb{S}^1 \times \Gamma$ on $\mathbb{S}^1 \times Y$. Fix a copy of $\mathbb{Z}$ in $\mathbb{S}^1$ generated by an irrational rotation; then $\mathbb{Z} \times \Gamma$ is dense in $\mathbb{S}^1 \times \Gamma$. By Theorem 3.2.1, we have that $[\mathcal{R}_{\mathbb{Z}\times\Gamma}]$ is dense in $[\mathcal{R}_G]$.

The equivalence relation $\mathcal{R}_{\mathbb{Z}\times\Gamma}$ has cost 1, by Proposition *VI*.23 of [Gab00] (note that the proof only uses that Γ_1 acts freely). So we can apply Theorem 1.7 in [LM15] to get the existence of an aperiodic $T \in [\mathcal{R}_{\mathbb{Z}\times\Gamma}]$ such that

$$\left\{U \in [\mathcal{R}_{\mathbb{Z}\times\Gamma}] : \overline{\langle T, U\rangle}^{d_u} = [\mathcal{R}_{\mathbb{Z}\times\Gamma}] \text{ and } \langle T, U\rangle \cong \mathbf{F}_2\right\} \subset [\mathcal{R}_{\mathbb{Z}\times\Gamma}]$$

is a dense subset of $[\mathcal{R}_{\mathbb{Z}\times\Gamma}]$ with respect to the uniform topology. This concludes the proof since by Theorem 2.3.4, the conjugacy class of T is dense in $[\mathcal{R}_G]$ for the topology of convergence in measure. □

3.2.1 Suitable Actions

We will prove Theorem 3.2.1 under a weaker hypothesis in the context of Polish group actions. Recall however, that we have already discussed a counterexample of Theorem 3.2.1 in Example 2.2.15. Indeed there is a Borel probability measure on $\{0,1\}^{\mathbb{N}}$ such that the full group generated by the finitely supported permutations is not dense in the orbit full group of the Polish group of all permutations of $\mathbb{N}$ acting by shift on $\{0,1\}^{\mathbb{N}}$. The fact is that this action is not *suitable*.

Definition 3.2.3 (Becker, [Bec13, Definition 1.2.7])**.** Let G be a Polish group. A Borel, measure preserving action of G on the probability space (X,μ) is **suitable** if for all Borel subsets $A, B \subset X$, one of the following two conditions holds:

(1) for any open neighborhood of the identity $O \subset G$, there is $g \in O$ such that $\mu(A \cap gB) > 0$;

(2) there are Borel subsets $A' \subset A$ and $B' \subset B$ of full measure in A and B and an open neighborhood O of the identity in G such that $(OA') \cap B' = \emptyset$.

We will prove the following.

Theorem 3.2.4. *Let G be a Polish group. For every Borel, measure preserving, ergodic suitable action of G on the probability space (X,μ) and for every dense subgroup $H \subset G$, the orbit full group $[\mathcal{R}_H]$ is dense in $[\mathcal{R}_G]$.*

Becker proved in Theorem 1.2.9 of [Bec13], that all measure preserving actions of locally compact Polish groups are suitable, so Theorem 3.2.4 implies Theorem 3.2.1

3.2.2 An equivalent statement

From now on, we will use the notations of Section 2.2.2. For every Borel, measure preserving action of G on the probability space (X,μ), we denote by $\widetilde{[\mathcal{R}_G]}_D \subset \widetilde{[\mathcal{R}_G]}$ the subset of function with countable (essential) image and we put $[\mathcal{R}_G]_D := \Phi(\widetilde{[\mathcal{R}_G]}_D)$, which is just $[G]_D$ of Definition 2.2.7.

Theorem 3.2.4 follows form the following weaker theorem.

Theorem 3.2.5. *Let G be a Polish group. For every Borel, measure preserving ergodic suitable action of G on the probability space (X,μ), we have that $\widetilde{[\mathcal{R}_G]}_D \subset \widetilde{[\mathcal{R}_G]}$ is a dense subgroup.*

In order to show that Theorem 3.2.5 implies Theorem 3.2.4, we will need the following proposition.

Proposition 3.2.6. *Consider an ergodic, measure preserving action of a Polish group G on the probability space (X,μ). Let A be a measurable subset and let $f : A \to G$ be a measurable function such that the map $\Phi(f) : A \to X$ defined by $\Phi(f)(x) := f(x)x$ is a measure preserving injective map. Then there exists a function $f' \in \widetilde{[\mathcal{R}_G]}$ such that*

- $f'\big|_A = f$ *almost everywhere,*
- $f'(X \setminus A)$ *is countable.*

Proof. Let $\Gamma \leq G$ be a countable dense subgroup of G. Since Γ and G have the same weak closure in $\mathrm{Aut}(X,\mu)$, it follows that Γ also acts ergodically. So one can find an element of the pseudo-full group $[[\mathcal{R}_\Gamma]]$ which maps $X \setminus A$ to $X \setminus B$ (see [KM04, Lemma 7.10]). This element defines a map $\tilde{f} : X \setminus A \to G$ which has countable range and by gluing this map together with f, we obtain a function f' satisfying the desired assumptions. □

Proof of Theorem 3.2.4. Let G be a Polish group and let H be a dense subgroup. Consider a Borel, measure preserving suitable action of G on the probability space (X,μ). We will prove that $\widetilde{[\mathcal{R}_H]}_D \subset \widetilde{[\mathcal{R}_G]}_D$ is dense. Then we can use Theorem 3.2.5 to end the proof.

Fix a compatible, right-invariant metric d_G on G bounded by 1, fix $\varepsilon > 0$ and take $f \in \widetilde{[\mathcal{R}_G]}_D$. There are $k \in \mathbb{N}$, a finite subset $\{g_1,\dots,g_k\} \subset G$ and a finite partition $\{A_0,\dots,A_k\}$ of X such that $\mu(A_0) \leq \varepsilon/2$ and for every $i \geq 1$, we have $f(A_i) = \{g_i\}$. By density and weak-continuity of the action, there exists $\{h_1,\dots,h_k\} \subset H$ such that for every $i \in \{1,\dots,k\}$, we have that $d_G(g_i,h_i) \leq \varepsilon$ and $\mu(g_i(A_i)\Delta h_i(A_i)) \leq \varepsilon/2k$. Put

$$B_i := h_i^{-1}(g_i(A_i) \cap h_i(A_i)) \subset A_i,$$

and observe that $\mu(\cup_{i=1}^k B_i) \geq 1-\varepsilon$. Since the subsets $\{h_iB_i\}$ are disjoint, Proposition 3.2.6 implies that there is $f' \in \widetilde{[\mathcal{R}_H]}$ with countable image such that $\tilde{d}_G(f,f') \leq 2\varepsilon$. □

3.2.3 Proof of Theorem 3.2.5

Definition 3.2.7. Fix $f \in \widetilde{[\mathcal{R}_G]}$ and a neighborhood of the identity $N \subset G$. We say that a couple (A,g) is (N-)**good** if

1. $A \subset X$ is a measurable subset of positive measure and $g : A \to G$ is a measurable function with countable image,
2. for every $x \in A$, we have $f(x)g(x)^{-1} \in N$,
3. the map $\Phi(g) : A \to X$ defined by $\Phi(g)(x) = g(x)x$ is injective and measure preserving.

We note that for a fixed $f \in \widetilde{[\mathcal{R}_G]}$ the existence of a good couple is not a trivial fact and it is exactly where we will use the hypothesis that the action is suitable.

The proof of the theorem will be a measurable version of the Hall's marriage theorem and it will follow the same strategy of Hudson in [Hud93]. For a fixed f as in Definition 3.2.7, using Zorn's lemma, we will construct for every $\varepsilon > 0$ and neighborhood of the identity $N \subset G$ a good couple such that $\mu(A) > 1-\varepsilon$ in three steps.

Step 1

In the first step, we will use the hypothesis that the action is suitable.

Proposition 3.2.8. *Let $f \in \widetilde{[\mathcal{R}_G]}$ and let $N \subset G$ be a neighborhood of the identity. For every $B \subset X$ of positive measure, there is a good couple (A, g) such that $A \subset B$ has positive measure and $\Phi(g)(A) \subset \Phi(f)(B)$.*

Proof. Consider a neighborhood of the identity $O \subset G$ such that $O = O^{-1}$ and $O^2 \subset N$. Let $f(x_0)$ be an element of the support of the pushforward measure $f_*\mu\big|_B$ and put $A := B \cap f^{-1}(Of(x_0))$. For every neighborhood of the identity O' in G, set $C_{O'} := f^{-1}(O'f(x_0)) \cap B$. Note that $C_{O'} \subset A$, whenever $O' \subset O$. By definition of the support of $f_*\mu\big|_B$ the Borel set $C_{O'}$ has positive measure.

Let us show that condition (2) of Definition 3.2.3 is not satisfied for the two Borel sets $\Phi(f)(A)$ and $f(x_0)A$. Indeed, $\Phi(f)^{-1}$ and $f(x_0)^{-1}$ are measure preserving so if condition (2) holds, then there is a full measure subset $A' \subseteq A$ such that $\Phi(f)(A')$ and $O'f(x_0)A'$ are disjoint. This is a contradiction because $\Phi(f)(A')$ and $O'f(x_0)A'$ contain $\Phi(f)(A' \cap C_{O'})$ which has positive measure.

Since the action is suitable, (1) (of Definition 3.2.3) has to hold. So there is $h \in O$ such that

$$\mu(\Phi(f)(A) \cap hf(x_0)A) > 0.$$

Set $\overline{A} := A \cap f(x_0)^{-1}h^{-1}\Phi(f)(A)$ and for every $x \in \overline{A}$ put $g(x) := hf(x_0)$. The couple $(\overline{A}, g)$ is good, because for every $x \in \overline{A}$ we have that $f(x)f(x_0)^{-1} \in O$ and

$$f(x)g(x)^{-1} = f(x)f(x_0)^{-1}h \in O^2 \subset N. \qquad \square$$

Step 2

For a neighborhood N of the identity in G and $\varepsilon > 0$, we now define the order on the family of N-good couples associated to a function $f \in \widetilde{[\mathcal{R}_G]}$.

Definition 3.2.9. Let (A_1, g_1) and (A_2, g_2) be two good couples. We say that $(A_1, g_1) \prec (A_2, g_2)$ if $A_2 \supseteq A_1$ almost everywhere and if

$$\mu(\{x \in A_1 : \ g_1(x) \neq g_2(x)\}) \leq \frac{1}{\varepsilon}(\mu(A_2) - \mu(A_1)).$$

Lemma 3.2.10. *The relation $\prec$ is an order relation on the set of good couples.*

Proof. The only non-trivial fact to prove is that $\prec$ is transitive. For this suppose that

$$(A_1, g_1) \prec (A_2, g_2) \prec (A_3, g_3),$$

then

$$\{x \in A_1 : \ g_1(x) \neq g_3(x)\} \subset \{x \in A_1 : \ g_1(x) \neq g_2(x)\} \cup \{x \in A_2 : \ g_2(x) \neq g_3(x)\},$$

so we get

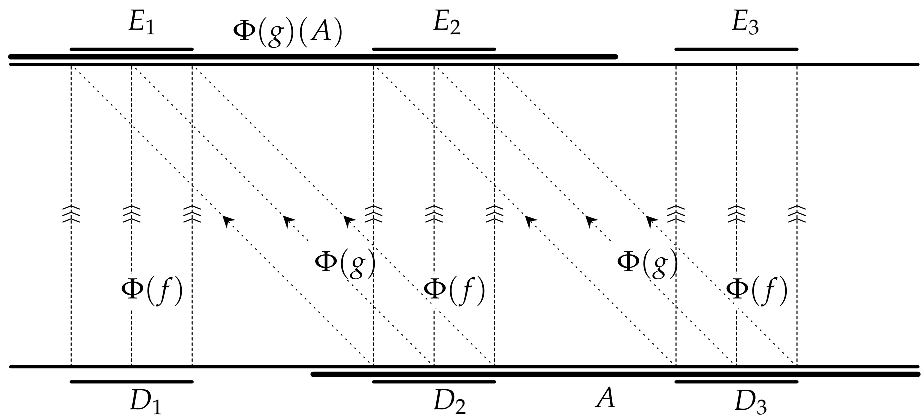
$$\begin{aligned}
&\mu(\{x \in A_1 : g_1(x) \neq g_3(x)\}) \\
&\qquad \leq \mu(\{x \in A_1 :\ g_1(x) \neq g_2(x)\}) + \mu(\{x \in A_2 :\ g_2(x) \neq g_3(x)\}) \\
&\qquad \leq \frac{1}{\varepsilon}(\mu(A_2) - \mu(A_1)) + \frac{1}{\varepsilon}(\mu(A_3) - \mu(A_2)) \\
&\qquad = \frac{1}{\varepsilon}(\mu(A_3) - \mu(A_1)).
\end{aligned}$$

□

The following proposition is the core of the proof of Theorem 3.2.5.

Proposition 3.2.11. *For every good couple (A, g) with $\mu(A) < 1 - \varepsilon$, there exists a good couple (A', g') such that $(A, g) \prec (A', g')$ and $\mu(A' \setminus A) > 0$.*

We would like to say that for every good couple (A, g) there is $B \subset X \setminus A$ such that $\Phi(f)(B) \cap \Phi(g)(A) = \emptyset$. When this is the case, we can conclude using Proposition 3.2.8. The problem is that this is not always possible, but it is possible in a finite number of steps.



In the figure $\Phi(f)$ acts vertically and $\Phi(g)$ acts diagonally.
Since $\Phi(f)(X \setminus A) \subset \Phi(g)(A)$, we can not use Proposition 3.2.8 directly.

Lemma 3.2.12. *There are $k \in \mathbb{N}$ with $k \leq 1/\varepsilon$ and two sequences $\{D_i\}_{i \leq k}$ and $\{E_i\}_{i \leq k}$ of measurable subsets of X of positive measure such that*

(a) the $\{D_i\}_{i \leq k}$ are pairwise disjoint as are the $\{E_i\}_{i \leq k}$,

(b) $D_1 \subset X \setminus A$ and $D_i \subset A$ for $i > 1$,

(c) $E_k \subset X \setminus \Phi(g)(A)$ and $E_i \subset \Phi(g)(A)$ for $i < k$,

(d) $\Phi(f)(D_k) = E_k$ and $E_{k-1} = \Phi(g)(D_k)$.

Proof. Set $B_1 := X \setminus A$ and $C_1 := \Phi(f)(B_1)$. For $i \geq 2$ define recursively

$$B_i := \Phi(g)^{-1}(C_{i-1} \cap \Phi(g)(A)) \text{ and } C_i := \Phi(f)(B_i).$$

Observe that $\{B_i\}_i$ are pairwise disjoint as are the $\{C_i\}_i$. Suppose now that for $l \geq 1$, we have that $C_i \subset \Phi(g)(A)$ for all $i \leq l$. Since $\Phi(g)$ and $\Phi(f)$ preserve the measure, we have that $\mu(C_i) = \mu(B_1)$ for all $i \leq l$ and hence we have that $l\mu(B_1) \leq 1 - \mu(B_1)$. By hypothesis $\mu(B_1) \geq \varepsilon$, so $l \leq 1/\varepsilon - 1$. Therefore there exists $k \leq 1/\varepsilon$, such that C_k is not contained in $\Phi(g)(A)$ and $C_i \subset \Phi(g)(A)$ for every $i < k$.

Put $E_k := C_k \setminus \Phi(g)(A)$ and set $D_k := \Phi(f)^{-1}(E_k)$. Observe that $D_k \subset B_k$ and define recursively $E_i := \Phi(g)(D_{i+1})$ and $D_i := \Phi(f)^{-1}(E_i)$. ☐

Proof of Proposition 3.2.11. Consider the families $\{D_i\}_{i\leq k}$ and $\{E_i\}_{i\leq k}$ defined in the previous lemma. By Proposition 3.2.8, there exists a good couple (A_1, g_1) such that $A_1 \subset D_1$ and $\Phi(g_1)(A_1) \subset \Phi(f)(A_1) \subset E_1$. For $i \in \{2, \ldots, k\}$, whenever A_{i-1} is defined, we set

$$A'_i := \Phi(g)^{-1}(\Phi(g_{i-1})(A_{i-1})) \subset D_i.$$

For every i such that A'_i is defined, Proposition 3.2.8 implies that there is a good couple (A_i, g_i) such that $A_i \subset A'_i$ is non-negligible and $\Phi(g_i)(A_i) \subset \Phi(f)(A_i) \subset E_i$. Put $B_k := A_k$. For $i \in \{1, \ldots, k-1\}$, we define recursively $B_i := \Phi(g_i)^{-1}(\Phi(g)(B_{i+1}))$.

Set $A' := A \cup B_1$ and define

$$g'(x) := \begin{cases} g(x) & \text{if } x \in A \setminus \cup_{i\geq 2} B_i, \\ g_i(x) & \text{if } x \in B_i. \end{cases}$$

By construction, $\Phi(g') : A' \to X$ is injective and preserves the measure. Moreover (A', g') is obtained *cutting and pasting* N-good couples, so it is a N-good couple. Let us finally check that $(A, g) \prec (A', g')$. Clearly we have $A' \supset A$ and $\mu(A' \setminus A) = \mu(B_1) > 0$. Moreover

$$\mu(\{x \in A : \ g(x) \neq g'(x)\}) \leq \mu\left(\cup_{i\geq 2} B_i\right) \leq k\mu(B_1) \leq \frac{1}{\varepsilon}(\mu(A') - \mu(A)).$$ ☐

Step 3

We verify now that we can apply Zorn's Lemma to the set of good couples.

Proposition 3.2.13. *Every chain for $\prec$ has an upper bound.*

Proof. Let us assume for the moment that $\{(A_n, g_n)\}_n$ is a countable chain of good couples. For every $n \in \mathbb{N}$ set

$$B_n := \{x \in A_n : \ g_n(x) = g_{n+1}(x)\}, \quad C_n := \cap_{k\geq n} B_n \quad \text{and} \quad A := \cup_n C_n.$$

Clearly $A \subset \cup_n A_n$ and we now check that the two measurable subsets have the same measure. In fact, since $\{A_n\}_n$ and $\{C_n\}_n$ are increasing sequences, for every $\eta > 0$, there is $K \in \mathbb{N}$ such that

$$\mu(\cup_n A_n) - \mu(A_K) < \eta \quad \text{and} \quad \mu(\cup_n C_n) - \mu(C_K) < \eta,$$

hence we have

$$\begin{aligned} \mu(\cup_n A_n) - \mu(A) \leq& 2\eta + \mu(A_K) - \mu(C_K) = 2\eta + \mu(A_K \setminus C_K) \\ =& 2\eta + \mu(A_K \cap (\cup_{k\geq K} X \setminus B_k)) = 2\eta + \mu(\cup_{k\geq K} A_K \setminus B_k) \\ \leq& 2\eta + \sum_{k\geq K} \mu(A_k \setminus B_k) \leq 2\eta + \frac{1}{\varepsilon} \sum_{k\geq K} \mu(A_{k+1} \setminus A_k) \\ \leq& 2\eta + \frac{1}{\varepsilon}\mu(\cup_{k\geq K+1} A_k \setminus A_K) \leq 2\eta + \frac{\eta}{\varepsilon}. \end{aligned}$$

As η is arbitrarily small, we get that $A = \cup_n A_n$ almost everywhere. For $x \in C_n$, observe that $g_n(x) = g_{n+j}(x)$ for every $j \geq 0$. We define

$$g(x) := g_n(x) \quad \text{if} \quad x \in C_n.$$

The couple (A, g) is obtained by cutting and pasting N-good couples so the couple is N-good. Moreover $A \supseteq \cup_n A_n$ almost everywhere and for every $n \in \mathbb{N}$, we have

$$\mu(x \in A_n : \ g_n(x) \neq g(x)) \leq \mu(A_n \setminus C_n) \leq \frac{1}{\varepsilon} \sum_{k \geq n} \mu(A_{k+1} - A_k) = \frac{1}{\varepsilon}(\mu(A) - \mu(A_n)).$$

Therefore the couple (A, g) is an upper bound for the countable chain. Consider now an arbitrary chain $\{(A_c, g_c)\}_{c \in C}$ and set $\lambda = \sup_{c \in C} \mu(A_c)$. If there is a good couple (A_c, g_c) such that $\mu(A_c) = \lambda$, then this couple is an upper bound of the chain and there is nothing to prove. Suppose that this is not the case and consider a subsequence $\{(A_n, g_n)\}_{n \in \mathbb{N}}$ of the chain such that $\lim_n \mu(A_n) = \lambda$. Let (A, g) be an upper bound for this sequence. Given any element of the chain (A_c, g_c) there exists n such that $\mu(A_c) \leq \mu(A_n)$ and hence $(A_c, g_c) \prec (A_n, g_n) \prec (A, \varphi)$. $\square$

End of the proof of Theorem 3.2.5

Let $f \in \widetilde{[\mathcal{R}_G]}$. By definition of the topology of convergence in measure, a base of neighborhoods of f is given by the open sets

$$\mathcal{U}_{\varepsilon,N} := \left\{ g \in \widetilde{[\mathcal{R}_G]} : \ \mu\left(\{x \in X : \ g(x) \in Nf(x)\}\right) > 1 - \varepsilon \right\},$$

where $\varepsilon > 0$ and $N \subset G$ is a neighborhood of the identity. For every neighborhood of the identity $N \subset G$, Proposition 3.2.8 implies that the set of good couples for f is not empty. For $\varepsilon > 0$, Proposition 3.2.13 tells us that there is a maximal good couple (A, g). The maximality of the couple and Proposition 3.2.11 imply that $\mu(A) > 1 - \varepsilon$. Hence we can use Proposition 3.2.6 to obtain an element $g' \in \widetilde{[\mathcal{R}_G]}_D \cap \mathcal{U}_{\varepsilon,N}$.

Chapter 4

Maximal amenable subalgebras of von Neumann algebras associated with hyperbolic groups

The following chapter is based on a joint work with Rémi Boutonnet.

We prove that for any infinite, maximal amenable subgroup H in a hyperbolic group G, the von Neumann subalgebra LH is maximal amenable inside LG. It provides many new, explicit examples of maximal amenable subalgebras in II_1 factors. We also prove similar maximal amenability results for direct products of relatively hyperbolic groups and orbit equivalence relations arising from measure-preserving actions of such groups.

Contents

Notations and conventions

All von Neumann algebras considered in the dissertion will be assumed to be separable. When a finite von Neumann algebra M is being considered, we will denote by τ a faithful normal trace on it. In this case we denote by $\|x\|_2 = \tau(x^*x)^{1/2}$ the 2-norm of an element $x \in M$. In contrast, we denote by $\|x\|_\infty$ the operator norm of x. Also we denote by $L^2(M, \tau)$ (or sometimes just $L^2(M)$) the Hilbert space obtained by completion of M with respect to the 2-norm. The left multiplication of M on itself extends to an action of M on $L^2(M)$. This representation is normal and often we will represent $M \subset B(L^2M)$ in this manner.

If M is a finite von Neumann algebra, and N is a von Neumann subalgebra of M, we denote by E_N the trace preserving conditional expectation.

If G is a discrete countable group, we denote by LG the associated von Neumann algebra; that is, the von Neumann subalgebra in $B(\ell^2G)$ generated by the left regular representation of G. We also denote by RG is commutant, which is also the von Neumann algebra generated by the right regular representation. Both LG and RG are finite and we choose τ to be the canonical trace on them, defined by $\tau(x) = \langle x\delta_e, \delta_e\rangle$ for all $x \in LG \subset B(\ell^2G)$ (here $\delta_e \in \ell^2G$ denotes the Dirac function at the identity element e of the group G). In this case, we have that $L^2(L\Gamma, \tau) \simeq \ell^2(\Gamma)$.

4.1 Preliminaries

4.1.1 Central sequences and group von Neumann algebras

In this section, we consider an inclusion of two countable discrete groups $H < G$. We denote by $LH \subset LG$ the associated von Neumann algebras and by u_g the canonical unitaries in LG that correspond to elements $g \in G$.

For a set $F \subset G$, we will by denote $P_F : \ell^2(G) \to \ell^2(F)$ the orthogonal projection onto $\ell^2(F)$.

As explained in the introduction, the proofs of our main results rely on an analysis of LH-central sequences. We describe here how the H-conjugacy action on G allows localizing the Fourier coefficients of LH-central sequences in terms of projections P_F, $F \subset G$.

Definition 4.1.1. Let $H < G$ be an inclusion of two countable groups. A set $F \subset G \setminus H$ is said to be *H-roaming* if there is an infinite sequence $(h_k)_{k\geq 0}$ of elements in H such that

$$h_k F h_k^{-1} \cap h_{k'} F h_{k'}^{-1} = \emptyset \text{ for all } k \neq k'.$$

Such a sequence $(h_k)_k$ is called a *disjoining sequence*.

The following standard lemma is the key of our proofs.

Lemma 4.1.2. *Let $H < G$ be an inclusion of two countable groups and denote by $LH \subset LG$ the associated von Neumann algebras. Assume that $(x_n)_n$ is a bounded LH-central sequence in LG.*

Then for any H-roaming set F we have that $\lim_n \|P_F(x_n)\|_2 = 0$.

Proof. Assume that F is an H-roaming set and consider a disjoining sequence $(h_k)_k \subset H$ for F. Since $(x_n)_n$ is LH-central, we have for all k

$$\limsup_n \|P_F(x_n)\|_2 = \limsup_n \|P_F(u_{h_k} x_n u_{h_k}^*)\|_2 = \limsup_n \|P_{h_k^{-1} F h_k}(x_n)\|_2. \tag{4.1}$$

But $P_{h_k^{-1}Fh_k}(x_n) \perp P_{h_{k'}^{-1}Fh_{k'}}(x_n)$ for all $k \neq k'$ and all n. Thus we get that for any $N \geq 0$ and $n \geq 0$,

$$\|x_n\|_\infty^2 \geq \|x_n\|_2^2 \geq \sum_{k \leq N} \|P_{h_k^{-1}Fh_k}(x_n)\|_2^2.$$

Applying (4.1), we deduce that $\sup_n \|x_n\|_\infty^2 \geq N \limsup_n \|P_F(x_n)\|_2^2$. Since N can be arbitrarily large, we get the result. $\square$

Proposition 4.1.3. *Let $H < G$ be an inclusion of two infinite countable groups. Assume that for any $s, t \in G \setminus H$, there is an H-roaming set $F \subset G \setminus H$ such that $sF^c t \cap F^c$ is finite.*

If LH has property Gamma, then it is maximal Gamma inside LG.

Proof. Assume that there is an intermediate von Neumann algebra P with property Gamma: $LH \subset P \subset LG$ and consider a central sequence $(v_n)_n \subset P$ of unitary elements which tends weakly to 0.

We have to show that any $a \in P \ominus LH$ is equal to 0. On the one hand, we have that $\lim_n \langle a v_n a^*, v_n \rangle = \|a\|_2^2$ because the unitaries v_n asymptotically commute with a. On the other hand we claim that this limit is 0.

Indeed, by a standard linearity/density argument, it is sufficient to check that for all $s, t \notin H$, we have $\lim_n \langle u_s v_n u_t, v_n \rangle = 0$.

So fix $s, t \in G \setminus H$. By assumption there is an H-roaming set F such that $K := sF^c t \cap F^c$ is finite. Since $(v_n)_n$ is LH-central and bounded, Lemma 4.1.2 implies that $\lim_n \|P_F(v_n)\|_2 = 0$. Noting that $u_s P_{F^c}(v_n) u_t$ is in the range of $P_{sF^c t}$ for all n, we obtain

$$\begin{aligned}
\limsup_n |\langle u_s v_n u_t, v_n \rangle| &= \limsup_n |\langle u_s P_{F^c}(v_n) u_t, P_{F^c}(v_n) \rangle| \\
&= \limsup_n |\langle u_s P_{F^c}(v_n) u_t, P_{sF^c t} \mathbb{S}^1 P_{F^c}(v_n) \rangle| \\
&\leq \limsup_n \|P_K(v_n)\|_2 = 0,
\end{aligned}$$

because $(v_n)_n$ tends weakly to 0 and K is finite. $\square$

If $H < G$ is an inclusion satisfying the assumption of Proposition 4.1.3, then H is *almost malnormal* in G in the sense that $sHs^{-1} \cap H$ is finite for all $s \notin H$ (or equivalently $sHt \cap H$ is finite for all $s, t \notin H$). In terms of von Neumann algebras this translates as follows.

Proposition 4.1.4. *A subgroup H of a group G is almost malnormal if and only if the von Neumann subalgebra $LH \subset LG$ is mixing, meaning that $\lim_n \|E_{LH}(a v_n b)\|_2 = 0$, for all $a, b \in LG \ominus LH$ and for any sequence $(v_n)_n \subset \mathcal{U}(LH)$ which tends weakly to* 0.

Proof. Assume that H is not almost malnormal inside G: there is $s \in G \setminus H$ such that $sHs^{-1} \cap H$ contains a sequence going to infinity $(h_n)_n$. For all n put $v_n := u_{h_n} \in \mathcal{U}(LH)$ and put $a^* = b = u_s \in LG \ominus LH$. Then $LH \subset LG$ is not mixing, because the sequence (v_n) goes weakly to 0, whereas

$$\limsup_n \|E_{LH}(a v_n b)\|_2 = \limsup_n \|E_{LH}(u_{s^{-1}h_n s})\|_2 = \limsup_n \|u_{s^{-1}h_n s}\|_2 = 1 \neq 0.$$

Conversely assume that H is almost malnormal inside G. Take a sequence $(v_n)_n \subset \mathcal{U}(LH)$ which tends weakly to 0. If $s, t \in G \setminus H$, then $K := s^{-1}Ht^{-1} \cap H$ is finite. Hence

$$\limsup_n \|E_{LH}(u_s v_n u_t)\|_2 = \limsup_n \|P_{s^{-1}Ht^{-1}}(v_n)\|_2 = \limsup_n \|P_K(v_n)\|_2 = 0.$$

This implies the mixing property, because $\{u_s\,,\, s \in G \setminus H\}$ spans a $\|\cdot\|_2$-dense subset of $LG \ominus LH$. $\square$

4.1.2 Relatively hyperbolic groups and their boundary

The contents of this section is taken from Bowditch [Bow12]. Let us fix first some terminology and notations about graphs.

Let K be a connected graph. Its vertex set and edge set are denoted by $V(K)$ and $E(K)$ respectively. A *path* of length n between two vertices x and y is a sequence $(x_0, x_1, \dots, x_n)$ of vertices such that $x_0 = x$ and $x_n = y$, and $(x_i, x_{i+1}) \in E(K)$ for all $i = 0, \dots, n-1$. The path $(x_0, \dots, x_n)$ is a *loop* if $x_0 = x_n$ and if $x_0, x_1, \dots, x_n$ are distinct. For a path $\alpha = (x_0, x_1, \dots, x_n)$, we put $\alpha(k) = x_k$, $k = 0, \dots, n$.

We endow K with the distance d given by the length of a shortest path between two points. A path α between two vertices x and y is a *geodesic* if its length equals to $d(x, y)$. We denote by $\mathcal{F}(x, y)$ the set of all geodesics between x and y.

More generally, for $r \geq 0$, a path α is an *r-quasi-geodesic* if all its vertices are distinct, and if for any finite subpath $\beta = (x_0, \dots, x_n)$ of α, the length of β is smaller than $d(x_0, x_n) + r$. Note that the geodesics are exactly the 0-quasi-geodesics. For $x, y \in V(K)$, denote by $\mathcal{F}_r(x, y)$ the set of r-quasi-geodesics between x and y.

We will also consider infinite paths $(x_0, x_1, \dots)$ or bi-infinite paths $(\dots, x_{-1}, x_0, x_1, \dots)$. For $r \geq 0$, such an infinite or bi-infinite path will be called r-quasi-geodesic if all its finite subpaths are r-quasi-geodesics.

In a graph K, a *geodesic triangle* is a set of three vertices $x, y, z \in V(K)$, together with geodesic paths $[x, y] \in \mathcal{F}(x, y)$, $[y, z] \in \mathcal{F}(y, z)$ and $[z, x] \in \mathcal{F}(z, x)$ connecting them. These paths are called the *sides* of the triangle.

Definition 4.1.5 (Gromov [Gro87])**.** A connected graph K is called *hyperbolic* if there is a constant $\delta > 0$ such that every geodesic triangle in K is *δ-thin*: each side of the triangle is contained in the δ-neighbourhood of the union of the other two, namely $[x, y] \subset B([y, z] \cup [z, x], \delta)$, and similarly for the other two sides.

Two infinite quasi-geodesics in a hyperbolic graph K are *equivalent* if their Hausdorff distance is finite. The *Gromov boundary* ∂K of K is the set of equivalence classes of infinite quasi-geodesics. The endpoints of a path $\alpha = (x_0, x_1, \dots)$ in a class x of ∂K are defined to be x_0 and x. Similarly, a bi-infinite path $\alpha = (\dots, x_{-1}, x_0, x_1, \dots)$ has endpoints $\alpha_- := [(x_0, x_{-1}, \dots)] \in \partial K$ and $\alpha_+ := [(x_0, x_1, \dots)] \in \partial K$. It turns out that for any two points $x, y \in K \cup \partial K$, for any $r \geq 0$, the set $\mathcal{F}_r(x, y)$ of r-quasi-geodesics connecting them is non-empty.

Recall that a *hyperbolic group* is a finitely generated group which admits a hyperbolic Cayley graph (this implies that all its Cayley graphs are hyperbolic). We will define similarly relatively hyperbolic groups, but we have to replace the Cayley graph by a graph in which some subgroups are “collapsed" to points.

Definition 4.1.6 ([Far98])**.** Consider a group G, with finite generating set S and denote by $\Gamma := \mathrm{Cay}(G, S)$ the associated Cayley graph. Let $\mathscr{G}$ be a collection of subgroups of G. The *coned-off graph of* Γ *with respect to* $\mathscr{G}$ is the graph $\widehat{\Gamma}$ with:

- vertex set $V(\widehat{\Gamma}) := V(\Gamma) \sqcup \bigsqcup_{H \in \mathscr{G}} G/H$;
- edge set $E(\widehat{\Gamma}) := E(\Gamma) \sqcup \{(gh, [gH]) \,|\, H \in \mathscr{G}, [gH] \in G/H, h \in H\}$.

In the sequel, we will identify $V(\Gamma)$ with G. The action of G on itself by left multiplication extends to an isometric action on $\widehat{\Gamma}$. The stabilizer of the vertex $[gH]$ is equal to gHg^{-1}.

Note that in the coned-off graph $\widehat{\Gamma}$, the distance between two elements g and gh is at most 2 whenever $h \in H$ for some $H \in \mathscr{G}$. Note also that this coned-off graph will not be locally finite in general. But it will sometimes satisfy the following *fineness* condition.

Definition 4.1.7 ([Bow12])**.** A graph Γ is called *fine* if each edge of Γ is contained in only finitely many loops of length n, for any given integer n.

Definition 4.1.8 ([Bow12])**.** A group G is said to be *hyperbolic relative to the family $\mathscr{G}$* if there is a finite generating set S of G such that the coned-off graph $\widehat{\Gamma}$ is fine and δ-hyperbolic (for some $\delta \geq 0$).

From this definition, usual hyperbolic groups appear as hyperbolic relative to the empty family. For any relatively hyperbolic group G with Cayley graph Γ, let us define a topology on $\Delta\Gamma := \widehat{\Gamma} \cup \partial\widehat{\Gamma}$.

Definition 4.1.9. Given $x \in \Delta\Gamma$ and a finite set $A \subset V(\widehat{\Gamma})$ such that $x \notin A$, we define

$$M(x, A) := \{y \in \Delta\Gamma : A \cap \alpha = \emptyset, \forall \alpha \in \mathcal{F}(x, y)\}.$$

Theorem 4.1.10 ([Bow12], section 8)**.** *The family $\{M(x, A)\}_{x,A}$ is a basis for a Hausdorff compact topology on $\Delta\Gamma$ such that $G \subset \Delta\Gamma$ is a dense subset, and every graph automorphism of $\widehat{\Gamma}$ extends to a homeomorphism of $\Delta\Gamma$.*

Actually, we will not use the fact that $\Delta\Gamma$ is compact. The proof of Theorem 4.1.10 relies on the following lemma, which will be our main tool in order to manipulate neighbourhoods in $\Delta\Gamma$.

Lemma 4.1.11 ([Bow12], Section 8)**.** *Let $r \geq 0$. The following facts are true.*

1. *For every $x, y \in \Delta\Gamma$, the graph $\bigcup_{\alpha \in \mathcal{F}_r(x,y)} \alpha$ is locally finite.*
2. *For every edge $e \in E(\widehat{\Gamma})$, there is a finite set $E_r(e) \subset E(\widehat{\Gamma})$ such that for all $x, y \in \Delta\Gamma$, and all $\alpha, \beta \in \mathcal{F}_r(x, y)$ with $e \in \alpha$, we have that $E_r(e) \cap \beta$ contains at least one edge.*
3. *For every $a \in V(\widehat{\Gamma}), x \in \Delta\Gamma$, with $x \neq a$, there is a finite set $V_{r,x}(a) \subset V(\widehat{\Gamma}) \setminus \{x\}$ such that for all $y \in \Delta\Gamma$, and all $\alpha, \beta \in \mathcal{F}_r(x, y)$ with $a \in \alpha$, we have that $\beta \cap V_{r,x}(a) \neq \emptyset$.*

Proof. The first two facts are Lemma 8.2 and Lemma 8.3 in [Bow12]. To derive the third fact from the others, fix $a \in V(\widehat{\Gamma})$ and $x \in \Delta\Gamma$. Denote by E_0 the set of edges e in the graph $\bigcup_{\alpha \in \mathcal{F}_r(a,x)} \alpha$ such that a is an endpoint of e. By (1), the set E_0 is finite. Now put $E := \bigcup_{e \in E_0} E_r(e)$, and define $V_{r,x}(a)$ to be the set of endpoints of E, from which we remove x if necessary. This is a finite set.

Now if $\alpha \in \mathcal{F}_r(x, y)$ goes through a, then it will contain an edge in E_0. Thus any $\beta \in \mathcal{F}_r(x, y)$ contains an edge in E, and we are done by the definition of $V_{r,x}(a)$. ☐

Lemma 4.1.11 will always be used via the following easy corollary.

Corollary 4.1.12. *Let $r > 0$, $x \in \Delta\Gamma$ and $A \subset V(\widehat{\Gamma}) \setminus \{x\}$ finite. Then there is a set $V_{r,x}(A)$ containing A such that,*

1. *if $y \notin M(x,A)$, then any r-quasi-geodesic $\alpha \in \mathcal{F}_r(x,y)$ intersects $V_{r,x}(A)$,*
2. *if $y \in M(x, V_{r,x}(A))$, then no r-quasi-geodesic from y to x intersect A.*

Proof. The set $V_{r,x}(A) := A \cup \bigcup_{a\in A} V_{r,x}(a)$ does the job. □

Now we describe a way of constructing quasi-geodesic paths. The following lemma is well known but we include a proof for convenience.

Lemma 4.1.13. *There is a constant $r_0 \geq 0$, only depending on the hyperbolicity constant of the graph $\widehat{\Gamma}$, with the following property: for any geodesic paths α, β sharing exactly one endpoint a, if a is the closest point of α to each point of β, then $\alpha \cup \beta$ is an r_0-quasi-geodesic.*

Proof. We can take $r_0 = 8\delta + 8$. Indeed, given $x \in \alpha$ and $y \in \beta$, we denote with $[x,y] \in \mathcal{F}(x,y)$ a geodesic between x and y. Since the triangle $\{x,y,a\}$ is δ-thin, there is $z \in [x,y]$ and $u \in \alpha$, $v \in \beta$ such that $d(z,u) \leq \delta + 1$ and $d(z,v) \leq \delta + 1$. By hypothesis on a, we have that $d(u,a) \leq 2\delta + 2$ and hence $d(v,a) \leq 4\delta + 4$. Finally,

$$\begin{aligned} d(x,a) + d(y,b) \leq& (d(x,z) + d(z,u) + d(u,a)) + (d(a,v) + d(v,z) + d(z,y)) \\ \leq& d(x,z) + d(y,z) + 8\delta + 8 = d(x,y) + 8\delta + 8. \end{aligned}$$ □

Definition 4.1.14. Consider $x,y,z \in \Delta\Gamma$ and let $\alpha \in \mathcal{F}(x,y)$ be a geodesic. A point $z_0 \in \alpha$ which minimizes the distance from z to α, is called a *projection of z on α*. Such a z_0 splits the path α into two geodesic paths $\alpha_x \in \mathcal{F}(x,z_0)$ and $\alpha_y \in \mathcal{F}(z_0,y)$. Given any geodesic $\beta \in \mathcal{F}(z,z_0)$, we can join β and α_x or α_y to get two paths that are r_0-quasi geodesic by Lemma 4.1.13.

We end this section with a lemma that we will need later and which is essentially contained Section 8 of [Bow12]. Its proof illustrates well how to use the tools introduced above.

Lemma 4.1.15. *For every $x \in \Delta\Gamma$ and for every finite subset $A \subset V(\widehat{\Gamma}) \setminus \{x\}$, there is a finite subset $C \subset V(\widehat{\Gamma}) \setminus \{x\}$ such that for every $y \in M(x,C)$,*

$$M(y,C) \subset M(x,A).$$

Proof. Let $r_0 \geq 0$ be given by Lemma 4.1.13, and set $C := V_{r_0,x}(V_{r_0,x}(A))$ (see Corollary 4.1.12). We will show that the conclusion of the lemma holds for this C.

If $y = x$, we see that $M(x,C) \subset M(x,A)$ because $A \subset V_{r_0,x}(A) \subset C$. Now let $y \in M(x,C)$, with $y \neq x$, and take $z \notin M(x,A)$. We will show that $z \notin M(y,C)$.

Let α be a geodesic between y and z. Consider a projection x_0 of x on α as in Definition 4.1.14 and let $\beta \in \mathcal{F}(x,x_0)$. We denote with α_y (resp. α_z) the subgeodesic of α between x_0 and y (resp. x_0 and z). Then, by Lemma 4.1.13 the paths $\beta \cup \alpha_y \in \mathcal{F}_{r_0}(x,y)$ and $\beta \cup \alpha_z \in \mathcal{F}_{r_0}(x,z)$ are r_0-quasi-geodesics.

Since $z \notin M(x,A)$, Corollary 4.1.12(1) implies that $\beta \cup \alpha_z$ intersects $V_{r_0,x}(A)$. If the intersection point lied on $\beta \cup \alpha_y$, then Corollary 4.1.12(2) would contradict our assumption that $y \in M(x,C)$. Hence the intersection point lies on $\alpha_z \subset \alpha$. We have found a geodesic between z and y which intersects a point of $V_{r_0,x}(A) \subset C$, which means precisely that $z \notin M(y,C)$. □

4.2 Hyperbolic case

Suppose that G is a hyperbolic group and that H is an infinite maximal amenable subgroup of G. We want to apply Proposition 4.1.3 to prove the following theorem.

Theorem 4.2.1. *Consider a hyperbolic group G and an infinite, maximal amenable subgroup $H < G$. Then the group von Neumann algebra LH is maximal amenable inside LG.*

As mentioned in Section 4.1.2, G is hyperbolic relative to the empty family and $\widehat{\Gamma} = \Gamma$, for any Cayley graph Γ of G. Thus $\Delta\Gamma := \Gamma \cup \partial\Gamma$ is the usual Gromov compactification of Γ, with boundary $\partial\Gamma$, endowed with the topology generated by the sets $\{M(x, A)\}_{x,A}$. As before, we identify G with $V(\Gamma)$.

Recall that the action of G by left multiplication on itself extends to a continuous action on $\Delta\Gamma$. The amenable subgroup H has a particular form by [GdlH90, Théorème 8.29, Théorème 8.37].

Lemma 4.2.2. *Let $H < G$ be an infinite maximal amenable subgroup. The following facts are true.*

(i) there are two points $a, b \in \partial\Gamma$ such that H is the stabiliser of the couple $\{a, b\}$, that is $H = \mathrm{Stab}_G(\{a, b\})$.

(ii) Any H-invariant probability measure on $\partial\Gamma$ is of the form $t\delta_a + (1-t)\delta_b$ for some $t \in [0, 1]$.

(iii) Any element $g \in G \setminus H$ is such that $g \cdot \{a, b\} \cap \{a, b\} = \emptyset$.

Proof. By [GdlH90, Théorème 8.37], H is virtually cyclic. Denote by $h \in H$ an element of infinite order. Then by [GdlH90, Théorème 8.29], h is a hyperbolic element: h acts on $\partial\Gamma$ with a north-south dynamics. Denote by a and b the attractive and repulsive points of h.

(i) Take $s \in H$. Then shs^{-1} is a hyperbolic element with fixed points $s \cdot a$ and $s \cdot b$. If $\{a, b\} \cap \{a \cdot a, s \cdot b\} = \emptyset$, then by the ping-pong lemma, the group generated by h and shs^{-1} contains a free group. Since H is amenable, this is not possible and hence h and shs^{-1} have at least a fixed point in common, say a. By [GdlH90, Théorème 8.30] they also both fix b. Hence $\{s \cdot a, s \cdot b\} = \{a, b\}$ and so $H \subset \mathrm{Stab}_G(\{a, b\})$. The action of G on the boundary is amenable [Ada94], so the equality follows from maximal amenability.

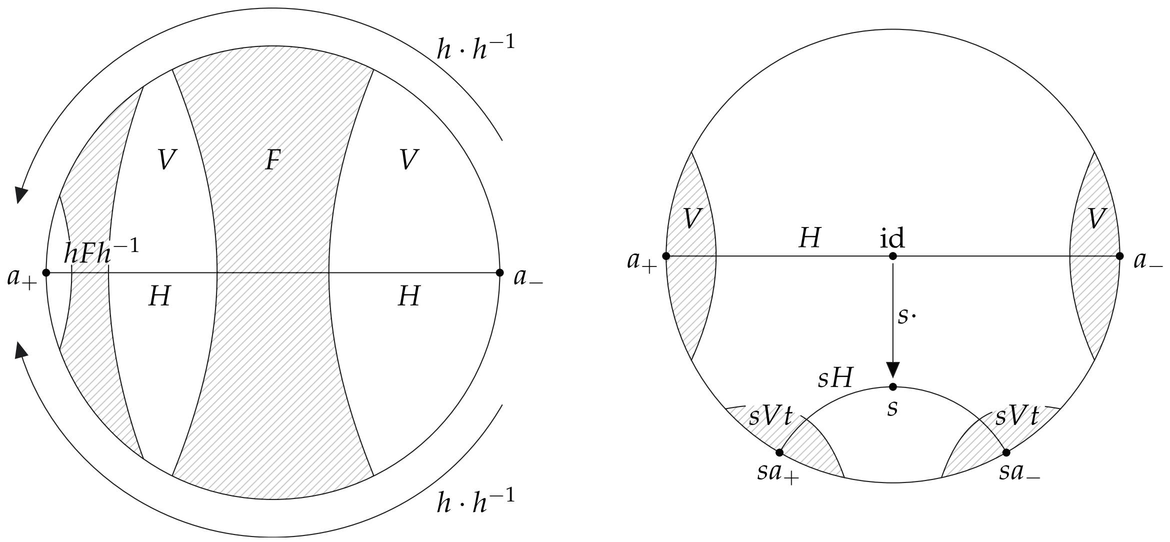
(ii) This is a consequence of the north-south dynamics action of $h \in H$.

(iii) The key result is [GdlH90, Théorème 8.30], which implies that any element which fixes one of the points a or b is in H. Take $g \in G$ such that $g \cdot a = b$. Assume first that there is $s \in H$ which exchanges a and b. Then sg fixes a and so $g \in H$. If all elements in H fix a and b, then gsg^{-1} fixes b and $g^{-1}sg$ fixes a (so they both belong to H), for all $s \in H$. In that case, g normalizes H so $g \in H$ by maximal amenability. ☐

For an amenable subgroup $H \subset G$, we will denote by $a_\pm \in \partial\Gamma$ the two fixed point of H. Up to relabeling, we can suppose that $\lim_{n\to+\infty} a^n x = a_+$ and $\lim_{n\to-\infty} a^n x = a_-$, for any $x \in \Delta\Gamma$ (so in particular a_+ is the unique cluster point of the sequence $\{a^n\}_{n\geq 0}$).

The action of G on itself by right multiplication also extends to a continuous action on $\Delta\Gamma$, in such a way that any element $g \in G$ acts trivially on $\partial\Gamma$ (see for instance [BO08, Proposition 5.3.18]).

In order to find an H-roaming set as in Proposition 4.1.3, we need to understand geometrically the conjugacy action of H on G. We start by collecting properties of left and right actions of H on $\Delta\Gamma$ separately, in the following two lemmas. Combining these lemmas, we will see that the conjugacy action of H has a uniform *"north-south dynamics" out of* H, as shown in Figure 4.1a.



(a) Conjugacy action of H on G.

(b) The subsets V and sVt are disjoint.

Figure 4.1 – The action of G and a good neighborhood V of $\{a_+, a_-\}$.

The following fact is certainly known, but we include a proof for the sake of completeness. Let us recall that the sets $M(a_\pm, A)$ are neighbourhoods of $a_\pm$.

Lemma 4.2.3. *For any finite sets $A, B \subset V(\Gamma)$, there is $n \in \mathbb{Z}$ such that*

$$G \cap (a^n \cdot M(a_-, B)^c) \subset M(a_+, A).$$

Proof. First note that we can (and we will) assume that $a_- \notin M(a_+, A)$. By Lemma 4.1.15 there is a finite set $C \subset V(\Gamma)$ such that for all $y \in M(a_+, C)$ we have $M(y, C) \subset M(a_+, A)$. In particular, for all $y \in M(a_+, C)$ and $z \notin M(a_+, A)$ there is a geodesic between y and z which intersects C.

Choose $n \in \mathbb{Z}$ such that $a^n B \subset M(a_+, C)$ and such that the distance between points of C and $a^n B$ is larger than the diameter D of $V_{0,a_-}(C)$. We claim that this n satisfies the conclusion of the lemma.

Assume by contradiction that there is $z \in G = V(\Gamma)$ such that $z \notin a^n M(a_-, B)$ and $z \notin M(a_+, A)$. Since $z \notin a^n M(a_-, B) = M(a_-, a^n B)$, there is a geodesic $\alpha \in \mathcal{F}(a_-, z)$ which contains a point $y \in a^n B \subset M(a_+, C)$. Let us denote α_{a_-} the sub-geodesic of α between a_- and y and with α_z the sub-geodesic between y and z.

Since $a_- \notin M(a_+, A)$, there is a geodesic between a_- and y which intersects C. By Corollary 4.1.12, the geodesic α_{a_-} meets $V_{0,a_-}(C)$ at a vertex x_1. Moreover $z \notin M(a_+, A)$, so replacing α_z by another geodesic between y and z if necessary, we can assume that α_z meets $C \subset V_{0,a_-}(C)$ at a vertex x_2 (while $\alpha = \alpha_{a_-} \cup \alpha_z$ is still a geodesic). But then

$$d(x_1, x_2) \leq \operatorname{diam}(V_{0,a_-}(C)) = D.$$

On the other hand, the length of α between these two points is equal to $d(x_1, y) + d(y, x_2)$, while $d(x_1, y) > D$ because $x_1 \in C$ and $y \in a^n B$. This is absurd. $\square$

Lemma 4.2.4. *For any $A \subset V(\Gamma)$ finite, there is a finite $B \subset V(\Gamma)$ such that for any $k \in \mathbb{Z}$,*

$$(M(a_+, B) \cap (G \setminus H))a^k \subset M(a_+, A).$$

Proof. We want to construct a neighbourhood $M(a_+, B)$ such that for any $y \in M(a_+, B)$, the sequence $(ya^k)_k$ stays inside $M(a_+, A)$. It is helpful to think about $(ya^k)_k$ as a quasi-geodesic, from ya_- to ya_+.

The proof goes in two steps. We firstly find an intermediate neighbourhood $M(a_+, B')$ such that if the sequence $(ya^k)_k$ leaves $M(a_+, A)$ for some $y \in M(a_+, B')$, then it has to go through a fixed finite set, which we can assume to be B'. We will conclude by choosing $M(a_+, B) \subset M(a_+, B')$ which does not intersects the sequences $(ya^k)_k$ for $y \in B'$.

Step 1. there is a finite set $B' \subset V(\Gamma)$ such that if $y \in M(a_+, B') \cap G$ is such that $ya^k \notin M(a_+, A)$ for some $k \in \mathbb{Z}$, then there is $m \in \mathbb{Z}$ such that $ya^m \in B'$.

By [GdlH90, Proposition 8.21], there is a finite constant $r > 0$ such that for any $p \in \mathbb{Z}$, all geodesics between the neutral element e and a^p are contained in the r-neighbourhood of the sequence $\{a^k, k \in \mathbb{Z}\}$.

By Lemma 4.1.15 there is a finite set $C \subset V(\Gamma)$ such that for all $y \in M(a_+, C)$ we have $M(y, C) \subset M(a_+, A)$. Put $B' := B(C, r)$, the r-neighbourhood of C.

Take $y \in M(a_+, B') \cap G \subset M(a_+, C)$ such that $ya^k \notin M(a_+, A)$ for some $k \in \mathbb{Z}$. Then $ya^k \notin M(y, C)$, so there is a geodesic α between y and ya^k which meets C at a point c. Then $y^{-1}c$ belongs to a geodesic between e and a^k, so it is at distance less than r to some a^m. In other words, $ya^m \in B(C, r) = B'$, which proves Step 1.

Step 2. Choice of B.

Observe that the set of cluster points of the sequences $(ya^k)_k$, with $y \in B' \setminus H$ is finite and contained in $\partial\Gamma \setminus \{a_+, a_-\}$. So there is B such that

$$M(a_+, B) \subset M(a_+, B') \quad \text{and} \quad M(a_+, B) \cap \{ba^k \mid b \in B' \setminus H, k \in \mathbb{Z}\} = \emptyset.$$

The subset B satisfies the conclusion of the lemma. Indeed, if $y \in M(a_+, B) \cap (G \setminus H)$ is such that $ya^k \notin M(a_+, A)$ for some $k \in \mathbb{Z}$, then by the claim there is h such that $y \in B'a^{-h}$. But in this case we would have $y \in M(a_+, B) \cap \{ba^p \mid b \in B' \setminus H, p \in \mathbb{Z}\}$, which was assumed to be empty. Therefore $ya^k \in M(a_+, A)$ for any k. □

Now we can deduce a relevant property of the conjugacy action of H, as shown in Figure 4.1a.

Proposition 4.2.5. *For every $s, t \in G \setminus H$, there is an H-roaming set $F \subset G \setminus H$ such that $sF^c t \cap F^c$ is finite.*

Proof. Choose a neighbourhood V_0 of $\{a_+, a_-\}$ such that V_0 is disjoint from sV_0. Since the right action of t on $\Delta\Gamma$ is continuous, we can find a $V \subset V_0$ such that V and sVt are disjoint (see Figure 4.1b). We observe that $sVt \cap H$, $sHt \cap V$ and $sHt \cap H$ are finite because the only cluster points of H are in V and the only cluster points of sHt are in sVt.

Therefore the set $F := V^c \cap (G \setminus H)$ is such that $sF^c t \cap F^c$ is finite. To prove that it is H-roaming, let us construct a disjoining sequence $(h_k)_k$ inductively. First put $h_0 := e$.

Now assume that $h_0, \dots, h_{n-1}$ have been constructed, for some $n \geq 1$. We will construct h_n. Denote by $V_n := \bigcap_{i=0}^{n} h_i V h_i^{-1}$. It is a neighbourhood of $\{a_-, a_+\}$, by continuity of left and right actions of H. Now put $F_n := V_n^c \cap (G \setminus H)$.

Recall that the family $\{M(a_\pm, A)\}_A$ forms a basis of neighbourhoods of $a_\pm$. By Lemma 4.2.4, there is a neighbourhood V' of a_+ such that $(V' \cap (G \setminus H))a^k \subset V_n$ for all $k \in \mathbb{Z}$. By Lemma 4.2.3, there is $k_n \in \mathbb{Z}$ such that $G \cap a^{k_n} V^c \subset V'$ and in particular $a^{k_n} F \subset V'$. Note also that $(a^{k_n} F) \cap H = \emptyset$.

Altogether, we get that $a^{k_n} F a^{-k_n} \subset V_n$ is disjoint from F_n. But F_n contains all the $h_i F h_i^{-1}$, $i \leq n-1$. So we can define $h_n = a^{k_n}$. □

Now Theorem 4.2.1 follows from Proposition 4.1.3.

Remark 4.2.6. Note that in the proof of Proposition 4.2.5 the disjoining sequence that we construct is contained in the subgroup $H_0 := \langle a \rangle \subset H$. Then the proof of Theorem 4.2.1 actually shows that if $P \subset LG$ is an algebra with property Gamma such that $LH_0 \subset P$, then $P \subset LH$. Hence u_a is contained in a unique maximal amenable von Neumann subalgebra of M.

4.3 Relatively hyperbolic case

We now prove a more general version of Theorem 4.2.1 in the context of relatively hyperbolic groups.

Theorem 4.3.1. *Let G be a group which is hyperbolic relative to a family $\mathscr{G}$ of subgroups of G and consider an infinite subgroup $H \in \mathscr{G}$ such that LH has property Gamma. Then the group von Neumann algebra LH is maximal Gamma inside LG.*

Let G be a hyperbolic group relative to a family $\mathscr{G}$ of subgroups of G, and let $H \in \mathscr{G}$ be an infinite subgroup.

Consider a Cayley graph Γ of G such that the coned-off graph $\widehat{\Gamma}$ of Γ with respect to $\mathscr{G}$ is fine and hyperbolic. Denote by $\Delta\Gamma$ its Gromov compactification, endowed with the topology generated by the sets $\{M(x, A)\}_{x,A}$. We still identify G with $V(\Gamma) \subset V(\widehat{\Gamma})$.

Now denote by $c = [H] \in V(\widehat{\Gamma})$ the vertex associated with $[H] \in G/H$. This point is not in the boundary $\partial\Gamma$, but it is represented out of Γ, as in Figure 4.1.

We will show that for any neighbourhood V of c, the set $F := V^c \cap (G \setminus H)$ (Figure 4.2a) is H-roaming in the sense of Definition 4.1.1. Then we will show that if V is small enough (Figure 4.2b), F satisfies the condition of Proposition 4.1.3, hence proving Theorem 4.3.1.

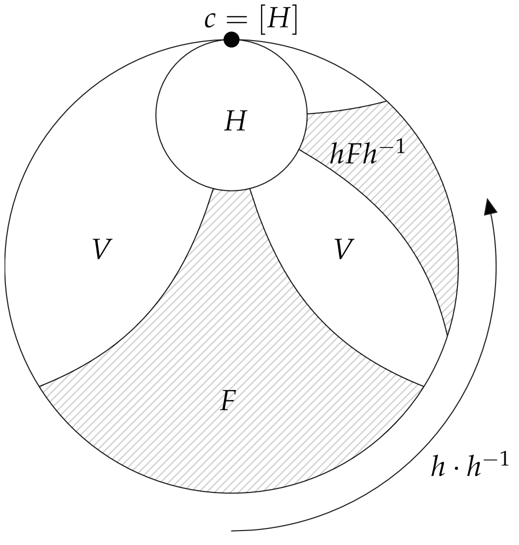
In this section, we will write V_r instead of $V_{r,c}$, $r \geq 0$ (see Lemma 4.1.11).

Remark that since c shares an edge with all the points in H (and only with them), any geodesic between c and a point $x \in \Delta\Gamma$ contains exactly one element in H. In particular one has the following simple lemma.

Lemma 4.3.2. *The family $\{M(c, A)\}_{A \subset H}$ is a basis of neighbourhoods of c.*

Proof. Let $B \subset V(\widehat{\Gamma})$ be a finite subset, for every $b \in B$ choose a geodesic α_b from c to b. Set $A := \{\alpha_b(1)\}_{b \in B}$ and observe that $M(c, A) \subset M(c, B)$. □

Remark 4.3.3. In the same way, if $A \subset H$ is finite and $r \geq 0$, the set $V_r(A)$ from Corollary 4.1.12 can be assumed to be contained in H. Indeed one can replace $V_r(A)$ by the finite set of points in H which lie on an r quasi-geodesic from $V_r(A)$ to c.



(a) Conjugacy action of H on G.

(b) The subsets V and sVt are disjoint.

Figure 4.2 – The action of G and a good neighborhood V of $c = [H]$.

To give an idea about the topology near c, let us mention that any sequence $(h_n)_n$ in H which goes to infinity converges to c.

As in the hyperbolic case, we will study geometrically the conjugacy action of H on G. We will treat left and right actions separately. First, the left multiplication of G on itself extends to an isometric action on $\widehat{\Gamma}$, and hence extends to a continuous action on $\Delta\Gamma$. Let us extend also the right action.

Definition 4.3.4. The *right action* of G on $\Delta\Gamma$ is the action whose restriction to G is equal to the right multiplication by G on itself, and which is trivial on $\Delta\Gamma \setminus G$. This action is *a priori* not continuous, and it clearly commutes with the left action.

The following lemma is contained in Proposition 12 of [Oza06], which actually shows the continuity of the right action on $\Delta\Gamma$.

Lemma 4.3.5. *The right action of G on $\Delta\Gamma$ is continuous at c.*

Proof. Let $g \in G$, and let (x_n) be a sequence converging to c. We want to prove that $(x_n g)$ converges to c. Since the right action is trivial on $\Delta\Gamma \setminus G$, we can assume that $x_n \in G$ for all n. Fix a finite set $A \subset H$. We will show that $x_n g \in M(c, A)$ for n large enough.

By Lemma 4.1.15, there is a finite set $C \subset V(\widehat{\Gamma}) \setminus \{c\}$ such that for all $y \in M(c, C)$ we have $M(y, C) \subset M(c, A)$. So if $y \in M(c, C)$ and $z \notin M(c, A)$, then there is a geodesic between y and z which intersects C.

Assume by contradiction that there are infinitely many indices n for which $x_n g \notin M(c, A)$. By assumption $x_n \in M(c, C)$ for n large enough, which implies that there is a geodesic $\alpha_n \in \mathcal{F}(x_n, x_n g)$ which intersects C for infinitely many n's. Then $x_n^{-1}\alpha_n$ belongs to $\mathcal{F}(e, g)$ and the set $X := \bigcup_{\alpha \in \mathcal{F}(e,g)} V(\alpha)$ is finite by Lemma 4.1.11(1). Altogether we get that $x_n^{-1} C \cap X \neq \emptyset$ for infinitely many n's. Taking a subsequence if necessary, we find an element $c' \in C$ and $x \in X$ such that $x_n^{-1} c' = x$ for all n.

This implies that $x_p^{-1}x_n \in \mathrm{Stab}_G(x)$ for all $p, n \in \mathbb{N}$. Since there are infinitely many distinct elements x_n, we get that x has to be a conic point, and for all fixed p, the sequence $(x_p^{-1}x_n)_n$ converges to x. But by continuity of the left action, the sequence also converges to $x_p^{-1}c$.

Therefore $c = x_p x = c'$. This contradicts our assumption that $c \notin C$. □

We now collect properties of left and right actions of H on $\Delta\Gamma$. Note that the left action of H stabilizes c (and $H = \mathrm{Stab}(c)$).

Lemma 4.3.6. *For any finite subsets $A, B \subset H$, there is $h \in H$ such that*

$$hM(c, A)^c \subset M(c, B).$$

Proof. By Remark 4.3.3, we may assume that $V_0(A) \subset H$. Let $h \in H$ be such that $hV_0(A) \cap B = \emptyset$. Let $x \in M(c, A)^c$ and let α be any geodesic between c and hx, $\alpha \in \mathcal{F}(c, hx)$. By Corollary 4.1.12(1), $h^{-1}\alpha \in \mathcal{F}(c, x)$ contains a point $a \in V_0(A)$. Thus ha is the unique point of H which is on α. In particular α contains no point of B. □

Lemma 4.3.7. *For any $A \subset V(\widehat{\Gamma})$ finite, there is a finite $B \subset V(\widehat{\Gamma})$ such that for any $h \in H$,*

$$(M(c, B) \cap (G \setminus H))h \subset M(c, A).$$

Proof. By Lemma 4.3.2, we can assume that $A \subset H$. Consider an element $x \in G \setminus H$ such that $x \notin M(c, A)$ and take $h \in H$. We will show that $xh \notin M(c, V_2(A))$.

Let α be a geodesic from c to x that meets A and put $a := \alpha(1) \in \alpha \cap A$. Note that since $xh \notin H$, we have $d(xh, c) \geq 2$, and at the same time $d(xh, x) \leq 2$, because xh and x lie in the same coset xH. Hence one can choose a projection z_0 of xh on α to be different from c. Thus the path from xh to c through z_0 constructed as in Definition 4.1.14 is a 2-quasi-geodesic and it contains $a = \alpha(1) \in A$. By Corollary 4.1.12(2), this implies that $xh \notin M(c, V_2(A))$. Thus $B := V_2(A)$ satisfies the conclusion of the lemma. □

As in the hyperbolic case, we deduce the following property of the conjugacy action of H on G, see Figure 4.2a.

Proposition 4.3.8. *For every $s, t \in G \setminus H$, there is an H-roaming set $F \subset G \setminus H$ such that $sF^c t \cap F^c$ is finite.*

Proof. We proceed as in Proposition 4.2.5. By continuity of left and right action at c (Lemma 4.3.5), there is a neighbourhood V of c such that V and sVt are disjoint (see Figure 4.2b). We observe that $sVt \cap H$, $sHt \cap V$ and $sHt \cap H$ are finite because the cluster point of H lies in V and the cluster point of sHt lies in sVt.

Therefore, the set $F := V^c \cap (G \setminus H)$ is such that $sF^c t \cap F^c$ is finite. One can deduce that this set is H-roaming from Lemma 4.3.7 and Lemma 4.3.6, as in Proposition 4.2.5. □

Now Theorem 4.3.1 follows from Proposition 4.1.3.

Remark 4.3.9. For later use, note that the set F in Proposition 4.3.8 can be chosen such that $sF^c t \cap F^c \subset H$ and hence $s(F \cup H)^c t \subset F \cup H$.

4.3.1 Hyperbolic groups relative to a family of amenable groups

We will now show how the following corollary can be deduced from Theorem 4.3.1.

Corollary 4.3.10. *Let G be a group which is hyperbolic relative to a family $\mathscr{G}$ of amenable subgroups and H be an infinite maximal amenable subgroup of G. Then the group von Neumann algebra LH is maximal amenable inside LG.*

Assume that G is hyperbolic relative to a family $\mathscr{G}$ of amenable subgroups, and consider an infinite maximal amenable subgroup $H < G$. We will show that G is hyperbolic relative to $\mathscr{G} \cup \{H\}$. Then Theorem 4.3.1 will directly allow to conclude that LH is maximal amenable inside LG.

The argument relies on Osin's work [Osi06b, Osi06a].

Definition 4.3.11. An element $g \in G$ is said to be *hyperbolic* if it has infinite order and is not contained in a conjugate of a group in $\mathscr{G}$.

Definition 4.3.12. A subgroup K of G is said to be *elementary* if it is either finite, or contained in a conjugate of a group in $\mathscr{G}$, or if it contains a finite index cyclic subgroup $\langle g \rangle$, for some hyperbolic element g.

The (Gromov-)Tukia's strong Tits alternative (see [Tuk94, Theorem 2T, Theorem 3A] using [Bow12, Definition 1]) states that a non-elementary subgroup K of G contains a copy of the free group on two generators.

In particular, our amenable subgroup H is elementary. If it is contained in a conjugate aH_ia^{-1} of a group in $\mathscr{G}$, then it is equal to aH_ia^{-1} by maximal amenability, and Theorem 4.3.1 gives the result.

Now assume that H contains a finite index cyclic subgroup $\langle g \rangle$, for some hyperbolic element g. Osin showed in [Osi06a, Section 3] that such a hyperbolic element g is contained in a unique maximal elementary subgroup $E(g)$ (thus $H = E(g)$, by maximal amenability). Moreover he showed [Osi06a, Corollary 1.7] that G is hyperbolic relative to $\mathscr{G} \cup \{E(g)\}$. This is what we wanted to show.

4.4 Product case

We will now prove the following theorem.

Theorem 4.4.1. *Let $n \geq 1$, and consider for all $i = 1, \dots, n$ an inclusion of groups $H_i < G_i$ as in Theorem 13. Put $G := G_1 \times \cdots \times G_n$ and $H := H_1 \times \cdots \times H_n$.*

Then for any trace-preserving action of G on a finite amenable von Neumann algebra (Q, τ), the crossed-product $Q \rtimes H$ is maximal amenable inside $Q \rtimes G$.

Observe that if $H_i < G_i$, for $i = 1, 2$, are infinite maximal amenable subgroups, then the von Neumann subalgebra $L(H_1 \times H_2) < L(G_1 \times G_2)$ is neither maximal Gamma nor mixing as soon as $H_1 \neq G_1$.

Therefore to treat the product case, we will have to deal with relative notions. We could consider a relative notion of property Gamma and proceed as in Section 4.1.1. We choose instead to apply directly the work of C. Houdayer and the *relative asymptotic orthogonality property*, [Hou14b]. Note that in the case of virtually abelian subgroups H_1, H_2 we could also use [CFRW10, Theorem 2.8].

Definition 4.4.2. Let $A \subset N \subset (M, \tau)$ be finite von Neumann algebras. The inclusion $N \subset M$ is said to be *weakly mixing through A* if there is a sequence of unitaries $(v_n)_n \subset \mathcal{U}(A)$ such that

$$\lim_n \|E_N(xv_ny)\|_2 = 0, \ \forall x, y \in M \ominus N.$$

Example 4.4.3. If $H < G$ is an inclusion of groups satisfying the assumption of Proposition 4.1.3 (*e.g.* if H and G are as in Theorem 4.3.1), then for any trace-preserving action $G \curvearrowright (Q, \tau)$ on a finite von Neumann algebra, the inclusion $Q \rtimes H \subset Q \rtimes G$ is weakly mixing through LH. The proof is the same as the proof of Proposition 4.1.4.

Definition 4.4.4 ([Hou14b], Definition 5.1)**.** Let $A \subset N \subset (M, \tau)$ be an inclusion of finite von Neumann algebras. We say that $N \subset M$ has the *asymptotic orthogonality property relative to A* if for every $\| \cdot \|_\infty$-bounded sequences $(x_n)_n$ and $(y_n)_n$ in $M \ominus N$ which asymptotically commute with A in the $\| \cdot \|_2$-norm, we have that

$$\lim_n \langle ax_nb, y_n \rangle = 0, \text{ for all } a, b \in M \ominus N.$$

Theorem 4.4.5 ([Hou14b], Theorem 8.1)**.** *Let $A \subset N \subset (M, \tau)$ be an inclusion of finite von Neumann algebras. Assume the following:*

1. *A is amenable.*
2. *The inclusion $N \subset M$ is weakly mixing through A.*
3. *The inclusion $N \subset M$ has the relative asymptotic orthogonality property relative to A.*

Then any amenable von Neumann subalgebra of M containing A is automatically contained in N.

From now on, we consider the crossed-product von Neumann algebras $Q \rtimes G$ associated to a trace-preserving actions $G \curvearrowright (Q, \tau)$. As for group von Neumann algebras, denote by u_g the unitaries of $Q \rtimes G$ corresponding to elements $g \in G$ and for any set $F \subset G$ denote by $P_F : L^2(Q, \tau) \otimes \ell^2(G) \to L^2(Q, \tau) \otimes \ell^2(F)$ the orthogonal projection.

Proposition 4.4.6. *Let $H < G$ be an inclusion of two infinite groups, with H amenable. Consider an action $G \curvearrowright (Q, \tau)$ of G on a tracial von Neumann algebra, and assume that for any $s, t \in G \setminus H$, there is an H-roaming set $F \subset G \setminus H$ such that $s(F \cup H)^c t \subset F \cup H$.*

Then the inclusion $Q \rtimes H \subset Q \rtimes G$ has the asymptotic orthogonality property relative to LH.

Proof. Consider two $\| \cdot \|_\infty$-bounded sequences $(x_n)_n$ and $(y_n)_n$ in $(Q \rtimes G) \ominus (Q \rtimes H)$ which asymptotically commute with LH. By linearity and density it is sufficient to check that for any $s, t \notin H$,

$$\lim_n \langle u_s x_n u_t, y_n \rangle = 0.$$

Fix $s, t \in G \setminus H$. There exists an H-roaming set F such that $s(F \cup H)^c t \subset F \cup H$. Proceeding as in the proof of Lemma 4.1.2, it is easy to show that $\lim_n \|P_F(x_n)\|_2 = \lim_n \|P_F(y_n)\|_2 = 0$. Note also that for all n, we have $x_n = P_{H^c}(x_n)$ and $y_n = P_{H^c}(y_n)$. Therefore

$$\begin{aligned} \lim_n \langle u_s x_n u_t, y_n \rangle &= \lim_n \langle u_s P_{F^c}(x_n) u_t, P_{F^c}(y_n) \rangle \\ &= \lim_n \langle u_s P_{(F \cup H)^c}(x_n) u_t, P_{(F \cup H)^c}(y_n) \rangle = 0, \end{aligned}$$

because $s(F \cup H)^c t \subset F \cup H$. This ends the proof of the proposition. □

Proof of Theorem 4.4.1. For $i = 1, \dots, n$, let G_i be a hyperbolic group relative to a family $\mathscr{G}_i$ of subgroups and let $H_i \in \mathscr{G}_i$ be an infinite amenable group. Consider the inclusion

$$H := H_1 \times \cdots \times H_n < G := G_1 \times \cdots \times G_n.$$

Let (Q, τ) be a finite amenable von Neumann algebra and consider a trace-preserving action $G \curvearrowright (Q, \tau)$ of G. Put $N := Q \rtimes H$ and $M := Q \rtimes G$.

Assume that P is an intermediate amenable von Neumann subalgebra: $N \subset P \subset M$. We have to show that $P = N$. In order to do so, we will show that for all $i = 1, \dots, n$, we have

$$P \subset N_i := Q \rtimes (G_1 \times \cdots \times G_{i-1} \times H_i \times G_{i+1} \times \cdots \times G_n).$$

This is enough to conclude, because $N = \cap_{i=1}^n N_i$.

For $i \in \{1, \dots, n\}$, we set $A_i := LH_i$ and $Q_i := Q \rtimes \widehat{G}_i$, where $\widehat{G}_i$ is the direct product of all G_j, $j \neq i$. Then we have $N_i \simeq Q_i \rtimes H_i$ and $M \simeq Q_i \rtimes G_i$.

By Proposition 4.3.8 (and Remark 4.3.9), we see that $H_i \subset G_i$ satisfies the assumptions of Proposition 4.4.6 so that $N_i \subset M$ has the asymptotic orthogonality property relative to A_i. Moreover the Example 4.4.3 tells us that $N_i \subset M$ is (weakly) mixing through A_i.

By Theorem 4.4.5, one concludes that the amenable algebra P, which contains A_i, is contained in N_i. This ends the proof of Theorem 4.4.1. □

Chapter 5

Maximal amenable von Neumann subalgebras arising from maximal amenable subgroups

The following chapter is based on a joint work with Rémi Boutonnet.

We provide a general criterion to deduce maximal amenability of von Neumann subalgebras $L\Lambda \subset L\Gamma$ arising from amenable subgroups Λ of discrete countable groups Γ. The criterion is expressed in terms of Λ-invariant measures on some compact Γ-space. The strategy of proof is different from Popa's approach to maximal amenability via central sequences [Pop83], and relies on elementary computations in a crossed-product C*-algebra.

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5.1 Singular subgroups

In this chapter we will use the same notations and conventions of the previous chapter. We will however use Λ and Γ for discrete groups and we will use G for Lie groups.

Definition 5.1.1. Consider an amenable subgroup Λ of a discrete countable group Γ. Suppose that Γ acts continuously on the compact space X. We say that Λ is *singular* in Γ (with respect to X) if for any $\mu \in \mathrm{Prob}_\Lambda(X)$ and $g \in \Gamma \setminus \Lambda$, we have $g \cdot \mu \perp \mu$.

To illustrate this definition, let us mention the following trivial observation.

Lemma 5.1.2. *Consider a subgroup Λ of a discrete countable group Γ. Then Λ is maximal amenable inside Γ if and only if there is a continuous action $\Gamma \curvearrowright X$ on a compact space X such that for any $\mu \in \mathrm{Prob}_\Lambda(X)$ and $g \in \Gamma \setminus \Lambda$, we have $g \cdot \mu \neq \mu$.*

Proof. If such a space X exists, then Λ is clearly maximal amenable.

Conversely, assume that Λ is maximal amenable. For any $g \in \Gamma \setminus \Lambda$, the group $\langle \Lambda, g\rangle$ is not amenable: there is a compact Γ-space X_g such that $\mathrm{Prob}_{\langle \Lambda, g\rangle}(X_g) = \emptyset$. Replacing X_g by a minimal Γ-invariant subset if necessary, we can assume that X_g is a quotient of the Stone-Czech compactification $\beta\Gamma$ of Γ.

Then we see that $X = \beta\Gamma$ does the job: if $\mu \in \mathrm{Prob}_\Lambda(X)$ then its push forward on X_g is Λ-invariant, so it is not g-invariant by definition of X_g. Therefore μ is not g-invariant. $\square$

Remark 5.1.3. Observe that if Λ is singular in Γ with respect to the compact space X, then it is singular with respect to any closed Γ-invariant subset of X. Hence if Λ is singular in Γ, it is also singular with respect to a minimal compact space. Therefore, arguing as in the above proof, we can see that an amenable group Λ is singular inside Γ with respect to some compact Γ-space X if and only if it is singular with respect to the Stone-Czech compactification $\beta\Gamma$ of Γ.

Theorem 5.1.4. *Suppose that Γ is a discrete countable group admitting an amenable, singular subgroup Λ. Then for any trace preserving action $\Gamma \curvearrowright (Q, \tau)$ on a finite amenable von Neumann algebra, $Q \rtimes \Lambda$ is maximal amenable inside $Q \rtimes \Gamma$.*

Proof. We will denote by $M := Q \rtimes \Gamma \subset B(L^2(Q) \otimes \ell^2\Gamma)$ and by $N := Q \rtimes \Lambda$. Consider an intermediate amenable von Neumann algebra $N \subset A \subset M$. We will show that $A = N$.

Suppose that Λ is singular in Γ. Since A is amenable, there is an A-central state $\varphi : B(L^2(Q) \otimes \ell^2\Gamma) \to \mathbb{C}$ whose restriction to M coincides with the trace.

Take a unitary $u \in \mathcal{U}(A)$. Using the fact that u centralizes the state φ, we will show that $u \in N$.

Fix $\varepsilon > 0$. We denote by $\{v_g\}_{g\in\Gamma}$ the canonical unitaries implementing the action. Then by density, one can find $u_0 \in M$ of the form $u_0 = \sum_{g\in F} a_g v_g$, with $F \subset \Gamma$ finite and non-zero elements $a_g \in Q$ for all $g \in F$, such that $\|u^* - u_0\|_2 < \varepsilon$.

We observe that $1 \otimes \ell^\infty(\Gamma) \simeq C(\beta\Gamma)$. With this identification, any element $f \in C(\beta\Gamma)$ commutes with Q, and we have that $v_g f v_g^* = \sigma_g(f)$ for all $g \in \Gamma$, where we denote by σ_g the action induced by the canonical action of Γ on $\beta\Gamma$. Since φ is Λ-central, its restriction to $C(\beta\Gamma)$ is given by a Λ-invariant regular Borel probability measure μ on $\beta\Gamma$ such that $\varphi(f) = \int_{\beta\Gamma} f d\mu$, for all $f \in C(\beta\Gamma)$. By hypothesis Λ is singular in Γ and hence by Remark 5.1.3, it is singular with respect to $\beta\Gamma$, hence for all $g \in F \setminus \Lambda$, the measures μ and $(g^{-1} \cdot \mu)$ are singular with respect to each other.

So there are a compact set $K \subset \beta\Gamma$ and an open set $V \subset \beta\Gamma$ containing K such that:

- $\mu(K) > 1 - \varepsilon$;
- $\mu(gV) < (\varepsilon/(|F|\|a_g\|_\infty))^2$, for all $g \in F \setminus \Lambda$.

Urysohn's Lemma gives us a continuous function $f \in C(\beta\Gamma)$ supported on V such that $0 \leq f \leq 1$ and $f = 1$ on K.

On the one hand, since φ is u-central, one derives

$$\begin{aligned}|\varphi(u_0 f u)| &= |\varphi(u u_0 f)| \\ &\geq |\varphi(f)| - |\varphi((u u_0 - 1) f)| \\ &\geq \mu(K) - \|u_0 - u^*\|_2 \\ &> 1 - 2\varepsilon.\end{aligned}$$

For the third line above we used Cauchy-Schwarz inequality and the fact that $\varphi_{|M} = \tau$.

On the other hand, one computes

$$\begin{aligned}|\varphi(u_0 f u)| &\leq \left|\sum_{g \in F \cap \Lambda} \varphi(a_g v_g f u)\right| + \sum_{g \in F \setminus \Lambda} |\varphi(a_g \sigma_g(f) v_g u)| \\ &\leq |\varphi(E_N(u_0) f u)| + \sum_{g \in F \setminus \Lambda} \varphi(a_g \sigma_g(f f^*) a_g^*)^{1/2} \\ &\leq \|E_N(u_0)\|_2 + \sum_{g \in F \setminus \Lambda} \|a_g\|_\infty \mu(gV)^{1/2} \\ &< (\|E_N(u)\|_2 + \varepsilon) + \varepsilon.\end{aligned}$$

Altogether, we get

$$1 - 2\varepsilon < |\varphi(u_0 f u)| < \|E_N(u)\|_2 + 2\varepsilon,$$

so that $\|E_N(u)\|_2 \geq 1 - 4\varepsilon$. Since ε was arbitrary, this implies that $u \in N$. □

Remark 5.1.5. The above proof actually shows that any unitary $u \in L\Gamma$ which centralizes the C*-algebra generated by $Q \rtimes_r \Gamma$ and $1 \otimes \ell^\infty(\Gamma)$ has to be contained in N. Note that this C*-algebra is isomorphic to the crossed-product $(Q \otimes_r C(\beta\Gamma)) \rtimes_r \Gamma$ (where Γ acts diagonally on $Q \otimes_r C(\beta\Gamma)$). We will present several applications of this point of view in Section 5.3.

Before actually providing examples of singular subgroups, let us prove Ozawa's characterization of singular subgroups.

Theorem 5.1.6 (Ozawa). *Consider an amenable subgroup Λ of a discrete countable group Γ. The following are equivalent.*

(1) Λ is a singular subgroup of Γ;

(2) Every Λ-invariant state on $C_r^*(\Gamma)$ vanishes on $\lambda(\Gamma \setminus \Lambda)$;

(3) For every $g \in \Gamma \setminus \Lambda$, we have that $0 \in \overline{\mathrm{conv}}^{\|\cdot\|}(\{\lambda(tgt^{-1}), t \in \Lambda\}) \subset B(\ell^2\Gamma)$;

(4) For any net (ξ_n) of almost Λ-invariant unit vectors in $\ell^2\Gamma$ and all $g \in \Gamma \setminus \Lambda$, the inner product $\langle \lambda_g \xi_n, \xi_n \rangle$ goes to 0.

Proof. (1) $\Rightarrow$ (2). Assume that Λ is singular inside Γ and take a Λ-invariant state φ on $C_r^*(\Gamma)$. Fix $g \in G \setminus \Lambda$ and $\varepsilon > 0$. Since Λ is amenable there is a Λ-invariant state (also denoted by φ) on $B(\ell^2\Gamma)$ which extends φ. The restriction of φ to $\ell^\infty(\Gamma)$ is a Λ-invariant state. By singularity, proceeding as in the proof of Theorem 5.1.4, we find $f \in \ell^\infty(\Gamma)$ such that $0 \leq f \leq 1$ and $\varphi(1-f) \leq \varepsilon^2$ while $\varphi(g \cdot f) \leq \varepsilon^2$. Using Cauchy-Schwarz inequality, we get

$$|\varphi(\lambda_g)| \leq |\varphi(\lambda_g f)| + \varepsilon \leq \varphi(\lambda_g f \lambda_g^*)^{1/2} \varphi(f)^{1/2} + \varepsilon \leq 2\varepsilon.$$

As ε was arbitrary, we indeed see that $\varphi(\lambda_g) = 0$.

(2) $\Rightarrow$ (3). Assume that there is a $g \in \Gamma \setminus \Lambda$ for which the conclusion of (3) does not hold. By Hahn–Banach, there is $\varphi \in C_r^*(\Gamma)^*$ such that $\inf_{t\in\Lambda} \Re(\varphi(\lambda_{tgt^{-1}})) > 0$. By taking an average over $\{\varphi \mathbb{S}^1 \operatorname{Ad}(\lambda_t)\,,\, t \in \Lambda\}$, we may assume that φ is Λ-invariant. Jordan decomposition now gives a Λ-invariant state on $C_r^*(\Gamma)$ such that $\varphi(g) \neq 0$. This violates (3).

(3) $\Rightarrow$ (4). Take a net (ξ_n) of almost Λ-invariant unit vectors in $\ell^2\Gamma$ and $g \in \Gamma \setminus \Lambda$. Fix $\varepsilon > 0$. Assuming (3), we find $x \in \operatorname{conv}(\{\lambda(tgt^{-1})\,,\, t \in \Lambda\})$ such that $\|x\| \leq \varepsilon$. By almost Λ-invariance, we get

$$\limsup_n |\langle \lambda_g \xi_n, \xi_n \rangle| = \limsup_n |\langle x\xi_n, \xi_n \rangle| \leq \varepsilon.$$

(4) $\Rightarrow$ (1). Suppose that Λ is not singular inside Γ. Then we get a Λ-invariant state φ on $\ell^\infty(\Gamma)$ and $g \in \Gamma \setminus \Lambda$ such that φ is not singular with respect to $g \cdot \varphi$. Equivalently, this means that $\|\varphi - g \cdot \varphi\|_1 < 2$. Approximating φ with normal states and using Hahn-Banach Theorem, we find a net of positive, norm one elements $\eta_n \in \ell^1(\Gamma)$ which is asymptotically Λ-invariant and satisfies $\|\eta_n - g \cdot \eta_n\|_1 \to \|\varphi - g \cdot \varphi\|_1 < 2$.

Define now $\xi_n = \eta_n^{1/2} \in \ell^2(\Gamma)$. Then these unit vectors are asymptotically Λ-invariant and yet we have the following fact showing that (4) does not hold.

$$\langle \lambda_g \xi_n, \xi_n \rangle = \frac{1}{2}(2 - \|\xi_n - \lambda_g \xi_n\|_2^2) \geq \frac{1}{2}(2 - \|\eta_n - g \cdot \eta_n\|_1) \not\to 0. \qquad \square$$

5.2 Examples

As a first application of our criterion, we observe that Theorem 4.2.1 is now a direct consequence of Lemma 4.2.2. Note that one can also recover the results about *relatively* hyperbolic groups.

5.2.1 Amalgamated free products and HNN extensions

Using Bass-Serre theory, our criterion also applies to amalgamated free products.

Proposition 5.2.1. *Let Λ_1 and Λ_2 be discrete groups (not necessarily finitely generated) with a common subgroup Λ_0. Put $\Gamma := \Lambda_1 *_{\Lambda_0} \Lambda_2$. If Λ_1 is amenable and the index $[\Lambda_1 : \Lambda_0] = \infty$ then Λ_1 is singular in Γ. In particular $L\Lambda_1$ is maximal amenable inside $L\Gamma$.*

Proof. Let us first construct the compact Γ-space X for which we will verify the singularity property of $\Lambda_1 < \Gamma$. Assume that Γ is as in the statement of Proposition 5.2.1 and consider the Bass-Serre tree T of Γ. By definition the vertex set of T equals to $V(T) := \Gamma/\Lambda_1 \sqcup \Gamma/\Lambda_2$ and its edge set equals to $E(T) := \Gamma/\Lambda_0$, where the edge $g\Lambda_0$ relates $g\Lambda_1$ to $g\Lambda_2$. By assumption the vertex Λ_1 has infinitely many neighbours. In particular this tree is not locally finite. However every tree is by definition a uniformly fine hyperbolic graph in the sense of [Bow12, Section 8], so one can still consider its visual boundary ∂T and define a compact topology on $X := V(T) \cup \partial T$ as in the previous chapter. Let us recall the notations.

For $x, y \in X$ denote by $[x, y]$ the unique geodesic path between x and y. If $x \in X$ and $A \subset V(T)$ is finite set of vertices, define

$$M(x, A) := \{y \in X \mid [x, y] \cap A = \emptyset\}.$$

Then the family of sets $M(x, A)$ with $x \in X$, $A \subset V(T)$ finite, forms an open basis of a compact (Hausdorff) topology on X. Note that the action $\Gamma \curvearrowright X$ is continuous for this topology.

To prove Proposition 5.2.1, it is enough to show that the only Λ_1-invariant probability measure on X is the Dirac measure δ_{Λ_1}.

To that aim, assume that μ is a Λ_1-invariant probability measure on X. Note that since Λ_0 has infinite index in Λ_1, the vertex Λ_1 has infinitely many neighbors $\{g\Lambda_2\}_{g \in R}$, where $R \subset \Lambda$ is a section for the onto map $\Lambda_1 \to \Lambda_1/\Lambda_0$. Since T is a tree, the open sets $\{M(g\Lambda_2, \{\Lambda_1\})\}_{g \in R}$ are disjoint and moreover Λ_1 acts transitively on these open sets. Hence they must have measure 0 and therefore the probability measure μ has to be supported on their complement, namely $\{\Lambda_1\}$. □

With the same proof we also get the following result.

Proposition 5.2.2. *Assume that $\Gamma = \mathrm{HNN}(\Lambda, \Lambda_0, \theta)$ is an HNN extension, where $\Lambda_0 < \Lambda$ and $\theta : \Lambda_0 \hookrightarrow \Lambda$ is an injective morphism. If Λ is amenable and $[\Lambda : \Lambda_0] = [\Lambda : \theta(\Lambda_0)] = \infty$ then Λ is singular in Γ.*

Sketch of the proof. Assume that $\Gamma = \langle \Lambda, t | t^{-1}gt = \theta(g) \text{ for } g \in \Lambda_0 \rangle$. As before, we will show that Λ is singular in Γ with respect to the compactification of the Bass-Serre tree T of Λ. Let us describe the Bass-Serre tree, by definition the vertex set of T equals to $V(T) := \Gamma/\Lambda$ and the vertex set equals to $E(T) := \Gamma/\Lambda_0$, where the edge $g\Lambda_0$ connects $g\Lambda$ to $gt\Lambda$. The vertex Λ has for neighbors all the vertices in one of the two families:

- The collection $\{gt\Lambda\}_{g \in R_1}$, where $R_1 \subset \Lambda$ is a section for the onto map $\Lambda \to \Lambda/\Lambda_0$;
- The collection $\{gt^{-1}\Lambda\}_{g \in R_2}$, where $R_2 \subset \Lambda$ is a section for the map $\Lambda \to \Lambda/\theta(\Lambda_0)$.

These two families are infinite by our index assumptions. Moreover, Λ acts transitively on each of these two families. So one can proceed exactly as in the previous proposition to deduce that any Λ-invariant probability measure on the compactification has to be supported on the vertex Λ. □

In the finite index setting the result is false in general and the condition of Theorem 5.1.4 is never satisfied. For instance assume that $\Gamma = \mathrm{BS}(m, n) = \langle a, t \mid ta^n t^{-1} = a^m \rangle$ with $m, n \geq 2$, and that $\Lambda = \langle a \rangle$. Then the conjugacy action of Λ on $\Gamma \setminus \Lambda$ admits a finite orbit. Namely, tat^{-1} has an orbit with m elements, and so the element $x := \sum_{k=0}^{m-1} a^k tat^{-1} a^{-k} \in \mathbb{C}\Gamma$ commutes with $L\Lambda$. In this case, $L\Lambda$ is not even maximal abelian in $L\Gamma$. However, one can check using Lemma 5.1.2 that Λ is maximal amenable inside Γ as soon as $|m|, |n| \geq 3$ (but it is not true for $|n| = 2$ or $|m| = 2$).

5.2.2 Lattices in semi-simple groups

Finally, our criterion also allows to produce examples of a different kind, out of the (relatively)-hyperbolic world.

Proposition 5.2.3. *For $n \geq 2$, put $\Gamma := \mathrm{SL}_n(\mathbb{Z})$ and denote by Λ the subgroup of upper triangular matrices in Γ. Then Λ is singular in Γ. Moreover, $L\Lambda$ has a diffuse center.*

Proof. Put $G = \mathrm{SL}_n(\mathbb{R})$ and denote by $P < G$ be the subgroup of upper triangular matrices, so that $\Lambda = \Gamma \cap P$. We will show that $\Lambda < \Gamma$ is singular with respect to the action on the homogeneous space $B = G/P$. It is enough to prove that the unique Λ-invariant probability measure on B is the Dirac mass on $[P]$. Fix $\mu \in \mathrm{Prob}_\Lambda(B)$.

Denote by $N < P$ the subgroup of unipotent matrices and put $\Lambda_0 := \Gamma \cap N$. Then [Moo79, Proposition 2.6] implies that the support of μ is pointwise fixed by the Zariski closure of Λ_0, namely N. So we are left to check that N has only one fixed point on B. Note that a point $g[P] \in B$ is fixed by N if and only if $g^{-1}Ng \subset P$. So let us take $g \in G$ such that $g^{-1}Ng \subset P$ and show that $g \in P$.

For a matrix $h \in G$, we denote with $\sigma(h)$ the spectrum of h. Observe that given $p \in P$ we have

$$p \in N \text{ if and only if } \sigma(p) = \{1\}.$$

Since the spectrum is conjugacy invariant and $g^{-1}ng \in P$ for all $n \in N$, we have that $g^{-1}ng \in N$. Hence $g^{-1}Ng \subset N$, and this is even an equality because the nilpotent groups N and $g^{-1}Ng$ have the same dimension. But a simple induction shows that the normalizer of N in G is P, so $g \in P$, as wanted.

For the moreover part, denote by I the identity matrix and by $E_{1,n}$ the matrix with 0 entries except for the entry row 1/column n which is equal to 1. A simple calculation shows that the Λ-conjugacy class of $I + E_{1,n}$ is contained in $\{I \pm E_{1,n}\}$. Therefore the center of $L\Lambda$ contains the element $u + u^*$, where u is the unitary in $L\Lambda$ corresponding to the element $I + E_{1,n} \in \Lambda$. Note that $I + E_{1,n}$ has infinite order, so u generates a copy of $L\mathbb{Z}$. Finally $u + u^*$ generates a subalgebra of index 2, which implies that $L\Lambda$ has diffuse center. $\square$

Of course, the example given in the above proposition is not abelian (unless $n = 2$). We now turn to the question of existence of abelian, maximal amenable subalgebras in von Neumann algebras associated with lattices in semi-simple Lie groups.

Proposition 5.2.4. *Consider a lattice Γ in a connected semi-simple real algebraic Lie group G with finite center. Then there is a virtually abelian subgroup Λ in Γ which is singular in Γ.*

Proof. Before starting the proof, let us fix some notation. Denote with d the real rank of G and let $G = KAN$ be an Iwasawa decomposition of G, so that K is a maximal compact subgroup, $A \cong \mathbb{R}^d$ and N is nilpotent. Denote with M the centralizer of A in K. By [PR72, Theorem 2.8], replacing Γ by one of its conjugates if necessary, there is an abelian subgroup $H \subset MA$ (a so-called *Cartan subgroup*) such that $H \cap \Gamma$ is cocompact in H. Moreover H contains A, so it is co-compact in MA. Therefore $\Lambda_0 := MA \cap \Gamma$ is a co-compact lattice in MA and it contains the abelian subgroup $\Gamma \cap H$ as a finite index subgroup.

Let $P = MAN$ be a minimal parabolic subgroup. We will show that the normalizer $\Lambda := N_\Gamma(\Lambda_0)$ is singular in Γ with respect to the action $\Gamma \curvearrowright G/P$. Consider a measure $\mu \in \mathrm{Prob}_\Lambda(G/P)$.

Claim 1. μ is supported on the set $F := \{x \in G/P \mid ax = x, \forall a \in A\}$.

The measure μ is Λ_0-invariant. Put $\tilde{\mu} := \int_M (g \cdot \mu) dg$, where dg denotes the Haar probability measure on M. Given an element $ma \in \Lambda_0$, with $m \in M$, $a \in A$, we see that

$$a \cdot \tilde{\mu} = \int_M (ag \cdot \mu) dg = \int_M (ga \cdot \mu) dg = \int (gma \cdot \mu) dg = \tilde{\mu}$$

Hence $\tilde{\mu}$ is invariant under the projection of Λ_0 on A. But this projection is a lattice in A. Applying [Moo79, Proposition 2.6], we deduce that $\tilde{\mu}$ is supported on F. Since M commutes with A, the set F is globally M-invariant: $g \cdot \mu(F) = \mu(F)$ for all $g \in M$. Hence $1 = \tilde{\mu}(F) = \mu(F)$, as claimed.

Claim 2. For all $x \in F$, we have $\mathrm{Stab}_G(x) \cap \Gamma = \Lambda_0$.

To prove this claim take $x \in F$, written $x = gP \in G/P$. Note that $\mathrm{Stab}_G(x) = gPg^{-1}$, so that $A \subset gPg^{-1}$. By [BT65, Theorem 4.15], it follows that $gPg^{-1} = MAgNg^{-1}$. In particular MA (and Λ_0) fixes x.

Take now $\gamma \in \Gamma \cap \mathrm{Stab}_G(x) = \Gamma \cap gPg^{-1}$. Write $\gamma = man$ with $m \in M$, $a \in A$ and $n \in gNg^{-1}$. By [Zim84, Proposition 8.2.4] there is a sequence $(b_k)_k$ in MA such that $b_k n b_k^{-1}$ converges to the identity element 1_G. Since Λ_0 is a uniform lattice in MA, there is a subsequence $(b_{k_j})_j$ and a sequence $(c_j)_j \subset \Lambda_0$ such that $b_{k_j} c_j^{-1}$ converges in MA. It is easy to conclude that $c_j n c_j^{-1}$ converges to the identity. Now, $c_j \gamma c_j^{-1}$ lies in Γ and we have

$$c_j \gamma c_j^{-1} = c_j (man) c_j^{-1} = (c_j m c_j^{-1}) a (c_j n c_j^{-1}), \text{ for all } j.$$

But $c_j m c_j^{-1}$ belongs to the compact set M, so taking a subsequence if necessary, we see that $c_j \gamma c_j^{-1}$ converges to an element in MA. By discreteness of Γ, this implies that $c_j \gamma c_j^{-1} \in MA$ for j large enough. Therefore $\gamma \in MA \cap \Gamma = \Lambda_0$ which proves Claim 2.

To prove that Λ is singular in Γ, consider an element $g \in \Gamma$ such that $g \cdot \mu$ is not singular with respect to μ. Then $g \cdot F \cap F \neq \emptyset$, so there are two points $x, y \in F$ such that $y = gx$. This implies that $g\Lambda_0 g^{-1}$ fixes y, while $g^{-1}\Lambda_0 g$ fixes x. From Claim 2 we deduce that g normalizes Λ_0, so that Λ is indeed singular in Γ.

To complete the proof of the proposition, it remains to show that Λ_0 has finite index inside Λ, which will ensure that Λ is virtually abelian.

Assume that $g \in \Gamma$ normalizes Λ_0. Since A lies in the Zariski closure of Λ_0, we have $gAg^{-1} \subset MA$. But MA has a unique maximal $\mathbb{R}$-split torus, A. So $gAg^{-1} = A$ and g normalizes A. Moreover MA coincides with the centralizer $Z_G(A)$ of A in G. Now we have only to observe that the Weyl group $N_G(A)/Z_G(A)$ is finite, see [Kna02, Section VII.7, item 7.84] for instance. $\square$

Remark 5.2.5. For $\mathrm{SL}_3(\mathbb{Z})$, let Λ_0 be the group generated by the following two commuting matrices:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -16 \\ 0 & 1 & 8 \end{pmatrix} \text{ and } \begin{pmatrix} 81 & 4 & -4 \\ -36 & 17 & 68 \\ 4 & -4 & -15 \end{pmatrix}.$$

Then Λ_0 has finite index in a singular subgroup of $\mathrm{SL}_3(\mathbb{Z})$. We do not know whether Λ_0 itself is singular in $\mathrm{SL}_3(\mathbb{Z})$.

Remark 5.2.6. We remark here that whenever Γ is a co-compact and torsion free lattice of a real algebraic group G without compact factors, there is a free abelian subgroup of Γ that is singular in Γ. In fact the authors in [RS10] proved that under these hypothesis, the Cartan subgroup constructed in [PR03] $H \subset G$ is such that $\Lambda_0 := \Gamma \cap H$ is isomorphic to $\mathbb{Z}^{\mathrm{rk}_{\mathbb{R}}(G)}$ and Λ_0 is malnormal in Γ. So we can use this Cartan subgroup in the proof of the above proposition to get that Λ_0 has finite index in a singular subgroup of Γ and since Λ_0 is malnormal in Γ, then we must have that Λ_0 is singular in Γ.

Question 5.2.7. If Γ is a lattice of a simple Lie group, is it true that every maximal amenable subgroup of Γ is singular in Γ with respect to G/P?

Note that we showed that it is the case for the two extreme cases:

- if the maximal amenable subgroup is a lattice of a parabolic subgroups, (we proved this fact only for $\mathrm{SL}_n(\mathbb{Z})$ but the same proof works in the general setting),
- if the maximal amenable subgroup is a lattice of a Cartan subgroup.

5.3 Amenable subalgebras as stabilizers of measures on some compact space

As explained in the introduction, the key of the above results is to view maximal amenable subalgebras of a group von Neumann algebra $L\Gamma$ as centralizers of states on some reduced crossed-product C*-algebra $C(X) \rtimes_r \Gamma$. In this section we further develop this point of view and explain its link with more theoretical questions. What follows is largely inspired from the work of N. Ozawa on solidity [Oza04, Oza10].

Let us fix some notations. For a von Neumann algebra M, we denote by $J : L^2(M) \to L^2(M)$, the canonical anti-unitary that extends the map $x \in M \mapsto x^* \in M$. Note that if Γ is a countable group, then we have that $R\Gamma = JL\Gamma J$. When we want to view an element x of M inside $L^2(M)$, we will write $\widehat{x}$.

The following proposition is the main ingredient. Our initial argument for (iii) relied on [Oza04, Lemma 5] and used exactness of the group in a redundant way. We are grateful to S. Vaes for suggesting to us a cleaner approach and to one of the referees (of the submited paper) for emphasizing and correcting a gap in an earlier version.

Proposition 5.3.1. *Assume that Γ is a countable discrete group which acts continuously on a compact space X. Denote by $B := C(X) \rtimes_r \Gamma$ the reduced crossed-product C^*-algebra. Consider a state φ on B which coincides on $C_r^*(\Gamma)$ with the canonical trace τ. The following are true.*

(i) *Given $x \in L\Gamma$ and $T \in B$, for every bounded sequence $(x_n)_n$ in $C_r^*(\Gamma) \subset B$ which converges strongly to x, the sequence $(\varphi(x_nT))_n$ converges and the limit depends only on x and T. Therefore one can define $\varphi(xT) = \lim_n \varphi(x_nT)$ and similarly $\varphi(Tx)$.*

(ii) *The set $A_\varphi := \{x \in L\Gamma \mid \varphi(xT) = \varphi(Tx), \forall T \in B\}$ is a von Neumann subalgebra of $L\Gamma$.*

(iii) *If the action is topologically amenable in the sense of [AD87] (see also [BO08, Section 4.3] for more on this), then A_φ is amenable. Of course, every maximal amenable subalgebra of $L\Gamma$ arises this way.*

Proof. Before proceeding to the proof of (i)-(iii), let us fix some notations. Denote by $(\pi_\varphi, H_\varphi, \xi_\varphi)$ the GNS triplet associated with B and φ. Extend the state φ to a normal state $\tilde{\varphi}$ on $\tilde{B} := \pi_\varphi(B)''$ by the formula $\tilde{\varphi}(x) = \langle x\xi_\varphi, \xi_\varphi\rangle$. Denote also by $\tilde{C} := \pi_\varphi(C^*_r(\Gamma))'' \subset \tilde{B}$. Note that $\tilde{\varphi}$ is a normal trace on $\tilde{C}$. Consider the central projection $p \in \tilde{C}$ that supports this trace, so that $\tilde{\varphi}$ is a faithful normal trace on $p\tilde{C}$. Then we see that the map $\sigma_0 : C^*_r(\Gamma) \to p\tilde{C}$ defined by $\sigma_0(x) = p\pi_\varphi(x)$ for all $x \in C^*_r(\Gamma)$ is a trace-preserving $*$-morphism. Hence it extends to a normal $*$-isomorphism $\sigma : L\Gamma \to p\tilde{C} \subset p\tilde{B}p$.

(i) Take $x \in L\Gamma$ and a sequence $x_n \in C$ that converges strongly to x, then we have that $\lim_n \varphi(x_n T) = \tilde{\varphi}(\sigma(x)\pi_\varphi(T))$, which does not depend on the choice of the sequence $(x_n)_n$. Indeed, since $\tilde{\varphi}(p) = 1$, we see that for all n,

$$\varphi(x_n T) = \tilde{\varphi}(\pi_\varphi(x_n T)) = \tilde{\varphi}(p\pi_\varphi(x_n)\pi_\varphi(T)) = \tilde{\varphi}(\sigma(x_n)\pi_\varphi(T)).$$

So the desired convergence is a consequence of the normality of σ and $\tilde{\varphi}$.

(ii) From the formula obtained in (i), we see that

$$A_\varphi = \{x \in L\Gamma \mid \tilde{\varphi}(\sigma(x)\pi_\varphi(T)) = \tilde{\varphi}(\pi_\varphi(T)\sigma(x)), \forall T \in B\}.$$

Thus $\sigma(A_\varphi)$ is the intersection of $\sigma(L\Gamma) = p\tilde{C}$ with the centralizer of $\tilde{\varphi}$ in $p\tilde{B}p$. In particular $\sigma(A_\varphi)$ is a von Neumann algebra and so is A_φ.

(iii) Assume that the action is amenable. Then B is nuclear, and so $\tilde{B} = \pi_\varphi(B)''$ is injective. In particular $p\tilde{B}p$ is injective as well. Moreover, there is a $\tilde{\varphi}$-preserving conditional expectation $E : p\tilde{B}p \to \sigma(A_\varphi)$, because $\sigma(A_\varphi)$ centralizes $\tilde{\varphi}$. Hence A_φ is amenable. The existence of E follows from a standard argument that we include in the lemma below, for completeness. ☐

Lemma 5.3.2. *Consider a von Neumann algebra M with a state φ on it (not necessarily faithful). Take a von Neumann subalgebra $Q \subset M$ that centralizes φ and assume that φ is faithful and normal on Q (so that it is a faithful normal trace on Q). Then there is a φ-preserving conditional expectation $E : M \to Q$.*

Proof. Given $x \in M$, define a sesquilinear form $\mathcal{B}_x$ on $Q \times Q$ by the formula $\mathcal{B}_x(a, b) = \tilde{\varphi}(b^* x a)$.

Then the Cauchy-Schwarz inequality gives $|\mathcal{B}_x(a, b)| \leq \|x\|\|a\|_2\|b\|_2$. In particular $\mathcal{B}_x$ induces a sesquilinear form on $L^2(Q, \varphi) \times L^2(Q, \varphi)$ and there is a unique operator $T_x \in B(L^2(Q, \varphi))$ such that

$$\mathcal{B}_x(\xi, \eta) = \langle T_x(\xi), \eta\rangle, \forall \xi, \eta \in L^2(Q, \varphi) \qquad \text{and} \qquad \|T_x\| \leq \|x\|.$$

Now we check that $T_x \in Q$. Take $y \in Q$ and $a, b \in Q$. We have

$$\langle T_x(ay), b\rangle = \varphi(b^* x a y) = \varphi(y b^* x a) = \langle T_x(a), b y^*\rangle = \langle T_x(a) y, b\rangle.$$

Therefore T_x commutes with the right action of y. Since $y \in Q$ is arbitrary, we deduce that $T_x \in Q$. The desired conditional expectation is then defined by the formula $E(x) := T_x$, for all $x \in M$. ☐

Let us provide some applications of Proposition 5.3.1 to solidity and strong solidity for bi-exact groups [Oza04, OP10a, OP10b, CS13].

Definition 5.3.3 ([BO08], Section 15.2)**.** A discrete group Γ is *bi-exact* if there is a compactification X of Γ such that

(1) the left translation action of Γ on itself extends to a continuous action $\Gamma \curvearrowright X$ which is topologically amenable;

(2) the right translation action of Γ on itself extends continuously to an action on X which is trivial on the boundary $X \setminus \Gamma$.

For instance any hyperbolic group is bi-exact (because the Gromov compactification $\Delta\Gamma$ satisfies the above conditions).

Given a bi-exact group Γ, choose a compactification X as in Definition 5.3.3. Since it is a compactification, we have inclusions $c_0(\Gamma) \subset C(X) \subset \ell^\infty(\Gamma)$. Denote by λ and ρ respectively the left and right regular representations of Γ on $\ell^2\Gamma$, and define

$$B_\Gamma := C^*(C(X) \cup \lambda(\Gamma)) \subset B(\ell^2\Gamma).$$

By [BO08, Proposition 5.1.3], B_Γ is isomorphic to the reduced crossed product $C(X) \rtimes_r \Gamma$ by the left action of Γ. Moreover condition 5.3.3.(2) implies that B_Γ commutes with $C^*_\rho(\Gamma)$ modulo compact operators:

$$[B_\Gamma, C^*_\rho(\Gamma)] \subset C^*(\lambda(\Gamma) \cdot [C(X), \rho(\Gamma)]) \subset C^*(\lambda(\Gamma) \cdot c_0(\Gamma)) \subset K(\ell^2(\Gamma)). \tag{5.1}$$

We now show how solidity and strong solidity results can be deduced from Proposition 5.3.1.

Theorem 5.3.4 ([Oza04])**.** *If Γ is bi-exact, then $L\Gamma$ is solid, meaning that the relative commutant of any diffuse subalgebra of $L\Gamma$ is amenable.*

Proof. Consider a sequence of unitaries $(u_n) \subset \mathcal{U}(L\Gamma)$ which tends weakly to 0. We will show that the von Neumann algebra A of elements $x \in L\Gamma$ satisfying $\|[x, u_n]\|_2 \to 0$ is amenable.

Consider the state φ on $B(\ell^2\Gamma)$ defined by

$$\varphi(T) := \lim_{n\to\omega} \langle T\widehat{u}_n, \widehat{u}_n \rangle,$$

where ω is a free ultrafilter on $\mathbb{N}$. Note that $\varphi_{|L\Gamma} = \tau = \varphi_{|R\Gamma}$ and that φ vanishes on the compact operators because u_n tends weakly to 0. Applying Proposition 5.3.1, we get that $A_\varphi = \{x \in L\Gamma \mid \varphi(xT) = \varphi(Tx), \forall T \in B_\Gamma\}$ is an amenable von Neumann algebra. Let us show that $A \subset A_\varphi$.

Take $u \in \mathcal{U}(A)$. By definition of A, we have for any $T \in B(H)$

$$\varphi(Tu) = \lim_{n\to\omega} \langle T(uu_n), u_n \rangle = \lim_{n\to\omega} \langle T(u_n u), u_n \rangle = \varphi(TJu^*J).$$

Similarly, we have $\varphi(uT) = \varphi(Ju^*JT)$.

Fix a bounded sequence $(x_k) \subset C^*_\rho(\Gamma)$ which converges strongly to Ju^*J. Since $\varphi_{|R\Gamma}$ is normal, the Cauchy-Schwarz inequality implies that

- $\lim_k \varphi(Tx_k) = \varphi(TJu^*J)$ and
- $\lim_k \varphi(x_k T) = \varphi(Ju^*JT)$.

Now for each k, the operator $[T, x_k]$ is compact thanks to (5.1). Since φ vanishes on compact operators we get

$$\varphi(uT) = \varphi(Ju^*JT) = \lim_k \varphi(x_kT) = \lim_k \varphi(Tx_k) = \varphi(TJu^*J) = \varphi(Tu). \qquad \square$$

Theorem 5.3.5 ([OP10a, CS13])**.** *If Γ is bi-exact and weakly amenable (this is the case if Γ is hyperbolic, [Oza08]) then $L\Gamma$ is strongly solid, in the sense that the normalizer of a diffuse amenable subalgebra of $L\Gamma$ is amenable.*

Our proof still relies on the following weak compactness property due to Ozawa and Popa. The formulation is a combination of [OP10a, Theorem 3.5] and [Oza12, Theorem B] with the characterization of weak compactness given in [OP10a, Proposition 3.2(4)].

Theorem 5.3.6 ([OP10a, Oza12])**.** *Assume that Γ is weakly amenable. Then for any amenable subalgebra A of $L\Gamma$, there is a state φ on $B(\ell^2\Gamma)$ such that*

(i) $\varphi(xT) = \varphi(Tx)$ *for every* $T \in B(\ell^2\Gamma)$ *and* $x \in A$*;*

(ii) $\varphi(uJuJT) = \varphi(TuJuJ)$ *for every* $T \in B(\ell^2\Gamma)$ *and* $u \in \mathcal{N}_{L\Gamma}(A)$*;*

(iii) $\varphi(x) = \tau(x) = \varphi(Jx^*J)$ *for every* $x \in L\Gamma$*.*

The new part of the proof is the conclusion of strong solidity from the existence of such a state. It becomes extremely simple.

Proof of Theorem 5.3.5. Assume that A is a diffuse amenable subalgebra and consider a state φ on $B(\ell^2\Gamma)$ as in Theorem 5.3.6. By Proposition 5.3.1, it suffices to show that $\mathcal{N}_{L\Gamma}(A) \subset A_\varphi = \{x \in L\Gamma \mid \varphi(xT) = \varphi(Tx), \forall T \in B_\Gamma\}$.

First note that [OP10b, Lemma 3.3] implies that φ vanishes on compact operators because A is diffuse.

Take $u \in \mathcal{N}_{L\Gamma}(A)$ and $T \in B_\Gamma$. By definition of φ, we have $\varphi(uJuJTJu^*J) = \varphi(Tu)$.

Fix a bounded sequence $(x_k) \subset C^*_\rho(\Gamma)$ which converges strongly to Ju^*J. Since $\varphi_{|R\Gamma}$ is normal, the Cauchy-Schwarz inequality implies that

- $\lim_k \varphi(uJuJTx_k) = \varphi(uJuJTJu^*J)$ and
- $\lim_k \varphi(uJuJx_kT) = \varphi(uT)$.

Now for each k, the operator $uJuJ[T, x_k]$ is compact thanks to (5.1). Since φ vanishes on compact operators we get

$$\varphi(uT) = \lim_k \varphi(uJuJx_kT) = \lim_k \varphi(uJuJTx_k) = \varphi(uJuJTJu^*J) = \varphi(Tu). \qquad \square$$

Let us mention that one could also do a relative version of this strategy to prose relative strong solidity results. In particular, one could recover some of the results in [PV14a, PV14b]. Note that the proof given in [PV14b] also relies on bi-exactness explicitly.

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